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ON THE TRAJECTORIES OF $\mathrm{U}(1)$-KEPLER PROBLEMS

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Abstract. The classical $\mathrm{U}(1)$-Kepler problems at level $n \geq 2$ were formulated, and their trajectories are determined via an idea similar to the one used by Kustaanheimo and Stiefel in the study of Kepler problem. It is found that a non-colliding trajectory is an ellipse, a parabola or a branch of hyperbola according as the total energy is negative, zero or positive, and the complex orientation-preserving linear automorphism group of $\mathbb{C}^n$ acts transitively on both the set of elliptic trajectories and the set of parabolic trajectories.

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1. Introduction

The quantum $\mathrm{U}(1)$-Kepler problems, which are higher dimensional generalizations of the MICZ-Kepler problems [9, 16], have been introduced and studied [10] for quiet a while. Their intimate connection with representation theory [1], especially local theta-correspondence [3], has been demonstrated in [10] as well. However, their corresponding classical models, though not difficult to be formulated, seem to be difficult to solve, that is why there is a significant delay of the current work. The clue to solve these classical models finally came after a closer examination of [4, 7, 8] and [12–15].

To formulate these classical models, we start with the euclidean Jordan algebra $\mathrm{H}_n(\mathbb{C})$ of complex hermitian matrices of order n. (Euclidean Jordan algebras were initially introduced by Jordan [5], and were subsequently classified by Jordan, von Neuman and Wigner [6]. A good reference for euclidean Jordan algebras is [2].) Next, we introduce the space $\mathcal{C}_1$ of rank one semi-positive elements in $\mathrm{H}_n(\mathbb{C})$. Thirdly, we observe that there are two canonical structures on the space $\mathcal{C}_1$:

I. The **Kepler metric**

$$\mathrm{d}s_K^2 = \frac{\operatorname{tr} x}{n^2}\operatorname{tr}(\mathrm{d}x\ \bar{L}_x^{-1}(\mathrm{d}x)) \tag{1}$$

where tr is the matrix trace, d is the exterior differential operator, and $\bar{L}_x^{-1}$ denotes the inverse of the linear automorphism of the tangent space $T_x\mathcal{C}_1$ induced from the Jordan multiplication by x. (A detailed description of $\bar{L}_x^{-1}$, valid for any simple euclidean Jordan algebra, is given in the first paragraph of [11]. For more details on Kepler metric, one can consult [12, 14])

II. The **Kepler form**

$$\omega_K := -\mathrm{i}\frac{\operatorname{tr}(x\ \mathrm{d}x \wedge \mathrm{d}x)}{(\operatorname{tr} x)^3}\cdot \tag{2}$$

Here, the multiplication of matrices is the ordinary matrix multiplication and "i" is the imaginary unit.

Finally, for each real number μ, we introduce the symplectic form

$$\omega_\mu := \omega_{\mathcal{C}_1} + 2\mu\,\pi^*\omega_K$$

on $T^*\mathcal{C}_1$. Here, $\omega_{\mathcal{C}_1}$ is the canonical symplectic form on $T^*\mathcal{C}_1$, $\pi^*\omega_K$ is the pull-back of ω_K under the cotangent bundle projection map

$$\pi:\quad T^*\mathcal{C}_1 \longrightarrow \mathcal{C}_1.$$

The symplectic manifold $(T^*\mathcal{C}_1, \omega_\mu)$ will serve as the phase space of the $\mathrm{U}(1)$-Kepler problem with magnetic charge μ, and is denoted by M^μ hereafter.

Definition 1. *Let $n \geq 2$ be an integer and μ be a real number. The classical* $\mathrm{U}(1)$*-Kepler problem at level n with magnetic charge μ is the Hamiltonian system for which the phase space is M^μ and the Hamiltonian is*

$$H^\mu = \frac{1}{2}||P||^2 + \frac{n^2\mu^2}{2(\operatorname{tr} x)^2} - \frac{n}{\operatorname{tr} x} \tag{3}$$

where $||P||$ denotes the length of the cotangent vector P, measured in terms of the Kepler metric on $\mathcal{C}_1$, and $x = \pi(P)$.

