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ASPECTS OF LUSTERNIK-SCHNIRELMANN CATEGORY

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Abstract. In this paper we review certain Lusternik-Schnirelmann categorical notions that pertain especially to Eilenberg-Mac Lane spaces. Further, we introduce a new categorical invariant that allows a refined estimate for category. Finally, we mention some results about topological complexity and relate these to category.

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1. Introduction to LS Category

1.1. Definition and Basic Properties of LS Category

Definition 1. *The* Lusternik-Schnirelmann category *of a space X, denoted* $\mathrm{cat}(X)$*, is the smallest integer k so that X can be covered by $(k+1)$ open sets $U_0, U_1, \ldots, U_k$, each of which is contractible to a point in X. Such a covering is called a* categorical covering.

LS category is an important numerical invariant in algebraic topology, critical point theory and symplectic geometry (see [5, 12, 15]). Furthermore, various "forms" of category are now finding use in areas ranging from differential geometry [17] to robotics and motion planning [9–11]. In this paper, which is in part a survey, we concentrate on how LS category and its offshoots interact with the fundamental group. In particular, we will recall and reformulate in modern terms the approach to computing the LS category of Eilenberg-Mac Lane spaces. Although category may be defined quite easily, this belies the difficulty of its computation. Because it cannot be computed explicitly in most cases, we typically give lower and upper bounds in terms of other homotopy invariants that we hope are more computable. Therefore, in this spirit, we shall also introduce a new invariant called *universal cover category* and show that it, together with an older type of category, gives a new (and sharp) upper bound for category. In the final section we will discuss a newer problem that has roots in LS category and has implications for robotics.

The first calculations of category use some rather simple properties that, nevertheless, require fairly sophisticated algebraic-topological notions. Since we are interested in applications, we shall list these properties here without proof. Later we shall see how to build on these properties to derive new estimates of category.

Properties 2. The basic properties of LS category that we shall use are the following (see [5] for instance).

1. Category is a homotopy type invariant. This means that spaces X and Y with $X \simeq Y$ have $\mathrm{cat}(X) = \mathrm{cat}(Y)$.
2. The *cuplength* of a space X is the largest integer k such that there exists a product $x_1 \cdots x_k \neq 0$, with $x_i \in H^*(X; A)$. Here we use the fact that cohomology supports a product structure for a coefficient ring A. The coefficient ring A may vary and the cuplength may be considered for any (local) coefficients. The fundamental relation between cup length and category is $\mathrm{cup}(X) \leqslant \mathrm{cat}(X)$.
3. An upper bound for category is given by $\mathrm{cat}(X) \leqslant \dim(X)$ (where, for paracompact spaces more general than manifolds, $\dim(X)$ denotes the covering dimension of X). In fact, it is possible to show that, if $\pi_k(X) = 0$ for $0 \leqslant k \leqslant n-1$, then $\mathrm{cat}(X) \leqslant \dim(X)/n$.

4. **Fundamental Estimate**. Combining the previous two results gives

$$\operatorname{cup}(X) \leqslant \operatorname{cat}(X) \leqslant \frac{\dim(X)}{n}$$

where $\pi_j(X) = 0$ for $j = 1, \ldots, n-1$.

Example 3. *Here are some simple LS category calculations.*

1. $\operatorname{cat}(S^k) = 1$ *for any* k. *This follows because a sphere can be covered by (slightly fattened, so open) upper and lower hemispheres which are homeomorphic to disks and are thus contractible. Having two such sets means that category is equal to one.* (*The same proof shows that any space that is the suspension of another space has category one.*)
2. $\operatorname{cat}(T^k) = k$. *This follows because a* k*-torus is a product of* k *circles,* $T^k = S^1 \times \cdots \times S^1$ *and this means that the cohomology* $H^*(T^k; \mathbb{Z})$ *is an exterior algebra on* k *generators. There is then a product of length* k, *so* $\operatorname{cup}(T^k) = k$. *Since* $\dim(T^k) = k$ *as well, the Fundamental Estimate gives the result.*
3. $\operatorname{cat}(\mathbb{R}\mathrm{P}^n) = n$. *Recall that* $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) = \mathbb{Z}_2[x_1]/(x_1^{n+1})$, *a truncated polynomial algebra on a degree one generator* x_1. *Hence,* $\operatorname{cup}(\mathbb{R}\mathrm{P}^n) = n$ *and we have, by the Fundamental Estimate,* Property 2 (4)

$$n = \operatorname{cup}(\mathbb{R}\mathrm{P}^n) \leqslant \operatorname{cat}(\mathbb{R}\mathrm{P}^n) \leqslant \dim(\mathbb{R}\mathrm{P}^n) = n$$

so $\operatorname{cat}(\mathbb{R}\mathrm{P}^n) = n$.
4. $\operatorname{cat}(\mathbb{C}\mathrm{P}^k) = k$. *This follows because the cohomology of complex projective space is known to be* $H^*(\mathbb{C}\mathrm{P}^k; \mathbb{Z}) = \mathbb{Z}[x_2]/(x_2^{k+1})$, *a truncated polynomial algebra on a degree two generator* x_2. *Thus,* $\operatorname{cup}(\mathbb{C}\mathrm{P}^k) = k$. *Now,* $\pi_1(\mathbb{C}\mathrm{P}^k) = 0$, *so the upper bound in the Fundamental Estimate is* $\dim(\mathbb{C}\mathrm{P}^k)/2 = 2k/2 = k$, *so we obtain the result.*

2. A Reformulation of Category

We have seen that the definition of LS category and the simple cuplength-dimension estimate is sufficient to obtain some simple results. In order to apply category further, however, it is necessary to provide a homotopically more friendly, but more complicated equivalent definition due to T. Ganea. For details, see [5, Chapter 1].

2.1. The Reformulation Diagram

Let $PX = \{\gamma \colon I \to X \,;\, \gamma(0) = x_0\}$ be the contractible space of based paths with path fibration $p_0 \colon PX \to X$ given by $\gamma \mapsto \gamma(1)$. We inductively construct a diagram of fibrations

$$\begin{array}{ccccccc}
\Omega X & \to & F_1(X) & \to \cdots \to & F_k(X) & \to & \cdots \\
\downarrow & & \downarrow & \cdots & \downarrow & & \\
PX & \to & G_1(X) & \to \cdots \to & G_k(X) & \to & \cdots \\
p_0 \downarrow & & p_1 \downarrow & \cdots & p_k \downarrow & & \\
X & \stackrel{1_X}{\to} & X & \stackrel{1_X}{\to} \cdots \to & X & \to & \cdots .
\end{array}$$

where $G_{j+1}(X) = G_j(X) \cup C(F_j(X)) \simeq G_j(X)/F_j(X)$ is the mapping cone of the previous fibre inclusion. For instance, consider the first fibration $\Omega X \to PX \to X$. Take the mapping cone on the fibre inclusion $\Omega X \to PX$ to obtain $G_1(X)$. There is still a map to X and we replace this map by a homotopically equivalent *fibration* (which we still denote by) $G_1(X) \to X$. Iterate this process to obtain the diagram above. Through a rather long sequence of equivalences (delineated in [5]), we end up with the following characterization of LS category.

Theorem 4 (Definition-Theorem). $\mathrm{cat}(X) \leqslant n$ *if and only if there is a* (*homotopy*) *section* $s\colon X \to G_n(X)$. *That is, in the diagram above,* $p_n \circ s \simeq 1_X$.

Note that, in cohomology, we have $s^* \circ p_n^* = 1_{H^*}$, so p_n^* is injective. Also, it can be shown that each fibre $F_n(X)$ is a join of copies of the loop space $\Omega(X)$

$$F_n(X) = *^{n+1}\Omega(X).$$

Since taking joins increases connectivity (i.e., homotopy groups vanish up to higher and higher degrees with increasing n), the long exact sequence for the homotopy groups of a fibration gives isomorphisms between the homotopy groups of X and $G_k(X)$ through higher and higher degrees. This means that the $G_n(X)$ become more and more like X as n increases.

