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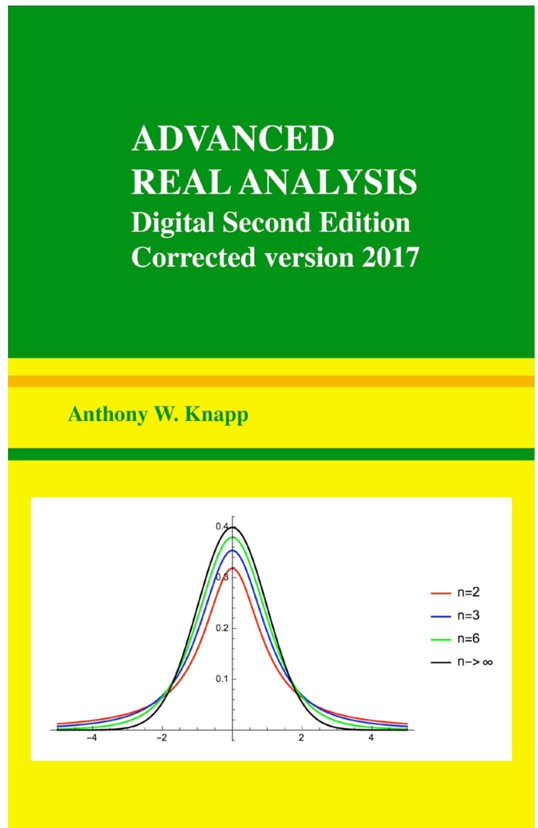

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CHAPTER VIII

Analysis on Manifolds

Abstract. This chapter explains how the theory of pseudodifferential operators extends from open subsets of Euclidean space to smooth manifolds, and it gives examples to illustrate the usefulness of generalizing the theory in this way.

Section 1 gives a brief introduction to differential calculus on smooth manifolds. The section defines smooth manifolds, smooth functions on them, tangent spaces to smooth manifolds, and differentials of smooth mappings between smooth manifolds, and it proves a version of the Inverse Function Theorem for manifolds.

Section 2 extends the theory of smooth vector fields and integral curves from open subsets of Euclidean space to smooth manifolds.

Section 3 develops a special kind of quotient space, called an "identification space," suitable for constructing general smooth manifolds, vector bundles and fiber bundles, and covering spaces out of local data. In particular, smooth manifolds may be defined as identification spaces without knowledge of the global nature of the underlying topological space; the only problem is in addressing the Hausdorff property.

Section 4 introduces vector bundles, including the tangent and cotangent bundles to a manifold. A vector bundle determines transition functions, and in turn the transition functions determine the vector bundle via the construction of the previous section. The manifold structures on the tangent and cotangent bundles are constructed in this way.

Sections 5–8 concern pseudodifferential operators, including aspects of the theory useful in solving problems in other areas of mathematics. The emphasis is on operators on scalar-valued functions. Section 5 introduces spaces of smooth functions and their topologies, and it defines spaces of distributions; the theory has to compensate for the lack of a canonical underlying measure on the manifold, hence for the lack of a canonical way to view a smooth function as a distribution. Section 5 goes on to study linear partial differential equations on the manifold; although the symbol of the differential operator is not meaningful, the principal symbol is intrinsically defined as a function on the cotangent bundle. The introduction of pseudodifferential operators on smooth manifolds requires new results for the theory in Euclidean space beyond what is in Chapter VII. Section 6 addresses this matter. A notion of transpose is needed, and it is necessary to understand the effect of diffeomorphisms on Euclidean pseudodifferential operators. To handle these questions, it is useful to enlarge the definition of pseudodifferential operator for Euclidean space and to redo the Euclidean theory from the new point of view. Once that program has been carried out, Section 7 patches together pseudodifferential operators in Euclidean space to obtain pseudodifferential operators on smooth separable manifolds. The notions of pseudolocal, properly supported, composition, and elliptic extend, and the theorems are what one might expect from the Euclidean theory. Again the principal symbol is well defined as a function on the cotangent bundle. Section 8 contains remarks about extending the theory to handle operators carrying sections of one vector bundle to sections of another vector bundle, about some other continuations of the theory, and about applications outside real analysis. The section concludes with some bibliographical material.

1. Differential Calculus on Smooth Manifolds

The goal of this chapter is to explain how aspects of the subject of linear partial differential equations extend from open subsets of Euclidean space to smooth manifolds. After an introduction to manifolds and their differential calculus, we shall see the extent to which definitions and theorems about distributions, differential operators, and pseudodifferential operators carry over from local facts about Euclidean space to global facts about smooth manifolds. We shall see also how certain important systems of differential equations can conveniently be expressed globally in terms of operators from one vector bundle to another.

The present section introduces smooth manifolds, smooth functions on them, tangent spaces to smooth manifolds, differentials of smooth mappings between smooth manifolds, and a version of the Inverse Function Theorem for manifolds.

We begin with the definition of smooth manifold. Let M be a Hausdorff topological space, and fix an integer $n \geq 0$. A **chart** on M of dimension n is a homeomorphism $\kappa : M_\kappa \to \widetilde{M}_\kappa$ of an open subset M_κ of M onto an open subset $\widetilde{M}_\kappa$ of $\mathbb{R}^n$; the chart κ is said to be **about** a point p in M if p is in the domain M_κ of κ. We say that M is a **manifold** if there is an integer $n \geq 0$ such that each point of M has a chart of dimension n about it.

A **smooth structure** of dimension n on a manifold M is a family $\mathcal{F}$ of n-dimensional charts with the following three properties:

(i) any two charts κ and κ' in $\mathcal{F}$ are smoothly **compatible** in the sense that $\kappa' \circ \kappa^{-1}$, as a mapping of the open subset $\kappa(M_\kappa \cap M_{\kappa'})$ of $\mathbb{R}^n$ to the open subset $\kappa'(M_\kappa \cap M_{\kappa'})$ of $\mathbb{R}^n$, is smooth and has a smooth inverse,

(ii) the system of compatible charts κ is an **atlas** in the sense that the domains M_κ together cover M,

(iii) $\mathcal{F}$ is maximal among families of compatible charts on M.

A **smooth manifold** of **dimension** n is a manifold together with a smooth structure of dimension n. In the presence of an understood atlas, a chart will be said to be **compatible** if it is compatible with all the members of the atlas.

Once we have an atlas of compatible n-dimensional charts for a manifold M, i.e., once (i) and (ii) are satisfied, then the family of all compatible charts satisfies (i) and (iii), as well as (ii), and therefore is a smooth structure. In other words, an atlas determines one and only one smooth structure. Thus, as a practical matter, we can construct a smooth structure for a manifold by finding an atlas satisfying (i) and (ii), and the extension of the atlas for (iii) to hold is automatic.

Let us make some remarks about the topology of manifolds. Let M be any manifold, let p be in M, and let $\kappa : M_\kappa \to \widetilde{M}_\kappa$ be a chart about p. Then $\widetilde{M}_\kappa$ is an open neighborhood of $\kappa(p)$. Since $\mathbb{R}^n$ is locally compact, we can find a compact subneighborhood N of $\kappa(p)$ contained in $\widetilde{M}_\kappa$. Then $\kappa^{-1}(N)$ is a compact neighborhood of p in M, and it follows that M is locally compact. Since M is by assumption Hausdorff, M is topologically regular. By the Urysohn Metrization Theorem[1] a separable Hausdorff regular space is metrizable; therefore the topology of a manifold is given by a metric if the manifold is separable.[2]

We shall not assume at any stage that M is connected, and until Section 5 we shall not assume that M is separable.

A simple example of a smooth manifold is $\mathbb{R}^n$ itself, with an atlas consisting of the single chart 1, where 1 is the identity function on $\mathbb{R}^n$. Another simple example is any nonempty open subset E of a smooth manifold M, which becomes a smooth manifold by taking all the compatible charts κ of M, replacing them by charts $\kappa\big|_{M_\kappa \cap E}$, and eliminating redundancies. In particular, any open subset of $\mathbb{R}^n$ becomes a smooth manifold since $\mathbb{R}^n$ itself is a smooth manifold.

Two less-trivial classes of examples are spheres and real projective spaces. They can be realized explicitly as metric spaces, and then one can specify an atlas and hence a smooth structure in each case. The details of these examples are discussed in Problems 1–2 at the end of the chapter.

Most manifolds, however, are constructed globally out of other manifolds or are pieced together from local data. The Hausdorff condition usually has to be checked, is often subtle, and is always important. We postpone a discussion of this matter for the moment.

Let us consider functions on smooth manifolds. If p is a point of the smooth n-dimensional manifold M, a compatible chart κ about p can be viewed as giving a **local coordinate system** near p. Specifically if the Euclidean coordinates in $\widetilde{M}_\kappa$ are $(u_1, \dots, u_n)$, then $q = \kappa^{-1}(u_1, \dots, u_n)$ is a general point of M_κ, and we define n real-valued functions $q \mapsto x_j(q)$ on M_κ by $x_j(q) = u_j$, $1 \le j \le n$. Then $\kappa = (x_1, \dots, x_n)$. To refer the functions x_j to Euclidean space $\mathbb{R}^n$, we use $x_j \circ \kappa^{-1}$, which carries $(u_1, \dots, u_n)$ to u_j.

The way that the functions x_j are referred to Euclidean space mirrors how a more general scalar-valued function on an open subset of M may be referred to Euclidean space, and then we can define the function to be smooth if it is smooth in the sense of Euclidean differential calculus when referred to Euclidean space. It will only occasionally be important whether our scalar-valued functions are real-valued or complex-valued. Accordingly, we shall follow the convention introduced in Chapter IV that $\mathbb{F}$ denotes the field of scalars, either $\mathbb{R}$ or $\mathbb{C}$; either field is allowed (consistently throughout) unless some statement is made to the contrary.

Therefore a **smooth function** $f : E \to \mathbb{F}$ on an open subset E of M is a function with the property, for each $p \in E$ and each compatible chart κ about p,

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[1] Theorem 10.45 of *Basic*.

[2] Some equivalent conditions for separability of a smooth manifold are given in Problem 3 at the end of the chapter.

that $f \circ \kappa^{-1}$ is smooth as a function from the open subset $\kappa(M_\kappa \cap E)$ of $\mathbb{R}^n$ into $\mathbb{F}$. A smooth function is necessarily continuous.

In verifying that a scalar-valued function f on an open subset E of M is smooth, it is sufficient, with each point in E, to check a condition for only one compatible chart about that point. The reason is the compatibility of the charts: if κ_1 and κ_2 are two compatible charts about p, then $f \circ \kappa_2^{-1}$ is the composition of the smooth function $\kappa_1 \circ \kappa_2^{-1}$ followed by $f \circ \kappa_1^{-1}$.

The space of smooth scalar-valued functions on the open set E will be denoted by $C^\infty(E)$; if we want to insist on a particular field of scalars, we write $C^\infty(E, \mathbb{R})$ or $C^\infty(E, \mathbb{C})$. The space $C^\infty(E)$ is an associative algebra under the pointwise operations, and it contains the constants. The **support** of a scalar-valued function is, as always, the closure of the set where the function is nonzero. We write $C^\infty_{\text{com}}(E)$ for the subset of $C^\infty(E)$ of functions whose support is a compact subset of E. The space $C^\infty_{\text{com}}(E)$, as well as the larger space $C^\infty(E)$, separates points of E as a consequence of the following lemma and proposition; the lemma makes essential use of the fact that the manifold is Hausdorff.

Lemma 8.1. If M is a smooth manifold, κ is a compatible chart for M, and f is a function in $C^\infty_{\text{com}}(M_\kappa)$, then the function F defined on M to equal f on M_κ and to equal 0 off M_κ is in $C^\infty_{\text{com}}(M)$ and has support contained in M_κ.

PROOF. The set $S = \text{support}(f)$ is a compact subset of M_κ and is compact as a subset of M since the inclusion of M_κ into M is continuous. Since M is Hausdorff, S is closed in M. The function F is smooth at all points of M_κ and in particular at all points of S, and we need to prove that it is smooth at points of the complement U of S in M. If p is in U, we can find a compatible chart κ' about p with $M_{\kappa'} \subseteq U$. The function F is 0 on $M_{\kappa'} \cap M_\kappa$ since $U \cap \text{support}(f) = \varnothing$, and it is 0 on $M_{\kappa'} \cap M_\kappa^c$ since it is 0 everywhere on M_κ^c. Therefore it is identically 0 on $M_{\kappa'}$ and is exhibited as smooth in a neighborhood of p. Thus F is smooth. □

Proposition 8.2. Suppose that p is a point in a smooth manifold M, that κ is a compatible chart about p, and that K is a compact subset of M_κ containing p. Then there is a smooth function $f : M \to \mathbb{R}$ with compact support contained in M_κ such that f has values in [0, 1] and f is identically 1 on K.

PROOF. The set $\kappa(K)$ is a compact subset of the open subset $\widetilde{M}_\kappa = \kappa(M_\kappa)$ of Euclidean space, and Proposition 3.5f produces a smooth function g in $C^\infty_{\text{com}}(\widetilde{M}_\kappa)$ with values in [0, 1] that is identically 1 on $\kappa(K)$. If f is defined to be $g \circ \kappa$ on M_κ, then f is in $C^\infty_{\text{com}}(M_\kappa)$. Extending f to be 0 on the complement of M_κ in M and applying Lemma 8.1, we see that the extended f satisfies the conditions of the proposition. □

EXAMPLE. This example shows what can go wrong if the Hausdorff condition is dropped from the definition of smooth manifold. Let X be the disjoint union of two copies of $\mathbb{R}$, say $(\mathbb{R}, +)$ and $(\mathbb{R}, -)$, with each of them open in X. Define an equivalence relation on X by requiring that every point be equivalent to itself and also that $(x, +)$ be equivalent to $(x, -)$ for $x \neq 0$. The quotient space M of X by this equivalence relation consists of the nonzero elements of one copy of $\mathbb{R}$, together with two versions of 0, which we denote by 0^+ and 0^-. The topological space M is not Hausdorff since 0^+ and 0^- cannot be separated by disjoint open sets. Let $\mathbb{R}^+ \subseteq M$ be the image of $(\mathbb{R}, +)$ under the quotient map, and define $\mathbb{R}^-$ similarly. Define $\kappa^+ : \mathbb{R}^+ \to \mathbb{R}^1$ and $\kappa^- : \mathbb{R}^- \to \mathbb{R}^1$ in the natural way, and then κ^+ and κ^- together behave like an atlas of compatible charts covering M. To proceed with a theory, it is essential to be able to separate points by smooth functions. Smooth functions are in particular continuous, and 0^+ and 0^- cannot be separated by continuous real-valued functions on M. Thus they cannot be separated by smooth functions, and Proposition 8.2 must fail. It is instructive, however, to see just exactly how it does fail. In the proposition let us take $p = 0^+$, $\kappa = \kappa^+$, and $K = \{0^+\}$. We can certainly construct a smooth function f on $\mathbb{R}^+$ with values in $[0, 1]$ that is 1 on $K = \{0^+\}$ and has compact support L as a subset of $\mathbb{R}^+$. However, L is not closed as a subset of M. When f is extended to be 0 off $\mathbb{R}^+$, the extended function is not continuous, much less smooth. To be continuous, it would have to be defined to be 1, rather than 0, at 0^-.

Corollary 8.3. Let p be a point of a smooth manifold M, let U be an open neighborhood of p, and let f be in $C^\infty(U)$. Then there is a function g in $C^\infty(M)$ such that $g = f$ in a neighborhood of p.

PROOF. Possibly by shrinking U, we may assume that U is the domain of some compatible chart κ about p. Let K be a compact neighborhood of p contained in U, and use Proposition 8.2 to find h in $C^\infty(M)$ with compact support in U such that h is identically 1 on K. Define g to be the pointwise product hf on U and to be 0 off U. Then g equals f on the neighborhood K of p, and Lemma 8.1 shows that g is everywhere smooth. $\square$

The Euclidean chain rule yields a necessary condition for a tuple of real-valued functions to provide a local coordinate system near a point, and the Inverse Function Theorem shows the sufficiency of the condition. The details are as in Proposition 8.4 below. Further results of this kind appear in Problems 6–7 at the end of the chapter.

Proposition 8.4. Let M be an n-dimensional smooth manifold, let p be in M, let κ be a chart about p, and let $f_1, \ldots, f_m$ be in $C^\infty(M_\kappa, \mathbb{R})$. In order for there to exist an open neighborhood V of p such that the restriction of $\kappa' = (f_1, \ldots, f_m)$ to V is a compatible chart, it is necessary and sufficient that

(a) $m = n$ and

(b) $\det\left[\dfrac{\partial(f_i \circ \kappa^{-1})}{\partial u_j}\right] \neq 0$ at the point $u = \kappa(p)$.

PROOF OF NECESSITY. Let $\kappa' = (f_1, \ldots, f_m)$. If κ' is a compatible chart about p when restricted to some neighborhood V of p, then $\kappa' \circ \kappa^{-1}$ and $\kappa \circ \kappa'^{-1}$ are smooth mappings on open sets in Euclidean space that are inverse to each other. By the chain rule the products of their Jacobian matrices in the two orders are the identity matrices of the appropriate size. Therefore $m = n$, and the determinant of the Jacobian matrix of $\kappa' \circ \kappa^{-1}$ at $\kappa(p)$ is not 0. $\square$

PROOF OF SUFFICIENCY. Let $m = n$. If (b) holds, then the Inverse Function Theorem produces an open neighborhood V' of $\kappa'(p)$ and an open neighborhood $U' \subseteq \widetilde{M}_\kappa$ of $\kappa(p)$ such that $\kappa' \circ \kappa^{-1}$ has a smooth inverse g mapping V' one-one onto U'. Let $V = \kappa^{-1}(U')$, and define $h = \kappa^{-1} \circ g$. Then h maps V' one-one onto V and satisfies $h \circ \kappa' = h \circ (\kappa' \circ \kappa^{-1}) \circ \kappa = \kappa^{-1} \circ (g \circ (\kappa' \circ \kappa^{-1})) \circ \kappa = \kappa^{-1} \circ \kappa = 1$. Thus $h = \kappa'^{-1}$ and $\kappa'\big|_V$ is a chart. To see that the chart $\kappa'\big|_V$ is compatible, let κ'' be a chart in the given atlas such that $V \cap M_{\kappa''} \neq \varnothing$. Then $\kappa' \circ \kappa''^{-1} = (\kappa' \circ \kappa^{-1}) \circ (\kappa \circ \kappa''^{-1})$ is smooth, and so is $\kappa'' \circ \kappa'^{-1} = \kappa'' \circ h = (\kappa'' \circ \kappa^{-1}) \circ g$. Hence the chart $\kappa'\big|_V$ is compatible. $\square$

A **smooth function** $F : E \to N$ from an open subset E of the n-dimensional smooth manifold M into a smooth k-dimensional manifold N is a continuous function with the property that for each $p \in E$, each compatible M chart κ about p, and each compatible N chart κ' about $F(p)$, the function $\kappa' \circ F \circ \kappa^{-1}$ is smooth from an open neighborhood of $\kappa(p)$ in $\kappa(M_\kappa \cap E) \subseteq \mathbb{R}^n$ into $\mathbb{R}^k$. The function $\kappa' \circ F \circ \kappa^{-1}$ is what F becomes when it is referred to Euclidean space. Let us examine $\kappa' \circ F \circ \kappa^{-1}$ further.

In a compatible M chart κ about p, we have used $(u_1, \ldots, u_n)$ as Euclidean coordinates within $\widetilde{M}_\kappa$, and the local coordinate functions on M_κ are the members x_j of $C^\infty(M_\kappa, \mathbb{R})$ such that $x_j \circ \kappa^{-1}(u_1, \ldots, u_n) = u_j$. In a compatible N chart κ' about $F(p)$, let us use $(v_1, \ldots, v_k)$ as Euclidean coordinates within $\widetilde{N}_{\kappa'}$, and let us denote the local coordinate functions on $N_{\kappa'}$ by y_i. The formula for y_i is $y_i \circ \kappa'^{-1}(v_1, \ldots, v_k) = v_i$. The function $\kappa' \circ F \circ \kappa^{-1}$ takes values of the form $(v_1, \ldots, v_k)$, and the way to extract the i^{th} coordinate function of $\kappa' \circ F \circ \kappa^{-1}$ is to follow it with $y_i \circ \kappa'^{-1}$. Thus when F is referred to Euclidean space, the i^{th} coordinate function of the result is $y_i \circ F \circ \kappa^{-1}$. We shall write F_i for this coordinate function.

If $F : M \to N$ is a smooth function between smooth manifolds and if F has a smooth inverse, then F is called a **diffeomorphism**.