Remark 2. *In the quantization of this model, $\frac{\omega_\mu}{2\pi}$ is required to represent a degree two integral cohomology class of $T\mathcal{C}_1$ (homotopy equivalent to $\mathbb{CP}^{n-1}$). Then μ must be a half of an integer.*

Note that, a trajectory is the path traced by a motion, so it is oriented by the velocity of the motion. By analyzing the trajectories of $\mathrm{U}(1)$-Kepler problems we shall show that a trajectory of the $\mathrm{U}(1)$-Kepler problem at level n with magnetic charge

μ is always the intersection of $\mathcal{C}_1$ with a real plane inside $\mathrm{H}_n(\mathbb{C})$, consequently, since

$$\mathcal{C}_1 = \{x \in \mathrm{H}_n(\mathbb{C})\,;\, x^2 = \mathrm{tr}\, x\, x,\ \mathrm{tr}\, x > 0\}$$

a trajectory is a quadratic curve. In fact, it will be shown that a non-colliding trajectory is an ellipse, a parabola or a branch of a hyperbola according as the total energy E is negative, zero or positive, moreover, the group $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(1)$ – the quotient group of $\mathrm{GL}(n, \mathbb{C})$ by the image of the diagonal imbedding of $\mathrm{U}(1)$ into $\underbrace{\mathrm{U}(1) \times \cdots \times \mathrm{U}(1)}_{n}$ – acts transitively on both the set of elliptic trajectories and the set of parabolic trajectories of the $\mathrm{U}(1)$-Kepler problems at level n.

Remark 3. *The* $\mathrm{U}(1)$*-Kepler problem at level two with magnetic charge* 0 *is just the Kepler problem. The group* $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(1)$ *is the complex-orientation preserving linear automorphism group of* $\mathbb{C}^n$*.*

1.1. Notations

If w is a complex number, then $\mathrm{Re}\, w$ and $\mathrm{Im}\, w$ denote the real and imaginary part of w respectively. We use $\bar{w}$ to denote the complex conjugate of w and $|w$ to denote the length of w. For example, if $w = 3 - 4\mathrm{i}$, then $\mathrm{Re}\, w = 3$, $\mathrm{Im}\, w = -4$, $\bar{w} = 3 + 4\mathrm{i}$, and $|w| = 5$. Note that, if z and w are two complex numbers, then $\overline{zw} = \bar{z}\bar{w}$. Now if

$$z = (z_1, \ldots, z_n)^t, \qquad w = (w_1, \ldots, w_n)^t$$

where each z_i and each w_i is a complex number, then

$$z \cdot w := z_1 w_1 + \cdots + z_n w_n, \qquad |z|^2 := z \cdot \bar{z} = \sum_i |z_i|^2.$$

2. A Local Description of the Model

In this section we shall count the row number and column number of a matrix from zero, so the top row of an $n \times n$-matrix x is $[x_{00}, x_{01}, \ldots]$. Note that, if x is semi-positive, then $x_{ii} \geq 0$ for all $0 \leq i < n$. For each $0 \leq i < n$, we introduce the dense open set

$$U_i := \{x \in \mathcal{C}_1\,;\, x_{ii} > 0\}.$$

It is clear that U_i's form an open cover for $\mathcal{C}_1$.