The notion of LS category can be extended from spaces to maps as follows.

Definition 5. *For* $f\colon Y \to X$, *the* category of the map f, $\mathrm{cat}(f)$ *is the least integer* n *such that there is a map* $s\colon Y \to G_n(X)$ *such that* $p_n \circ s \simeq f$.

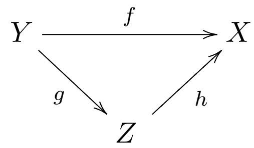
Properties 6. There are several properties of the reformulation diagram that will be important for us later.

1. $G_1(X) \simeq \Sigma\Omega(X)$, where ΣY denotes the suspension of the space Y. This follows since

   $$G_1(X) = G_0(X) \cup C(\Omega X) \simeq * \cup C(\Omega X)$$

   and attaching the cone $C(\Omega X)$ to a point crushes ΩX and gives the suspension of $\Omega(X)$.
2. $\mathrm{cat}(f) \leqslant \mathrm{cat}(X)$. This is immediate from the definitions. It is also true that $\mathrm{cat}(f) \leqslant \mathrm{cat}(Y)$ since f induces $G_k(f)\colon G_k(Y) \to G_k(X)$ with $p_k^X \circ G_k(f) = f \circ p_k^Y$.

3. If we have a commutative diagram



then $\mathrm{cat}(f) \leqslant \min\{\mathrm{cat}(g), \mathrm{cat}(h)\}$. This follows from Definition 5 once we note that a map $q\colon A \to B$ induces maps $G_k(A) \to G_k(B)$ compatible with the projections to A and B respectively.

2.2. Sectional Category

The notion of category can be extended in the following way (see [5] for instance).

Definition 7. *Suppose $F \to E \xrightarrow{p} B$ is a fibration. Then the* sectional category *of p, denoted $\mathrm{secat}(p)$, is the least integer n such that there exists an open covering, $U_0, \ldots, U_n$, of B and, for each U_i, a map $s_i\colon U_i \to E$ having $p \circ s_i = \mathrm{incl}_{U_i}$. (That is, s_i is a local section of p).*

Note that, in the reformulation fibration $\Omega(X) \to G_0(X) = PX \xrightarrow{p_0} X$, the total space PX is contractible. Therefore, if we have an open set $U \subset X$ in the definition of sectional category, then because it factors through PX, the inclusion $U \hookrightarrow X$ is nullhomotopic. That is, U contracts to a point in X. Hence, $\mathrm{cat}(X) \leqslant \mathrm{secat}(p_0)$. Since a basic property is that sectional category is always bounded above by the category of the base space (see (1) below), we actually have $\mathrm{cat}(X) = \mathrm{secat}(p_0)$. This is what we mean by extending the notion of category.

Properties 8. Consider a fibration $F \to E \xrightarrow{p} B$. Here are some properties of $\mathrm{secat}(p)$.

1. $\mathrm{secat}(p) \leqslant \mathrm{cat}(B)$. This follows since if $U \hookrightarrow B$ is nullhomotopic, then the Homotopy Lifting Property gives a local section of p over U.
2. If E is contractible, then $\mathrm{secat}(p) = \mathrm{cat}(B)$. (This is the case we discussed above for $G_0(X)$.)
3. If there are $x_1, \ldots, x_k \in \widetilde{H}^*(B; R)$ (any coefficient ring R) with

$$p^* x_1 = \ldots = p^* x_k = 0 \qquad \text{and} \qquad x_1 \cup \cdots \cup x_k \neq 0$$

then $\mathrm{secat}(p) \geqslant k$. This is an analogue of the cuplength estimate for ordinary category.

4. If $F \xrightarrow{i} E \xrightarrow{p} B$ arises as a pullback of a fibration $\widehat{p}\colon \widehat{E} \to \widehat{B}$ along a map $f\colon B \to \widehat{B}$, then

$$\mathrm{secat}(p) \leqslant \mathrm{secat}(\widehat{p}).$$

This follows because an open set $\widehat{U} \subset \widehat{B}$ with a local section $\widehat{s}\colon \widehat{U} \to \widehat{E}$ has an open inverse image $U = f^{-1}(\widehat{U}) \subset B$ and a map $\widehat{s} \circ f\colon U \to \widehat{E}$ with $\widehat{p} \circ \widehat{s} \circ f = f$. The pullback property then gives a map $s\colon U \to E$ with $p \circ s = \mathrm{incl}_U$.

If it is also the case that $\widehat{E}$ is contractible (such as for a principal bundle), then $\mathrm{secat}(p) = \mathrm{cat}(f)$. This says, for example, that the sectional category of principal bundles is precisely the category of the classifying map. (We will use this to characterize cat_1 below.)

There is a type of sectional category that will important to us when we discuss categorical estimates (see [16] for instance).

Definition 9. *The* 1-category *of a space X, denoted $\mathrm{cat}_1(X)$, is the least integer n so that X may be covered by open sets $U_0, \ldots, U_n$ having the property that, for each U_i, there is a local section $s_i\colon U_i \to \widetilde{X}$, where $p\colon \widetilde{X} \to X$ is the universal cover (so $p \circ s_i$ is homotopic to the inclusion $U_i \hookrightarrow X$). Thus,* $\mathrm{cat}_1(X) = \mathrm{secat}(p\colon \widetilde{X} \to X)$.

Before we can list properties of cat_1, we need to remind the reader about a certain construction. Assume X is a CW complex. That is, X is inductively constructed by attaching cells in a certain allowed fashion. Now, if $\pi_2(X) \neq 0$, then we may attach more 3-cells to X to obtain X_2 with $\pi_2(X_2) = 0$. Similarly, we may attach 4-cells to obtain X_3 with $\pi_3(X_3) = 0$. Continuing in this manner produces an *Eilenberg-Mac Lane space* with a fundamental group π and zero higher homotopy groups. This space is denoted by $K(\pi, 1)$ with $\pi = \pi_1(X)$. Note that the construction produces *an inclusion* $j_1\colon X \hookrightarrow K(\pi, 1)$ that induces an isomorphism on fundamental groups. This map j_1 classifies the universal cover of X, denoted $\widetilde{X} \to X$, in the sense that the universal cover is the pullback over j_1 of the path fibration $PK(\pi, 1) \to K(\pi, 1)$.

Properties 10. Here are some properties of cat_1 that prove useful.

1. $\mathrm{cat}_1(X) = 0$ if X is simply connected. This is clear from the definition or from the next property.
2. $\mathrm{cat}_1(X) = \mathrm{cat}(j_1\colon X \to K(\pi_1 X, 1))$. This follows because the universal cover $\widetilde{X} \to X$ is a pullback by the classifying map j_1 of the path fibration $PK(\pi_1 X, 1)) \to K(\pi_1 X, 1))$. Since $PK(\pi_1 X, 1))$ is contractible, Property 8 (4) applies. It also then follows that $\mathrm{cat}_1(X) \leqslant \mathrm{cat}(X)$.
3. If $j_1^*\colon H^k(K(\pi_1(X), 1); \mathcal{A}) \to H^k(X; \mathcal{A})$ is non-trivial (for some local coefficients $\mathcal{A}$), then
$$k \leqslant \mathrm{cat}_1(X).$$
The proof of this follows from the properties of an invariant called *category weight* derived from the reformulation diagram. See [5, 17] for example.

4. $\mathrm{cat}_1(X \times Y) \leqslant \mathrm{cat}_1(X) + \mathrm{cat}_1(Y)$. This property is, more or less, a general property of category-like invariants. For cat_1, see [16, 17].