If M and N are smooth manifolds, then the product $M \times N$ becomes a smooth manifold in a natural way by taking an atlas of $M \times N$ to consist of all products $\kappa \times \kappa'$ of compatible charts of M by compatible charts of N. With this definition of smooth structure for $M \times N$, the projections $M \times N \to M$ and $M \times N \to N$ are smooth and so are the inclusions $M \to M \times \{y\}$ and $N \to \{x\} \times N$ for any y in N and x in M.

Fix a point p in M. The "tangent space" to M at p will be defined shortly in a way so as to consist of all first-derivative operators on functions at p. Traditionally one uses only real-valued functions in making the definition, but we shall adhere to our convention and allow scalars from either $\mathbb{R}$ or $\mathbb{C}$ except when we need to make a choice. Construction of the tangent space can be done in a concrete fashion, using the coordinate functions x_j, or it can be done with a more abstract definition. The latter approach, which we follow, has the advantage of incorporating all the necessary analysis into the problem of sorting out the definition rather than into incorporating it into a version of the chain rule valid for manifolds. In other words the one result that will need proof will be a statement limiting the size of the tangent space, and the chain rule will become purely a formality.

To the extent that a tangent vector at p is a first derivative operator at p, its effect will depend only on the behavior of functions in a neighborhood of p. Within the abstract approach, there are then two subapproaches. One subapproach works with functions on a fixed but arbitrary open set containing p and looks at a kind of first-derivative-at-p operation on them. The other subapproach works simultaneously with all functions such that any two of them coincide on some neighborhood of p. Either subapproach will work in our present context of smooth manifolds. It turns out, however, that a similar formalism applies to other kinds of manifolds—particularly to complex manifolds and to real-analytic manifolds—and only the second subapproach works for them. We shall therefore introduce the idea of the tangent space to M at p by working simultaneously with all functions such that any two of them coincide on some neighborhood of p. The operative notion is that of a "germ" at p.

To emphasize domains, let us temporarily write (f, U) for a member of $C^\infty(U)$. We consider all such objects such that p lies in U, and we define (f, U) to be equivalent to (g, V) if $f = g$ on some subneighborhood about p of the common domain $U \cap V$. This notion of "equivalent" is readily checked to be an equivalence relation, and we let $\mathcal{C}_p(M)$ be the set of equivalence classes. An equivalence class is called a **germ** of a smooth scalar-valued function at p. The set of germs inherits addition and multiplication from that for functions. Specifically the germ of the sum $(f, U)+(g, V)$ is defined to be the germ of $\big((f\big|_{U\cap V})+(g\big|_{U\cap V}), U\cap V\big)$. One has to check that this definition is independent of the choice of representatives, but that is routine. Multiplication is handled similarly. Then one checks that the operations on germs have the usual properties of an associative algebra over $\mathbb{F}$.

Let us sketch the argument for associativity of addition. Let three germs be given, and let (f, U), (g, V), and (h, W) be representatives. A representative of the sum of the three is defined on the intersection $I = U \cap V \cap W$. On I, the restrictions to I satisfy $(f+g)+h = f+(g+h)$ because of associativity for addition of functions; hence the germs of the two sides of the associativity formula are equal, and addition is associative in $\mathcal{C}_p(M)$.

The algebra $\mathcal{C}_p(M)$ admits a distinguished linear function into the field of scalars $\mathbb{F}$, namely evaluation at p. If a germ is given and (f, U) is a representative, then the value $f(p)$ at p is certainly independent of the choice of representative; thus evaluation at p is well defined on $\mathcal{C}_p(M)$. We denote it by e. Although germs are not functions, we often use the same symbol for a germ as for a representative function in order to remind ourselves how germs behave. A **derivation** of $\mathcal{C}_p(M)$ is a linear function $L : \mathcal{C}_p(M) \to \mathbb{F}$ such that $L(fg) = L(f)e(g) + e(f)L(g)$. If the germ f is the class of a function (f, U), then we can define L on the function to be equal to L on the germ, and the formula for L on a product of two functions will be valid on the common domain of the two representative functions.

Any derivation L of $\mathcal{C}_p(M)$ has to satisfy $L(1) = L(1 \cdot 1) = L(1)1 + 1L(1) = 2L(1)$ and thus must annihilate the constant functions and their germs. The derivations of $\mathcal{C}_p(M)$ are also called **tangent vectors** to M at p, and the space of these derivations is called the **tangent space** to M at p and is denoted by $T_p(M)$.

For $M = \mathbb{R}^n$, evaluation of a first partial derivative at p is an example. More generally we can obtain examples for any M as follows: Let κ be a compatible chart with p in M_κ. The specific derivations of $\mathcal{C}_p(M)$ that we construct will depend on the choice of κ. We obtain n examples $\big[\frac{\partial}{\partial x_j}\big]_p$ of derivations of $\mathcal{C}_p(M)$, one for each j with $1 \le j \le n$, by the definition

$$\Big[\frac{\partial f}{\partial x_j}\Big]_p = \frac{\partial (f \circ \kappa^{-1})}{\partial u_j}(\kappa(p)) = \frac{\partial (f \circ \kappa^{-1})}{\partial u_j}\Big|_{(u_1,\dots,u_n)=(x_1(p),\dots,x_n(p))}.$$

For $f = x_i$, we have

$$\Big[\frac{\partial x_i}{\partial x_j}\Big]_p = \frac{\partial (x_i \circ \kappa^{-1})}{\partial u_j}(x_1(p), \dots, x_n(p)) = \frac{\partial u_i}{\partial u_j}(x_1(p), \dots, x_n(p)) = \delta_{ij}.$$

Consequently the n derivations $\big[\frac{\partial}{\partial x_j}\big]_p$ of $\mathcal{C}_p(M)$ are linearly independent.

Proposition 8.5. Let M be a smooth manifold of dimension n, let p be in M, and let κ be a compatible chart about p. Then the n derivations $\big[\frac{\partial}{\partial x_j}\big]_p$ of $\mathcal{C}_p(M)$ form a basis for the tangent space $T_p(M)$ of M at p, and any derivation L of $\mathcal{C}_p(M)$ satisfies

$$L = \sum_{j=1}^{n} L(x_j)\Big[\frac{\partial}{\partial x_j}\Big]_p.$$

PROOF. We know that the n explicit derivations are linearly independent. To prove spanning, let L be a derivation of $\mathcal{C}_p(M)$, and let (f, E) represent a member of $\mathcal{C}_p(M)$. Without loss of generality, we may assume that $E \subseteq M_\kappa$ and that $\kappa(E)$ is an open ball in $\mathbb{R}^n$. Put $u_0 = (u_{0,1}, \dots, u_{0,n}) = \kappa(p)$, let q be a variable point in E, and define $u = (u_1, \dots, u_n) = \kappa(q)$. Taylor's Theorem[3] applied to $f \circ \kappa^{-1}$ on $\kappa(E)$ gives

$$\begin{aligned} f \circ \kappa^{-1}(u) = f \circ \kappa^{-1}(u_0) &+ \sum_{j=1}^{n} (u_j - u_{0,j}) \tfrac{\partial(f \circ \kappa^{-1})}{\partial u_j}(u_0) \\ &+ \sum_{i,j} (u_i - u_{0,i})(u_j - u_{0,j}) R_{ij}(u) \end{aligned}$$

with R_{ij} in $C^\infty(\kappa(E))$. Referring this formula to M, we obtain

$$\begin{aligned} f(q) = f(p) &+ \sum_{j=1}^{n} (x_j(q) - x_j(p)) \big[\tfrac{\partial f}{\partial x_j}\big]_p \\ &+ \sum_{i,j} (x_i(q) - x_i(p))(x_j(q) - x_j(p)) r_{ij}(q) \end{aligned}$$

on E, where $r_{ij} = R_{ij} \circ \kappa$ on E. Because L annihilates constants and has the derivation property, application of L yields

$$\begin{aligned} L(f) &= \sum_{j=1}^{n} L(x_j) \big[\tfrac{\partial f}{\partial x_j}\big]_p + \sum_{i,j} \big(L(x_i)(e(x_j) - x_j(p)) e(r_{ij}) \\ &\qquad + (e(x_i) - x_i(p)) L(x_j) e(r_{ij}) \ + \ (e(x_i) - x_i(p))(e(x_j) - x_j(p)) L(r_{ij}) \big) \\ &= \sum_{j=1}^{n} L(x_j) \big[\tfrac{\partial f}{\partial x_j}\big]_p, \end{aligned}$$

as asserted. □

A smooth function $F : E \to N$ as above has a "differential" that carries the tangent space to M at p linearly to the tangent space to N at $F(p)$. We shall define the differential, find its matrix relative to local coordinates, and establish a version of the chain rule for smooth manifolds. Let L be in $T_p(M)$, and let g be in $\mathcal{C}_{F(p)}(M)$. Regard g as a smooth function defined on some open neighborhood of $F(p)$, and define $(dF)_p(L)$ to be the member of $T_{F(p)}(N)$ given by $(dF)_p(L)(g) = L(g \circ F)$. To see that $(dF)_p(L)$ is indeed in $T_{F(p)}(N)$, we need to check that $L(g \circ F)$ depends only on the germ of g and not on the choice of representative function; also we need to check the derivation property.

[3] In the form of Theorem 3.11 of *Basic*.

To check these things, let g and g^* be functions representing the same germ at $F(p)$. Then $g = g^*$ in a neighborhood of $F(p)$, and the continuity of F ensures that $g \circ F = g^* \circ F$ in a neighborhood of p. The derivation L depends only on a germ at p, and thus $(dF)_p(L)(g)$ depends only on the germ of g. For the derivation property we have

$$\begin{aligned}(dF)_p(L)(g_1 g_2) &= L((g_1 g_2) \circ F) = L((g_1 \circ F)(g_2 \circ F)) \\ &= L(g_1 \circ F)(g_2(F(p))) + (g_1(F(p)))L(g_2 \circ F) \\ &= (dF)_p(L)(g_1)(g_2(F(p))) + (g_1(F(p)))(dF)_p(L)(g_2),\end{aligned}$$

and thus $(dF)_p(L)$ is in $T_{F(p)}(N)$.

The mapping $(dF)_p : T_p(M) \to T_{F(p)}(N)$ is evidently linear, and it is called the **differential** of F at p. We may write dF_p for it if there is no ambiguity; later we shall denote it by $dF(p)$ as well. Proposition 8.5 gives us bases of $T_p(M)$ and $T_{F(p)}(N)$, and we shall determine the matrix of dF_p relative to these bases.

Proposition 8.6. Let M and N be smooth manifolds of respective dimensions n and k, and let $F : M \to N$ be a smooth function. Fix p in M, let κ be an M chart about p, and let κ' be an N chart about $F(p)$. Relative to the bases $\Big[\frac{\partial}{\partial x_j}\Big]_p$ of $T_p(M)$ and $\Big[\frac{\partial}{\partial y_i}\Big]_{F(p)}$ of $T_{F(p)}(N)$, the matrix of the linear function $dF_p : T_p(M) \to T_{F(p)}(N)$ is $\Big[\frac{\partial F_i}{\partial u_j}\Big|_{(u_1,\dots,u_n)=(x_1(p),\dots,x_n(p))}\Big]$.

REMARK. In other words it is the Jacobian matrix of the set of coordinate functions of the function obtained by referring F to Euclidean space. Hence the differential is the object for smooth manifolds that generalizes the multivariable derivative for Euclidean space. Accordingly, let us make the definition

$$\Big[\frac{\partial F_i}{\partial x_j}\Big]_p = \Big[\frac{\partial F_i}{\partial u_j}\Big|_{(u_1,\dots,u_n)=(x_1(p),\dots,x_n(p))}\Big].$$

PROOF. Application of the definitions gives

$$\begin{aligned}dF_p\Big(\Big[\frac{\partial}{\partial x_j}\Big]_p\Big)(y_i) &= \Big[\frac{\partial}{\partial x_j}\Big]_p (y_i \circ F) \\ &= \frac{\partial(y_i \circ F \circ \kappa^{-1})}{\partial u_j}(x_1(p), \dots, x_n(p)) \\ &= \frac{\partial F_i}{\partial u_j}\Big|_{(u_1,\dots,u_n)=(x_1(p),\dots,x_n(p))}.\end{aligned}$$

The formula in Proposition 8.5 allows us to express any member of $T_{F(p)}(N)$ in terms of its values on the local coordinate functions y_i, and therefore

$$dF_p\Big(\Big[\frac{\partial}{\partial x_j}\Big]_p\Big) = \sum_{i=1}^{k} \frac{\partial F_i}{\partial u_j}\bigg|_{(u_1,\dots,u_n)=(x_1(p),\dots,x_n(p))} \Big[\frac{\partial}{\partial y_i}\Big]_p.$$

Thus the matrix is as asserted. $\square$

Proposition 8.7 (chain rule). Let M, N, and R be smooth manifolds, and let $F : M \to N$ and $G : N \to R$ be smooth functions. If p is in M, then

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p.$$

PROOF. If L is in $T_p(M)$ and h is in $\mathcal{C}_{G(F(p))}(R)$, then the definitions give

$$d(G \circ F)_p(L)(h) = L(h \circ G \circ F) = dF_p(L)(h \circ G) = dG_{F(p)}(dF_p(L)(h)),$$

as asserted. $\square$

2. Vector Fields and Integral Curves

A **vector field** on an open subset U of $\mathbb{R}^n$ was defined in Chapter IV of *Basic* as a function $X : U \to \mathbb{R}^n$. The vector field is **smooth** if X is a smooth function. In classical notation, X is written $X = \sum_{j=1}^n a_j(x_1,\dots,x_n)\,\frac{\partial}{\partial x_j}$, and the function carries $(x_1,\dots,x_n)$ to $(a_1(x_1,\dots,x_n),\dots,a_n(x_1,\dots,x_n))$. The traditional geometric interpretation of X is to attach to each point p of U the vector $X(p)$ as an arrow based at p. This interpretation is appropriate, for example, if X represents the velocity vector at each point in space of a time-independent fluid flow.

Taking the interpretation with arrows into account and realizing that the use of arrows implicitly takes $\mathbb{F} = \mathbb{R}$, we see that an appropriate generalization in the case of a smooth manifold M is this: a vector field attaches to each p in M a member of the tangent space $T_p(M)$. Let us make this definition more precise.

If M is a smooth n-dimensional manifold, let

$$T(M) = \{(p, L) \mid p \in M \text{ and } L \in T_p(M)\},$$

and let $\pi : T(M) \to M$ be the projection to the first coordinate. A **vector field** X on an open subset U of M is a function from U to $T(M)$ such that $\pi \circ X$ is the identity on U; so X is indeed a function whose value at any point p is a tangent vector at p. The value of X at p will be written X_p.

We shall be mostly interested in vector fields that are "smooth." Ultimately this smoothness will be defined by making $T(M)$ into a smooth manifold known as the **tangent bundle** of M. The local structure of this smooth manifold is easily accessible via Proposition 8.5. That proposition shows that having a chart κ of M singles out an ordered basis of the tangent space at each point in M_κ. Identifying all these tangent spaces with $\mathbb{F}^n$ by means of this ordered basis, we obtain an identification of $\{(p, L) \mid p \in M_\kappa \text{ and } L \in T_p(M)\}$ with $M_\kappa \times \mathbb{F}^n$ and hence with $\widetilde{M}_\kappa \times \mathbb{F}^n$. The result is a chart for $T(M)$ that we shall include in our atlas. It will be fairly easy to see how these charts are to be patched together compatibly. The problem in obtaining the structure of a smooth manifold is in proving that $T(M)$ is Hausdorff. Although the Hausdorff property may look evident at first glance, it perhaps looks equally evident for the example with $\mathbb{R}^+$ and $\mathbb{R}^-$ in the previous section, and there the Hausdorff property fails. Thus some care is appropriate. We shall study this matter more carefully in Section 3 and complete the construction of the smooth structure on the tangent bundle in Section 4.

For now we shall proceed with a more utilitarian definition of smoothness of a vector field. A vector field X on M carries $C^\infty(U)$, for any open subset U of M, to a space of functions on M by the rule $(Xf)(p) = X_p(f)$. We say that the vector field X on M is **smooth** if Xf is in $C^\infty(U)$ whenever U is open in M and f is in $C^\infty(U)$.

Proposition 8.8. Let X be a vector field on a smooth n-dimensional manifold M. If $\kappa = (x_1, \dots, x_n)$ is a compatible chart and if f is in $C^\infty(M_\kappa)$, then

$$Xf(p) = \sum_i \frac{\partial f}{\partial x_i}(p)\,(Xx_i)(p) \qquad \text{for } p \in M_\kappa.$$

The vector field X is smooth if and only if Xx_i is smooth for each coordinate function x_i of each compatible chart on M.

PROOF. The displayed formula is immediate from Proposition 8.5. To see that if X is smooth, then Xx_i is smooth on M_κ, let q be a point of M_κ and choose, by Proposition 8.2, a function g in $C^\infty(M)$ such that $g = x_i$ in a neighborhood of q. Then $\frac{\partial g}{\partial x_j}(p) = \delta_{ij}$ identically for p in that neighborhood of q. The displayed formula shows that $Xg(p) = Xx_i(p)$ for p in that neighborhood. Since Xg is smooth everywhere, Xx_i must be smooth in that neighborhood of q.

Conversely suppose that each Xx_i is smooth. Let f be in $C^\infty(M)$. Since $\frac{\partial f}{\partial x_i}(p)$ means $\frac{\partial(f\circ\kappa^{-1})}{\partial u_i}\Big|_{u=\kappa(p)}$ and since $f\circ\kappa^{-1}$ is in $C^\infty(\widetilde{M}_\kappa)$, the function $p \mapsto \frac{\partial f}{\partial x_i}(p)$ is in $C^\infty(U)$. Since each Xx_i is in $C^\infty(M_\kappa)$ by assumption, $Xf\big|_{M_\kappa}$ is in $C^\infty(M_\kappa)$. Then Xf must be $C^\infty(M)$ because the compatible chart κ is arbitrary. $\square$

A smooth **curve** $c(t)$ on the smooth manifold M is a smooth function c from an open interval of $\mathbb{R}^1$ into M. The smooth curve $c(t)$ is an **integral curve** for a smooth real-valued vector field X if $X_{c(t)} = dc_t\big(\frac{d}{dt}\big)$ for all t in the domain of c. Integral curves in open subsets of Euclidean space were discussed in Section IV.2 of *Basic*. We shall now transform those results into results about integral curves on smooth manifolds.

Let M be a smooth manifold of dimension n, let $\kappa = (x_1, \dots, x_n)$ be a compatible chart, and let $X = \sum_{j=0}^{n} a_j(x) \frac{\partial}{\partial x_j}$ be the local expression from Proposition 8.8 for a smooth real-valued vector field X on M within M_κ, so that a_j is in $C^\infty(M_\kappa, \mathbb{R})$. Let $c(t)$ be a smooth curve on U. Define $b_j(y) = a_j(\kappa^{-1}(y))$ for $y \in \widetilde{M}_\kappa \subseteq \mathbb{R}^n$, and let $y(t) = (y_1(t), \dots, y_n(t)) = \kappa(c(t))$, so that $y(t)$ is a smooth curve on $\widetilde{M}_\kappa$. Then we have

$$\begin{aligned} X_{c(t)} f &= \sum_{i=1}^{n} \Big[a_i(x)\,\frac{\partial f}{\partial x_i}\Big]_{c(t)} = \sum_{i=1}^{n} (a_i \circ \kappa^{-1}) \circ (\kappa(c(t))\Big[\frac{\partial f}{\partial x_i}\Big]_{c(t)} \\ &= \sum_{i=1}^{n} b_i(y(t))\Big[\frac{\partial f}{\partial x_i}\Big]_{c(t)} \end{aligned}$$

and

$$\begin{aligned} dc_t\Big(\frac{d}{dt}\Big)(f) &= \frac{d}{dt}(f \circ c)(t) = \frac{d}{dt}(f \circ \kappa^{-1} \circ y)(t) \\ &= \sum_{i=1}^{n} \Big[\frac{\partial (f \circ \kappa^{-1})}{\partial u_i}\Big]_{u=y(t)}\Big[\frac{dy_i(t)}{dt}\Big]_t = \sum_{i=1}^{n} \Big[\frac{dy_i(t)}{dt}\Big]_t\Big[\frac{\partial f}{\partial x_i}\Big]_{c(t)}. \end{aligned}$$

The two left sides are equal for all f, i.e., $c(t)$ is an integral curve for X on M_κ in M, if and only if the two right sides are equal for all f, i.e., $y(t)$ satisfies

$$\frac{dy_j}{dt} = b_j(y) \qquad \text{for } 1 \le j \le n.$$

The latter condition is the condition for $y(t)$ to be an integral curve for the vector field $\sum_{j=0}^{n} b_j(y) \frac{\partial}{\partial y_j}$ on $\widetilde{M}_\kappa$ in $\mathbb{R}^n$. Applying Proposition 4.4 of *Basic*, which in turn is an immediate consequence of the standard existence-uniqueness results for systems of ordinary differential equations, we obtain the following generalization to manifolds.