We shall work out a local description for the model on each U_i. In fact, due to symmetry, it suffices to do it on U_0. For $x \in U_0$, we introduce coordinate $(r, z^1, \ldots, z^{n-1})$

$$z^i := \frac{x_{i0}}{x_{00}}, \qquad r := \frac{x_{00}}{n}(1 + |z|^2).$$

Since $x \in \mathcal{C}_1$, we have $\mathrm{tr}\, x\, x = x^2$, so $\mathrm{tr}\, x\, x_{00} = \sum_i x_{0i} x_{i0} = \sum_i |x_{0i}|^2$, then

$$r = \frac{\mathrm{tr}\, x}{n}\cdot$$

In terms of this coordinate, the Kepler form can be written as

$$\omega_K = \mathrm{i}\left(\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{1+|z|^2} - \frac{(\bar{z}\cdot \mathrm{d}z)\wedge(z\cdot \mathrm{d}\bar{z})}{(1+|z|^2)^2}\right)$$

and the Kepler metric can be written as

$$\mathrm{d}s_K^2 = \mathrm{d}r^2 + 4r^2\frac{(1+|z|^2)|\mathrm{d}z|^2 - |\bar{z}\cdot \mathrm{d}z|^2}{(1+|z|^2)^2}\cdot \tag{4}$$

The key step in verifying (4) is to verify the identity

$$\mathrm{tr}\left(\mathrm{d}x\, \bar{L}_x^{-1}(\mathrm{d}x)\right) = 4\left(|\mathrm{d}Z|^2 - \left(\frac{\mathrm{Im}(\bar{Z}\cdot \mathrm{d}Z)}{|Z|}\right)^2\right) \tag{5}$$

for $x = ZZ^\dagger$. The proof of equation (5) is omitted here because it is very similar to the detailed proof of identity (2.3) in [11] for $\mathrm{H}_n(\mathbb{R})$.

The coordinate functions r, z^i, $\bar{z}^i$ on U_0 induce the coordinate functions r, z^i, $\bar{z}^i$, P_r, P_{z^i}, $P_{\bar{z}^i}$ on T^*U_0. One can check that $\omega_K = \mathrm{d}A$ with

$$A = \frac{\mathrm{Im}(\bar{z}\cdot \mathrm{d}z)}{1+|z|^2} =: A_z \cdot \mathrm{d}z + A_{\bar{z}}\cdot \mathrm{d}\bar{z} \tag{6}$$

consequently, on $T^*\mathcal{C}_1|_{U_0} = T^*U_0$, we have

$$\omega_\mu = \mathrm{d}p_r \wedge \mathrm{d}r + p_z \cdot \mathrm{d}z + p_{\bar{z}}\cdot \mathrm{d}\bar{z}$$

where $p_r = P_r$, $p_z = P_z + 2\mu A_z$ and $p_{\bar{z}} = P_{\bar{z}} + 2\mu A_{\bar{z}} = \overline{p}_z$. Therefore, on $T^*\mathcal{C}_1|_{U_0}$, the only nontrivial basic Poisson relations are

$$\{r, p_r\} = 1 \quad \text{and} \quad \{z^i, p_{z^i}\} = \{\bar{z}^i, p_{\bar{z}^i}\} = 1 \quad \text{for each } 1 \le i < n. \tag{7}$$

In physics, $p := p_r\, \mathrm{d}r + p_z \cdot \mathrm{d}z + p_{\bar{z}}\cdot \mathrm{d}\bar{z}$ is called the *canonical momentum* because of above canonical Poisson relations.

Proposition 4. *On $T^*\mathcal{C}_1|_{U_0} = T^*U_0$, the Hamiltonian* (3) *can be written as*

$$H_\mu = \frac{1}{2}p_r^2 + \frac{(1+|z|^2)}{2r^2}\left(|p_{\bar{z}} - 2\mu A_{\bar{z}}|^2 + |\bar{z}\cdot(p_{\bar{z}} - 2\mu A_{\bar{z}})|^2\right) + \frac{\mu^2}{2r^2} - \frac{1}{r}\cdot \tag{8}$$

Proof: From equation (4) we know that the nontrivial metric tensor components are

$$g_{rr} = 1, \qquad g_{\bar{z}^i z^j} = 2r^2\frac{(1+|z|^2)\delta_{ij} - z^i\bar{z}^j}{(1+|z|^2)^2} = g_{z^j\bar{z}^i}.$$

Since $r = \dfrac{\operatorname{tr} x}{n}$, all we need to verify is that the nontrivial tensor components for the inverse of the metric are

$$g^{rr} = 1, \qquad g^{z^i \bar{z}^j} = \frac{1+|z|^2}{2r^2}(\delta_{ij} + z^i \bar{z}^j) = g^{\bar{z}^j z^i}.$$

But this can be easily verified.