Example 11. *We have* $\mathrm{cat}_1(T^n \times X) = n$ *if* X *is simply connected. To see this, we use* Property 10. *First, note that* $j_1 \colon T^n \times X \to T^n$ *is the classifying map since* $T^n = K(\mathbb{Z}^n, 1)$. *But then, because the map on cohomology is injective, we have* $n \leqslant \mathrm{cat}_1(T^n \times X)$. *But we also have* $\mathrm{cat}_1(T^n \times X) \leqslant \mathrm{cat}_1(T^n) \times \mathrm{cat}_1(X)$ *and* $\mathrm{cat}_1(X) = 0$. *Further, because* $T^n = K(\mathbb{Z}^n, 1)$, *we know* $\mathrm{cat}_1(T^n) = \dim(T^n) = n$. *Thus, we have the result.*

3. LS Category and Eilenberg-Mac Lane Spaces

3.1. Group Cohomology

A discrete group π may be looked at both algebraically and geometrically. We have already seen that we may construct an *Eilenberg-Mac Lane space* $K(\pi, 1)$ from any complex X with fundamental group π (together with a "classifying" map $j_1 \colon X \to K(\pi, 1)$). Therefore, any homotopy invariant qualities of $K(\pi, 1)$ translate into algebraic properties of the group π. On the other hand, we can study π algebraically directly by understanding its "representation theory". Much of the following is from the beautiful book [2].

Recall that if R is a ring and M is an R-module, then $\mathrm{projdim}_R(M) \leqslant n$ if there is a projective resolution of M of length n

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

where at the right end we mean $H_0(P) = P_0/\mathrm{im}(P_1 \to P_0) = M$. Of course, the word "resolution" means that $H_i(P) = 0$ for $i \geqslant 1$. If π is a discrete group, then the ring we take is the group ring $\mathbb{Z}\pi$ and the module we take is $\mathbb{Z}$ with the trivial module structure.

Definition 12. *Define the* cohomological dimension *of* π *to be*

$$\begin{aligned}\mathrm{cd}(\pi) &= \mathrm{projdim}_{\mathbb{Z}\pi}(\mathbb{Z}) \\ &= \inf\{n;\, \mathbb{Z} \text{ admits a resolution as a } \mathbb{Z}\pi\text{−module of length } n\}.\end{aligned}$$

Given such a resolution $0 \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$, we can take any $\mathbb{Z}\pi$-module $\mathcal{A}$ and form the complex (where the subscript π denotes "as $\mathbb{Z}\pi$-modules")

$$\mathrm{Hom}_\pi(\mathbb{Z}, \mathcal{A}) \to \cdots \to \mathrm{Hom}_\pi(P_i, \mathcal{A}) \to \mathrm{Hom}_\pi(P_{i+1}, \mathcal{A}) \to \cdots .$$

The homology of this complex then defines the *cohomology of* π *with coefficients in* $\mathcal{A}$

$$H^i(\pi; \mathcal{A}) = H_i(\mathrm{Hom}(P, \mathcal{A})).$$

It can be shown that the cohomological dimension is independent of the particular resolution P chosen. Furthermore, we have the following.

Proposition 13.

$$\begin{aligned}\operatorname{cd}(\pi) &= \inf\{n;\ H^i(\pi;\mathcal{A}) = 0 \text{ for } i > n \text{ and all } \mathcal{A}\}\\ &= \sup\{n;\ H^n(\pi;\mathcal{A}) \neq 0 \text{ for some } \mathcal{A}\}.\end{aligned}$$

The process of constructing a $K(\pi,1)$ that we gave before Properties 10 is not the only way that $K(\pi,1)$'s arise. In geometry, for instance, Hadamard's theorem says that a compact manifold of negative sectional curvature is a $K(\pi,1)$. Also, any compact surface except for the sphere S^2 and the projective space $\mathbb{R}\mathrm{P}^2$ is a $K(\pi,1)$. Higher dimensional examples often appear as quotients of Lie groups. For instance, it is known that a nilpotent Lie group N is diffeomorphic to some Euclidean space $\mathbb{R}^n$. This means that the quotient N/π of N by any co-compact discrete subgroup π has universal cover the contractible space $\mathbb{R}^n$ and, hence, is a $K(\pi,1)$. All of these examples are $K(\pi,1)$'s of finite dimension as manifolds and as CW complexes.

Definition 14. *Define the* geometric dimension *of the group* π*, denoted* $\dim(\pi)$*, to be the smallest integer* n *such that there is a* $K(\pi,1)$ *of dimension* n (*as a* CW *complex*).

Now let $X = K(\pi,1)$ and note that the universal cover $\widetilde{X}$ is contractible. The projection $p\colon \widetilde{X} \to X$ may be chosen to preserve cell structures, so in particular, the inverse image of a k-skeleton is the k-skeleton. This has the following algebraic consequences. First, since $\widetilde{X}$ is path-connected, we can take it to have a single 0-cell, so the zero chain module is $C_0(\widetilde{X}) = \mathbb{Z}$. Secondly, each i-chain module $C_i(\widetilde{X})$ is a free $\mathbb{Z}\pi$-module generated by the lift of an i-cell in X. Therefore, we have a free $\mathbb{Z}\pi$-complex

$$\cdots \to C_n(\widetilde{X}) \xrightarrow{\partial_n} C_{n-1}(\widetilde{X}) \to \cdots \to C_1(\widetilde{X}) \to \mathbb{Z} \to 0.$$

Now, because $\widetilde{X}$ is contractible, this complex is a $\mathbb{Z}\pi$-resolution of $\mathbb{Z}$ (as a trivial $\mathbb{Z}\pi$-module). Therefore, we can compute the cohomology of π with any coefficients $\mathcal{A}$ as

$$H^i(\pi;\mathcal{A}) = H_i(\operatorname{Hom}_\pi(C_*(\widetilde{X}),\mathcal{A})).$$

Properties 15. The following are immediate.

1. If $\mathcal{A} = \mathbb{Z}\pi$, then $H^i(\pi;\mathbb{Z}\pi) = H_c^*(\widetilde{X};\mathbb{Z})$, where the subscript c denotes "cohomology with compact supports".
2. If $\mathcal{A} = \mathbb{Z}$ (as a trivial $\mathbb{Z}\pi$-module), then $H^i(\pi;\mathbb{Z}) = H^*(X;\mathbb{Z})$.

3. If $\dim(X) = n$, then $C_i(\widetilde{X}) = 0$ for $i > n$, so $H^i(\pi; \mathcal{A}) = 0$ for $i > n$ and all $\mathbb{Z}\pi$-modules $\mathcal{A}$. Therefore, we have

$$\mathrm{cd}(\pi) \leqslant \dim(\pi).$$

Here is a sample calculation (see [2]).

Proposition 16.

$$H^i(\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{for } i \text{ even} \\ 0 & \text{for } i \text{ odd}. \end{cases}$$

Corollary 17. *If $X = K(\pi, 1)$ is finite dimensional, then π is torsionfree.*

Proof: If π is not torsionfree, then there exists an element of order p for some prime. The subgroup $\mathbb{Z}/p\mathbb{Z} \subseteq \pi$ gives a covering space $K(\mathbb{Z}/p\mathbb{Z}, 1) \to X$. Now, because X is finite dimensional, the same is true for $\widetilde{X}$ *and* every covering space of X. But $H^*(K(\mathbb{Z}/p\mathbb{Z}, 1); \mathbb{Z}) = H^*(\mathbb{Z}/p\mathbb{Z}; \mathbb{Z})$ is non-zero in an infinite number of degrees, so $K(\mathbb{Z}/p\mathbb{Z}, 1)$ cannot be finite dimensional. This contradiction proves the result. ∎

A natural question arises when $\mathrm{cd}(\pi) = n$. Namely, how much cohomology does π have. An answer is provided in the following result from [6].

