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ON HOLOMORPHICALLY PROJECTIVE MAPPINGS OF EQUIDISTANT PARABOLIC KÄHLER SPACES

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Abstract. In this paper we construct holomorphically projective mappings of equidistant parabolic Kähler spaces. We discus fundamental equations of these mappings as well.

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1. Introduction

First we note the general dependence of holomorphically-projective mappings of parabolic Kähler manifolds in dependence on the smoothness class of the metric. We present well known facts, which were proved by M. Shiha, J. Mikeš *et al*, see [2, 3, 9, 14, 15, 19, 21–23].

The similar problems have been studied for holomorphically-projective mappings of Kähler spaces cite [4–7, 9–11, 14, 16, 18, 27].

Finally, we construct holomorphically-projective mappings of equidistant parabolic Kähler spaces. For equidistant Kähler spaces were those spaces constructed in [12, 13].

2. Parabolic Kähler Manifolds

In the following definition we introduce generalizations of Kähler manifolds [8], see [9,10,14]. A basis on this definition see monography by V. Vishnevskii, A. Shirokov and V. Shurigin [26].

Definition 1. An n-dimensional (pseudo-)Riemannian manifold (M, g) is called an *m-parabolic Kähler manifold* $K_n^{o(m)}$, if beside the metric tensor g, a tensor field F of a rank $m \geq 2$ of type $(1,1)$ is given on the manifold M_n, called a *structure* F, such that the following conditions hold

$$F^2 = 0, \quad g(X, FX) = 0, \qquad \nabla F = 0 \tag{1}$$

where X is an arbitrary vector of TM_n, and ∇ denotes the covariant derivative in $K_n^{o(m)}$.

We remind, that Kähler spaces, were characterized by conditions $F^2 = -\operatorname{Id}$, $g(X, FX) = 0$, $\nabla F = 0$, were first considered by Shirokov [25]. Independently they were studied by E. Kähler [8]. Hyperbolic Kähler space (also *para Kähler space*, see D. Alekseevsky [1]) characterized by $F^2 = \operatorname{Id}$, $g(X, FX) = 0$, $\nabla F = 0$, were considered by P. Rashevskij [9].

3. Holomorphically Projective Mappings Theory Between Parabolic Kähler Spaces

Assume that we have two parabolic Kähler manifolds $K_n^{o(m)} = (M, g, F)$ and $\bar{K}_n^{o(\bar{m})} = (\bar{M}, \bar{g}, \bar{F})$ with metrics g and $\bar{g}$, structures F and $\bar{F}$, Levi-Civita connections ∇ and $\bar{\nabla}$, respectively. Here $\bar{K}_n$, $\bar{K}_n \in C^1$, i.e. $g, \bar{g} \in C^1$ which means that their components $g_{ij}, \bar{g}_{ij} \in C^1$. Likewise, as in [19,21] we introduce the following notations, this is an analogy by [16], see [10, p. 240].

Definition 2. A curve ℓ in K_n which is given by the equation $\ell = \ell(t)$, $\lambda = \mathrm{d}\ell/\mathrm{d}t \ (\neq 0), t \in I$, where t is a parameter is called *analytical planar*, if under the parallel translation along the curve, the tangent vector λ belongs to the two-dimensional distribution $D = \operatorname{Span}\{\lambda, F\lambda\}$ generated by λ and its conjugate $F\lambda$, that is, it satisfies

$$\nabla_t \lambda = a(t)\lambda + b(\lambda)F\lambda$$

where $a(t)$ and $b(t)$ are some functions of the parameter t. Particularly, in the case $b(t) = 0$, an analytical planar curve is a geodesic.

On an analytical planar curve, it is possible to locally find parameter t, for which $a(t) \equiv 0$. It is clear to see too, that vector λ and $F\lambda$ are orthogonal in $K_n^{o(m)}$.

Definition 3. A diffeomorphism f: $K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$ is called a *holomorphically-projective mapping* of $K_n^{o(m)}$ onto $\bar{K}_n^{o(\bar{m})}$ if f maps any analytical planar curve in $K_n^{o(m)}$ onto an analytical planar curve in $\bar{K}_n^{o(\bar{m})}$.