■

3. Conformal Kepler Problems

To continue the discussion, we need to introduce also Iwai's [4] conformal Kepler problem.

Definition 5. *The n-th complex conformal Kepler problem is a dynamic problem with configuration space $\mathbb{C}^n_*$ and Lagrangian*

$$L = 2|Z|^2|\dot{Z}|^2 + \frac{1}{|Z|^2} \tag{9}$$

where $|Z|^2 = Z \cdot \bar{Z}$ and $|\dot{Z}|^2 = \dot{Z} \cdot \dot{\bar{Z}}$.

Since the Lagrangian in equation (9) is clearly invariant under the $\mathrm{U}(1)$ action on $\mathbb{C}^n_*$, via Noether's theorem, the $\mathrm{i}\mathbb{R}$-valued

$$\mathscr{M} := |Z|^2(\bar{Z} \cdot \dot{Z} - Z \cdot \dot{\bar{Z}}) \tag{10}$$

on $T\mathbb{C}^n_*$ must be a constant of motion. As we shall see in the proof of Proposition 7 that $\mathrm{Im}\mathscr{M}$ can be identified with the magnetic charge, so $\mathscr{M}$ is refereed to as the magnetic momentum. The total energy is

$$E = 2|Z|^2|\dot{Z}|^2 - \frac{1}{|Z|^2} \tag{11}$$

and the equation of motion is

$$\left(|Z|^2 \frac{\mathrm{d}}{\mathrm{d}t}\right)^2 Z = \frac{E}{2} Z. \tag{12}$$

The following proposition from [11], is adapted for this article.

Proposition 6. 1) *If $E < 0$, then the solution to equation* (12) *is*

$$Z(t) = \cos\tau\, u + \sin\tau\, v \tag{13}$$

for some $u \in \mathbb{C}^n_*$ and $v \in \mathbb{C}^n$. Here τ is an increasing function of t implicitly defined via equation

$$t = \sqrt{2(|u|^2+|v|^2)}\left(\frac{|u|^2+|v|^2}{2}\tau + \frac{|u|^2-|v|^2}{4}\sin(2\tau) + \frac{\mathrm{Re}(\bar u\cdot v)}{2}\left(1-\cos(2\tau)\right)\right).$$

Moreover, for this solution we have

$$\mathscr{M} = \mathrm{i}\sqrt{\frac{2}{|u|^2+|v|^2}}\mathrm{Im}(\bar u\cdot v), \qquad E = -\frac{1}{|u|^2+|v|^2}.$$

2) *If $E = 0$, then the solution to equation* (12) *is*

$$Z(t) = u + \tau v \tag{14}$$

for some $u \in \mathbb{C}^n_*$ and $v \in \mathbb{C}^n$ with $|v|^2 = \frac{1}{2}$. Here τ is an increasing function of t implicitly defined via equation

$$t = |u|^2\tau + \mathrm{Re}(\bar u\cdot v)\,\tau^2 + \frac{1}{6}\tau^3.$$

Moreover, for this solution we have

$$\mathscr{M} = 2\mathrm{i}\,\mathrm{Im}(\bar u\cdot v).$$

3) *If $E > 0$, then the solution to equation* (12) *is*

$$Z(t) = \cosh\tau\, u + \sinh\tau\, v \tag{15}$$

for some $u \in \mathbb{C}^n_*$ and $v \in \mathbb{C}^n$ with $|v|^2 > |u|^2$. Here τ is an increasing function of t implicitly defined via equation

$$t = \sqrt{2(|v|^2-|u|^2)}\left(\frac{|u|^2-|v|^2}{2}\tau + \frac{|u|^2+|v|^2}{4}\sinh(2\tau) + \frac{\mathrm{Re}(\bar u\cdot v)}{2}\left(\cosh(2\tau)-1\right)\right).$$