Proposition 8.9. Let X be a smooth real-valued vector field on a smooth manifold M, and let p be in M. Then there exist an $\varepsilon > 0$ and an integral curve $c(t)$ defined for $-\varepsilon < t < \varepsilon$ such that $c(0) = p$. Any two integral curves c and d for X having $c(0) = d(0) = p$ coincide on the intersection of their domains.

As in the Euclidean case, the interest is not only in Proposition 8.9 in isolation but also in what happens to the integral curves when X is part of a family of vector fields.

Proposition 8.10. Let $X^{(1)}, \dots, X^{(m)}$ be smooth real-valued vector fields on a smooth n-dimensional manifold M, and let p be in M. Let V be a bounded open neighborhood of 0 in $\mathbb{R}^m$. For λ in V, put $X_\lambda = \sum_{j=1}^m \lambda_j X^{(j)}$. Then there exist an $\varepsilon > 0$ and a system of integral curves $c(t, \lambda)$, defined for $t \in (-\varepsilon, \varepsilon)$ and $\lambda \in V$, such that $c(\,\cdot\,, \lambda)$ is an integral curve for X_λ with $c(0, \lambda) = p$. Each curve $c(t, \lambda)$ is unique, and the function $c : (-\varepsilon, \varepsilon) \times V \to M$ is smooth. If $m = n$, if the vectors $X^{(1)}(p), \dots, X^{(n)}(p)$ are linearly independent, and if δ is any positive number less than ε, then $c(\delta, \,\cdot\,)$ is a diffeomorphism from an open subneighborhood of 0 (depending on δ) onto an open subset of M, and its inverse defines a chart about p.

PROOF. All but the last sentence is just a translation of Proposition 4.5 of *Basic* into the setting with manifolds. For the last sentence, Proposition 4.5 of *Basic* establishes that the the Jacobian matrix at $\lambda = 0$ of the function $\lambda \mapsto c(\delta, \lambda)$ transferred to Euclidean space is nonsingular, and the rest follows from Proposition 8.4. $\square$

3. Identification Spaces

We saw in a 1-dimensional example in Section 1 that the Hausdorff condition is subtle (and does not always hold) when one tries to build a smooth manifold out of smooth charts. In Section 2 we saw that it would be desirable to obtain a smooth manifold structure on the tangent bundle of a smooth manifold in order to make the definition of smoothness of vector fields more evident from the smooth structure, and the natural way of proceeding was to piece the structure together from charts that were products of charts for the smooth manifold by mappings on whole Euclidean spaces. The example in Section 1 serves as a reminder, however, that we should not take the Hausdorff condition for granted in working with the tangent bundle.

In fact, the construction in both instances appears in a number of important situations in mathematics. One is in constructing "vector bundles" and more general "fiber bundles" out of local data, and another is in constructing covering spaces in the theory of fundamental groups. Still a third is in the construction of restricted direct products[4] in Problem 30 in Chapter IV.

[4] In fairness it should be said that restricted direct products, which involve a direct limit, are more easily handled by the method in Chapter IV than by a construction analogous to that of the tangent bundle.

For a clearer picture of what is happening, let us abstract the situation. The idea is to build complicated topological spaces out of simpler ones by piecing together local data. For lack of a better name for the abstract construction, we shall call the result an "identification space." A simple example of the use of charts in defining manifold structures will point the way to the general definition.

EXAMPLE. Suppose, by way of being concrete, that we have overlapping open sets U_1 and U_2 in $\mathbb{R}^n$. We take U_1 and U_2 as completely understood, and we want to describe $U_1 \cup U_2$ as a topological space. Let X be the **disjoint union** of U_1 and U_2, which we write as $X = U_1 \sqcup U_2$. By definition, X as a set is the set of all pairs (x, i) with x in U_i, and i takes on the values 1 and 2. We identify $U_1 \subseteq U_1 \sqcup U_2$ with the set of pairs $(x, 1)$ and $U_2 \subseteq U_1 \sqcup U_2$ with the set of pairs $(y, 2)$. A subset E of X is defined to be open if $E \cap U_1$ is open in U_1 and $E \cap U_2$ is open in U_2. The resulting collection of open sets is a topology for X, and the embedded copies of U_1 and U_2 in X are open. We define $(x, 1) \sim (y, 2)$ if $x = y$ as members of $\mathbb{R}^n$, and the identification space is $X/\sim$. We give $X/\sim$ the quotient topology, and it is not hard to see that $X/\sim$ is homeomorphic to the union $U_1 \cup U_2$ as a topological subspace of the metric space $\mathbb{R}^n$.

Let us come to the general definition. We are given a set of topological spaces W_i for i in some nonempty index set I, and we assume, for each ordered pair (i, j), that we have a homeomorphism ψ_{ji} of an open subset W_{ji} of W_i onto an open subset W_{ij} of W_j (possibly with W_{ji} and W_{ij} both empty) such that

(i) ψ_{ii} is the identity on $W_{ii} = W_i$,
(ii) $\psi_{ij} \circ \psi_{ji}$ is the identity on W_{ji}, and
(iii) $W_{ki} \cap W_{ji} = \psi_{ij}(W_{kj} \cap W_{ij})$, and $\psi_{kj} \circ \psi_{ji} = \psi_{ki}$ on this set.

We form the disjoint union $X = \bigsqcup_i W_i$, i.e., the set of pairs (x, i) with x in W_i. We topologize X by requiring that each W_i be open in X. Then we introduce a relation $\sim$ on X by saying that $(x, i) \sim (y, j)$ if $\psi_{ji}(x) = y$. The three properties (i), (ii), and (iii) show that $\sim$ is an equivalence relation, and $X/\sim$ is called an **identification space**. It is given the quotient topology.

Let us see the effect of this construction in the special case that we reconstruct a general smooth n-dimensional manifold out of an atlas of its charts. If κ_i is a chart in the atlas, we take W_i to be the image $\widetilde{M}_{\kappa_i}$ of κ_i. With two such charts κ_i and κ_j, define

$$W_{ji} = \kappa_i(\widetilde{M}_{\kappa_i} \cap \widetilde{M}_{\kappa_j}), \qquad W_{ij} = \kappa_j(\widetilde{M}_{\kappa_i} \cap \widetilde{M}_{\kappa_j}), \qquad \psi_{ji} = \kappa_j \circ \kappa_i^{-1}.$$

It is a routine matter to check (i), (ii), and (iii). The disjoint union $\bigsqcup_i \kappa_i^{-1}$ of the maps κ_i^{-1} is a continuous open function from $X = \bigsqcup_i W_i$ onto M. Let $q : X \to X/\sim$ be the quotient map. If $(x, i) \sim (y, j)$, then $\psi_{ji}(x) = y$ and hence $\kappa_j \circ \kappa_i^{-1}(x) = y$ and $\kappa_i^{-1}(x) = \kappa_j^{-1}(y)$. Thus equivalent points in X map to the same point in M, and we obtain a factorization $\bigsqcup_i \kappa_i^{-1} = \varphi \circ q$ for a continuous open map $\varphi : X/\sim \to M$. Since the only identifications in M are the ones determined by the charts, i.e., the ones of the form $(x, i) \sim (y, j)$ as above, φ is one-one and consequently is a homeomorphism. We can recover the charts of M as well, since the restriction of q to a single W_i is one-one. The i^{th} chart is the function $q^{-1} \circ \varphi^{-1}\big|_{M_{\kappa_i}} : M_{\kappa_i} \to \widetilde{M}_{\kappa_i}$.

Thus an identification space is a suitable device for reconstructing a smooth manifold from its charts. We can therefore try to use identification spaces to build new smooth manifolds out of what ought to be their charts. Proposition 8.11 below simplifies the checking of the Hausdorff condition. Proposition 8.12 shows, under natural additional assumptions, that the identification space is a smooth manifold if it has been shown to be Hausdorff.

Proposition 8.11. In the situation of an identification space formed from a disjoint union $X = \bigsqcup_i W_i$ and an equivalence relation $\sim$, the quotient mapping $q : X \to X/\sim$ is necessarily open. Consequently the identification space $X/\sim$ is Hausdorff if and only if the set of equivalent pairs in $X \times X$ is closed.

REMARKS. In applications we may expect that the given topological spaces W_i are Hausdorff, and then their disjoint union X will be Hausdorff, and so will $X \times X$. In this case the theory of nets becomes a handy tool for deciding whether the set of equivalent pairs within $X \times X$ is closed. Thus suppose we have nets with $x_\alpha \sim y_\alpha$ in X and that $x_\alpha \to x_0$ and $y_\alpha \to y_0$. We are to prove that $x_0 \sim y_0$. Let x_0 be in W_i, and let y_0 be in W_j. Since W_i and W_j are open in X, x_α is eventually in W_i and y_α is eventually in W_j. In other words, the Hausdorff condition depends on only two sets W_i at a time and is as follows: We may assume that x_α and x_0 are in W_i with $x_\alpha \to x_0$, that y_α and y_0 are in W_j with $y_\alpha \to y_0$, and that $x_\alpha \sim y_\alpha$ for all α. What needs proof is that $x_0 \sim y_0$.

PROOF. The second statement follows from the first in view of Proposition 10.40 of *Basic*. Thus we have only to show that the quotient map is open. If U is open in X, we are to show that $q^{-1}(q(U))$ is open in X. The direct image of a function respects arbitrary unions, and thus $q(U) = \bigcup_j q(U \cap W_j)$. Hence $q^{-1}(q(U)) = \bigcup_j q^{-1}(q(U \cap W_j))$, and it is enough to prove that a single $q^{-1}(q(U \cap W_j))$ is open. Since X is the disjoint union of the open sets W_i, it is enough to prove that each $W_i \cap q^{-1}(q(U \cap W_j))$ is open. This intersection is the subset of elements in W_i that get identified with elements in $U \cap W_j$, namely $\psi_{ij}(U \cap W_{ij})$. Since ψ_{ij} is a homeomorphism of W_{ij} with W_{ji}, the set $\psi_{ij}(U \cap W_{ij})$ is open in W_{ji}. Since W_{ji} is open in W_i, $\psi_{ij}(U \cap W_{ij})$ is open in W_i. $\square$

Proposition 8.12. Let the topological space M be obtained as an identification space from a disjoint union $X = \bigsqcup_i W_i$ in which each W_i is an open subset of $\mathbb{R}^n$. Suppose that each identification $\psi_{ji} : W_{ji} \to W_{ij}$ is a smooth function, and suppose that $q : X \to M$ denotes the quotient mapping. Assume that the set of equivalent pairs in $X \times X$ is a closed subset, so that M is a Hausdorff space. Then M becomes a smooth n-dimensional manifold under the following definition of an atlas of compatible charts: For each i, let $U_i = q(W_i)$, and define $\kappa_i : U_i \to W_i$ to be the inverse of $q\big|_{W_i} : W_i \to U_i$. The charts of the atlas are the maps κ_i.

PROOF. The mapping q is open according to Proposition 8.11. Since W_i is open in X, $U_i = q(W_i)$ is open in M. To see that q is one-one from W_i to U_i, suppose that two members of W_i are equivalent. We know that the members of W_i are of the form (w, i), and the equivalence relation is given by the statement

$$(w_i, i) \sim (w_j, j) \quad \text{if and only if} \quad \psi_{ji}(w_i) = w_j. \tag{$*$}$$

In particular w_i must be in the domain of ψ_{ji}, which is W_{ji}. Then two members of W_i, say (w, i) and (w', i), can be equivalent only if $\psi_{ii}(w) = w'$. Since ψ_{ii} is the identity function, $w = w'$. Therefore q is one-one on W_i and is a homeomorphism of W_i onto the open subset U_i of M. Consequently κ_i is well defined as a homeomorphism of the open subset U_i of M with the open subset W_i of Euclidean space $\mathbb{R}^n$.

We have to check the compatibility of the charts. We have

$$\begin{aligned} U_i \cap U_j &= q(W_i) \cap q(W_j) \\ &= \big\{\text{classes of } \{q(w_i, i) \mid \psi_{ji} \text{ is defined on } w_i\}\big\} = q(W_{ji}). \end{aligned}$$

Then

$$\kappa_i(U_i \cap U_j) = \kappa_i\big((q\big|_{W_i})(W_{ji})\big) = W_{ji},$$

and similarly $\kappa_j(U_i \cap U_j) = W_{ij}$. Hence $\kappa_j \circ \kappa_i^{-1}$ carries W_{ji} onto W_{ij}. If (w_i, i) is a member of W_{ji}, we show that

$$\kappa_j(\kappa_i^{-1}((w_i, i))) = (\psi_{ji}(w_i), j). \tag{$**$}$$

If we drop the second entries of our pairs, which are present only to emphasize that X is a disjoint union, equation $(**)$ says that $\kappa_j \circ \kappa_i^{-1}$ equals ψ_{ji} on W_{ji}. Since ψ_{ji} is smooth by assumption, the verification of $(**)$ will therefore complete the proof of the proposition. Taking $(*)$ into account, we have

$$\kappa_i^{-1}((w_i, i)) = q((w_i, i)) = q((\psi_{ji}(w_i), j)) = \kappa_j^{-1}((\psi_{ji}(w_i), j)).$$

Application of κ_j to both sides of this identity yields $(**)$ and thus completes the proof. $\square$

4. Vector Bundles

In this section we introduce general vector bundles over a smooth manifold M. Of particular interest are the tangent and cotangent bundles. The tangent bundle as a set is to be identifiable with $\bigcup_{p\in M} T_p(M)$, and one realization of the cotangent bundle as a set will be the same kind of union of the dual vector spaces $T_p^*(M)$ to $T_p(M)$. To construct these bundles as manifolds, we shall form them as identification spaces in the sense of Section 3, and that step will be carried out in this section.

We continue with the convention of writing $\mathbb{F}$ for the field of scalars, which is to be $\mathbb{R}$ or $\mathbb{C}$. The fiber of any vector bundle will be $\mathbb{F}^n$ for some n, and we speak of real and complex vector bundles in the two cases.

Let M be a smooth manifold of dimension m, and let $\{\kappa\}$ be an atlas of compatible charts, where κ is the map $\kappa : M_\kappa \to \widetilde{M}_\kappa$. Denote by $GL(n, \mathbb{F})$ the general linear group of all n-by-n nonsingular matrices with entries in $\mathbb{F}$. A smooth **coordinate vector bundle** of **rank** n over M relative to this atlas consists of a smooth manifold B called the **bundle space**, a smooth mapping π of B onto M called the **projection** from the bundle space to the **base space** M, and diffeomorphisms $\phi_\kappa : M_\kappa \times \mathbb{F}^n \to \pi^{-1}(M_\kappa)$ called the **coordinate functions** such that

(i) $\pi\phi_\kappa(p, v) = p$ for $p \in M_\kappa$ and $v \in \mathbb{F}^n$,

(ii) the smooth maps $\phi_{\kappa,p} : \mathbb{F}^n \to \pi^{-1}(M_\kappa)$ defined for p in M_κ by $\phi_{\kappa,p}(v) = \phi_\kappa(p, v)$ are such that $\phi_{\kappa',p}^{-1} \circ \phi_{\kappa,p} : \mathbb{F}^n \to \mathbb{F}^n$ is in $GL(n, \mathbb{F})$ for each κ and κ' and for all p in $M_\kappa \cap M_{\kappa'}$,

(iii) the map $g_{\kappa'\kappa} : M_\kappa \cap M_{\kappa'} \to GL(n, \mathbb{F})$ defined by $g_{\kappa'\kappa}(p) = \phi_{\kappa',p}^{-1} \circ \phi_{\kappa,p}$ is smooth.

The maps $p \mapsto g_{\kappa'\kappa}(p)$ will be called the **transition functions**[5] of the coordinate vector bundle.

An atlas of compatible charts of the coordinate vector bundle may be taken to consist of the maps $(\kappa \times 1) \circ \phi_\kappa^{-1} : \pi^{-1}(M_\kappa) \to \widetilde{M}_\kappa \times \mathbb{F}^n$. The dimension of B is $m + n$ if $\mathbb{F} = \mathbb{R}$ and is $m + 2n$ if $\mathbb{F} = \mathbb{C}$.

EXAMPLE. Data for the tangent bundle. Although we have not yet introduced the topology on the bundle space in this instance, we can identify the functions ϕ_κ, $\phi_{\kappa'}$, and $g_{\kappa'\kappa}$ explicitly. Let the local expressions for κ and κ' be $\kappa = (x_1, \dots, x_n)$ and $\kappa' = (y_1, \dots, y_n)$. Let $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ and $d = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$ be members of $\mathbb{F}^n$. The set $\pi^{-1}(M_\kappa)$ is to consist of all tangent vectors at points of M_κ, and Proposition

[5] The terms **coordinate transformations** and **transition matrices** are used also.

8.5 shows that these are all expressions $\sum_{j=1}^n c_j\big[\frac{\partial}{\partial x_j}\big]_p$, where $\big[\frac{\partial f}{\partial x_j}\big]_p$ concretely means $\frac{\partial(f\circ\kappa^{-1})}{\partial u_j}(\kappa(p))$. The formulas for ϕ_κ and $\phi_{\kappa'}$ are then

$$\phi_{\kappa,p}(c)=\sum_{j=1}^n c_j\big[\tfrac{\partial}{\partial x_j}\big]_p$$

and

$$\phi_{\kappa',p}(d)=\sum_{j=1}^n d_j\big[\tfrac{\partial}{\partial y_j}\big]_p.$$

The other relevant formula is the formula for the matrix of the differential of a smooth mapping relative to compatible charts in the domain and range. The formula is given in Proposition 8.6 and is

$$dF_p\big(\big[\tfrac{\partial}{\partial x_j}\big]_p\big)=\sum_{i=1}^n \big[\tfrac{\partial F_i}{\partial x_j}\big]_p\big[\tfrac{\partial}{\partial y_i}\big]_p.$$

We apply this formula with F equal to the identity mapping, whose local expression is $\kappa'\circ\kappa^{-1}$ and therefore has $F_i=y_i\circ\kappa^{-1}$. Since the differential of the identity is the identity, we have

$$\big[\tfrac{\partial}{\partial x_j}\big]_p=\sum_{i=1}^n \big[\tfrac{\partial y_i}{\partial x_j}\big]_p\big[\tfrac{\partial}{\partial y_i}\big]_p.$$

Substituting into the formula for $\phi_{\kappa,p}(c)$, we obtain

$$\phi_{\kappa,p}(c)=\sum_{i=1}^n\Big(\sum_{j=1}^n c_j\big[\tfrac{\partial y_i}{\partial x_j}\big]_p\Big)\big[\tfrac{\partial}{\partial y_i}\big]_p.$$

Therefore $\phi_{\kappa',p}^{-1}\phi_{\kappa,p}(c)=d$, where $d_i=\sum_{j=1}^n c_j\big[\frac{\partial y_i}{\partial x_j}\big]_p=\Big(\big[\frac{\partial y_i}{\partial x_j}\big]_p c\Big)_i$, and we conclude that

$$g_{\kappa'\kappa}(p)=\big[\tfrac{\partial y_i}{\partial x_j}\big]_p.$$

Returning to case of a general coordinate vector bundle, let us observe a simple property of the transition functions.

Proposition 8.13. Let M be an m-dimensional smooth manifold M, fix an atlas $\{\kappa\}$ for M, and let $\pi: B\to M$ be a smooth vector bundle of rank n with transition functions $p\mapsto g_{\kappa'\kappa}(p)$. Then

$$g_{\kappa''\kappa'}(p)g_{\kappa'\kappa}(p)=g_{\kappa''\kappa}(p)\qquad\text{for all } p\in M_\kappa\cap M_{\kappa'}\cap M_{\kappa''}.$$

Consequently the transition functions satisfy the identities $g_{\kappa\kappa}(p)=1$ for $p\in M_\kappa$ and $g_{\kappa\kappa'}(p)=(g_{\kappa'\kappa}(p))^{-1}$ for $p\in M_\kappa\cap M_{\kappa'}$.

PROOF. We have $g_{\kappa''\kappa'}(p)g_{\kappa'\kappa}(p)=\phi_{\kappa'',p}^{-1}\phi_{\kappa',p}\phi_{\kappa',p}^{-1}\phi_{\kappa,p}=\phi_{\kappa'',p}^{-1}\phi_{\kappa,p}=g_{\kappa''\kappa}(p)$. Putting $\kappa=\kappa'=\kappa''$ yields $g_{\kappa\kappa}(p)g_{\kappa\kappa}(p)=g_{\kappa\kappa}(p)$; thus $g_{\kappa\kappa}(p)=1$. Putting $\kappa=\kappa''$ yields $g_{\kappa\kappa'}(p)g_{\kappa'\kappa}(p)=g_{\kappa\kappa}(p)=1$. $\square$

The main abstract result about vector bundles for our purposes will be a converse to Proposition 8.13, enabling us to construct a vector bundle from an atlas of M and a system of smooth functions $p \mapsto g_{\kappa'\kappa}(p)$ defined on $M_\kappa \cap M_{\kappa''}$ if these functions satisfy the conditions of the proposition. This result will be given as Proposition 8.14 below. In the case of the tangent bundle, we saw above that $g_{\kappa'\kappa}(p)$ is given by $g_{\kappa'\kappa}(p) = \big[\frac{\partial y_i}{\partial x_j}\big]_p$. The identity $g_{\kappa''\kappa'}(p)g_{\kappa'\kappa}(p) = g_{\kappa''\kappa}(p)$ follows from the chain rule, and thus the abstract result will complete the construction of the tangent bundle as a smooth manifold. We shall construct the cotangent bundle similarly.