Proposition 18. *If $\mathrm{cd}(\pi) = n$, then there exist coefficients $\mathcal{A}_j$, $j = 0, ..., n-1$, such that $H^{n-j}(\pi; \mathcal{A}_j) \neq 0$.*

Proof: Because $\mathrm{cd}(\pi) = n$, there exists some $\mathcal{A}_0$ such that $H^n(\pi; \mathcal{A}_0) \neq 0$. Now it is a standard result of homological algebra that any module can be embedded in an injective module, so take $\mathcal{A}_0 \hookrightarrow I_0$ with I_0 injective. Let the quotient module be $\mathcal{A}_1$. Because $\mathrm{Hom}(-, I)$ is an exact functor when I is injective, this means that $H^*(\pi; I) = 0$ for any injective module. But now the long exact cohomology sequence associated to the exact sequence $0 \to A_0 \to I_0 \to A_1 \to 0$ of $\mathbb{Z}\pi$-modules shows that

$$H^i(\pi; \mathcal{A}_1) \cong H^{i+1}(\pi; \mathcal{A}_0).$$

Hence, since $H^n(\pi; \mathcal{A}_0) \neq 0$, we see that $H^{n-1}(\pi; \mathcal{A}_1) \neq 0$ as well. Continue in this manner to obtain the result. ∎

Corollary 19 ([7])**.** *Suppose M^n and N^n are closed oriented n-manifolds and $\pi_1(M)$ is a free group. If there exists a map of degree one, $f\colon M \to N$, then $\pi_1(N)$ is also a free group.*

Proof: Suppose $\pi_1(N)$ is not free. By [21, 22], we then have that $\mathrm{cd}(\pi_1(N)) = n > 1$. By Proposition 18, $H^2(\pi_1(N); \mathcal{A}) \neq 0$ for some $\mathbb{Z}\pi_1(N)$-module $\mathcal{A}$.

Consider the following homotopy commutative diagram.

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ {\scriptstyle j_1^M}\big\downarrow & & \big\downarrow{\scriptstyle j_1^N} \\ K(\pi_1(M),1) & \xrightarrow{f_\#} & K(\pi_1(N),1). \end{array}$$

By our construction of $K(\pi,1)$'s, we see that $C_j(N) = C_j(K(\pi_1(N),1))$ for $j = 1,2$. This leads to the fact that, for any coefficients $\mathcal{A}$, $H^2(\pi_1(N);\mathcal{A}) \to H^2(N;\mathcal{A})$ is injective. But, by Poincaré duality with local coefficients, we also have that $f^*\colon H^2(N;\mathcal{A}) \to H^2(M;\mathcal{A})$ is injective for all $\mathcal{A}$ since f is degree 1. The homotopy commutativity of the diagram then shows that

$$(f_\#)^*\colon H^2(\pi_1(N);\mathcal{A}) \to H^2(\pi_1(M);\mathcal{A})$$

is non-zero. This contradicts the fact that $\pi_1(M)$ is free since then $\mathrm{cd}(\pi_1(M)) = 1$. ∎

3.2. Geometry of $K(\pi,1)$'s

Now let us try to connect the algebraic results above with geometric ones gleaned from LS category. First, let us note two facts.

Properties 20. The following hold.

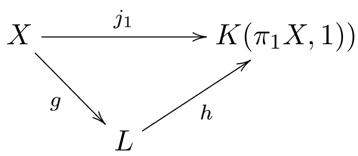
1. If $X = K(\pi,1)$, then in the reformulation diagram $G_1(X) \simeq \vee S^1$. This follows because $\pi_k(\Omega(X)) \cong \pi_{k+1}(X)$ for all $k \geqslant 0$, so $\pi_k(\Omega(X)) = 0$ for all $k \geqslant 1$ and $\pi_0(\Omega(X))$ is in bijection with $\pi_1(X) = \pi$. So, up to homotopy, $\Omega(X)$ is a set of discrete points. But then what is the suspension? The discrete set of points gives a set of disjoint intervals and these are then all smashed to a point at the top and bottom. The result, homotopically, is a set of circles all touching at a single point (i.e., a wedge of circles) $\vee S^1$.
2. Similarly, for $X = K(\pi,1)$, then $G_k(X)$ is homotopy k-dimensional. The proof here is much the same using the properties of the join. For instance, we have seen that $G_1(X) \simeq \vee S^1$. But $F_1(X) = \Omega(X) * \Omega(X)$ is the join of a discrete set of points with itself and this also is a wedge of circles. The mapping cone of $F_1(X) \hookrightarrow G_1(X)$ then attaches two-cells to $G_1(X)$ to obtain $G_2(X)$ which is then homotopy two-dimensional.

Now let us prove a result of Eilenberg and Ganea from this modern viewpoint.

Theorem 21 (Eilenberg-Ganea [8])**.** *Let X be a CW complex (which is not necessarily a $K(\pi,1)$). Then $\mathrm{cat}_1(X) \leqslant n$ if and only if there exists an n-dimensional complex L such that there is a map $f\colon X \to L$ inducing an isomorphism*

$$f_*\colon \pi_1(X) \stackrel{\cong}{\to} \pi_1(L).$$

Proof: Let us suppose that $\mathrm{cat}_1(X) = k \leqslant n$. Because $\mathrm{cat}_1(X) = \mathrm{cat}(j_1\colon X \to K(\pi_1 X, 1))$ by Properties 10, we have a lifting $X \to G_k(K(\pi_1 X, 1))$ by Properties 6. If $k = 1$, then since $G_1(K(\pi_1 X, 1)) \simeq \vee S^1$ by Properties 20, it's fundamental group is a free group $\mathcal{F}$. But the lifting induces maps on fundamental groups $\pi_1(X) \to \mathcal{F} \to \pi_1(X)$ whose composition is the identity. This then displays $\pi_1(X)$ as a retract of $\mathcal{F}$ and hence it is free. Thus $K(\pi_1 X, 1) \simeq \vee S^1 = L$.
If $k \geqslant 2$, then because $G_k(K(\pi_1 X, 1))$ is homotopy k-dimensional by Properties 20, we may take $L = G_k(K(\pi_1 X, 1))$. Because $*^{k+1}\Omega K(\pi_1 X, 1)$ is $(k-1)$-connected and $k \geqslant 2$, then $X \to L = G_k(K(\pi_1 X, 1))$ induces an isomorphism of fundamental groups.
In the other direction, for an L as in the condition, there is a commutative triangle



where all maps induce isomorphisms on fundamental groups. But then, by Properties 6) we have

$$\mathrm{cat}(j_1) \leqslant \min\{\mathrm{cat}(g), \mathrm{cat}(h)\} \leqslant \min\{\dim(X), \dim(L)\} \leqslant \dim(L).$$

■

Notice that the first part of the proof shows the following.

Corollary 22. *$\pi_1(X)$ is free if and only if $\mathrm{cat}_1(X) = 1$.*

Now, by [21, 22], we know that a group π is free if and only if $\mathrm{cd}(\pi) = 1$. This is our first hint that category and cohomological dimension are linked. Before we see more connections, let us understand the relationship between $\mathrm{cat}(X)$ and $\mathrm{cat}_1(X)$ when $X = K(\pi, 1)$.

Proposition 23. *If $X = K(\pi, 1)$, then $\mathrm{cat}(X) = \mathrm{cat}_1(X)$.*

Proof: By Properties 10, we know $\mathrm{cat}_1(X) \leqslant \mathrm{cat}(X)$. Let $\mathrm{cat}_1(X) = k$. Hence, there is an open cover of X, $\{U_0, \ldots, U_k\}$ such that the inclusion of each U_j into X has a lifting to the universal cover $\widetilde{X}$. But $\widetilde{X}$ is contractible, so each inclusion $U_j \hookrightarrow X$ is nullhomotopic. Hence, the cover is categorical and $\mathrm{cat}(X) \leqslant k = \mathrm{cat}_1(X)$. ■

Finally, we come to the LS-categorical connection between the algebra and geometry of discrete groups.

Proposition 24. *If* $X = K(\pi, 1)$*, then* $\mathrm{cat}(X) = \mathrm{cd}(\pi)$.