Moreover, for this solution we have

$$\mathscr{M} = \mathrm{i}\sqrt{\frac{2}{|v|^2-|u|^2}}\mathrm{Im}(\bar u\cdot v), \qquad E = \frac{1}{|v|^2-|u|^2}.$$

4. Solving Equation of Motion for $\mathrm{U}(1)$-Kepler Problems

The equation of motion for the Kepler problem was ingeniously solved by Kustaanheimo and Stiefel in [7] in which the nonlinear equation of motion was transformed into a linear ordinary differential equation (ODE). This transformation, referred to as the KS *transformation* in literatures, is based on the quadratic map from $\mathbb{C}^2 \to \mathbb{R}^3$: $z \mapsto z^\dagger \vec{\sigma} z$, where $\vec{\sigma} = \sigma_1 \vec{i} + \sigma_2 \vec{j} + \sigma_3 \vec{k}$ with σ_i being the Pauli matrices.

We shall use a similar idea to solve the equation of motion for $\mathrm{U}(1)$-Kepler problems. The similar transformation that we shall use, which turns the equation of motion into a linear ODE, is based on the following quadratic map

$$q : \mathbb{C}^n \to \mathrm{H}_n(\mathbb{C}), \qquad Z \mapsto nZZ^\dagger \tag{16}$$

where $Z^\dagger$ is the complex hermitian conjugate of the column vector Z and $ZZ^\dagger$ is the matrix multiplication of Z with $Z^\dagger$. Map q, when restricted to $\mathbb{C}^n_* := \mathbb{C}^n \setminus \{\mathbf{0}\}$, becomes a principal $\mathrm{U}(1)$-bundle over $\mathcal{C}_1$

$$\bar{q} : \quad \mathbb{C}^n_* \longrightarrow \mathcal{C}_1. \tag{17}$$

One can check that the $\mathrm{i}\mathbb{R}$-valued differential one-form

$$\Theta = \frac{\bar{Z} \cdot \mathrm{d}Z - Z \cdot \mathrm{d}\bar{Z}}{2|Z|^2} \tag{18}$$

on $\mathbb{C}^n_*$ is a connection form on this principal bundle, and the curvature form

$$\mathrm{d}\Theta = \frac{\mathrm{d}\bar{Z} \wedge \mathrm{d}Z}{|Z|^2} - \frac{(Z \cdot \mathrm{d}\bar{Z}) \wedge (\bar{Z} \cdot \mathrm{d}Z)}{|Z|^4}$$

on $\mathbb{C}^n_*$ is the pullback of ω_K in (2) under the bundle projection map (17).

Proposition 7. 1) *Let $Z(t)$ be a solution to equation* (12) *with magnetic momentum $\mathscr{M}$. Then $q(Z(t))$ is a solution to the equation of motion of the* $\mathrm{U}(1)$ *Kepler problem at level n with magnetic charge $-\mathrm{i}\mathscr{M}$.*

2) *Any solution to the equation of motion of the* $\mathrm{U}(1)$ *Kepler problem at level n with magnetic charge μ is of the form $q(Z(t))$ for some solution $Z(t)$ to equation* (12) *with magnetic momentum $\mathrm{i}\mu$.*