One can equally construct other vector bundles of interest in analysis, as we shall see, but we shall omit the details for most of these. It is fairly clear from the example above that one can make local calculations with vector bundles by working with the transition functions. Here is an example.

EXAMPLE. Suppose for a particular coordinate vector bundle that we have a system of functions $f_\kappa : \widetilde{M}_\kappa \times \mathbb{F}^n \to S$ with range equal to some set S independent of κ. Let us determine the circumstances under which the system $\{f_\kappa\}$ is the local form of some globally defined function $f : B \to S$. A necessary and sufficient condition is that whenever $(x, v) \in \widetilde{M}_\kappa \times \mathbb{F}^n$ and $(y, v') \in \widetilde{M}_{\kappa'} \times \mathbb{F}^n$ correspond to the same point of B, then $f_\kappa(x, v) = f_{\kappa'}(y, v')$. The maps from $\widetilde{M}_\kappa \times \mathbb{F}^n$ and $\widetilde{M}_{\kappa'} \times \mathbb{F}^n$ into B are $\phi_\kappa \circ (\kappa^{-1} \times 1)$ and $\phi_{\kappa'} \circ (\kappa'^{-1} \times 1)$. Thus (x, v) and (y, v') correspond to the same member of B if and only if $\phi_\kappa(\kappa^{-1}x, v) = \phi_{\kappa'}(\kappa'^{-1}y, v')$. We must have $\kappa^{-1}x = \kappa'^{-1}y$ for this equality. In this case let us put $p = \kappa^{-1}x = \kappa'^{-1}y$, and then it is necessary and sufficient that $\phi_{\kappa,p}(v) = \phi_{\kappa',p}(v')$, hence that $\phi_{\kappa',p}^{-1} \circ \phi_{\kappa,p}(v) = v'$, hence that $g_{\kappa'\kappa}(p)(v) = v'$. Thus (x, v) and (y, v') correspond to the same point in B if and only if $y = \kappa'\kappa^{-1}x$ and $g_{\kappa'\kappa}(\kappa^{-1}x)(v) = v'$. Consequently the functions f_κ define a global f if and only if

$$\boxed{f_\kappa(x, v) = f_{\kappa'}\big(\kappa'\kappa^{-1}x, g_{\kappa'\kappa}(\kappa^{-1}x)(v)\big)}$$

whenever $\kappa'\kappa^{-1}x$ is defined. In the case of the tangent bundle, we saw in the previous example that $g_{\kappa'\kappa} = \big[\frac{\partial y_i(x)}{\partial x_j}\big]$. Thus the condition is that

$$f_\kappa(x, v) = f_{\kappa'}\big(y, \big[\tfrac{\partial y_i(x)}{\partial x_j}\big](v)\big)$$

whenever $y = \kappa'\kappa^{-1}(x)$; here the fiber dimension n is also the dimension of the base manifold M.

Before getting to the converse result to Proposition 8.13, let us address the question of when, for given n, $\mathbb{F}$, M, B, and π, we get the "same" coordinate vector bundle from a different but compatible atlas $\{\lambda\}$ and different coordinate functions ϕ_λ. The condition that we impose, which is called **strict equivalence**, is that if we set up the transition functions corresponding to a member κ of the first atlas and a member λ of the second atlas, namely

$$\bar{g}_{\lambda\kappa}(p) = {\phi'_{\lambda,p}}^{-1} \circ \phi_{\kappa,p} \qquad \text{for } p \in M_\kappa \cap M_\lambda,$$

then each $\bar{g}_{\lambda\kappa}(p)$ lies in $GL(n, \mathbb{F})$ and the function $p \mapsto \bar{g}_{\lambda\kappa}(p)$ is smooth from $M_\kappa \cap M_\lambda$ into $GL(n, \mathbb{F})$. In other words, strict equivalence means that the union of the two atlases, along with the associated data, is to make $\pi : B \to M$ into a coordinate vector bundle. Strict equivalence is certainly reflexive and symmetric. Since we can discard some charts from the construction of a coordinate vector bundle as long as the remaining charts cover M, strict equivalence is transitive. An equivalence class of strictly equivalent coordinate vector bundles is called a **vector bundle**, real or complex according as $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

With the definition of smooth structure for a smooth manifold, we were able to make the atlas canonical by assuming that it was maximal. Every atlas of compatible charts could be extended to one and only one maximal such atlas, and therefore smooth manifolds were determined by specifying any atlas of compatible charts, not necessarily a maximal one. We do not have to address the corresponding question about vector bundles—whether the atlas of M used in defining a coordinate vector bundle can be enlarged to a maximal atlas of M and still define a coordinate vector bundle. The reason is that the specific vector bundles with which we work are all definable immediately by maximal atlases of M.

Now let us proceed with the converse result.

Proposition 8.14. If a smooth m-dimensional manifold M is given, along with an atlas $\{\kappa\}$ of compatible charts and a system of smooth functions $g_{\kappa'\kappa} : M_\kappa \cap M_{\kappa'} \to GL(n, \mathbb{F})$ satisfying the property $g_{\kappa''\kappa'}(p)g_{\kappa'\kappa}(p) = g_{\kappa''\kappa}(p)$ for all p in $M_\kappa \cap M_{\kappa'} \cap M_{\kappa''}$, then there exists a coordinate vector bundle $\pi : B \to M$ with the functions $g_{\kappa'\kappa}$ as transition functions. The result is unique in the following sense: if $\pi : B \to M$ and $\pi' : B' \to M$ are two such coordinate vector bundles, with coordinate functions ϕ_κ and ϕ'_κ, then there exists a diffeomorphism $h : B \to B'$ such that $\pi' \circ h = \pi$ and $\phi'_\kappa = h \circ \phi_\kappa$ for all charts κ in the atlas.

PROOF OF UNIQUENESS OF COORDINATE VECTOR BUNDLE UP TO FUNCTION h. Define a diffeomorphism $h_\kappa : \pi^{-1}(M_\kappa) \to \pi'^{-1}(M_\kappa)$ by $h_\kappa = \phi'_\kappa \circ \phi_\kappa^{-1}$, so that $h_\kappa \circ \phi_\kappa = \phi'_\kappa$. Evaluating both sides at $(p, \mathbb{F}^n)$ with p in M_κ, we obtain $h_\kappa(\pi^{-1}(p)) = \pi'^{-1}(p)$. Thus $\pi' \circ h_\kappa = \pi$ on $\pi^{-1}(M_\kappa)$.

Since the map $h_{\kappa,p} = h_\kappa\big|_{\pi^{-1}(p)}$ carries $\pi^{-1}(p)$ to $\pi'^{-1}(p)$, we can write $h_{\kappa,p} \circ \phi_{\kappa,p} = \phi'_{\kappa,p}$. If p is also in $M_{\kappa'}$, then we have $h_{\kappa',p} \circ \phi_{\kappa',p} = \phi'_{\kappa',p}$

as well. Since B and B' are assumed to have the same transition functions, $g_{\kappa'\kappa}(p) = {\phi_{\kappa',p}}^{-1}\phi_{\kappa,p} = {\phi'_{\kappa',p}}^{-1}\phi'_{\kappa,p}$; in other words, $\phi_{\kappa',p}g_{\kappa'\kappa}(p) = \phi_{\kappa,p}$ and $\phi'_{\kappa',p}g_{\kappa'\kappa}(p) = \phi'_{\kappa,p}$. Therefore

$$h_{\kappa,p}\phi_{\kappa,p} = \phi'_{\kappa,p} = \phi'_{\kappa',p}g_{\kappa'\kappa}(p) = h_{\kappa',p}\phi_{\kappa',p}g_{\kappa'\kappa}(p) = h_{\kappa',p}\phi_{\kappa,p},$$

and we obtain $h_{\kappa,p} = h_{\kappa',p}$. Thus the functions h_κ are consistently defined on their common domains and fit together as a global diffeomorphism of B onto B'. □

PROOF OF EXISTENCE OF COORDINATE VECTOR BUNDLE. Let us construct B as an identification space. We are writing $\widetilde{M}_\kappa$ for $\kappa(M_\kappa)$, and we put $\widetilde{M}_{\kappa'\kappa} = \kappa(M_\kappa \cap M_{\kappa'})$. Define $W_\kappa = \widetilde{M}_\kappa \times \mathbb{F}^n$ and $W_{\kappa'\kappa} = \widetilde{M}_{\kappa'\kappa} \times \mathbb{F}^n$, and let

$$\psi_{\kappa'\kappa}(\widetilde{m}, v) = \big(\kappa'\kappa^{-1}(\widetilde{m}), g_{\kappa'\kappa}(\kappa^{-1}(\widetilde{m}))(v)\big) \qquad \text{for } (\widetilde{m}, v) \in W_{\kappa'\kappa}.$$

We shall prove that $X = \bigsqcup_\kappa W_\kappa$, together with the functions $\psi_{\kappa'\kappa}$, defines an identification space $B = X/\sim$. We have to check (i), (ii), and (iii) in Section 3. For (i), we need that $\psi_{\kappa\kappa}$ is the identity on $W_{\kappa\kappa} = W_\kappa$, and the computation is

$$\psi_{\kappa\kappa}(\widetilde{m}, v) = \big(\widetilde{m}, g_{\kappa\kappa}(\kappa^{-1}(\widetilde{m}))(v)\big) = (\widetilde{m}, v)$$

since $g_{\kappa\kappa}(\,\cdot\,)$ is identically the identity matrix. For (ii), we need that $\psi_{\kappa\kappa'}\psi_{\kappa'\kappa}$ is the identity on $W_{\kappa'\kappa}$. The composition on $(\widetilde{m}, v)$ is

$$\begin{aligned}
&\psi_{\kappa\kappa'}\big(\kappa'\kappa^{-1}(\widetilde{m}), g_{\kappa'\kappa}(\kappa^{-1}(\widetilde{m}))(v)\big)\\
&\qquad = \big(\kappa\kappa'^{-1}\kappa'\kappa^{-1}(\widetilde{m}),\ g_{\kappa\kappa'}(\kappa'^{-1}(\kappa'\kappa^{-1}(\widetilde{m})))g_{\kappa'\kappa}(\kappa^{-1}(\widetilde{m}))(v)\big)\\
&\qquad = \big(\widetilde{m},\ g_{\kappa\kappa'}(\kappa^{-1}(\widetilde{m}))g_{\kappa'\kappa}(\kappa^{-1}(\widetilde{m}))(v)\big).
\end{aligned}$$

The second member of the right side collapses to v since $g_{\kappa\kappa'}(p)g_{\kappa'\kappa}(p) = 1$ for all p in M_κ. This proves (ii). For (iii), we need that $\psi_{\kappa''\kappa'} \circ \psi_{\kappa'\kappa} = \psi_{\kappa''\kappa}$ on the set $W_{\kappa''\kappa} \cap W_{\kappa'\kappa} = \psi_{\kappa\kappa'}(W_{\kappa''\kappa'} \cap W_{\kappa\kappa'})$, and the composition on $(\widetilde{m}, v)$

$$\begin{aligned}
&= \psi_{\kappa''\kappa'}\big((\kappa'\kappa^{-1}(\widetilde{m}), g_{\kappa'\kappa}(\kappa^{-1}(\widetilde{m})(v)\big)\\
&= \big(\kappa''\kappa'^{-1}(\kappa'\kappa^{-1}(\widetilde{m})), g_{\kappa''\kappa'}(\kappa'^{-1}(\kappa'\kappa^{-1}(\widetilde{m})))g_{\kappa'\kappa}(\kappa^{-1}(\widetilde{m}))(v)\big)\\
&= \big(\kappa''\kappa^{-1}(\widetilde{m}), g_{\kappa''\kappa'}(\kappa^{-1}(\widetilde{m}))g_{\kappa'\kappa}(\kappa^{-1}(\widetilde{m}))(v)\big)\\
&= \big(\kappa''\kappa^{-1}(\widetilde{m}), g_{\kappa''\kappa}(\kappa^{-1}(\widetilde{m}))(v)\big)\\
&= \psi_{\kappa''\kappa}(\widetilde{m}, v).
\end{aligned}$$

This proves (iii) and completes the construction of B.

To prove that B is Hausdorff, we apply Proposition 8.11 and its remark. Thus suppose that we have nets with $x_\alpha \sim y_\alpha$ in X, that $x_\alpha \to x_0$ and $y_\alpha \to y_0$, and that x_α and x_0 are in W_κ and y_α and y_0 are in $W_{\kappa'}$. We are to prove that $x_0 \sim y_0$. Write $x_\alpha = (\widetilde{m}_\alpha, v_\alpha)$, $x_0 = (\widetilde{m}_0, v_0)$, $y_\alpha = (\widetilde{m}'_\alpha, v'_\alpha)$, and $y_0 = (\widetilde{m}'_0, v'_0)$. The assumed convergence says that $\widetilde{m}_\alpha \to \widetilde{m}_0$, $v_\alpha \to v_0$, $\widetilde{m}'_\alpha \to \widetilde{m}'_0$, and $v'_\alpha \to v'_0$. The assumed equivalence $x_\alpha \sim y_\alpha$ says that $\psi_{\kappa'\kappa}(\widetilde{m}_\alpha, v_\alpha) = (\widetilde{m}'_\alpha, v'_\alpha)$, i.e.,

$$\kappa'\kappa^{-1}(\widetilde{m}_\alpha) = \widetilde{m}'_\alpha \qquad \text{and} \qquad g_{\kappa'\kappa}(\kappa^{-1}(\widetilde{m}_\alpha))(v_\alpha) = v'_\alpha,$$

and we are to prove that

$$\kappa'\kappa^{-1}(\widetilde{m}_0) = \widetilde{m}'_0 \qquad \text{and} \qquad g_{\kappa'\kappa}(\kappa^{-1}(\widetilde{m}_0))(v_0) = v'_0.$$

The functions $\kappa'\kappa^{-1}$, $g_{\kappa'\kappa}$, and κ^{-1} are continuous, and the only question is whether $\widetilde{m}_0$ is in the domain of $\kappa'\kappa^{-1}$ and $\kappa^{-1}(\widetilde{m}_0)$ is in the domain of $g_{\kappa'\kappa}$, i.e., whether $\widetilde{m}_0$ is in the subset $\widetilde{M}_{\kappa'\kappa} = \kappa(M_\kappa \cap M_{\kappa'})$ of $\widetilde{M}_\kappa = \kappa(M_\kappa)$. Assume the contrary. Then $\widetilde{m}_0$ is on the boundary of $\kappa(M_\kappa \cap M_{\kappa'})$ in $\kappa(M_\kappa)$ and $\widetilde{m}'_0$ is on the boundary of $\kappa'(M_\kappa \cap M_{\kappa'})$ in $\kappa'(M_{\kappa'})$. So $\kappa^{-1}(\widetilde{m}_0)$ is on the boundary of $M_\kappa \cap M_{\kappa'}$ in M_κ, and $\kappa'^{-1}(\widetilde{m}'_0)$ is on the boundary of $M_\kappa \cap M_{\kappa'}$ in $M_{\kappa'}$. This implies that $\kappa^{-1}(\widetilde{m}_0)$ is in M_κ but not $M_{\kappa'}$ while $\kappa'^{-1}(\widetilde{m}'_0)$ is in $M_{\kappa'}$ but not M_κ. Consequently $\kappa^{-1}(\widetilde{m}_0) \neq \kappa'^{-1}(\widetilde{m}'_0)$. Since M is Hausdorff, we can find disjoint open neighborhoods V and V' of $\kappa^{-1}(\widetilde{m}_0)$ and $\kappa'^{-1}(\widetilde{m}'_0)$ in M. Since κ^{-1} is continuous, $\kappa^{-1}(\widetilde{m}_\alpha)$ is eventually in V; since κ'^{-1} is continuous, $\kappa'^{-1}(\widetilde{m}'_\alpha)$ is eventually in V'. Then we cannot have $\kappa^{-1}(\widetilde{m}_\alpha) = \kappa'^{-1}(\widetilde{m}'_\alpha)$ eventually, hence cannot have $\kappa'\kappa^{-1}(\widetilde{m}_\alpha) = \widetilde{m}'_\alpha$ eventually, contradiction. We conclude that B is Hausdorff.

To complete the proof, we exhibit B as a coordinate vector bundle. Let $q : X \to B$ be the quotient map. Application of Proposition 8.12 produces a manifold structure on B, the charts being of the form $\kappa^\# = (q\big|_{W_\kappa})^{-1}$ with domain $q(W_\kappa)$. If p_κ denotes the projection of W_κ on $\widetilde{M}_\kappa$, we define $\pi : q(W_\kappa) \to M$ to be the composition $\kappa^{-1} p_\kappa \kappa^\#$. To have $\pi : B \to M$ be globally defined, we have to check consistency from chart to chart. Thus suppose that $b = q(w_\kappa) = q(w_{\kappa'})$ with $w_\kappa = (\widetilde{m}_\kappa, v_\kappa)$ in W_κ and $w_{\kappa'} = (\widetilde{m}_{\kappa'}, v_{\kappa'})$ in $W_{\kappa'}$. We are to check that $\kappa^{-1} p_\kappa(w_\kappa) = \kappa'^{-1} p_{\kappa'}(w_{\kappa'})$, hence that $\kappa^{-1}(\widetilde{m}_\kappa) = \kappa'^{-1}(\widetilde{m}_{\kappa'})$. The condition $q(w_\kappa) = q(w_{\kappa'})$ means that $w_\kappa \sim w_{\kappa'}$, which means that $\psi_{\kappa'\kappa}(w_\kappa) = w'_\kappa$ and therefore that $\big(\kappa'\kappa^{-1}(\widetilde{m}_\kappa), g_{\kappa'\kappa}(\kappa^{-1}(\widetilde{m}_\kappa))(v_\kappa)\big) = (\widetilde{m}_{\kappa'}, v_{\kappa'})$. Examining the first entries shows that $\kappa^{-1}(\widetilde{m}_\kappa) = \kappa'^{-1}(\widetilde{m}_{\kappa'})$. Therefore π is well defined.

The diffeomorphism $\phi_\kappa : M_\kappa \times \mathbb{F}^n \to \pi^{-1}(M_\kappa)$ is given by $\phi_\kappa = q \circ (\kappa \times 1)$. If p is in $M_\kappa \cap M_{\kappa'}$, write $v' = \phi_{\kappa',p}^{-1}(\phi_{\kappa,p}(v))$. Then $\phi_{\kappa',p}(v') = \phi_{\kappa,p}(v)$, and hence $q(\kappa'(p), v') = q(\kappa(p), v)$. Thus $(\kappa'(p), v') \sim (\kappa(p), v)$, and

$$(\kappa'(p), v') = \psi_{\kappa'\kappa}(\kappa(p), v) = \big(\kappa'\kappa^{-1}(\kappa(p)), g_{\kappa'\kappa}(\kappa^{-1}(\kappa(p)))(v)\big).$$

Examining the equality of the second coordinates, we see that $v' = g_{\kappa'\kappa}(p)(v)$. Therefore $\phi_{\kappa',p}^{-1} \circ \phi_{\kappa,p} = g_{\kappa'\kappa}(p)$, and the transition functions match the given functions. This completes the proof. $\square$

As we mentioned after Proposition 8.13, Proposition 8.14 enables us to introduce the structure of a vector bundle on the **tangent bundle** $T(M)$, since the product formula for the transition functions $g_{\kappa'\kappa}(p) = \big[\frac{\partial y_i}{\partial x_j}\big]_p$ follows from the chain rule. The transition functions $g_{\kappa'\kappa}(p) = \big[\frac{\partial y_i}{\partial x_j}\big]_p$ are real-valued and thus can be regarded as in $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$. Thus $T(M)$, in our construction, can be regarded as having fiber $\mathbb{R}^n$ or $\mathbb{C}^n$, whichever is more convenient in a particular context. We can speak of the **real tangent bundle** $T(M, \mathbb{R})$ and the **complex tangent bundle** $T(M, \mathbb{C})$ in the two cases.[6]

We shall make use also of the **cotangent bundle** $T^*(M)$, and again we shall allow this to be real or complex. Members of the cotangent bundle will be called **cotangent vectors**. We give two slightly different realizations of $T^*(M)$, one starting from $T(M)$ as the object of primary interest and the other proceeding directly to $T^*(M)$. In both cases, $T^*(M)$ and $T(M)$ will be fiber-by-fiber duals of one another, and the transition functions will be transpose inverses of one another.