Assume that we have a holomorphically-projective mapping f: $K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$. Since f is a diffeomorphism, we can suppose local coordinate charts on M or $\bar{M}$, respectively, such that locally, $f : K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$ maps points onto points with the same coordinates, and $\bar{M} = M$. A manifold $K_n^{o(m)}$ admits a holomorphically-projective mapping onto $\bar{K}_n^{o(\bar{m})}$ if and only if the following equations [19, 21]

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X + \varphi(X)FY + \varphi(Y)FX \tag{2}$$

hold for any tangent fields X, Y and where ψ is a gradient-like form and $\psi(X) = \varphi(FX)$. If $\varphi \equiv 0$ than f is *affine* or *trivially holomorphically-projective* . Moreover, structures F and $\bar{F}$ are preserved, i.e. $\bar{F} = F$, and $\bar{m} = m$. This fact implies from the theory of F-planar mappings, see [10, pp. 219-220]. In local form

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i\delta^h_j + \psi_j\delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i, \qquad \psi_i = \varphi_j F^j_i$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are the Christoffel symbols of K_n and $\bar{K}_n$, ψ_i, F^h_i are components of ψ, F and δ^h_i is the Kronecker delta

$$\psi_i = \frac{\partial \Psi}{\partial x^i}, \qquad \Psi = \frac{1}{2(n+2)} \ln\left|\frac{\det \bar{g}}{\det g}\right|.$$

Here and in the following we will use the conjugation operation of indices in the way

$$A_{\dots\,\bar{i}\,\dots} = A_{\dots\,k\,\dots} F^k_i.$$

Equations (2) are equivalent to the following equations

$$\begin{aligned}\nabla_Z \bar{g}(X,Y) = 2\psi(Z)\bar{g}(X,Y) + \psi(X)\bar{g}(Y,Z) + \psi(Y)\bar{g}(X,Z) \\ -\varphi(F)\bar{g}(Y,FZ) - \varphi(F)\bar{g}(FX,Z).\end{aligned} \tag{3}$$

In local form

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi \bar{g}_{ik} - \varphi_i \bar{g}_{\bar{j}k} - \varphi_j \bar{g}_{\bar{i}k}$$

where " , " denotes the covariant derivative on $K_n^{o(m)}$. In the local coordinate system $\psi_i \equiv \varphi_{\bar{i}}$ holds.

M. Shiha [19, 21] proved that equations (2) and (3) are equivalent to

$$\begin{aligned}\nabla_Z a(X,Y) = {} & \lambda(X)g(Y,Z) + \lambda(Y)g(X,Z) \\ & +\theta(X)g(Y,FZ) + \theta(Y)g(X,FZ).\end{aligned} \tag{4}$$

In local form

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} - \theta_i g_{\bar{j}k} - \theta_j g_{\bar{i}k}$$

where

$$a_{ij} = \mathrm{e}^{2\Psi}\bar{g}^{\alpha\beta}g_{\alpha i}g_{\beta j}, \quad \lambda_i = -\mathrm{e}^{2\Psi}\bar{g}^{\alpha\beta}g_{\beta i}\psi_\alpha, \quad \theta_i = -\mathrm{e}^{2\Psi}\bar{g}^{\alpha\beta}g_{\beta i}\varphi_\alpha. \tag{5}$$

From (4) follows that λ_i is gradient-like vector and it holds

$$\lambda_i = \partial_i\Lambda, \qquad \Lambda = 1/4 a_{\alpha\beta}g^{\alpha\beta}. \tag{6}$$

Moreover, from the condition $\psi_i \equiv \varphi_{\bar{i}}$ follows, that

$$\lambda_i = \theta_{\bar{i}}.$$

On the other hand [10]

$$\bar{g}_{ij} = \mathrm{e}^{2\Psi}\tilde{g}_{ij}, \qquad \Psi = \frac{1}{2}\ln\left|\frac{\det\tilde{g}}{\det g}\right|, \qquad \|\tilde{g}_{ij}\| = \|g^{i\alpha}g^{j\beta}a_{\alpha\beta}\|^{-1}. \tag{7}$$

The above formulas (4) with a regular tensor a are the criterion for holomorphically-projective mappings $K_n^{o(m)} \to \bar{K}_n^{o(m)}$, globally as well as locally.