Proof: For each $0 \le i < n$, take U_i to be the i-th dense open sets of $\mathcal{C}_1$ introduced in Section 2, and let $\tilde{U}_i$ be the inverse image of U_i under the map $\bar{q}$ in equation (17). Then the $\tilde{U}_i$'s form an open cover for $\mathbb{C}^n_*$. Let

$$\bar{q}_i := \bar{q}|_{\tilde{U}_i} : \tilde{U}_i \to U_i.$$

1) Assume that $Z(t)$ is a solution to equation (12) with magnetic momentum $\mathscr{M}$. To verify that $q(Z(t))$ is a solution to the $\mathrm{U}(1)$ Kepler problem at level n with magnetic charge $-\mathrm{i}\mathscr{M}$, we just need to do it on each U_i. Due to symmetry, we just

need to do it on U_0 only. For $Z := (Z_0, Z_1, \ldots)^t \in \tilde{U}_0$, we introduce coordinate $(g, r, z^1, \ldots, z^{n-1})$

$$g = \mathrm{e}^{\mathrm{i}\frac{\alpha}{2}} := \frac{Z_0}{|Z_0|}, \qquad r := |Z|^2, \qquad z^i := \frac{Z_i}{Z_0}.$$

One can check that, under the map $\bar{q}_0$, a point in $\tilde{U}_0$ with coordinates $(g, r, z^1, \ldots, z^{n-1})$ is mapped into a point in U_0 with coordinate $(r, z^1, \ldots, z^{n-1})$. Moreover, in terms of coordinates $(\alpha, r, z^1, \ldots, z^{n-1})$, Lagrangian (9) can be written as

$$L = \frac{1}{2}\dot{r}^2 + 2r^2 \frac{(1+|z|^2)|\dot{z}|^2 - |\bar{z}\cdot\dot{z}|^2}{(1+|z|^2)^2} + 2r^2\left(\frac{\dot{\alpha}}{2} + \frac{\mathrm{Im}(\bar{z}\cdot\dot{z})}{1+|z|^2}\right)^2 + \frac{1}{r}$$

so the conjugate momentums are

$$p_\alpha = 2r^2\left(\frac{\dot{\alpha}}{2} + \frac{\mathrm{Im}(\bar{z}\cdot\dot{z})}{1+|z|^2}\right) = -\mathrm{i}\mathscr{M}, \qquad p_r = \dot{r}$$
$$p_{\bar{z}} = 2r^2\frac{(1+|z|^2)\dot{z} - (\bar{z}\cdot\dot{z})z}{(1+|z|^2)^2} + 2p_\alpha A_{\bar{z}}.$$

Then the Hamiltonian, obtained from the Legendre transform of L, can be written as

$$H = \frac{1}{2}p_r^2 + \frac{(1+|z|^2)}{2r^2}(|P_{\bar{z}}|^2 + |\bar{z}\cdot P_{\bar{z}}|^2) + \frac{p_\alpha^2}{2r^2} - \frac{1}{r} \tag{19}$$

where

$$P_{\bar{z}} = p_{\bar{z}} - 2p_\alpha A_{\bar{z}}.$$

By comparing with the Hamiltonian H_μ in Proposition 4, in view of the fact that under the map $\bar{q}_0$, a point in $\tilde{U}_0$ with coordinate $(g, r, z^1, \ldots, z^{n-1})$ is mapped to a point in U_0 with coordinate $(r, z^1, \ldots, z^{n-1})$, we conclude that, for those solutions to equation (12) with with magnetic momentum $\mathscr{M}$, equation (12) becomes the equation of motion of the $\mathrm{U}(1)$ Kepler problem at level n with magnetic charge $-\mathrm{i}\mathscr{M}$, augmented with one more equation for g

$$2r^2\left(\dot{g}g^{-1} + \frac{\bar{z}\cdot\dot{z} - z\cdot\dot{\bar{z}}}{2(1+|z|^2)}\right) = \mathscr{M}. \tag{20}$$

Therefore, if $Z(t)$ is a solution to equation (12) with magnetic momentum $\mathscr{M}$, then $q(Z(t))$ is a solution to the equation of motion of the $\mathrm{U}(1)$ Kepler problem at level n with magnetic charge $-\mathrm{i}\mathscr{M}$ for those t such that $Z(t)$ in $\tilde{U}_0$, hence in any $\tilde{U}_i$ due to symmetry.