For the first construction we shall identify the dual of $T_p(M)$ in terms of differentials as defined in Section 1. Let M be n-dimensional, let κ be a compatible chart about p, and let $f \in C^\infty(U)$ be a smooth function in a neighborhood of p. By definition from Section 1, the differential $(df)_p$ is the linear function $(df)_p : T_p(M) \to T_{f(p)}(\mathbb{F})$ given by

$$(df)_p(L)(g) = L(g \circ f).$$

Let us take $g_0 : \mathbb{F} \to \mathbb{F}$ to be the function $g_0(t) = t$. Since

$$(df)_p\big[\tfrac{\partial}{\partial x_j}\big]_p(g_0) = \tfrac{\partial(g_0 \circ f)}{\partial x_j}(p) = g_0'(f(p))\,\tfrac{\partial f}{\partial x_j}(p) = \tfrac{\partial f}{\partial x_j}(p),$$

we see that $(df)_p(L)(g_0) = Lf$ for all L in $T_p(M)$. In particular, each differential $(df)_p$ acts as a linear functional on $T_p(M)$. Moreover, the elements $(dx_i)_p$, namely the differentials for $f = x_i$, are the members of the dual basis to the basis $\big[\frac{\partial}{\partial x_j}\big]_p$ of $T_p(M)$, and we can use them to write

$$(df)_p = \sum_{i=1}^n \tfrac{\partial f}{\partial x_i}(p)\,(dx_i)_p.$$

We postpone a discussion of the bundle structure on $T^*(M)$ until after the second construction.

[6]Traditionally the words "tangent bundle" refer to what is being called the real tangent bundle, and the traditional notation for it is $T(M)$.

For the second construction we use the algebra $\mathcal{C}_p$ of germs at p. Evaluation at p is well defined on germs at p, and we let $\mathcal{C}_p^0$ be the vector subspace of germs whose value at p is 0. Inside $\mathcal{C}_p^0$, we wish to identify the vector subspace $\mathcal{C}_p^1$ of germs that vanish at least to second order at p. These are[7] germs of functions f with the property that $|f(q)-f(p)|$ is dominated by a multiple of $|\kappa(q)-\kappa(p)|^2$ in any chart κ about p when q is in a sufficiently small neighborhood of p.

Within the second construction the cotangent space $T_p^*(M)$ is defined as the vector space quotient $\mathcal{C}_p^0/\mathcal{C}_p^1$. To introduce a vector-bundle structure on $T^*(M) = \bigcup_p T_p^*(M)$ by means of Proposition 8.14, we need to set up the local expression for a member of the cotangent space and understand how it changes when we pass from one compatible chart κ to another κ'. We begin by observing for any open neighborhood U of p that there is a well-defined linear map $f \mapsto df(p)$ of $C^\infty(U)$ onto $T_p^*(M)$ given by passing from f to $f - f(p)$ in $\mathcal{C}_p^0$ and then to the coset representative of $f - f(p)$ in $T_p^*(M) = \mathcal{C}_p^0/\mathcal{C}_p^1$.

Proposition 8.15. Let M be a smooth manifold of dimension n, let p be in M, and let $\kappa = (x_1, \dots, x_n)$ be a compatible chart about p. In either construction of $T_p^*(M)$, the n quantities $dx_i(p)$ form a vector-space basis of $T_p^*(M)$, and any smooth function f defined in a neighborhood of p has

$$df(p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)\,dx_i(p).$$

PROOF. We have already obtained this formula for the first construction. For the second construction, we observe as in the proof of Proposition 8.5 that Taylor's Theorem yields an expansion for f in the chart κ about p as

$$\begin{aligned} f(q) = f(p) &+ \sum_{i=1}^n (x_i(q) - x_i(p))\tfrac{\partial f}{\partial x_i}(p) \\ &+ \sum_{i,j} (x_i(q) - x_i(p))(x_j(q) - x_j(p))r_{ij}(q), \end{aligned}$$

from which we obtain

$$df(p) = \sum_{i=1}^n \tfrac{\partial f}{\partial x_i}(p)\,dx_i(p).$$

This establishes the asserted expansion and shows that the $dx_i(p)$ span the vector space $T_p^*(M)$. For the linear independence suppose that $\sum_{i=1}^n c_i dx_i(p) = 0$ with

[7] If we allow ourselves to peek momentarily at the tangent space, we see that $\mathcal{C}_p^1$ is the subspace of all members of $\mathcal{C}_p^0$ on which all tangent vectors at p vanish.

the constants c_i not all 0. If we define $f = \sum_{i=1}^n c_i x_i$ in M_κ, then computation gives $\frac{\partial f}{\partial x_i}(p) = c_i$ and hence $df(p) = \sum_{i=1}^n c_i\, dx_i(p) = 0$. Thus $f - f(p)$ vanishes at least to order 2 at p. Since $f - f(p)$ is linear, we conclude that $f - f(p)$ vanishes identically near p. Hence all coefficients c_i are 0. This proves the linear independence. □

When p moves within the compatible chart κ, we can express all members of the spaces $T_q^*(M)$ for q in that neighborhood as $\sum_{i=1}^{n} \xi_i(q)\, dx_i(q)$, but the functions $\xi_i(q)$ need not always be of the form $\frac{\partial f}{\partial x_i}(q)$ for a single function f. Nevertheless, we can use the transformation properties of $df(p)$ for special f's to introduce a natural vector-bundle structure on $T^*(M)$ by means of Proposition 8.14.

EXAMPLE. Direct construction of bundle structure on cotangent bundle. Continuing with the direct analysis of $T^*(M)$, let us form the coordinate functions and charts. Define $T^*(M_\kappa) = \bigcup_{p \in M_\kappa} T_p^*(M)$. Using Proposition 8.15, we associate to a member (p, ξ) of $T^*(M_\kappa)$ the coordinates

$$(x_1(p), \dots, x_n(p); \xi_1, \dots, \xi_n),$$

where $\kappa(p) = (x_1(p), \dots, x_n(p))$ and $\xi = \sum_{i=1}^{n} \xi_i\, dx_i(p)$. The coordinate function ϕ_κ is given in this notation as a composition carrying $(p; \xi_1, \dots, \xi_n)$ first to $(x_1(p), \dots, x_n(p); \xi_1, \dots, \xi_n)$ and then to $\sum_{i=1}^{n} \xi_i\, dx_i(p)$. That is,

$$\phi_\kappa(p; \xi_1, \dots, \xi_n) = \sum_{i=1}^{n} \xi_i\, dx_i(p).$$

If p lies in another chart $\kappa' = (y_1, \dots, y_n)$, then we similarly have

$$\phi_{\kappa'}(p; \eta_1, \dots, \eta_n) = \sum_{i=1}^{n} \eta_i\, dy_i(p).$$

The formula of Proposition 8.15 shows that

$$dx_i(p) = \sum_{j=1}^{n} \frac{\partial x_i}{\partial y_j}(p)\, dy_j(p).$$

Therefore

$$\phi_\kappa(p; \xi_1, \dots, \xi_n) = \sum_{i=1}^{n} \xi_i\, dx_i(p) = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \xi_i \frac{\partial x_i}{\partial y_j}(p) \Big)\, dy_j(p),$$

and

$$\phi_{\kappa'}^{-1}\phi_\kappa(p;\xi_1,\dots,\xi_n) = \Big(p;\ \sum_{i=1}^n \xi_i \tfrac{\partial x_i}{\partial y_1}(p),\dots,\sum_{i=1}^n \xi_i \tfrac{\partial x_i}{\partial y_n}(p)\Big).$$

In other words,

$$\phi_{\kappa'}^{-1}\phi_\kappa(p;\xi_1,\dots,\xi_n) = (p;\eta_1,\dots,\eta_n)$$

with $\eta_j = \sum_{i=1}^n \xi_i \frac{\partial x_i}{\partial y_j}(p)$. This says that the row vector $(\eta_1 \ \cdots \ \eta_n)$ is the product of the row vector $(\xi_1 \ \cdots \ \xi_n)$ by the matrix $\big[\frac{\partial x_i}{\partial y_j}(p)\big]$. Taking the transpose of this matrix equation, we see that the transition functions for the cotangent bundle are to be

$$g_{\kappa'\kappa}(p) = \big[\tfrac{\partial x_i}{\partial y_j}(p)\big]^{\mathrm{tr}},$$

i.e., the transpose inverses of the transition functions for the tangent bundle. In view of the boxed formula earlier in this section, a system of functions $f_\kappa : \widetilde{M}_\kappa \times \mathbb{F}^n \to S$ arises from a globally defined function on the cotangent bundle if and only if

$$f_\kappa(x,\xi) = f_{\kappa'}\big(y(x), \big[\tfrac{\partial x_i(y)}{\partial y_j}\big]^{\mathrm{tr}}(\xi)\big),$$

i.e., if and only if

$$f_\kappa\big(x(y), \big(\big[\tfrac{\partial x_i(y)}{\partial y_j}\big]^{-1}\big)^{\mathrm{tr}}(\eta)\big) = f_{\kappa'}(y,\eta).$$

If $\pi : B \to M$ is a smooth vector bundle, a **section** of B is a function $s : M \to B$ such that $\pi(s(p)) = p$ for all $p \in M$, and the section is a **smooth section** if s is smooth as a function between smooth manifolds.

Proposition 8.16. Let $\pi : B \to M$ be a smooth vector bundle of rank n, let $s : M \to B$ be a section, and let κ be a compatible chart for M. Then the coordinate function ϕ_κ has the property that $\phi_\kappa^{-1} \circ s(p) = (p, v_\kappa(p))$ for p in M_κ and for a function $v_\kappa(\,\cdot\,) : M_\kappa \to \mathbb{F}^n$. Moreover, the section s is smooth if and only if the function $p \mapsto v_\kappa(p)$ is smooth for every chart κ in an atlas.

PROOF. Let $P_\kappa : M_\kappa \times \mathbb{F}^n \to M_\kappa$ be projection to the first coordinate. Let us check that $P_\kappa \circ \phi_\kappa^{-1} = \pi$ on $\pi^{-1}(M_\kappa)$. Suppose that p is in M_κ and $\phi_\kappa(p, v) = b$. Applying π gives $\pi(b) = \pi\phi_\kappa(p, v) = p$ by the defining property (i) of ϕ_κ. Therefore $\phi_\kappa^{-1}(b) = (p, v)$ and $P_\kappa\phi_\kappa^{-1}(b) = p = \pi(b)$. Since b is arbitrary in $\pi^{-1}(M_\kappa)$, $P_\kappa \circ \phi_\kappa^{-1} = \pi$.

For a section s, the condition $\pi \circ s = 1$ on M therefore implies that $P_\kappa \circ \phi_\kappa^{-1} \circ s = 1$ on M_κ. Hence $\phi_\kappa^{-1} \circ s(p) = (p, v_\kappa(p))$ for p in M_κ and for some function $v_\kappa : M_\kappa \to \mathbb{F}^n$. Since each $\phi_\kappa : M_\kappa \times \mathbb{F}^n \to \pi^{-1}(M_\kappa)$ is a diffeomorphism, s is smooth if and only if each function $\phi_\kappa^{-1} \circ s$ is smooth for κ in an atlas, and this condition holds if and only if each v_κ is smooth. $\square$

EXAMPLES.

(1) Vector fields. A **vector field** on M is a section of the tangent bundle. In the first example in this section, we obtained the formula $\phi_\kappa(p, v) = \sum_{i=1}^n v_i \big[\frac{\partial}{\partial x_i}\big]_p$ if p is in M_κ and $v = (v_1, \dots, v_n)$. Applying ϕ_κ to the formula of Proposition 8.16, we see that $s(p) = \phi_\kappa(p, v(p)) = \sum_{i=1}^n v_i(p)\big[\frac{\partial}{\partial x_i}\big]_p$ if the function $v(p)$ is $(v_1(p), \dots, v_n(p))$. On the other hand, Proposition 8.8 shows that any vector field X acts by $Xf(p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)(Xx_i)(p)$. If we regard X as our section s, we see therefore that $v_i(p) = (Xx_i)(p)$. Since s is smooth if and only if all $v_i(p)$ are smooth, s is smooth if and only if all $(Xx_i)(p)$ are smooth. In view of Proposition 8.8, we conclude that a vector field is smooth as a section if and only if it is smooth in the sense of Section 2.

(2) Differential 1-forms. A **differential 1-form** on M is a section of the cotangent bundle. Just before Proposition 8.16 we obtained the formula $\phi_\kappa(p, \xi) = \sum_{i=1}^n \xi_i \, dx_i(p)$ if p is in M_κ and $\xi = (\xi_1, \dots, \xi_n)$. Applying ϕ_κ to the formula of Proposition 8.16, we see that $s(p) = \phi_\kappa(p, \xi(p)) = \sum_{i=1}^n \xi_i(p) \, dx_i(p)$ if the function $\xi(p)$ is $(\xi_1(p), \dots, \xi_n(p))$. Proposition 8.16 shows that s is smooth if and only if all the $\xi_i(p)$ are smooth, and thus a differential 1-form is smooth if and only if in each of its local expressions $\sum_{i=1}^n \xi_i(p) \, dx_i(p)$, all the coefficient functions $\xi_i(p)$ are smooth. In particular Proposition 8.15 gives the formula $df(p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) \, dx_i(p)$ whenever f is a smooth function on M_κ, and therefore df is a smooth differential 1-form on M whenever f is in $C^\infty(M)$.

5. Distributions and Differential Operators on Manifolds

The goal of Sections 5–7 is to describe the framework for extending the method of pseudodifferential operators, as introduced in Section VII.6, from open subsets of Euclidean space to smooth manifolds. Just as in Section VII.6 a number of lengthy verifications are involved, and we omit them.

Several sources of examples with $\mathbb{F} = \mathbb{R}$ are worth mentioning. All of them come about in the context of some smooth manifold with some additional structure. All of them involve differential operators, as opposed to general pseudodifferential operators, at least initially. From this point of view, the reason for introducing pseudodifferential operators is to have tools for working with differential operators.

The first source is the subject of "Lie groups." A **Lie group** G is a smooth manifold that is a group in such a way that multiplication and inversion are smooth functions. Closed subgroups of $GL(n, \mathbb{F})$ furnish examples, but not in an obvious way. In any event, if a tangent vector at the identity is moved to arbitrary points of G by the differentials of the right translations of G, the result is a vector field that can be shown to be smooth and to have an invariance property relative to left translation. We can regard this left-invariant vector field as a first-order differential operator on G. Out of such operators we can form further differential operators by forming compositions, sums, and so on.

A related and larger source is quotient spaces of Lie groups. Any Lie group G is a locally compact group in the sense of Chapter VI. If H is a closed subgroup, then the quotient G/H turns out to have a smooth structure such that the group action $G \times G/H \to G/H$ is smooth. The quotient G/H may admit differential operators that are invariant under the action of G. For example the Laplacian makes sense on the unit sphere S^{n-1} and is invariant under rotations. The sphere S^{n-1} is the quotient of rotation groups $SO(n)/SO(n-1)$, and thus the Laplacian on the sphere falls into the category of an invariant differential operator on a quotient space of a Lie group.

A third source, overlapping some with the previous two, is Riemannian geometry. A **Riemannian manifold** M is a smooth manifold with an inner product specified on each tangent space $T_p(M)$ so as to vary smoothly with p. The additional structure on M is called a **Riemannian metric** and can be formalized, by the same process as for the tangent bundle itself, as a smooth section of a suitable vector bundle over M. A Riemannian manifold carries a natural Laplacian operator and other differential operators of interest that capture aspects of the geometry. One way of creating Riemannian manifolds is by embedding a smooth manifold of interest in a Riemannian manifold. For example one can embed any compact orientable 2-dimensional smooth manifold in $\mathbb{R}^3$, and $\mathbb{R}^3$ carries a natural Riemannian metric. The inclusion of the manifold into $\mathbb{R}^3$ induces an inclusion of tangent spaces, and the Riemannian metric of $\mathbb{R}^3$ can be restricted to the manifold.

A fourth source is the field of several complex variables. The Cauchy–Riemann operator, consisting of $\frac{\partial}{\partial \bar{z}_j}$ in each complex variable z_j, makes sense on any open set, and the functions annihilated by it are the holomorphic functions. If a bounded open subset of $\mathbb{C}^n$ has a smooth boundary, then the tangential component of the Cauchy–Riemann operator makes sense on smooth functions defined on the boundary. The significance of the tangential Cauchy–Riemann operator is that the functions annihilated by it are the ones that locally have extensions to holomorphic functions in a neighborhood of the boundary. The Lewy example, mentioned in Section VII.2, ultimately comes from such a construction using the unit ball in $\mathbb{C}^2$.

The subject being sufficiently rich with examples, let us establish the framework. Let M be an n-dimensional smooth manifold. It is customary to assume that M is separable. This condition is satisfied in all examples of interest, and in particular every compact manifold is separable. With the assumption of separability, we automatically obtain an **exhausting sequence** $\{K_j\}_{j=1}^{\infty}$ of compact sets such that $M = \bigcup_j K_j$ and $K_j \subseteq K_{j+1}^o$.

We have already introduced the associative algebras $C^\infty(M)$ and $C^\infty_{\text{com}}(M)$, and these spaces of functions need to be topologized. For $C^\infty(M)$, the topology is to be given by a countable separating family of seminorms, and convergence is to be uniform convergence of the function and all its derivatives on each compact subset of M. The exact family of seminorms will not matter, but we need to see that it is possible to specify one. Fix K_j. To each point p of K_j, associate a chart κ_p about p and associate also a compact neighborhood N_p of p lying within M_{κ_p}. For p in K_j, the interiors N_p^o of the N_p's cover K_j, and we select a finite subcover $N_{p_1}^o, \dots, N_{p_r}^o$. Let $\kappa_{p_1}, \dots, \kappa_{p_r}$ be the corresponding charts. If φ is in $C^\infty(M)$, the seminorms of φ relating to K_j will be indexed by a multi-index α and an integer i with $1 \le i \le r$, the associated seminorm being $\sup_{x \in N_{p_i}} |D^\alpha(\varphi \circ \kappa_{p_i}^{-1})|$. When j is allowed to vary, the result is that $C^\infty(M)$ is a complete metric space with a metric given by countably many seminorms. If we construct seminorms by starting from a different exhausting sequence, then there is no difficulty in seeing that any seminorm in either construction is $\le$ a positive linear combination of seminorms from the other construction. Thus the identity mapping of $C^\infty(M)$ with the one metric to $C^\infty(M)$ with the other metric is uniformly continuous.

For $C^\infty_{\text{com}}(M)$, we use the inductive limit construction of Section IV.7 relative to the sequence of compact subsets K_j. That is, we let $C^\infty_{K_j}$ be the vector subspace of functions in $C^\infty_{\text{com}}(M)$ with support in K_j, we give $C^\infty_{K_j}$ the relative topology from $C^\infty(M)$, and then we form the inductive limit. Again the topology is independent of the exhausting sequence, and $C^\infty_{\text{com}}(M)$ is an LF space in the sense of Section IV.7.

The next step is to introduce distributions on manifolds, and there we encounter an unpleasant surprise. In Euclidean space the effect $\langle T, \varphi \rangle$ of a distribution on a function was supposed to generalize the effect $\langle f, \varphi \rangle = \int f \varphi \, dx$ of integration with a function f. The dx in the Euclidean case refers to Lebesgue measure. To get such an interpretation in the case of a manifold M, we have to use a measure on M, and there may be no canonical one. If we drop any insistence that distributions generalize integration with a function, then we encounter a different problem. The problem is that the three global notions—smooth function, distribution, and linear functional on smooth functions—each have to satisfy certain transformation rules as we move from chart to chart, and these transformation rules are not compatible with having the space of distributions coincide with the space of linear functionals on smooth functions.