Proof: Suppose $\mathrm{cat}(X) = n$. Then there is a map $s\colon X \to G_n(X)$ with $p_n \circ s = 1_X$. Hence, $p_n^*\colon H^*(X; \mathcal{A}) \to H^*(G_n(X); \mathcal{A})$ is injective for any coefficients $\mathcal{A}$. Because $G_n(X)$ is homotopy n-dimensional, we must have $H^i(G_n(X); \mathcal{A}) = 0$ for $i > n$. Hence, $H^i(X; \mathcal{A}) = 0$ for $i > n$ for all $\mathcal{A}$, so $\mathrm{cd}(\pi) \leqslant n = \mathrm{cat}(X)$.
Now suppose that $\mathrm{cd}(\pi) = n$. Consider the fibration

$$*^{n+1}\Omega X \to G_n(X) \to X.$$

Now, the obstructions to finding a section $X \to G_n(X)$ lie in the cohomology groups $H^{i+1}(X; \pi_i(*^{n+1}\Omega X))$. But these groups are all trivial because $*^{n+1}\Omega X$ is $(n-1)$-connected and $H^j(X; \mathcal{A}) = 0$ for all $\mathcal{A}$ and all $j > n$. Hence, a section exists and this means $\mathrm{cat}(X) \leqslant n = \mathrm{cd}(\pi)$. ■

3.3. The Eilenberg-Gamea Theorem and Conjecture

Now, both $\mathrm{cat}(X)$ and $\mathrm{cd}(\pi)$ are bounded above by geometric dimension $\dim(X)$. The exact relationship is given by the following beautiful result (see [8] or [2] for a detailed proof).

Theorem 25. *Let* π *be a discrete group and let* $n = \max\{\mathrm{cd}(\pi), 3\}$*. Then there exists an* n*-dimensional* $K(\pi, 1)$*. That is, for* $\mathrm{cd}(\pi) \geqslant 3$*, it is always the case that* $\dim(\pi) = \mathrm{cd}(\pi)$.

Note that the Stallings-Swan theorem [21,22] says that a group has $\mathrm{cd} = 1$ exactly when it is free and any free group has a one-dimensional Eilenberg-Mac Lane complex given by a wedge of circles. Therefore, only the case $\mathrm{cd} = 2$ is still unknown. We have the following.

Conjecture 26 (Eilenberg-Ganea Conjecture)**.** *If* $\mathrm{cd}(\pi) = 2$*, then there is a two-dimensional* $K(\pi, 1)$.

In fact, this conjecture is part of a larger one. In [23], C. T. C. Wall proved the following result.

Theorem 27. *For* $n \neq 2$*, a finite CW complex* X *is homotopic to a finite* n*-dimensional complex if and only if*

1. $H_i(\widetilde{X}; \mathbb{Z}) = 0$ *for* $i > n$*, where* $\widetilde{X}$ *is the universal cover of* X
2. $H^{n+1}(X; \mathcal{A}) = 0$ *for all local coefficients* $\mathcal{A}$.

Notice that the case $n = 2$ is explicitly omitted. In fact, we have the following.

Conjecture 28 (Wall Conjecture)**.** *The theorem holds for $n = 2$ as well.*

Proposition 29. *If π has a finite $K(\pi, 1)$, then the Wall Conjecture holds if and only if the Eilenberg-Ganea Conjecture holds.*

Proof: Suppose Wall's Conjecture holds, $\mathrm{cd}(\pi) = 2$ and $X = K(\pi, 1)$ is finite. Then $\widetilde{X} \simeq *$, so $H_i(\widetilde{X}; \mathbb{Z}) = 0$ for $i > 0$. Furthermore, $H^3(X, \mathcal{A}) = 0$ for all $\mathcal{A}$ because $\mathrm{cd}(\pi) = 2$. Therefore, Wall's conditions are satisfied for $n = 2$ and X may be taken to be two-dimensional.

Now suppose the Eilenberg-Ganea Conjecture holds and $X = K(\pi, 1)$ is finite. If Wall's second condition holds for $n = 2$, then $H^3(\pi; \mathcal{A}) = H^3(X; \mathcal{A}) = 0$ for all $\mathcal{A}$. But the definition of cohomological dimension then says that $\mathrm{cd}(\pi) = 2$. The Eilenberg-Ganea Conjecture then gives a two-dimensional $K(\pi, 1)$. ■

There is another famous conjecture about dimension two which is related to the two conjectures above. A $K(\pi, 1)$ is sometimes known as an *aspherical* space.

Conjecture 30 (Whitehead Conjecture)**.** *Every connected subcomplex of a two-dimensional aspherical CW complex is also aspherical.*

On the face of it, it is hard to see the relation between this conjecture and the others. However, using a clever argument together with a general Morse-type theory for complexes, Bestvina and Brady were able to show the following sobering result.

Theorem 31 ([1])**.** *Either the Eilenberg-Ganea Conjecture or the Whitehead Conjecture (or both!) are false.*

Now let us move away from algebra and back to topology.

4. Universal Cover Category

In this section, we define a new numerical homotopy invariant which, together with cat_1, can be used to obtain an upper bound for LS category.

4.1. Basics

Definition 32. *Let $p\colon \widetilde{X} \to X$ denote the universal covering of X. The* universal covering Lusternik-Schnirelmann category *of a space X, denoted $\widetilde{\mathrm{cat}}(X)$, is the smallest integer k so that X can be covered by open sets $U_0, U_1, \ldots, U_k$, such that each pullback $p^{-1}(U_j)$ is contractible to a point in $\widetilde{X}$. Such a covering is called a* uc-categorical covering.

We will need the following lemma from [16] in order to "mix" two notions of category to obtain our upper bound.

Lemma 33 ([16]). *Let X be a normal space with two open covers*

$$\mathcal{U} = \{U_0, U_1, \ldots, U_k\} \qquad \textit{and} \qquad \mathcal{V} = \{V_0, V_1, \ldots, V_m\}$$

such that each set of $\mathcal{U}$ satisfies Property (A)*, and each set of $\mathcal{V}$ satisfies* Property (B). *Assume that* Properties (A) *and* (B) *are preserved under taking open subsets and disjoint unions. Then X has an open cover*

$$\mathcal{W} = \{W_0, W_1, \ldots, W_{k+m}\}$$

by open sets satisfying both Property (A) *and* Property (B).

Now let us turn to another "dimensional" version of category. As a particularization of the general definition of $\mathcal{A}$-category due to Clapp and Puppe [3], define $\mathrm{cat}^n(X)$ to be the least integer k such that there exists an open cover $\mathcal{U} = \{U_0, U_1, \cdots, U_k\}$ of X where each inclusion $i_j\colon U_j \hookrightarrow X$ factors through the n-skeleton of X up to homotopy. Note that $\mathrm{cat}^m(X) \leqslant \mathrm{cat}^n(X)$ if $m \geqslant n$ since it is easier to deform into a larger skeleton. These notions of category were combined in [16] to produce the following estimate of category.

Theorem 34. *If X is a normal space, then*

$$\mathrm{cat}(X) \leqslant \mathrm{cat}_1(X) + \mathrm{cat}^1(X).$$

Our goal here is to improve this estimate. Before we prove the main theorem, let us prove some simple, but important, properties of $\widetilde{\mathrm{cat}}(X)$. First, we need to know that $\widetilde{\mathrm{cat}}$ is a homotopy invariant.

Proposition 35. *Universal cover category is a based homotopy invariant.*

As in the case of ordinary category, this proposition follows from a domination lemma. Recall that $f\colon Y \to X$ is a based domination if there exists a based map $g\colon X \to Y$ such that fg is based homotopic to 1_X.