2) Assume that $x(t)$ is a solution to the equation of motion of the $\mathrm{U}(1)$ Kepler problem at level n with magnetic charge μ. Without loss of generality, we may

assume that $x(t_0) \in U_0$. Let v be the unique point in $T\tilde{U}$ such that, i) $T\bar{q}_0(v) = \dot{x}(t_0)$, ii) if $(g, r, z, \dot{g}, \dot{r}, \dot{z})$ is the local coordinate for v, then

$$g = 1, \qquad 2r^2 \left(\dot{g} g^{-1} + \frac{\bar{z} \cdot \dot{z} - z \cdot \dot{\bar{z}}}{2(1 + |z|^2)} \right) = \mathrm{i}\mu.$$

Now, if we let $Z(t)$ be the unique solution to the conformal Kepler problem with initial condition $(Z(t_0), \dot{Z}(t_0)) = v$, then the analysis in part 1) of this proof says that the magnetic momentum for $Z(t)$ is $\mathrm{i}\mu$ and $q(Z(t))$ is a solution to the equation of motion of $\mathrm{U}(1)$ Kepler problem at level n with magnetic charge μ, moreover, since $q(Z(t))$ and $x(t)$ have the same initial condition at t_0, we have $x(t) = q(Z(t))$. ■

The analysis in Section 3, when combined with Proposition 7 here, yields all solutions to the equation of motion of the $\mathrm{U}(1)$-Kepler problem at level n with magnetic charge μ, though the dependence on time t is only implicitly given. Moreover, for any solution $Z(t)$ to the equation of motion of the complex conformal Kepler problem we have obtained in Section 3, one can check that the total trace of $q(Z(t))$, i.e., the trajectory of the motion represented by $q(Z(t))$, always lies inside a real plane inside $\mathrm{H}_n(\mathbb{C})$. Therefore, results in Section 3 and Proposition 7 together imply

Theorem 8. *For the* $\mathrm{U}(1)$*-Kepler problem at level* n *with magnetic charge* μ*, the followings are true.*

1) *A trajectory is always the intersection of the space* $\mathcal{C}_1$ *with a real plane inside* $\mathrm{H}_n(\mathbb{C})$*, and it is bounded or unbounded according as the total energy* E *is negative or not.*

2) *A bounded trajectory can be parametrized as* $\alpha(\tau) = q(\cos\tau\, u + \sin\tau\, v)$ *for some* $u \in \mathbb{C}^n_*$ *and* $v \in \mathbb{C}^n$ *with*

$$\mu = \sqrt{\frac{2}{|u|^2 + |v|^2}}\, \mathrm{Im}(\bar{u} \cdot v).$$

Moreover, any parametrized curve of this form is a bounded trajectory with negative total energy $E = -\dfrac{1}{|u|^2 + |v|^2}$.

3) *An unbounded trajectory with zero total energy can be parametrized as* $\alpha(\tau) = q(u + v\tau)$ *for some* $u \in \mathbb{C}^n_*$ *and* $v \in \mathbb{C}^n$ *with* $|v|^2 = \frac{1}{2}$ *and*

$$\mu = 2\, \mathrm{Im}(\bar{u} \cdot v).$$

Moreover, any parametrized curve of this form is a trajectory with zero total energy.

4) *An unbounded trajectory with positive total energy can be parametrized as* $\alpha(\tau) = q(\cosh\tau\, u + \sinh\tau\, v)$ *for some* $u \in \mathbb{C}^n_*$ *and* $v \in \mathbb{C}^n$ *with* $|v|^2 > |u|^2$

and

$$\mu = \sqrt{\frac{2}{|v|^2 - |u|^2}}\, \mathrm{Im}(\bar{u} \cdot v).$$

Moreover, any parametrized curve of this form is a trajectory with positive total energy $E = \dfrac{1}{|v|^2 - |u|^2}$.