There are several ways of handling this problem, and we use one of them. What we shall do is fix a global but noncanonical notion of integration on M satisfying some smoothness properties. Thus we are constructing a positive linear functional λ on $C_{\text{com}}(M)$. We suppose given relative to each chart $\kappa = (x_1, \dots, x_n)$ a positive smooth function $g_\kappa(x)$ on $\widetilde{M}_\kappa$ such that $\lambda(\varphi) = \int_{\widetilde{M}_\kappa} \varphi(\kappa^{-1}(x))g_\kappa(x)\,dx$ whenever φ is in $C_{\text{com}}(M_\kappa)$. Let $\kappa' = (y_1, \dots, y_n)$ be a second chart, and put $M_{\kappa,\kappa'} = M_\kappa \cap M_{\kappa'}$. If φ is in $C_{\text{com}}(M_{\kappa,\kappa'})$, then we require that

$$\int_{\kappa(M_{\kappa,\kappa'})} \varphi(\kappa^{-1}(x))g_\kappa(x)\,dx = \int_{\kappa'(M_{\kappa,\kappa'})} \varphi(\kappa'^{-1}(y))g_{\kappa'}(y)\,dy.$$

Substituting $y = \kappa'(\kappa^{-1}(x))$ on the right side, we can transform the right side into $\int_{\kappa(M_{\kappa,\kappa'})} \varphi(\kappa^{-1}(x))g_{\kappa'}(\kappa'(\kappa^{-1}(x)))\big|\det\big[\frac{\partial y_i}{\partial x_j}(x)\big]\big|\,dx$ by the change-of-variables formula for multiple integrals. Thus the compatibility condition for the functions g_κ is that

$$g_\kappa(x) = g_{\kappa'}(y(x))\Big|\det\Big[\frac{\partial y_i}{\partial x_j}(x)\Big]\Big| \qquad \text{for } x \in \kappa(M_{\kappa,\kappa'}),\ y(x) = \kappa'(\kappa^{-1}(x)).$$

Conversely if this compatibility condition on the system of g_κ's is satisfied, we can use a smooth partition of unity[8] to define λ consistently and obtain a measure on M. This measure is a positive smooth function times Lebesgue measure in the image of any chart, and we refer to it as a **smooth measure** on M. We denote it by μ_g. The key formula for computing with it is

$$\int_M \varphi\,d\mu_g = \int_{\widetilde{M}_\kappa} \varphi(\kappa^{-1}(x))g_\kappa(x)\,dx$$

for all Borel functions $\varphi \geq 0$ on M that equal 0 outside M_κ.

One can prove that a smooth measure always exists,[9] and there are important cases in which a distinguished smooth measure exists. With Lie groups, for example, a left Haar measure is distinguished. With the quotient of a Lie group by a closed subgroup, Theorem 6.18 gives a necessary and sufficient condition for the existence of a nonzero left-invariant Borel measure, and that is distinguished. With a Riemannian manifold, there always exists a distinguished smooth measure that is definable directly in terms of the Riemannian metric.

[8] Smooth partitions of unity are discussed in Problem 5 at the end of the chapter.

[9] If every connected component of M is orientable, there is a positive smooth differential n-form, and it gives such a measure. All components are open; any nonorientable component has an orientable double cover with such a measure, and this can be pushed down to the given manifold.

The smooth measure is not unique, but any two smooth measures μ_g and μ_h are absolutely continuous with respect to each other. By the Radon–Nikodym Theorem we can therefore write $d\mu_g = F\,d\mu_h$ for a positive Borel function F; the function F may be redefined on a set of measure 0 so as to be in $C^\infty(M)$, as we see by examining matters in local coordinates. Conversely if F is any everywhere-positive member of $C^\infty(M)$, then $F\,d\mu_g$ is another smooth measure.

If we fix a smooth measure μ_g, we can define spaces $L^1_{\text{com}}(M, \mu_g)$ and $L^1_{\text{loc}}(M, \mu_g)$ as follows: the first is the vector subspace of all members of $L^1(M, \mu_g)$ with compact support, and the second is the vector space of all functions, modulo null sets, whose restriction to each compact subset of M is in $L^1_{\text{com}}(M, \mu)$. It will not be necessary for us to introduce a topology on $L^1_{\text{com}}(M, \mu_g)$ or on $L^1_{\text{loc}}(M, \mu_g)$. If we replace μ_g by another smooth measure $d\mu_h = F\,d\mu_g$, then it is evident that $L^1_{\text{com}}(M, \mu_h) = L^1_{\text{com}}(M, \mu_g)$ and $L^1_{\text{loc}}(M, \mu_h) = L^1_{\text{loc}}(M, \mu_g)$.

We define $\mathcal{D}'(M)$ and $\mathcal{E}'(M)$ in the expected way: $\mathcal{D}'(M)$, which is the space of all **distributions** on M, is the vector space of all continuous linear functionals on $C^\infty_{\text{com}}(M)$, and $\mathcal{E}'(M)$ is the vector space of all continuous linear functionals on $C^\infty(M)$. The effect of a distribution T on a function φ continues to be denoted by $\langle T, \varphi\rangle$. The **support** of a distribution is the complement of the union of all open subsets U of M such that the distribution vanishes on $C^\infty_{\text{com}}(U)$. We omit the verification that $\mathcal{E}'(M)$ is exactly the subspace of members of $\mathcal{D}'(M)$ of compact support. It will not be necessary for us to introduce a topology on $\mathcal{D}'(M)$ or $\mathcal{E}'(M)$.

With the smooth measure μ_g fixed, we can introduce distributions T_f corresponding to certain functions f. If f is in $L^1_{\text{loc}}(M, \mu_g)$, we define T_f by

$$\langle T_f, \varphi\rangle = \int_M f\varphi\,d\mu_g \qquad \text{for } \varphi \in C^\infty_{\text{com}}(M).$$

This is a member of $\mathcal{D}'(M)$. If f is in $L^1_{\text{com}}(M, \mu_g)$, we define T_f by

$$\langle T_f, \varphi\rangle = \int_M f\varphi\,d\mu_g \qquad \text{for } \varphi \in C^\infty(M).$$

This is a member of $\mathcal{E}'(M)$.

As we did in the Euclidean case in Section V.2, we want to be able to pass from certain continuous linear operators L on smooth functions to linear operators on distributions. With μ_g replacing Lebesgue measure, the procedure is unchanged. We have a definition of L on functions, and we identify a continuous **transpose** operator L^{tr} on smooth functions satisfying the defining condition

$$\int_M L(f)\varphi\,d\mu_g = \int_M f\,L^{\text{tr}}(\varphi)\,d\mu_g.$$

Then we let

$$\langle L(T), \varphi \rangle = \langle T, L^{\mathrm{tr}}(\varphi) \rangle.$$

For example, if L is the operator given as multiplication by the smooth function ψ, then $L^{\mathrm{tr}} = L$ on smooth functions because we have $\int_M L(f)\varphi \, d\mu_g = \int_M (\psi f)(\varphi)\, d\mu_g = \int_M (f)(\psi\varphi)\, d\mu_g = \int_M f L(\varphi)\, d\mu_g$. Thus the definition is

$$\langle \psi T, \varphi \rangle = \langle T, \psi\varphi \rangle.$$

A **linear differential operator** L of order $\leq m$ on a manifold M is a continuous linear operator from $C^\infty(M)$ into itself with the property that for each point p in M, there is some compatible chart κ about p and there are functions a_α in $C^\infty(M_\kappa)$ such that the operator takes the form $Lf(q) = \sum_{|\alpha|\leq m} a_\alpha(q) D^\alpha f(q)$ for all f in $C^\infty(M_\kappa)$. Here if $\kappa = (x_1, \ldots, x_n)$, then $D^\alpha f(q)$ is by definition the Euclidean expression $D^\alpha(f \circ \kappa^{-1})(x_1, \ldots, x_n)$ evaluated at $\kappa(q)$.

If we have an expansion $Lf(q) = \sum_{|\alpha|\leq m} a_\alpha(q) D^\alpha f(q)$ in the chart κ about p and if κ' is another compatible chart about p, then a Euclidean change of variables shows that $Lf(q)$ is of the form $\sum_{|\beta|\leq m} d_\beta(q) D^\beta f(q)$ in the chart κ' for suitable smooth coefficient functions d_β.

The operator L carries the vector subspace $C^\infty_{\mathrm{com}}(M)$ of $C^\infty(M)$ into itself and is continuous as a mapping of $C^\infty_{\mathrm{com}}(M)$ into itself. One says that L has **order** m if in some compatible chart, some coefficient function a_α is not identically 0.

Let us compute how the transpose of a linear differential operator of order m acts on smooth functions. The claim is that this transpose is again a linear differential operator of order m. Since linear differential operators on open subsets of Euclidean space are mapped to other such operators by diffeomorphisms, it is enough to make a computation in a neighborhood of a point p within a compatible chart κ about p. Evidently the operation of taking the transpose is linear and reverses the order of operators, and we saw that multiplication by a smooth function is its own transpose. Thus it is enough to verify that the transpose of $\frac{\partial}{\partial x_j}$ is a linear differential operator.

To simplify the notation in the verification, let us abbreviate $\langle T_f, \varphi \rangle$ as $\langle f, \varphi \rangle$ when f and φ are smooth functions on M and at least one of them has compact support. That is, we set $\langle f, \varphi \rangle = \int_M f\varphi \, d\mu_g$. Let φ and ψ be in $C^\infty(M_\kappa)$, and assume that one of φ and ψ has compact support. With $\{g_\kappa\}$ as the system of functions defining the smooth measure μ_g, we have

$$\int_{\widetilde{M}_\kappa} \tfrac{\partial}{\partial x_j}\big((\psi \circ \kappa^{-1})(\varphi \circ \kappa^{-1}) g_\kappa\big)\, dx = 0.$$

Expanding the derivative and setting $h_\kappa = g_\kappa \circ \kappa$ gives

$$\begin{aligned} \big\langle \big(\tfrac{\partial}{\partial x_j}\big)^{\mathrm{tr}}\varphi,\ \psi \big\rangle &= \big\langle \varphi,\ \tfrac{\partial \psi}{\partial x_j} \big\rangle \\ &= \int_{\widetilde{M}_\kappa} \varphi(\kappa^{-1}(x))\, \tfrac{\partial}{\partial x_j}(\psi \circ \kappa^{-1})(x)\, g_\kappa(x)\, dx \end{aligned}$$

$$= -\int_{\widetilde{M}_\kappa} \psi(\kappa^{-1}(x))\, \tfrac{\partial}{\partial x_j}\big((\varphi\circ\kappa^{-1})g_\kappa\big)(x)\ dx$$
$$= -\int_{\widetilde{M}_\kappa} g_\kappa(x)^{-1}\psi(\kappa^{-1}(x))\, \tfrac{\partial}{\partial x_j}\big((\varphi\circ\kappa^{-1})g_\kappa\big)(x)\, g_\kappa(x)\, dx$$
$$= -\int_{\widetilde{M}_\kappa} (h_\kappa\circ\kappa^{-1})(x)^{-1}(\psi\circ\kappa^{-1})(x)\, \tfrac{\partial}{\partial x_j}\big((\varphi\circ\kappa^{-1})(h_\kappa\circ\kappa^{-1})\big)(x)\, g_\kappa(x)\, dx.$$

Therefore $\big(\frac{\partial}{\partial x_j}\big)^{\mathrm{tr}}\varphi = (h_\kappa)^{-1}\psi\frac{\partial}{\partial x_j}(\varphi h_\kappa)$, and $\big(\frac{\partial}{\partial x_j}\big)^{\mathrm{tr}}$ is exhibited as a linear differential operator in local coordinates.

Certainly transpose does not increase the order of a linear differential operator. Applying transpose twice reproduces the original operator, and it follows that the transpose differential operator has the same order as the original.

If L is a linear differential operator acting on $C^\infty_{\mathrm{com}}(M)$ or $C^\infty(M)$, we are now in a position to extend the definition of L to distributions. To do so, we form the linear differential operator L^{tr} such that $\langle L\varphi, \psi\rangle = \langle\varphi, L^{\mathrm{tr}}\psi\rangle$ whenever φ and ψ are smooth on M and at least one of them has compact support. If T is in $\mathcal{D}'(M)$, we define $L(T)$ in $\mathcal{D}'(M)$ by $\langle L(T), \varphi\rangle = \langle T, L\varphi\rangle$ for φ in $C^\infty_{\mathrm{com}}(M)$. If T is in $\mathcal{E}'(M)$, then we can allow φ to be $C^\infty(M)$, and the consequence is that $L(T)$ is in $\mathcal{E}'(M)$. Thus L carries $\mathcal{D}'(M)$ to itself and $\mathcal{E}'(M)$ to itself.

Recall from Section VII.6 that a linear differential operator $\sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$ of order m has, by definition, full symbol $\sum_{|\alpha|\le m} a_\alpha(x)(2\pi i)^{|\alpha|}\xi^\alpha$ and principal symbol $\sum_{|\alpha|=m} a_\alpha(x)(2\pi i)^{|\alpha|}\xi^\alpha$, with the factors of $2\pi i$ reflecting the way that the Euclidean Fourier transform is defined in this book. When we try to extend this definition in a coordinate-free way to smooth manifolds M, we find no ready generalization of the full symbol, but we shall see that the principal symbol extends to be a certain kind of function on the cotangent bundle of M.

Let L be a linear differential operator on M of order m. Fix a point p in M, let $\kappa = (x_1, \dots, x_n)$ be a compatible chart about p, and let φ be in $C^\infty(M_\kappa)$. Suppose that D^α makes a contribution to L in this chart. For $t > 0$ and f in $C^\infty(M_\kappa)$, consider the expression

$$t^{-m}e^{-2\pi it\varphi}D^\alpha(e^{2\pi it\varphi}f) \qquad \text{evaluated at } p.$$

We are interested in this expression in the limit $t\to\infty$. When $D^\alpha(e^{2\pi it\varphi}f)$ is expanded by the Leibniz rule, each derivative that is applied to $e^{2\pi it\varphi}$ yields a factor of t, and each derivative that is applied to f yields no such factor. Moreover, the exponentials cancel after the differentiations. The surviving dependence on t in each term is of the form t^{-r}, where $r \ge m - |\alpha|$. Thus our expression has limit 0 if $|\alpha| < m$. If $|\alpha| = m$, we get a nonzero contribution only when all the derivatives from the Leibniz rule are applied to f. Thus the limit of our expression with $|\alpha| = m$ is of the form $cD^\alpha f(p)$, where c is a constant depending on α and the germ of φ at p.

Meanwhile, our expression is unaffected by replacing φ by $\varphi - \varphi(p)$, and its dependence on φ is therefore as a member of $\mathcal{C}_p^0$. A little checking shows that our expression is unchanged if a member of $\mathcal{C}_p^1$ is added to φ. Consequently our expression, for α fixed with $|\alpha| = m$, is a function on $\mathcal{C}_p^0/\mathcal{C}_p^1 = T_p^*(M)$.

Let us write a general member of $T_p^*(M)$ as (p, ξ). We define the **principal symbol** of the linear differential operator L of order m to be the scalar-valued function $\sigma_L(p, \xi)$ on the real cotangent bundle $T^*(M, \mathbb{R})$ given by

$$\sigma_L(p, \xi) f(p) = \lim_{t\to\infty} t^{-m} e^{-2\pi i t\varphi(p)} L(e^{2\pi i t\varphi} f)(p),$$

where φ is chosen so that $d\varphi(p) = \xi$. Reviewing the construction above, we see that this definition is independent of f and of any choice of local coordinates.

We can compute the principal symbol explicitly if an expression for L is given in local coordinates. With our chart $\kappa = (x_1, \dots, x_n)$ as above, we know from Proposition 8.15 that the differentials $dx_1(p), \dots, dx_n(p)$ form a basis of $T_p^*(M)$. Let the expansion of the given cotangent vector ξ in this basis be $\xi = \sum_i \xi_i \, dx_i(p)$, and define $\varphi(x) = \sum_i \xi_i (x_i - x_i(p))$. This function has $d\varphi(p) = \xi$ by Proposition 8.15, and direct computation gives

$$\sigma_L(p, \xi) = \sum_{|\alpha|=m} a_\alpha(x)(2\pi i)^{|\alpha|} \xi^\alpha \qquad \text{if } L = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha.$$

In particular, $\sigma_L(p, \xi)$ is homogeneous of degree m in the ξ variable.[10]

6. More about Euclidean Pseudodifferential Operators

Before introducing pseudodifferential operators on an n-dimensional separable smooth manifold M, it is necessary to supplement the Euclidean theory as presented in Section VII.6. We need to understand the effect of transpose on a Euclidean pseudodifferential operator and also the effect of a diffeomorphism.

First let us consider transpose. If G is a pseudodifferential operator on $U \subseteq \mathbb{R}^n$, we know that

$$\langle G^{\mathrm{tr}} \psi, \varphi \rangle = \langle \psi, G\varphi \rangle = \int_{\mathbb{R}^n} \int_U \int_U e^{2\pi i (x-y)\cdot\xi} g(x, \xi) \psi(x) \varphi(y) \, dy \, dx \, d\xi$$

for φ and ψ in $C^\infty_{\mathrm{com}}(U)$. If we interchange x and y and replace ξ by $-\xi$, we obtain

$$\langle G^{\mathrm{tr}} \psi, \varphi \rangle = \int_{\mathbb{R}^n} \int_U \int_U e^{2\pi i (x-y)\cdot\xi} g(y, -\xi) \psi(y) \varphi(x) \, dy \, dx \, d\xi.$$

[10]A function $\sigma(p, \xi)$ is **homogeneous of degree** m in the ξ variable if $\sigma(p, r\xi) = r^m \sigma(p, \xi)$ for all $r > 0$ and all $\xi \neq 0$.

The function that ought to play the role of the symbol of G^{tr} is $g(y, -\xi)$. It has a nontrivial y dependence, unlike what happens with pseudodifferential operators as defined in Section VII.6. Thus we cannot tell from this formula whether G^{tr} coincides with a pseudodifferential operator. Although it is possible to cope with this problem directly, a tidier approach is to enlarge the definition of pseudodifferential operator to allow dependence on y, as well as on x and ξ, in the function playing the role of the symbol. Then the transpose of one of the new operators will again be an operator of the same kind, and one can develop a theory for the enlarged class of operators.[11] Remarkably, as we shall see, the new class of operators turns out to be not so much larger than the original class.

Accordingly, let $S^m_{1,0,0}(U\times U)$ be the set of all functions g in $C^\infty(U\times U\times\mathbb{R}^n)$ such that for each compact set $K\subseteq U\times U$ and each triple of multi-indices (α,β,γ), there exists a constant $C = C_{K,\alpha,\beta,\gamma}$ with

$$|D^\alpha_\xi D^\beta_x D^\gamma_y g(x,y,\xi)| \le C(1+|\xi|)^{m-|\alpha|} \qquad \text{for } (x,y)\in K \text{ and } \xi\in\mathbb{R}^n.$$

Then $D^\alpha_\xi D^\beta_x D^\gamma_y g$ will be a symbol in the class $S^{m-|\alpha|}_{1,0,0}(U\times U)$. Let $S^{-\infty}_{1,0,0}(U\times U)$ be the intersection of all $S^{-n}_{1,0,0}(U\times U)$ for $n\ge 0$. A function $g(x,y,\xi)$ in $S^m_{1,0,0}(U\times U)$ is called an **amplitude**, and the **generalized pseudodifferential operator** that is associated to it is given by[12]

$$G\varphi(x) = \int_{\mathbb{R}^n}\int_U e^{2\pi i(x-y)\cdot\xi} g(x,y,\xi)\varphi(y)\,dy\,d\xi$$

for φ in $C^\infty_{\mathrm{com}}(U)$. Such an operator is continuous from $C^\infty_{\mathrm{com}}(U)$ into $C^\infty(U)$. The transposed operator G^{tr} such that $\langle G\varphi,\psi\rangle = \langle\varphi, G^{\mathrm{tr}}\psi\rangle$ for φ and ψ in $C^\infty_{\mathrm{com}}(U)$ is given by

$$G^{\mathrm{tr}}\varphi(x) = \int_{\mathbb{R}^n}\int_U e^{2\pi i(y-x)\cdot\xi} g(y,x,\xi)\varphi(y)\,dy\,d\xi,$$

which becomes an operator of the same kind when we change ξ into $-\xi$. Because of the displayed formula for $G^{\mathrm{tr}}\varphi(x)$, we are led to define

$$\langle Gf,\varphi\rangle = \Big\langle f, \int_{\mathbb{R}^n}\int_U e^{2\pi i(y-(\,\cdot\,))\cdot\xi} g(y,\,\cdot\,,\xi)\varphi(y)\,dy\,d\xi\Big\rangle$$

[11]The theory for the new operators is the "tidier and faster" approach to Euclidean pseudodifferential operators that was mentioned just before the statement of Theorem 7.20.

[12]The use of the word "generalized" here is not standard terminology. It would be more standard to use some distinctive notation for the class of operators of this kind, but we have introduced no notation for it at all.

for $f \in \mathcal{E}'(U)$ and $\varphi \in C^\infty_{\text{com}}(U)$. Then Gf is in $\mathcal{D}'(U)$. In the special case that g is independent of its second variable, the above formula for $\langle Gf, \varphi\rangle$ reduces to the formula for $\langle Gf, \varphi\rangle$ in Section VII.6 as a consequence of Theorem 5.20 and an interchange of limits.[13]

If the amplitude of G is in $S^{-\infty}_{1,0,0}(U \times U)$, then the generalized pseudodifferential operator G carries $\mathcal{E}'(U)$ into $C^\infty(U)$, and it is consequently said to be a **smoothing operator**.