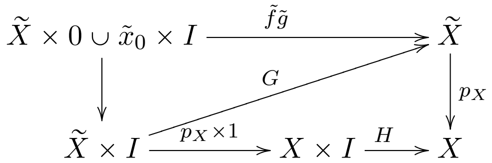
Lemma 36. *Suppose $f\colon Y \to X$ is a based domination. Then $\widetilde{\mathrm{cat}}(X) \leqslant \widetilde{\mathrm{cat}}(Y)$.*

Proof: Suppose $\widetilde{\mathrm{cat}}(Y) = n$ with corresponding cover of Y, $U_0, \ldots, U_n$. Then X has a cover $\{V_i = g^{-1}(U_i)\}$. Let $p_Y\colon \widetilde{Y} \to Y$, $p_X\colon \widetilde{X} \to X$ denote the respective universal coverings. Choose basepoints in the covers with $p_Y(\tilde{y}_0) = y_0$ and $p_X(\tilde{x}_0) = x_0$. Once these points have been chosen, there are unique lifts

$$\tilde{f}\colon \widetilde{Y} \to \widetilde{X}, \qquad \tilde{g}\colon \widetilde{X} \to \widetilde{Y}$$

with $\tilde{f}(\tilde{y}_0) = \tilde{x}_0$ and $\tilde{g}(\tilde{x}_0) = \tilde{y}_0$ and $p_X\tilde{f} = fp_Y$ and $p_y\tilde{g} = gp_X$. We claim that $\tilde{f}$ is also a domination. To see this, let $H\colon X \times I \to X$ be a homotopy with

$H_0 = fg$ and $H_1 = 1_X$. Now consider the following diagram



in which G exists by the homotopy lifting property and $G_1(\tilde{x}_0) = \tilde{x}_0$. Moreover, $G_0 = \tilde{f}\tilde{g}$ and $p_X G_1 = p_X$. But then G_1 covers 1_X and agrees with $1_{\widetilde{X}}$ at the point $\tilde{x}_0$. Therefore, by uniqueness of lifts, $G_1 = 1_{\widetilde{X}}$. Of course, G_1 is also based homotopic to $G_0 = \tilde{f}\tilde{g}$, so $\tilde{f}$ is a based domination.

Now consider the composition $p_X^{-1}(V_i) \hookrightarrow \widetilde{X} \xrightarrow{\tilde{g}} p_Y^{-1}(U_i) \hookrightarrow \widetilde{Y}$. Let $K\colon p_Y^{-1}(U_i) \times I \to \widetilde{Y}$ denote the contraction of $p_Y^{-1}(U_i)$ to a point: $K_0(\tilde{u}, 0) = \tilde{u}$, $K_1(\tilde{u}, 1)\tilde{y}_0$. Let $L\colon p_X^{-1}(V_i) \times I \to \widetilde{X}$ be defined by $L = \tilde{f} \circ K \circ (\tilde{g} \times 1_I)$. Then we have

$$L_0(\tilde{v}) = \tilde{f}(K_0(\tilde{g}(\tilde{v}))) = \tilde{f}(\tilde{g}(\tilde{v}))$$
$$L_1(\tilde{v}) = \tilde{f}(K_1(\tilde{g}(\tilde{v}))) = \tilde{f}(\tilde{y}_0) = \tilde{x}_0.$$

Therefore, since $L_0 = \tilde{f}\tilde{g} \simeq 1_{\widetilde{X}}$, we see that V_i contracts to a point in $\widetilde{X}$. Hence, $\widetilde{\mathrm{cat}}(X) \leqslant n = \widetilde{\mathrm{cat}}(Y)$. ■

Now, with a view toward improving Theorem 34, let us compare $\mathrm{cat}^1(X)$ and $\widetilde{\mathrm{cat}}(X)$.

Proposition 37. $\widetilde{\mathrm{cat}}(X) \leqslant \mathrm{cat}^1(X)$.

Proof: Let $U \subset X$ be a cat^1-open set. Therefore, U deforms into the one-skeleton of X. Now, since $p\colon \widetilde{X} \to X$ is a covering, we can always arrange cell structures so that $p^{-1}(X_1) \subseteq \widetilde{X}_1$, where the subscript 1 denotes the one-skeleton. Let $H\colon U_j \times I \to X$ be the homotopy that deforms U_j into the one-skeleton X_1 and compose with p to obtain a homotopy $G\colon p^{-1}(U_j) \times I \to X$. The homotopy lifting theorem then lifts this homotopy to $\widetilde{G}\colon p^{-1}(U_j) \times I \to \widetilde{X}$ with $\widetilde{G}_1(p^{-1}(U_j)) \subset \widetilde{X}_1$. But $\widetilde{X}$ is simply connected, so $\widetilde{X}_1$ is contractible in $\widetilde{X}$. Therefore $p^{-1}(U_j)$ is contractible in $\widetilde{X}$ and we see that a cat^1-open cover is a $\widetilde{\mathrm{cat}}$-open cover. ■

In fact, we also have a result for cat^n.

Proposition 38. *Suppose* $\pi_1(X) \neq 0$ *and* $\pi_j(X) = 0$ *for* $1 < j \leqslant n$. *Then* $\widetilde{\mathrm{cat}}(X) \leqslant \mathrm{cat}^n(X)$.

Proof: Let $U \subset X$ be a cat^n-open set. Therefore, U deforms into the n-skeleton of X. As above, we arrange cell structures so that $p^{-1}(X_n) \subseteq \widetilde{X}_n$, where the subscript n denotes the n-skeleton. Let $H\colon U_j \times I \to X$ be the homotopy that deforms U_j into the n-skeleton X_1 and compose with p to obtain a homotopy $G\colon p^{-1}(U_j) \times I \to X$. The homotopy lifting theorem then lifts this homotopy to $\widetilde{G}\colon p^{-1}(U_j) \times I \to \widetilde{X}$ with $\widetilde{G}_1(p^{-1}(U_j)) \subset \widetilde{X}_n$. But $\widetilde{X}$ is n-connected, so $\widetilde{X}_n$ is contractible in $\widetilde{X}$. Therefore $p^{-1}(U_j)$ is contractible in $\widetilde{X}$ and we see that a cat^n-open cover is a $\widetilde{\mathrm{cat}}$-open cover. ■

The next result is immediate, but gives a link to the fundamental result Theorem 25.

Proposition 39. *Suppose X has the homotopy type of a CW complex. Then $X = K(\pi, 1)$ if and only if $\widetilde{\mathrm{cat}}(X) = 0$.*

Proof: If $X = K(\pi, 1)$, then the universal cover $\widetilde{X}$ is contractible. Hence we see that $p^{-1}(X) = \widetilde{X}$ is contractible and $\widetilde{\mathrm{cat}}(X) = 0$. Conversely, if $\widetilde{\mathrm{cat}}(X) = 0$, then, by definition, $p^{-1}(X)$ is contractible. But the covering $p\colon \widetilde{X} \to X$ is surjective, so $p^{-1}(X) = \widetilde{X}$, so $X = K(\pi, 1)$. ■

4.2. A New Estimate for Category

We now wish to apply our results about open covers to obtain an estimate for LS category which, by Proposition 37, improves the estimate Theorem 34.

Theorem 40. *If X is a normal space, then*

$$\mathrm{cat}(X) \leqslant \mathrm{cat}_1(X) + \widetilde{\mathrm{cat}}(X).$$

Proof: Suppose $\mathrm{cat}_1(X) = k$ and $\widetilde{\mathrm{cat}}(X) = m$. Let $\mathcal{U} = \{U_0, \ldots, U_k\}$ be a cat_1-open cover of X and let $\mathcal{V} = \{V_0, \ldots, V_m\}$ be a $\widetilde{\mathrm{cat}}$-open cover of X. Thus, each inclusion $U_j \hookrightarrow X$ lifts through the universal cover $\widetilde{X}$, while each V_j gives a categorical open set $p^{-1}(V_j)$ in $\widetilde{X}$. By Lemma 33, we have an open cover $\mathcal{W} = \{W_0, \ldots, W_{k+m}\}$ such that each inclusion $W_j \hookrightarrow X$ satisfies both properties.