5. Non-Colliding Trajectories

The interesting trajectories are the non-colliding ones, i.e., the ones such that in their parametrization $\alpha(\tau)$ given in Theorem 8, $\alpha(\tau) \neq \mathbf{0} \in \mathrm{H}_n(\mathbb{C})$ for any $\tau \in \mathbb{R}$. It is evident that if v is a complex scalar multiple of u in theorem 8, then $\alpha(\tau) = \mathbf{0}$ for some finite value of τ and it is not hard to check that the converse is also true. Therefore, being applied to non-colliding trajectories, Theorem 8 becomes

Theorem 9. *For a non-colliding trajectory of a* $\mathrm{U}(1)$*-Kepler problem at level* n*, the followings are true.*

1) *It is an ellipse, a parabola or a branch of hyperbola according as the total energy* E *is negative, zero or positive.*

(*We assume in the next three statements that the variable* τ *runs over the entire* $\mathbb{R}$.)

2) *If it is an ellipse then it can be parametrized as* $\alpha(\tau) = q(\cos\tau\, u + \sin\tau\, v)$ *for some complex linearly independent* $u, v \in \mathbb{C}^n$ *with*

$$\mu = \sqrt{\frac{2}{|u|^2 + |v|^2}}\, \mathrm{Im}(\bar{u} \cdot v).$$

Moreover, any parametrized curve of this form is an elliptic trajectory with negative total energy $E = -\dfrac{1}{|u|^2 + |v|^2}$.

3) *If it is a parabola then it can be parametrized as* $\alpha(\tau) = q(u + v\tau)$ *for some complex linearly independent* $u, v \in \mathbb{C}^n$ *with*

$$\mu = \frac{\sqrt{2}}{|v|}\, \mathrm{Im}(\bar{u} \cdot v).$$

Moreover, any parametrized curve of this form is a parabolic trajectory with zero total energy.

4) *If it is a branch of hyperbola then it can be parametrized as* $\alpha(\tau) = q(\cosh\tau\, u + \sinh\tau\, v)$ *for some complex linearly independent* $u, v \in \mathbb{C}^n$ *with* $|v|^2 > |u|^2$ *and*

$$\mu = \sqrt{\frac{2}{|v|^2 - |u|^2}}\, \mathrm{Im}(\bar{u} \cdot v).$$

Moreover, any parametrized curve of this form is a hyperbolic trajectory with positive total energy $E = \dfrac{1}{|v|^2 - |u|^2}$.

Note that, in statement 3) of Theorem 9 the condition $|v|^2 = \frac{1}{2}$ is no longer needed because one can rescale v due to the fact that $\tau \in \mathbb{R}$. Let $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(1)$ be the quotient group of $\mathrm{GL}(n, \mathbb{C})$ by the image of the diagonal imbedding of $\mathrm{U}(1)$ into $\underbrace{\mathrm{U}(1) \times \cdots \times \mathrm{U}(1)}_{n}$. Since the standard linear action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$ $(n \geq 2)$ acts transitively on the set of complex linearly independent pairs of vectors in $\mathbb{C}^n$, Theorem 9 implies the following

Corollary 10. *For the* $\mathrm{U}(1)$*-Kepler problems at level* n*, the group* $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(1)$ *acts transitively on both set of elliptic trajectories and the set of parabolic trajectories.*

Since

$$\mathrm{SL}(2, \mathbb{C}) \times \mathbb{R}_+ \longrightarrow \mathrm{GL}(2, \mathbb{C})/\mathrm{U}(1), \qquad (A, c) \mapsto [cA]$$

is a two-to-one covering map, and $\mathrm{SL}(2, \mathbb{C})$ is the double cover of the identity component of the Lorentz group $\mathrm{O}(3, 1)$, this corollary for $n = 2$ is just a restatement of parts 3) and 4) in Theorem 2 in [15].

Finally we note that $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(1)$ is the orientation-preserving linear automorphism group of $\mathbb{C}^n$.

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