Following the pattern of the development in Section VII.6, we define a linear functional $\mathcal{G}$ on $C^\infty_{\text{com}}(U \times U)$ by the formula

$$\langle \mathcal{G}, w\rangle = \int_{\mathbb{R}^n} \Big[\int_{U\times U} e^{2\pi i(x-y)\cdot\xi} g(x, y, \xi) w(x, y)\, dx\, dy \Big]\, d\xi.$$

Then $\mathcal{G}$ is continuous and hence is a member of $\mathcal{D}'(U \times U)$. The formal expression

$$\mathcal{G}(x, y) = \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} g(x, y, \xi)\, d\xi$$

is called the **distribution kernel** of G; again it is not to be regarded as a function but as an expression that defines a distribution.

With the insertion of the word "generalized" in front of "pseudodifferential operator," Theorem 7.19 remains true word for word; the distribution kernel is a smooth function off the diagonal in $U \times U$, and the operator is **pseudolocal**.

We extend the definition of **properly supported** from pseudodifferential operators to the generalized operators. Examining the extended definition along with the formula for the distribution kernel, we see that G is properly supported if and only if G^{tr} is properly supported. The main theorem concerning generalized pseudodifferential operators is as follows.

Theorem 8.17. For U open in $\mathbb{R}^n$, let G be the generalized pseudodifferential operator corresponding to an amplitude $g(x, y, \xi)$ in $S^m_{1,0,0}(U \times U)$, and suppose that G is properly supported. Then

(a) G is the pseudodifferential operator with symbol

$$g(x, \xi) = e^{-2\pi i x\cdot\xi} G(e^{2\pi i(\,\cdot\,)\cdot\xi}) \qquad \text{in } S^m_{1,0}(U),$$

(b) the symbol $g(x, \xi)$ has asymptotic series

$$g(x, \xi) \sim \sum_\alpha \frac{(2\pi i)^{-|\alpha|}}{\alpha!} D^\alpha_\xi D^\alpha_y g(x, y, \xi)\big|_{y=x}.$$

[13] This discussion therefore completes the justification of the definition of $\langle Gf, \varphi\rangle$ in Section VII.6.

In (a) of Theorem 8.17, the fact that G is properly supported implies that G extends to be defined on $C^\infty(U)$, and $e^{2\pi i(\,\cdot\,)\cdot\xi}$ is a member of this space. The operator $\widetilde{G}$ with symbol $g(x,\xi)$ as in (a) is given by

$$\widetilde{G}\varphi(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} g(x,\xi)\widehat{\varphi}(\xi)\,d\xi = \int_{\mathbb{R}^n} G(e^{2\pi i(\,\cdot\,)\cdot\xi})\widehat{\varphi}(\xi)\,d\xi,$$

and the assertion in (a) is that this equals $G\varphi(x)$. Consequently the assertion is that if G is applied to the formula $\varphi(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\widehat{\varphi}(\xi)\,d\xi$, then G may be moved under the integral sign. This interchange of limits is almost handled pointwise for each x by Problem 5 in Chapter V, but we cannot take the compact metric space K in that problem to be all of $\mathbb{R}^n$. Instead, we take K to be a large ball in $\mathbb{R}^n$, apply the result of Problem 5, and do a passage to the limit.

The proof of (b) is long but reuses some of the omitted proof of Theorem 7.20. In the course of the argument, one obtains as a byproduct a conclusion that does not make use of the hypothesis "properly supported." Theorem 8.18 may be regarded as an extension of Theorem 7.22a to the present setting.

Theorem 8.18. For U open in $\mathbb{R}^n$, let G be the generalized pseudodifferential operator corresponding to an amplitude in $S^m_{1,0,0}(U\times U)$. Then there exist a pseudodifferential operator G_1 with symbol in $S^m_{1,0}(U)$ and a generalized pseudodifferential operator G_2 corresponding to an amplitude in $S^{-\infty}_{1,0,0}(U\times U)$ such that $G = G_1 + G_2$.

In any event, Theorem 8.17 is the heart of the theory of generalized pseudodifferential operators in Euclidean space, and most other results are derived from it. It is immediate from Theorem 8.17 that if G is a properly supported pseudodifferential operator as in Chapter VII with symbol $g(x,\xi)$ in $S^m_{1,0}(U)$, then so is G^{tr}, and furthermore the symbol $g^{\mathrm{tr}}(x,\xi)$ has asymptotic series

$$g^{\mathrm{tr}}(x,\xi) \sim \sum_\alpha \frac{(2\pi i)^{-|\alpha|}}{\alpha!} D^\alpha_\xi D^\alpha_x g(x,-\xi).$$

In the treatment of composition, the result is unchanged from Theorem 7.22b, but the use of amplitudes greatly simplifies the proof. In fact, let G and H be two properly supported pseudodifferential operators with respective symbols g and h, and let h^{tr} be the symbol of H^{tr}. Since $H = (H^{\mathrm{tr}})^{\mathrm{tr}}$, we have

$$H\varphi(x) = \int_{\mathbb{R}^n}\int_U e^{2\pi i(x-y)\cdot\xi} h^{\mathrm{tr}}(y,-\xi)\varphi(y)\,dy\,d\xi \qquad \text{for } \varphi\in C^\infty_{\mathrm{com}}(U).$$

Using Fourier inversion, we recognize this formula as saying that $\widehat{H\varphi}(\xi) = \int_U e^{-2\pi i y\cdot\xi} h^{\mathrm{tr}}(y,-\xi)\varphi(y)\,dy$. Substituting $\psi = H\varphi$ in the formula $G\psi(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} g(x,\xi)\widehat{\psi}(\xi)\,d\xi$ therefore gives

$$GH\varphi(x) = \int_{\mathbb{R}^n}\int_U e^{2\pi i(x-y)\cdot\xi} g(x,\xi)h^{\mathrm{tr}}(y,-\xi)\varphi(y)\,dy\,d\xi.$$

We conclude that GH is the generalized pseudodifferential operator with amplitude $g(x,\xi)h^{\mathrm{tr}}(y,-\xi)$. Applying Theorem 8.17b and sorting out the asymptotic series that the theorem gives, we obtain a quick proof of Theorem 7.22b.

We turn to the effect of diffeomorphisms on Euclidean pseudodifferential operators. Let $\Phi : U \to U^{\#}$ be a diffeomorphism between open subsets of $\mathbb{R}^n$, and suppose that a generalized pseudodifferential operator $G : C^\infty_{\mathrm{com}}(U) \to C^\infty(U)$ is given by

$$G\varphi(x) = \int_{\mathbb{R}^n}\int_U e^{2\pi i(x-y)\cdot\xi} g(x,y,\xi)\varphi(y)\,dy\,d\xi$$

for φ in $C^\infty_{\mathrm{com}}(U)$. We define $G^{\#}$ to be the operator carrying $C^\infty_{\mathrm{com}}(U^{\#})$ to $C^\infty(U^{\#})$ and given by

$$G^{\#}\psi = (G(\psi\circ\Phi))\circ\Phi^{-1} \qquad \text{for } \psi \in C^\infty_{\mathrm{com}}(U^{\#}).$$

Our objectives are to see that $G^{\#}$ is a generalized pseudodifferential operator, to obtain a formula for an amplitude of it, and to examine the effect on symbols.

Let us put $x^{\#} = \Phi(x)$ and $y^{\#} = \Phi(y)$. Put $\Phi_1 = \Phi^{-1}$. Direct use of the change-of-variables formula for multiple integrals gives

$$\begin{aligned} G^{\#}\psi(x^{\#}) &= G(\psi\circ\Phi)(x) = \int_{\mathbb{R}^n}\int_U e^{2\pi i(x-y)\cdot\xi} g(x,y,\xi)\psi(\Phi(y))\,dy\,d\xi \\ &= \int_{\mathbb{R}^n}\int_{U^{\#}} e^{2\pi i(\Phi_1(x^{\#})-\Phi_1(y^{\#}))\cdot\xi} g(\Phi_1(x^{\#}),\Phi_1(y^{\#}),\xi)\psi(y^{\#})|\det((\Phi_1)'(y^{\#}))|\,dy^{\#}d\xi. \end{aligned}$$

The hard part in showing that the expression on the right side is a generalized pseudodifferential operator is to handle the exponential factor. The starting point is the formula

$$\Phi_1(x^{\#}) - \Phi_1(y^{\#}) = \int_0^1 (\Phi_1)'(tx^{\#} + (1-t)y^{\#})(x^{\#}-y^{\#})\,dt,$$

which is valid if the line segment from $x^{\#}$ to $y^{\#}$ lies in $U^{\#}$ and which follows from the directional derivative formula and the Fundamental Theorem of Calculus. From that, one derives the following lemma.

Lemma 8.19. About each point $X = (p^{\#}, q^{\#})$ of $U^{\#}\times U^{\#}$, there exist an open neighborhood N_X and a smooth function $J_X : N_X \to GL(n,\mathbb{F})$ such that

$$\Phi_1(x^{\#}) - \Phi_1(y^{\#}) = J_X(x^{\#},y^{\#})(x^{\#}-y^{\#})$$

for every $(x^{\#}, y^{\#})$ in N_X.

The lemma allows us to write $e^{2\pi i(\Phi_1(x^\#)-\Phi_1(y^\#))\cdot\xi} = e^{2\pi i(x^\#-y^\#)\cdot J_X(x^\#,y^\#)^{\mathrm{tr}}(\xi)}$ for $(x^\#, y^\#)$ in N_X. Thus locally we can convert the integrand for $G^\#\psi(x^\#)$ into the integrand of a generalized pseudodifferential operator. It is just a question of fitting the pieces together. Using an exhausting sequence for $U^\#$ and a smooth partition of unity,[14] one can find a sequence of points X_j and smooth functions h_j with values in $[0, 1]$ such that h_j has compact support in N_{X_j}, such that each point of $U^\# \times U^\#$ has a neighborhood in which only finitely many h_j are nonzero, and such that $\sum_j h_j$ is identically 1. Let J_j be the function J_{X_j} of the lemma. Sorting out the details leads to the following result.

Theorem 8.20. If $\Phi : U \to U^\#$ is a diffeomorphism between open sets in $\mathbb{R}^n$, if $G : C^\infty_{\mathrm{com}}(U) \to C^\infty(U)$ is the generalized pseudodifferential operator with amplitude $g(x, y, \xi)$ in $S^m_{1,0,0}(U \times U)$, and if $G^\#$ is defined by $G^\#\psi = (G(\psi \circ \Phi)) \circ \Phi^{-1}$, then $G^\#$ is the generalized pseudodifferential operator on $U^\#$ with amplitude

$$\begin{aligned} g^\#(x^\#, y^\#, \eta) &= |\det(\Phi^{-1})'(x^\#)| \\ &\quad\times \Big(\sum_j h_j(x^\#, y^\#)|\det J_j(x^\#, y^\#)|^{-1} g(x, y, (J_j(x^\#, y^\#)^{-1})^{\mathrm{tr}}(\eta))\Big) \end{aligned}$$

in $S^m_{1,0,0}(U^\# \times U^\#)$, where $x = \Phi^{-1}(x^\#)$ and $y = \Phi^{-1}(y^\#)$. If G is properly supported, then so is $G^\#$.

Under the assumption that G and $G^\#$ are properly supported and G has symbol $g(x, \xi)$, let us use Theorem 8.17 to compute the symbol of $G^\#$, starting from the formula in Theorem 8.20. For that computation all that is needed is the values of $g^\#(x^\#, y^\#, \eta)$ for $(x^\#, y^\#)$ in any single neighborhood of the diagonal, however small the neighborhood.

In Lemma 8.19, one can arrange for a single N_X, say the one for $X = X_1$, to contain the entire diagonal of $U^\# \times U^\#$. The point X_1 can be one of the points used in forming the partition of unity, and the corresponding function h_1 can be arranged to be identically 1 in a neighborhood of the diagonal. Thus for purposes of computing the symbol, we may drop all the terms for $j \neq 1$ and write the formula of Theorem 8.20 as

$$g^\#(x^\#, y^\#, \eta) \approx |\det(\Phi^{-1})'(x^\#)||\det J_1(x^\#, y^\#)|^{-1} g(x, (J_1(x^\#, y^\#)^{-1})^{\mathrm{tr}}(\eta)).$$

Theorem 8.17b says that $g^\#(x, \eta) \sim \sum_\alpha \frac{(2\pi i)^{-|\alpha|}}{\alpha!} D^\alpha_\eta D^\alpha_{y^\#} g^\#(x^\#, y^\#, \eta)\big|_{y^\#=x^\#}$. The term for $\alpha = 0$ in Theorem 8.17 comes from taking $y^\# = x^\#$ in $g^\#(x^\#, y^\#, \eta)$. The function J_1 simplifies for this calculation and gives $J_1(x^\#, x^\#) = (\Phi^{-1})'(x^\#)$. Let us summarize.

[14]Smooth partitions of unity are discussed in Problem 5 at the end of the chapter.

Corollary 8.21. If $\Phi : U \to U^\#$ is a diffeomorphism between open sets in $\mathbb{R}^n$, if $G : C^\infty_{\text{com}}(U) \to C^\infty(U)$ is a properly supported pseudodifferential operator with symbol $g(x,\xi)$ in $S^m_{1,0}(U)$, and if $G^\#$ is defined by $G^\#\psi = (G(\psi \circ \Phi)) \circ \Phi^{-1}$, then $G^\#$ is a properly supported pseudodifferential operator on $U^\#$, and its symbol $g^\#(x^\#, \eta)$ has the property that

$$g^\#(x^\#, \eta) - g\big(\Phi^{-1}(x^\#), (((\Phi^{-1})'(x^\#))^{-1})^{\text{tr}}(\eta)\big)$$

is in $S^{m-1}_{1,0}(U^\#)$.

7. Pseudodifferential Operators on Manifolds

With the Euclidean theory and the necessary tools of manifold theory in place, we can now introduce pseudodifferential operators on manifolds. Let M be an n-dimensional separable smooth manifold. A typical compatible chart will be denoted by $\kappa : M_\kappa \to \widetilde{M}_\kappa$, where M_κ is open in M and $\widetilde{M}_\kappa$ is open in $\mathbb{R}^n$. Fix a smooth measure μ_g on M as in Section 5, and let $\langle \varphi_1, \varphi_2 \rangle = \int_M \varphi_1 \varphi_2 \, d\mu_g$ whenever φ_1 and φ_2 are in $C^\infty(M)$ and at least one of them has compact support.

A pseudodifferential operator on M is going to be a certain kind of continuous linear operator G from $C^\infty_{\text{com}}(M)$ into $C^\infty(M)$. The operator $G^{\text{tr}} : C^\infty_{\text{com}}(M) \to C^\infty(M)$ such that $\langle G\varphi_1, \varphi_2 \rangle = \langle \varphi_1, G^{\text{tr}}\varphi_2 \rangle$ for φ_1 and φ_2 in $C^\infty_{\text{com}}(M)$ will be another continuous linear operator of the same kind, and therefore the definition

$$\langle G(T), \varphi \rangle = \langle T, G^{\text{tr}}(\varphi) \rangle \qquad \text{for } \varphi \in C^\infty_{\text{com}}(M) \text{ and } T \in \mathcal{E}'(M)$$

extends our G to a linear function $G : \mathcal{E}'(M) \to \mathcal{D}'(M)$ in a natural way.

For any continuous linear operator $G : C^\infty_{\text{com}}(M) \to C^\infty(M)$, the scalar-valued function $\langle G\varphi_1, \varphi_2 \rangle$ on $C^\infty_{\text{com}}(M) \times C^\infty_{\text{com}}(M)$ is continuous and linear in each variable when the other variable is held fixed, and it follows from a result known as the Schwartz Kernel Theorem[15] that there exists a unique distribution $\mathcal{G}$ in $\mathcal{D}'(M \times M)$ such that

$$\langle G\varphi_1, \varphi_2 \rangle = \langle \mathcal{G}, \varphi_1 \otimes \varphi_2 \rangle \qquad \text{for } \varphi_1 \in C^\infty_{\text{com}}(M) \text{ and } \varphi_2 \in C^\infty_{\text{com}}(M),$$

where $\varphi_1 \otimes \varphi_2$ is the function on $M \times M$ with $(\varphi_1 \otimes \varphi_2)(x, y) = \varphi_1(x)\varphi_2(y)$. We call $\mathcal{G}$ the **distribution kernel** of G. The distribution kernel $\mathcal{G}^{\text{tr}}$ of G^{tr} is obtained from the distribution kernel $\mathcal{G}$ by interchanging x and y.

In analogy with the Euclidean situation, we say that G is **properly supported** if the subset support($\mathcal{G}$) of $M \times M$ has compact intersection with $K \times M$ and with

[15]A special case of the Schwartz Kernel Theorem is proved in Problems 14–19 at the end of Chapter V. This special case is at the heart of the matter in the general case.

$M \times K$ for every compact subset K of M. In this case it follows for each compact subset K of M that there exists a compact subset L of M such that $G(C_K^\infty) \subseteq C_L^\infty$. Concretely the set L is $p_1\big((M \times K) \cap \operatorname{support}(\mathcal{G})\big)$, where $p_1(x, y) = x$. Then it is immediate that G carries $C_{\text{com}}^\infty(M)$ into $C_{\text{com}}^\infty(M)$ and is continuous as such a map. The same thing is true of G^{tr} since the definition of proper support is symmetric in x and y, and therefore the definition

$$\langle G(T), \varphi\rangle = \langle T, G^{\text{tr}}(\varphi)\rangle \qquad \text{for } \varphi \in C_{\text{com}}^\infty(M) \text{ and } T \in \mathcal{D}'(M)$$

extends the properly supported G to a linear function $G : \mathcal{D}'(M) \to \mathcal{D}'(M)$ in a natural way.

A **pseudodifferential operator** of order $\leq m$ on M is a continuous linear operator $G : C_{\text{com}}^\infty(M) \to C^\infty(M)$ with the property, for every compatible chart κ, that the operator $G_\kappa : C_{\text{com}}^\infty(\widetilde{M}_\kappa) \to C^\infty(\widetilde{M}_\kappa)$ given by

$$G_\kappa(\psi) = G(\psi \circ \kappa)\big|_{M_\kappa} \circ \kappa^{-1} \qquad \text{for } \psi \in C_{\text{com}}^\infty(\widetilde{M}_\kappa)$$

is a generalized pseudodifferential operator on $\widetilde{M}_\kappa$ defined by an amplitude in $S_{1,0,0}^m(\widetilde{M}_\kappa \times \widetilde{M}_\kappa)$. Theorem 8.20 shows that this condition about all compatible charts is satisfied if it holds for all charts in an atlas.

For such an operator the distribution kernel is automatically a smooth function away from the diagonal of $M \times M$, as a consequence of the same fact about Euclidean pseudodifferential operators. One has only to realize that if two distinct points of M are given, then one can find compatible charts about the points whose domains are disjoint and whose images are disjoint; then the union of the charts is a compatible chart, and the fact about Euclidean operators can be applied.

For a distribution on a smooth manifold, it makes sense to speak of the **singular support** as the union of all open sets on which the distribution is a smooth function, and the above fact about the distribution kernel implies that any pseudodifferential operator G on M is **pseudolocal** in the sense that the singular support of $G(T)$ is contained in the singular support of T for every T in $\mathcal{E}'(M)$.

The composition of two properly supported pseudodifferential operators on M is certainly defined as a continuous linear operator from $C_{\text{com}}^\infty(M)$ into itself, but a little care is needed in checking that the composition, when referred to a compatible chart κ, is a generalized pseudodifferential operator on $\widetilde{M}_\kappa$. The reason is that when G is properly supported on M, it does not follow that the restriction of G to M_κ, i.e., to $C_{\text{com}}^\infty(M_\kappa)$, is properly supported, not even if M is an open subset of $\mathbb{R}^n$. To handle this problem, we start from this observation: if G is any pseudodifferential operator on M, if V is open in M, and if ψ_1 and ψ_2 are in $C_{\text{com}}^\infty(V)$, then the operator defined for φ in $C_{\text{com}}^\infty(V)$ by $\varphi \mapsto \psi_1 G(\psi_2\varphi)$ is a properly supported pseudodifferential operator on V; in fact, the distribution kernel of this operator is supported in the compact subset $\operatorname{support}(\psi_2) \times \operatorname{support}(\psi_1)$ of $V \times V$.

This observation, the device used above for showing that distribution kernels are smooth off the diagonal, and an argument with a partition of unity yield a proof of the following lemma.

Lemma 8.22. If L is a properly supported pseudodifferential operator on M of order $\le m$ and K is a compact subset of M_κ for some compatible chart κ of M, then there exist compatible charts $\kappa_0, \kappa_1, \dots, \kappa_r$ with $\kappa_0 = \kappa$, with each M_{κ_i} containing K and, for each $i \ge 0$, with a properly supported pseudodifferential operator L_i on M_{κ_i} such that $L(\varphi) = \sum_{i=0}^{r} L_i(\varphi)$ for every φ in C_K^∞.