M. Shiha [19, 21] proved the following theorem

Theorem 4. *A diffeomorphism $f : K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$ is a holomorphically-projective mapping if and only if there exist a solution of the following linear Cauchy-like system*

$$\begin{aligned} a)\ \ a_{ij,k} &= \lambda_i g_{jk} + \lambda_j g_{ik} + \theta_i g_{j\bar{k}} + \theta_j g_{i\bar{k}} \\ b)\ \ \theta_{i,j} &= \tau g_{i\bar{j}} + a_{\alpha\beta}M^{\alpha\beta}_{1|ij} \\ c)\ \ \tau_{,i} &= \theta_\alpha M^{\alpha}_{2|i} + a_{\alpha\beta}M^{\alpha\beta}_{3|i} \end{aligned} \tag{8}$$

for the unknown tensor a_{ij} ($a_{ij} = a_{ji},\ a_{\bar{i}\bar{j}} + a_{i\bar{j}} = 0,\ \det a_{ij} \neq 0$), a vector λ_i, and a function τ. Here $M^{\alpha\beta}_{1|ij},\ M^{\alpha}_{2|i},\ M^{\alpha\beta}_{3|i}$ are tensors determined from the metric and the structure tensors g_{ij} and F_i^h of the space $K_n^{o(m)}$.

Remark 5. This theorem was proved with assuming that $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$ belong to C^3 class. Assuming that $K_n^{o(m)}$ and $\bar{K}_n^{o(\bar{m})}$ belong to C^2 class, formula (8a) and (8b) hold.

Remark 6. We will prove in [17], that the Theorem 4 valides too if $K_n^{o(m)} \in C^r$, $r \geq 3$, and $\bar{K}_n^{o(\bar{m})} \in C^2$. Then it is true that also $\bar{K}_n^{o(\bar{m})} \in C^r$.

The system (8) has at most one solution for the initial values in a point x_0: $a_{ij}(x_0)$, $\lambda_i(x_0)$ and $\tau(x_0)$. Hence, the general solution of this system depends on no more than $(n+2)(n+1)/2 - m(n-m+1)$ essential parameters.

The integrability of conditions (8) and their differential prolongations are linear algebraic equations on the components of the unknown tensors $a_{ij},\ \lambda_{ij}$ and τ with coefficients from $K_n^{o(m)}$.

4. Homomorphically-Projective Mapping of Equidistant Parabolic Kähler Spaces

It is well-known, see [20], that the (pseudo-)Riemannian space is called *equidistant* if there exists a vector field ξ^h, for which

$$\xi^h_{,i} = \rho\, \delta^h_i \tag{9}$$

where ρ is a function.

K. Yano called such vector field *concircular* [28].

The equidistant parabolic Kählerian spaces were studied by Shiha and Mikeš [24]. In their work, the metrics of those spaces were found if $\rho \neq 0$. Moreover, it has been proven that ρ is a constant.

We have to mention that in 1956 Sinyukov [20] proved that equidistant spaces with $\rho \neq 0$ admit geodesic mappings. From this elementary follows that the above mentioned equidistant parabolic Kähler spaces admit geodesic mappings onto (pseudo-) Riemannian spaces $\bar{V}_n$. In the general case those spaces $\bar{V}_n$ are not Kähler.

We proved, that the following theorem holds.

Theorem 7. *The equidistant parabolic Kähler spaces for which $\rho \neq 0$ admit non-trivial holomorphically-projective mapping.*

Proof: Let the $K_n^{o(m)}$ be an equidistant parabolic Kähler space for which there exists the equidistant vector field ξ^h defined by formula (9) for which $\rho \neq 0$. By the following formula, we construct regular symmetric tensor field a_{ij}

$$a_{ij} = c_1 \cdot g_{ij} + c_2 \cdot \xi_{\bar{i}}\, \xi_{\bar{j}} \tag{10}$$

where c_1 and c_2 are constant for which $\det\|a_{ij}\| \neq 0$.

We will convince that for this tensor fields a_{ij} the following formula

$$a_{\bar{i}j} + a_{i\bar{j}} = 0$$

holds and for fundamental equations (4) of holomorphically-projective mapping of parabolic Kähler spaces as well.

Indeed, deriving the equation (10) along x^k and using formula (9), we have

$$a_{ij,k} = c_1\, \rho\, \xi_{\bar{i}}\, g_{\bar{j}k} + c_2\, \rho\, \xi_{\bar{j}}\, g_{\bar{i}k}. \tag{11}$$

Putting $\varphi_i = -c_2\, \rho\, \xi_{\bar{i}} \neq 0$ then we have $\lambda_i \equiv \varphi_{\bar{i}} = 0$. Thus the equation (11) have the form of fundamental equations (4) and the theorem is proved. ■

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References

[1] Alekseevsky D., *Pseudo-Kähler and Para-Kähler Symmetric Spaces*, In: Handbook of Pseudo-Riemannian Geometry and Supersymmetry, EMS, Zürich 2010, pp 703–729.

[2] Chudá H., Chodorová M. and Shiha M., *On Composition of Conformal and Holomorphically Projective Mappings Between Conformally Kählerian Spaces*, J. Appl. Math. Bratislava **5** (2012) 91–96.