We claim that any open subset of X that is both a cat_1-set and a $\widetilde{\mathrm{cat}}$-set is actually categorical. Consider the inclusion $i_j\colon W_j \hookrightarrow X$ and note that a partial section $s_j\colon W_j \to \widetilde{X}$ must have image in $p^{-1}(W_j)$ since $ps_j = i_j$. However, the inclusion $k\colon p^{-1}(W_j) \hookrightarrow \widetilde{X}$ is nullhomotopic, so $i_j = ps_j = pks_j \simeq *$. Hence i_j is nullhomotopic and, thus, W_j is categorical. Because $\mathcal{W} = \{W_0, \ldots, W_{k+m}\}$ is a categorical open cover, we then have

$$\mathrm{cat}(X) \leqslant k + m = \mathrm{cat}_1(X) + \widetilde{\mathrm{cat}}(X).$$

■

Now, by Proposition 37, we see that we have given another proof of Theorem 34. Furthermore, Proposition 38 also gives the following.

Corollary 41. *Suppose* $\pi_1(X) \neq 0$ *and* $\pi_j(X) = 0$ *for* $1 < j \leqslant n$. *Then*

$$\operatorname{cat}(X) \leqslant \operatorname{cat}_1(X) + \operatorname{cat}^n(X).$$

Remark 42. *In* [16]*, the "geography" of categories was proposed to be the set of* (m, n) *with* $\operatorname{cat}(X) \leqslant \operatorname{cat}_m(X) + \operatorname{cat}^n(X)$. *While elementary results were obtained, a general result was harder to come by. Here we see that natural homotopy conditions lead to a geographic point* $(1, n)$.

Now let us focus on three crucial examples that will illustrate the strengths and weaknesses of Theorem 40.

Example 43. *First, let us give an example where the inequality of* Theorem 40 *is actually an equality. By the cuplength and product inequalities for category, it is easy to see that* $\operatorname{cat}(S^2 \times T^2) = 3$. *As shown in* [17], $\operatorname{cat}_1(S^2 \times T^2) = 2$. *Now, observe that, if* H *denotes either the northern or southern hemisphere of* S^2 *union a small open collar, then for the covering map* $p\colon S^2 \times \mathbb{R}^2 \to S^2 \times T^2$, $p^{-1}(H \times T^2) = H \times \mathbb{R}^2$ *is contractible. Thus,* $\widetilde{\operatorname{cat}}(S^2 \times T^2) = 1$ *and*

$$\operatorname{cat}(S^2 \times T^2) = 3 = \operatorname{cat}_1(S^2 \times T^2) + \widetilde{\operatorname{cat}}(S^2 \times T^2)$$

Hence the estimate of Theorem 40 *is sharp.*

Remark 44. *The argument above generalizes to the case of* $X = K(\pi, 1) \times S^2$, *but requires some extra ingredients such as category weight. The result is that*

$$\operatorname{cat}(X) = \operatorname{cd}(\pi) + 1 = \operatorname{cat}_1(X) + \widetilde{\operatorname{cat}}(X)$$

while $\operatorname{cat}_1(X) = \operatorname{cd}(\pi)$.

Example 45. *This example provides another proof of the Eilenberg-Ganea identification of* $\operatorname{cat}(X)$ *with* $\operatorname{cat}_1(X)$ *for* $X = K(\pi, 1)$. *We have*

$$\operatorname{cat}_1(X) \leqslant \operatorname{cat}(X) \leqslant \operatorname{cat}_1(X) + \widetilde{\operatorname{cat}}(X).$$

By Proposition 39*, we also have that* $\widetilde{\operatorname{cat}}(X) = 0$ *and the result follows. This is a case where* $\operatorname{cat}_1 = \operatorname{cat}$ *with* $\widetilde{\operatorname{cat}} = 0$.

Example 46. *Here we shall describe a case where* $\operatorname{cat}_1 = \operatorname{cat}$ *with* $\widetilde{\operatorname{cat}} = 1$. *Consider a quotient* $X = S^n/G$ *where* G *is a finite group acting freely and preserving orientation (with* n *odd). In fact, we can reduce to the case where* $G = \mathbb{Z}/p\mathbb{Z}$, *so we only consider this situation. We can see that* $\operatorname{cat}_1(X) = n$ *as follows. By*

[5, Lemma 9.30], *we know that the classifying map* $X \to K(G,1)$ *induces a surjection* $H^*(K(G,1);G) \to H^*(X;G)$*. Therefore, because* $\dim(X) = n$*, the estimate of* Properties 10 *shows that we have* $n \leqslant \mathrm{cat}_1(X)$*. But we also have*

$$\mathrm{cat}_1(X) \leqslant \mathrm{cat}(X) \leqslant \dim(X) = n$$

so $\mathrm{cat}_1(X) = n$*. The inequality above also shows that* $\mathrm{cat}(X) = n$*. Now, to see what* $\widetilde{\mathrm{cat}}(X)$ *is, remove a small disk* D *from* X*, so that the inverse image under* $p\colon S^n \to S^n/G$ *consists of disjoint disks homeomorphic to* D*. This set is then contractible in the universal cover* S^n*. Now, the inverse image of* $X - \overline{D}$ *misses a point of* S^n*, so* $p^{-1}(X - \overline{D})$ *is contractible in* S^n*. Therefore,* $\widetilde{\mathrm{cat}}(X) = 1$*.*

5. Topological Complexity

5.1. Introduction

In this section, we describe a relatively recent addition to the family of LS category-type invariants. We will prove only elementary facts here and rather refer to works such as [9–11, 13, 14] for harder results and details.

A mechanical system $\mathcal{S}$ is described by its totality of states $X = X(\mathcal{S})$, this is the *configuration space* of $\mathcal{S}$.

Example 47. *A planar robot arm with* n *links has configuration space the* n*-torus* T^n *since the relevant parameters of the system are the* n *angles between consecutive links.*

In robotics, the fundamental problem is how to control a robot from any one configuration to any other configuration. Formally, we write the following.

Problem 48. Let X be the configuration space of a system $\mathcal{S}$. The *motion planning problem* is to algorithmically determine a continuous path $\gamma\colon I \to X$ with $\gamma(0) = A$ and $\gamma(1) = B$ for any $A, B \in X$.

A precise mathematical formulation of the problem is the following (see [9]). Let $\mathrm{ev}\colon X^I \to X \times X$ be the evaluation fibration $\mathrm{ev}(\gamma) = (\gamma(0), \gamma(1))$, where X^I is the space of all paths $\gamma\colon I \to X$. A *motion planning algorithm* is a continuous section

$$s\colon X \times X \to X^I, \qquad \mathrm{ev} \circ s = 1_{X \times X}.$$

Unfortunately, we have the following sobering result.

Proposition 49. *A motion planning algorithm* $s\colon X \times X \to X^I$ *exists if and only if* X *is contractible* (*i.e., deformable to a point*)*.*

Proof: If $X \simeq *$, then there is a homotopy $H\colon X \times I \to X$ with $H(x, 0) = x$ and $H(x, 1) = B_0$, for fixed B_0. Given A, B, define

$$\gamma(t) = \begin{cases} H(A, 2t) & 0 \leqslant t \leqslant 1/2 \\ H(B, 2-2t) & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Since this continuously determines a path $\gamma(t)$ for each pair (A, B), we obtain a motion planning algorithm. On the other hand, if a motion planning algorithm $s\colon X \times X \to X^I$ exists, define $H\colon X \times I \to X$ by $H(A, t) = s(A, B_0)(t)$. Then $H(A, 0) = A$ and $H(A, 1) = B_0$ because s is a section of ev. ■

So what can be done for more general spaces if motion planning algorithms only exist for contractible configuration spaces? Well, this is precisely the LS category approach to a space's complexity.

Definition 50. *The* Topological Complexity *of the motion planning algorithm problem for X is*

$$\mathrm{TC}(X) = \mathrm{secat}(\mathrm{ev}\colon X^I \to X \times X).$$

The idea behind this definition is that we decompose $X \times X$ into open sets U for which there *is* a motion planning algorithm in X.

The relation between LS category and topological complexity is expressed by the following inequalities.

Proposition 51. *Topological complexity is a homotopy invariant and the following estimates hold:*

$$\mathrm{cat}(X) \leqslant \mathrm{TC}(X) \leqslant \mathrm{cat}(X \times X) \leqslant 2\,\mathrm{cat}(X).$$

For the proof of homotopy invariance, see [9]. The estimate $\mathrm{TC}(X) \leqslant \mathrm{cat}(X \times X)$ is simply (1) of Property 8 while the estimate $\mathrm{cat}(X) \leqslant \mathrm{TC}(X)$ follows from (5) of Property 8 when we recognize that the (based) path fibration $PX \to X$ is a pullback of the evaluation fibration $\mathrm{ev}\colon X^I \to X \times X$ by the mapping $X \to X \times X$, $x \mapsto (x_0, x)$ for a fixed basepoint $x_0 \in X$. Here is the fundamental example (see [9]).

Proposition 52.

$$TC(S^n) = \begin{cases} 1 & n \text{ odd} \\ 2 & n \text{ even.} \end{cases}$$

Proof: Case n odd. let us break $S^n \times S^n$ into two open sets.

$$U = \{(x, y)\ ;\ y \neq -x\}, \qquad V = \{(x, y)\ ;\ y = -x\}.$$

(Note that the second set is not open, but we can take a small deformable neighborhood around the anti-diagonal to "make it open".) We need to define local sections on these sets. This means specifying a path from x to y in an algorithmic continuous fashion. For U, simply take the path that is the unique minimizing geodesic between x and y. This makes sense precisely because $y \neq -x$. For V this definition would be a problem since there are two minimizing geodesics. However, here we can use the fact that, since the Euler characteristic of an odd sphere is zero, there is a non-vanishing vector field on S^n. So, at x, take the vector field vector as the initial condition of a geodesic and take this path to $y = -x$. (Here we use the fact that geodesics are great circles.) Since we have local sections on U and V, we see that $\mathrm{TC}(S^n) \leqslant 1$. By Proposition 51, we also see that $1 = \mathrm{cat}(S^n) \leqslant \mathrm{TC}(S^n) \leqslant 1$, so $\mathrm{TC}(S^n) = 1$.

Case n even. So what is the only difference from the case n even? It is simply that the Euler characteristic of an even sphere is 2, not 0. So we cannot use the method on V to find an algorithmic path. However, we *do* know that there is a vector field on S^n with only a single zero x_0, say. So if we define U as above and

$$V = \{(x, y)\ ;\ y = -x,\ x \neq x_0\}, \qquad W = \{(x_0, -x_0)\}$$

then we can use the vector field on V as before. Again, we can take a small contractible neighborhood around $(x_0, -x_0)$ that deforms to $(x_0, -x_0)$, so we need only define a path for this single point. For this we can take any path from x_0 to $-x_0$. Therefore, since we cover with three sets $U,\ V,\ W$, $\mathrm{TC}(S^n) \leqslant 2$. To get a good lower bound, we use (3) of Property 8 by considering the element

$$\begin{aligned}\chi &= x \otimes 1 - 1 \otimes x \\ &\in (H^n(S^n;\mathbb{Q}) \otimes H^0(S^n;\mathbb{Q})) \oplus (H^0(S^n;\mathbb{Q}) \otimes H^n(S^n;\mathbb{Q})) \\ &\cong H^n(S^n \times S^n;\mathbb{Q})\end{aligned}$$

where $x \neq 0$ in $H^n(S^n;\mathbb{Q})$. Now, $\mathrm{ev}^*(\chi) = 0$ because $X^I \simeq X$ and $\mathrm{ev}^*(x \otimes 1 = x = \mathrm{ev}^*(1 \otimes x)$. Also, $\chi^2 \neq 0$ since graded commutativity of cohomology gives

$$\chi^2 = -x \otimes x - (-1)^{n^2} x \otimes x$$

and n is even. (Note that this argument would not work for n odd.) Hence, by (3) of Property 8, $\mathrm{TC}(S^n) \geqslant 2$. Thus, $\mathrm{TC}(S^n) = 2$. ∎

Proposition 52 begs the question of determining spaces with low topological complexity. In [13], these spaces were identified.

Theorem 53 (Grant-Lupton-Oprea)**.** *If* $\mathrm{TC}(X) = 1$*, then* X *is homotopy equivalent to some sphere of odd dimension. Moreover, if* X *is also a closed manifold, then* X *is homeomorphic to an odd sphere.*

This result has a rather complicated proof, but the starting point is the recognition that the inequality $\mathrm{cat}(X) \leqslant \mathrm{TC}(X) = 1$ implies that X is a co-H-space (i.e., a space with co-multiplication) since $\mathrm{cat}(X) = 1$ identifies these spaces. Now, there are co-H-spaces that are not spheres. In fact, every suspension ΣX is a co-H-space and $\mathrm{cat}(\Sigma X) = 1$ as can be seen by decomposing ΣX into its top and bottom cones. So Theorem 53 is giving yet another indication that TC is a more complicated invariant than LS category.

From our discussion of LS category and $K(\pi, 1)$'s, it should not be surprising that the following is a major problem in the subject.

Problem 54. Determine the topological complexity of a $K(\pi, 1)$.

As we have seen, Eilenberg and Ganea showed that, when $\mathrm{cd}(\pi) > 2$, then

$$\mathrm{cd}(\pi) = \mathrm{cat}(K(\pi, 1)) = \dim(K(\pi, 1)).$$

The problem for topological complexity, however, is much more delicate and at the moment no general answer is known. There *are* some determinations of TC for $K(\pi, 1)$'s which often use some type of auxiliary structure associated to the group π that enables an application of Property 8_3 (e.g. see [4]). Here is a more general result that centers on the subgroup structure of π [14].

Theorem 55 (Grant-Lupton-Oprea)**.** *If A and B are complementary subgroups of π (i.e., $AB = \pi$ and $A \cap B = \varnothing$), then*

$$\mathrm{cd}(A \times B) \leqslant \mathrm{TC}(K(\pi, 1)).$$

Using this result, we can recover lower bounds for TC for various types of $K(\pi, 1)$'s such as right-angled Artin groups and braid groups. Furthermore, using much harder arguments, we obtain the following result.

Corollary 56. *Let $\mathcal{H}$ denote the Higman group with presentation*

$$\langle x, y, z, w \,;\, xyx^{-1}y^{-2},\, yzy^{-1}z^{-2},\, zwz^{-1}w^{-2},\, wxw^{-1}x^{-2}\rangle.$$

Then $\mathrm{TC}(\mathcal{H}) = 4$.

It is known that the group $\mathcal{H}$ is acyclic (it has the same integer homology as a trivial group), and so $H^*(\mathcal{H}; k) = 0$ (in positive degrees) for every abelian group k. Moreover, $\mathcal{H}$ has no non-trivial finite quotients, so it has no non-trivial finite dimensional representations over any field. It then follows that if M is any coefficient $\mathcal{Z}[\mathcal{H}]$-module which is finitely generated as an abelian group, then $H^*(\mathcal{H}; M) = 0$. Thus the group $\mathcal{H}$ is difficult to distinguish from a trivial group using cohomological invariants.

On the other hand, since $\mathcal{H}$ is not a free group we have $\mathrm{cd}(\mathcal{H}) \geqslant 2$. The two-dimensional complex associated to the presentation P is known to be aspherical and it follows that $\mathrm{cat}(\mathcal{H}) = \mathrm{cd}(\mathcal{H}) = 2$. Thus the topological complexity of Higman's group satisfies $2 \leqslant \mathrm{TC}(\mathcal{H}) \leqslant 4$. A nontrivial argument using Bass-Serre theory shows the result that $\mathrm{TC}(\mathcal{H}) = 4$.

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