[3] Chudá H. and Shiha M., *Conformal Holomorphically Projective Mappings Satisfying a Certain Initial Condition*, Miskolc Math. Notes **14** (2013) 569–574.

[4] Domashev V. and Mikeš J., *Theory of Holomorphically Projective Mappings of Kählerian Spaces*, Math. Notes **23** (1978) 160–163.

[5] Hinterleitner I., *On Holomorphically Projective Mappings of e-Kähler Manifolds*, Arch. Math. Brno **48** (2012) 333–338.

[6] Hinterleitner I. and Mikeš J., *Geodesic Mappings of (Pseudo-)Riemannian Manifolds Preserve Class of Differentiability*, Miskolc Math. Notes **14** (2013) 575–582.

[7] Hinterleitner I. and Mikeš J., *Geodesic Mappings and Einstein Spaces*, In: Geometric Methods in Physics, Birkhäuser, Basel 2013, pp. 331–335.

[8] Kähler E., *Über Eine Bemerkenswerte Hermitesche Metrik*, Sem. Hamburg. Univ. **9** (1933) 173–786.

[9] Mikeš J. *et al*, *Special Mappings in Differential Geometry*, Palacky Univ. Press, Olomouc 2015.

[10] Mikeš J., Vanžurová A. and Hinterleitner I., *Geodesic Mappings and Some Generalizations*, Palacky Univ. Press, Olomouc 2009.

[11] Mikeš J., *On Holomorphically Projective Mappings of Kählerian Spaces*, Ukr. Geom. Sb. **23** (1980) 90–98.

[12] Mikeš J. *Equidistant Kähler Spaces*, Math. Notes **38** (1985) 855–858 .

[13] Mikeš J. *On Sasaki Spaces and Equidistant Kähler Spaces*, Sov. Math. Dokl. **34** (1987) 428–431.

[14] Mikeš J., *Holomorphically Projective Mappings and Their Generalizations*, J. Math. Sci. (New York) **89** (1998) 1334–1353.

[15] Mikeš J., Shiha M. and Vanžurová A., *Invariant Objects by Holomorhpically Projective Mappings of Parabolically Kähler Spaces*, J. Appl. Math. **2** (2009) 135–141.

[16] Otsuki T. and Tashiro Y., *On Curves in Kaehlerian Spaces*, Math. J. Okayama Univ. **4** (1954) 57–78.

[17] Peška P., Mikeš J., Chudá H. and Shiha M., *On Holomorphically Projective Mappings of Parabolic Kahler Manifolds*, Miskolc Math. Notes **17** (2016) 1011–1019.

[18] Prvanović M., *Holomorphically Projective Transformations in a Locally Product Space*, Math. Balk. **1** (1971) 195–213.

[19] Shiha M., *Geodesic and Holomorphically Projective Mappings of Parabolically Kählerian Spaces*, PhD Thesis, Moscow Ped. Inst., Moskow 1992.

[20] Sinyukov, S., *On Equidistant Spaces*, Vestn. Odessk. Univ. (1957) 133–135.

[21] Shiha, M., *On the Theory of Holomorphically-Projective Mappings of Parabolically-Kählerian Spaces*, Math. Publ. Silesian Univ. Opava **1** (1993) 157–160.

[22] Shiha M., Juklová L. and Mikeš J., *Holomorphically Projective Mappings Onto Riemannian Tangent-Product Spaces*, J. Appl. Math. Bratislava **5** (2012) 259–266.

[23] Shiha M. and Mikeš J., *On Holomorphically Projective Flat Parabolically-Kählerian Spaces*, In: Proc. Conf. Contemp. Geom. and Related Topics, Univ. Belgrade, Faculty of Mathematics, Belgrade 2006, pp. 467–474.

[24] Shiha M. and Mikeš J., *On Equidistant Parabolically Kählerian Spaces*, Tr. Geom. Semin. **22** (1994) 97–107.

[25] Shirokov P., *Selected Investigations on Geometry*, Kazan' Univ. Press, Kazan 1966.

[26] Vishnevskij V., Shirokov A. and Shurygin V., *Spaces Over Algebras*, Izd. Kazansk. Univ., Kazan 1985.

[27] Yano K., *Differential Geometry on Complex and Almost Complex Spaces*, Pergamon Press, Oxford 1965.

[28] Yano K., *Concircular Geometry*, Proc. Imp. Acad. Tokyo, **16** (1940) 195–200, 354–360, 442–448, 505–511.