PROOF. Choose K' compact such that $\varphi \in C_K^\infty$ implies $L(\varphi) \in C_{K'}^\infty$, and let $\psi \ge 0$ be a member of $C_{\text{com}}^\infty(M)$ that is 1 in a neighborhood of K'. Next choose open neighborhoods N, N', N'' of K such that $N'' \subseteq N''^{\text{cl}} \subseteq N' \subseteq N'^{\text{cl}} \subseteq N \subseteq N^{\text{cl}} \subseteq M_\kappa$ with N^{cl} compact. Finally choose $\psi_1 \in C_{\text{com}}^\infty(M)$ with values in [0, 1] that is 1 on N' and is 0 on N^c. Then $1 - \psi_1$ is 0 on N' and hence has support disjoint from K. Define $\psi_2 = (1 - \psi_1)\psi$.

For each x in the compact support of ψ_2, find a compatible chart containing x with domain V_x contained in N''^c. The sets V_x cover support(ψ_2), and there is a finite subcover $V_1, \dots, V_r$. Since each V_i with $i \ge 1$ is the domain of a compatible chart and since $V_i \cap N'' = \varnothing$, there exists a compatible chart κ_i with domain $V_i \cup N''$. Within the sets V_i, we can find open subsets W_i with W_i^{cl} compact in V_i such that the W_i cover support(ψ_2). Repeating this process, we can find open subsets X_i with X_i^{cl} compact in W_i such that the X_i cover support(ψ_2). By choosing, for each i, a smooth function on $\cup V_i$ with values in [0, 1] that is 1 on X_i and is 0 off W_i^{cl} and by then dividing by the sum of these and a smooth function that is positive on $\cup V_i - \cup W_i$ and is 0 in a neighborhood of support(ψ_2), we can produce smooth functions $\eta_1, \dots, \eta_r$ on $\cup V_i$, all ≥ 0, with sum identically 1 in a neighborhood of support(ψ_2) such that η_i has compact support in V_i. Then the operators $L_0(\varphi) = \psi_1 L(\psi_1\varphi)$ and, for $i \ge 1$, $L_i(\varphi) = \eta_i \psi_2 L(\psi_1 \varphi)$ have the required properties. □

If we have a composition $J = GH$ of properly supported pseudodifferential operators, we apply the lemma to H to write $GH(\varphi) = \sum_i G(H_i(\varphi))$. For each i, all members of $H_i(C_K^\infty)$ have support in some compact subset L_i of M_{κ_i}. Thus we can apply the lemma again to G and the set L_i to write G as a certain sum in a fashion depending on i. The result is that GH is exhibited on C_K^∞ as a sum of terms, each of which is the composition of properly supported operators within a compatible chart. Since compositions of properly supported generalized pseudodifferential operators in Euclidean space are again properly supported generalized pseudodifferential operators, each term of the sum is a pseudodifferential operator on M. Thus $J = GH$ is a pseudodifferential operator on M.

We turn to the question of symbols. As with linear differential operators, which were discussed in Section 5, we cannot expect a coordinate-free meaning for the symbol of a pseudodifferential operator on the smooth manifold M, even if the operator is properly supported. But we can associate a "principal symbol" to such an operator in many cases, generalizing the result for differential operators in Section 5. For a linear differential operator of order m, we saw that the principal symbol is a smooth function on the cotangent bundle $T^*(M, \mathbb{R})$ that is homogeneous of degree m in each fiber. For a pseudodifferential operator whose order is not a nonnegative integer, the homogeneity may disrupt the smoothness at the origin of each fiber, and we thus have to allow for a singularity. Accordingly, let $T^*(M, \mathbb{R})^\times$ denote the cotangent bundle with the zero section removed, i.e., the closed subset consisting of the 0 element of each fiber is to be removed. The **principal symbol** of order m for a properly supported pseudodifferential operator G of order $\leq m$ on M will turn out to be, in cases where it is defined, a smooth function on $T^*(M, \mathbb{R})^\times$ that is homogeneous of degree m in each fiber.

Let G be a pseudodifferential operator of order $\leq m$ on M, and let κ be a compatible chart. Let $G_\kappa(\psi) = G(\psi \circ \kappa)\big|_{M_\kappa} \circ \kappa^{-1}$ be the corresponding generalized pseudodifferential operator on $\widetilde{M}_\kappa$, and let $g_\kappa(x, y, \xi)$ be an amplitude for it, so that $g_\kappa(x, y, \xi)$ is in $S^m_{1,0,0}(\widetilde{M}_\kappa \times \widetilde{M}_\kappa)$. Suppose that $\sigma_\kappa(x, \xi)$ is a smooth function on $\widetilde{M}_\kappa \times (\mathbb{R}^n - \{0\})$ that is homogeneous of degree m in the ξ variable for each fixed x in $\widetilde{M}_\kappa$. The function $\sigma_\kappa(x, \xi)$ is not necessarily in $S^m_{1,0}(\widetilde{M}_\kappa)$ because of the potential singularity at $\xi = 0$, but the function $\tau(\ell_x(\xi))\sigma_\kappa(x, \xi)$ is in $S^m_{1,0}(\widetilde{M}_\kappa)$ if τ is a smooth scalar-valued function on $\mathbb{R}^n$ that is 0 in a neighborhood of 0 and is 1 for $|\xi|$ sufficiently large and if $x \mapsto \ell_x$ is a smooth function from $\widetilde{M}_\kappa$ into $GL(n, \mathbb{F})$. Moreover, for any two choices of τ and ℓ_x of this kind, the difference of the two symbols $\tau(\ell_x(\xi))\sigma_\kappa(x, \xi)$ is the symbol of a smoothing operator. Fix such a τ and ℓ_x. We say that G_κ has **principal symbol** $\sigma_\kappa(x, \xi)$ if there is some $\varepsilon > 0$ such that $g_\kappa(x, y, \xi) - \tau(\ell_x(\xi))\sigma_\kappa(x, \xi)$ is in $S^{m-\varepsilon}_{1,0,0}(\widetilde{M}_\kappa \times \widetilde{M}_\kappa)$. This condition is independent of τ and ℓ_x. We say that the given pseudodifferential operator G of order $\leq m$ **has a principal symbol**, namely the family $\{\sigma_\kappa(x, \xi)\}$ as κ varies, if this condition is satisfied for every κ and if ε can be taken to be independent of κ.

In this case we shall show that $\{\sigma_\kappa(x, \xi)\}$ is the system of local expressions for a scalar-valued function on the part of the cotangent bundle of M where $\xi \neq 0$, the dependence in the cotangent space being homogeneous of degree m at each point of M; consequently one refers also to this function on $T^*(M, \mathbb{R})^\times$ as the **principal symbol**. There is no assertion that a principal symbol exists, but it will be unique when it exists.[16] Moreover, this definition agrees with the definition

[16]Some authors define the principal symbol more broadly—the local expression being the coset of amplitudes for G modulo amplitudes in $S^{m-\varepsilon}_{1,0,0}(\widetilde{M}_\kappa \times \widetilde{M}_\kappa)$. This alternative definition, however,

in Section 5 in the case of a linear differential operator on M. To see that the functions $\sigma_\kappa(x,\xi)$ correspond to a single function on $T^*(M,\mathbb{R})^\times$, suppose that κ and κ' are compatible charts whose domains overlap. Let $\kappa=(x_1,\dots,x_n)$ and $\kappa'=(y_1,\dots,y_n)$. We write $y=y(x)$ for the function $\kappa'\circ\kappa^{-1}$ and $x=x(y)$ for the inverse function $\kappa\circ\kappa'^{-1}$. Theorem 8.18 shows that there is no loss of generality in assuming that the local expressions for G in the charts κ and κ' have symbols in $S^m_{1,0}(\widetilde{M}_\kappa)$ and $S^m_{1,0}(\widetilde{M}_{\kappa'})$. Let these be $g_\kappa(x,\xi)$ and $g_{\kappa'}(y,\eta)$. Corollary 8.21 shows that

$$g_{\kappa'}(y,\eta)-g_\kappa\big(x(y),\big(\big[\tfrac{\partial x_i(y)}{\partial y_j}\big]^{-1}\big)^{\mathrm{tr}}(\eta)\big)$$

is in $S^{m-1}_{1,0}(\kappa'(M_\kappa\cap M_{\kappa'}))$. Our construction shows that

$$g_{\kappa'}(y,\eta)-\tau_1(\eta)\sigma_{\kappa'}(y,\eta)$$

and

$$g_\kappa\big(x(y),\big(\big[\tfrac{\partial x_i(y)}{\partial y_j}\big]^{-1}\big)^{\mathrm{tr}}(\eta)\big)-\tau_2\big(\big[\tfrac{\partial x_i(y)}{\partial y_j}\big]^{-1}\big)^{\mathrm{tr}}(\eta)\big)\sigma_\kappa\big(x(y),\big(\big[\tfrac{\partial x_i(y)}{\partial y_j}\big]^{-1}\big)^{\mathrm{tr}}(\eta)\big)$$

are in $S^{m-\varepsilon}_{1,0}(\kappa'(M_\kappa\cap M_{\kappa'}))$. Therefore

$$\tau_2\big(\big[\tfrac{\partial x_i(y)}{\partial y_j}\big]^{-1}\big)^{\mathrm{tr}}(\eta)\big)\sigma_\kappa\big(x(y),\big(\big[\tfrac{\partial x_i(y)}{\partial y_j}\big]^{-1}\big)^{\mathrm{tr}}(\eta)\big)-\tau_1(\eta)\sigma_{\kappa'}(y,\eta)$$

is in $S^{m-\varepsilon'}_{1,0}(\kappa'(M_\kappa\cap M_{\kappa'}))$ for $\varepsilon'=\min(1,\varepsilon)$. For y fixed and $|\eta|$ sufficiently large, each term in this expression has the property that its value at $r\eta$ is r^m times its value at η if $r\geq 1$. Then the same thing is true of the difference. Since the condition of being in $S^{m-\varepsilon'}_{1,0}(\kappa'(M_\kappa\cap M_{\kappa'}))$ says that the absolute value of the difference at $r\eta$ has to be $\leq r^{m-\varepsilon'}$ times the absolute value of the difference at η, the difference has to be 0 for η sufficiently large. Therefore

$$\sigma_\kappa\big(x(y),\big(\big[\tfrac{\partial x_i(y)}{\partial y_j}\big]^{-1}\big)^{\mathrm{tr}}(\eta)\big)=\sigma_{\kappa'}(y,\eta)$$

for y in $\kappa'(M_\kappa\cap M_{\kappa'})$. According to a computation with $T^*(M)$ in Section 4, the family $\{\sigma_\kappa(x,\xi)\}$ satisfies the correct compatibility condition to be regarded as a scalar-valued function on $T^*(M,\mathbb{R})^\times$. In short, we can treat the principal symbol as a scalar-valued function on the cotangent bundle minus the zero section.

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does not reduce to the definition made in Section 5 for linear differential operators, and it seems wise in the present circumstances to avoid it.

The pseudodifferential operator G on M is said to be **elliptic** of order m if its principal symbol is nowhere 0 on $T^*(M, \mathbb{R})^\times$. It is a simple matter to check that ellipticity in this sense is equivalent to the condition that all the local expressions for the operator differ by smoothing operators[17] from operators that are elliptic of order m in the sense of Chapter VII.

Theorem 7.24 extends from Euclidean space to separable smooth manifolds: any properly supported elliptic operator G has a two-sided parametrix, i.e., a properly supported pseudodifferential operator H having $GH = 1 +$ smoothing and $HG = 1 +$ smoothing. The proof consists of using Theorem 7.24 for each member of an atlas and patching the results together by a smooth partition of unity. A certain amount of work is necessary to arrange that the local operators are properly supported. We omit the details.

As usual, the existence of the left parametrix implies a regularity result—that the singular support of Gf equals the singular support of f if f is in $\mathcal{E}'(M)$.

8. Further Developments

Having arrived at a point in studying pseudodifferential operators on manifolds comparable with where the discussion stopped for the Euclidean case, let us briefly mention some further aspects of the theory that have a bearing on parts of mathematics outside real analysis.

1. Quantitative estimates. Much of the discussion thus far has concerned the effect of pseudodifferential operators on spaces of smooth functions of compact support, and rather little has concerned distributions. Useful investigations of what happens to distributions under such operators require further tools that distinguish some distributions from others. A fundamental such tool is the continuous family of Sobolev spaces denoted by H^s, or more specifically by $H^s_{\text{com}}(M)$ or $H^s_{\text{loc}}(M)$, with s being an arbitrary real number.

The starting point is the family of Hilbert spaces $H^s(\mathbb{R}^n)$ that were introduced in Problems 8–12 at the end of Chapter III. The space $H^s(\mathbb{R}^n)$ consists of all tempered distributions $T \in \mathcal{S}(\mathbb{R}^n)$ whose Fourier transforms $\mathcal{F}(T)$ are locally square integrable functions such that $\int_{\mathbb{R}^n} |\mathcal{F}(T)|^2(1+|\xi|^2)^s\,d\xi$ is finite, the norm $\|T\|_{H^s}$ being the square root of this expression. These spaces get larger as s decreases. For K compact in $\mathbb{R}^n$, let H^s_K be the vector subspace of all members of $H^s(\mathbb{R}^n)$ with support in K; this subspace is closed and hence is complete. If U is open in $\mathbb{R}^n$, the space $H^s_{\text{com}}(U)$ is the union of all spaces H^s_K with K compact

[17] This condition takes into account Theorem 8.18, which says that the given operator differs by a smoothing operator from an operator with a symbol. If the local operator is defined by an amplitude and not a symbol, then ellipticity has not yet been defined for it.

in U, and it is given the inductive limit topology from the closed vector subspaces H^s_K. The space $H^s_{\text{loc}}(U)$ is the space of all distributions T on U such that φT is in $H^s_{\text{com}}(U)$ for all φ in $C^\infty_{\text{com}}(U)$; this space is topologized by the separating family of seminorms $T \mapsto \|\varphi T\|_{H^s}$, and a suitable countable subfamily of these seminorms suffices.

For U open in $\mathbb{R}^n$, it is a consequence of Theorem 5.20 that each member of $\mathcal{E}'(U)$ lies in $H^s_{\text{com}}(U)$ for some s. There is no difficulty in defining $H^s_{\text{com}}(M)$ and $H^s_{\text{loc}}(M)$ for a separable smooth manifold M in a coordinate-free way, and the result persists that $\mathcal{E}'(M)$ is the union of all the spaces $H^s_{\text{com}}(M)$ for s real.

We have seen that any generalized pseudodifferential operator on M carries $\mathcal{E}'(M)$ into $\mathcal{D}'(M)$. The basic quantitative refinement of this result is that any generalized pseudodifferential operator of order $\leq m$ carries $H^s_{\text{com}}(M)$ continuously into $H^{s-m}_{\text{loc}}(M)$.

2. Local existence for elliptic operators. We have seen that a properly supported elliptic pseudodifferential operator on a manifold has a two-sided parametrix. The existence of the left parametrix implies the regularity result that the elliptic operator maintains singular support. With the aid of the Sobolev spaces in subsection (1), one can prove that the existence of a right parametrix for an elliptic differential operator L with smooth coefficients implies a local existence theorem for the equation $L(u) = f$.

3. Pseudodifferential operators on sections of vector bundles. The theory presented above concerned pseudodifferential operators that mapped scalar-valued functions on a manifold into scalar-valued functions on the manifold. The first step of useful generalization is to pseudodifferential operators carrying vector-valued functions to vector-valued functions; these provide a natural setting for considering systems of differential equations. The next step of useful generalization is to pseudodifferential operators carrying sections of one vector bundle to sections of another vector bundle. The prototype is the differential operator d on a manifold, which carries smooth scalar-valued functions to smooth differential 1-forms. The latter, as we know from Section 4, are not to be considered as vector-valued functions on the manifold but as sections of the cotangent bundle. The ease of adapting our known techniques to handling the operator d in this setting illustrates the ease of handling the overall generalization of pseudodifferential operators to sections. In considering the equation $df = 0$, for example, we can use local coordinates and write $df(p) = \sum_i \frac{\partial f}{\partial x_i}(p)\, dx_i(p)$, regarding $\frac{\partial f}{\partial x_i}$ as a coefficient function for a basis vector. If $df = 0$, then each coefficient must be 0. So the partial derivatives of f in local coordinates must vanish, and f must be constant in local coordinates. Thus we have solved the equation in local coordinates. When we pass from one local coordinate system to another, aligning the basis vectors dx_i requires taking the bundle structure into account, but that is a separate problem from understanding d locally. For a pseudodifferential operator carrying sections of one vector bundle to sections of another, the formalism is completely analogous. Locally we can regard the operator as a matrix of generalized pseudodifferential operators of the kind considered earlier in this section. One can introduce appropriate generalizations of the various notions considered in this section and work with them without difficulty. In particular, one can define principal symbol and ellipticity and can follow through the usual kind of theory of parametrices for elliptic operators, obtaining the usual kind of regularity result. In place of $H^s_{\mathrm{com}}(M)$ and $H^s_{\mathrm{loc}}(M)$, one works with spaces of sections $H^s_{\mathrm{com}}(M, E)$ and $H^s_{\mathrm{loc}}(M, E)$, E being a vector bundle.

4. Pseudodifferential operators on sections when the manifold is compact. Of exceptional interest for applications is the situation in subsection (3) above when the underlying smooth manifold is compact. Here every pseudodifferential operator is of course properly supported, and the subscripts "com" and "loc" for Sobolev spaces mean the same thing. Three fundamental tools in this situation are the theory of "Fredholm operators," a version of **Sobolev's Theorem**, saying that the members of $H^s(M, E)$ have k continuous derivatives if $s > [\frac{1}{2}\dim M] + k + 1$, and **Rellich's Lemma**, saying that the inclusion of $H^s(M, E)$ into $H^t(M, E)$ if $t < s$ carries bounded sets into sets with compact closure. An important consequence is that the kernel of an elliptic operator of order m carrying $H^s(M, E)$ to $H^{s-m}(M, F)$ is finite dimensional, the dimension being independent of s; moreover, the image of $H^s(M, E)$ in $H^{s-m}(M, F)$ has finite codimension independent of s. The difference of the dimension of the kernel and the codimension of the image is called the **index** of the elliptic operator and plays a role in subsection (5) below.

5. Applications of the theory with sections over a compact manifold M. In this discussion we shall freely use some terms that have not been defined in the text, putting many of them in quotation marks or boldface at their first occurrence.

5a. A prototype of the theory of subsection (4) is **Hodge theory**, which involves "higher-degree differential forms." The operator d carries smooth forms of degree k to smooth forms of degree $k + 1$, hence is an operator from sections of one vector bundle to sections of another. If M is Riemannian, then the space of differential forms of each degree acquires an inner product, and there is a well-defined Laplacian $dd^* + d^*d$ carrying the space of forms of each degree into itself. Forms annihilated by this Laplacian are called harmonic. Roughly speaking, the theory shows that the kernel of d on the space of forms of degree k is the direct sum of the harmonic forms of degree k and the image under d of the space of forms of degree $k - 1$. Consequently "de Rham's Theorem" allows one to identify the space of harmonic forms with the cohomology of M with coefficients in the field of scalars $\mathbb{F}$.

5b. For any complex manifold M, there is an operator $\bar{\partial}$ on smooth differential forms that plays the same role for the partial derivative operators $\frac{\partial}{\partial \bar{z}_j}$ that d plays for the operators $\frac{\partial}{\partial x_j}$. The same kind of analysis as in subsection (5a), when done for a compact complex manifold with a Hermitian metric and a Laplacian of the form $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, identifies, roughly speaking, a suitable space of harmonic forms as a vector-space complement to the image of $\bar{\partial}$ in a kernel for $\bar{\partial}$.

5c. For a Riemann surface M, a holomorphic-line-bundle version of subsection (5b) leads to a proof[18] of the **Riemann–Roch Theorem**, a result allowing one to compute the dimensions of various spaces of meromorphic sections on the Riemann surface. For a compact complex manifold a holomorphic-vector-bundle version of subsection (5b) leads to Hirzebruch's generalization of the Riemann–Roch Theorem.

5d. In place of d or $\bar{\partial}$, one may use a version of a "Dirac operator" in the above kind of analysis. The result is one path that leads to the **Atiyah–Singer Index Theorem**, which relates a topological formula and an analytic formula for the index of an elliptic operator from sections of one vector bundle over the compact manifold to sections of another such bundle. This theorem has a number of applications relating topology and analysis, and the Hirzebruch–Riemann–Roch Theorem may be regarded as a special case.

BIBLIOGRAPHICAL REMARKS. There are several books on pseudodifferential operators, and the treatment here in Chapters VII and VIII has been influenced heavily by three of them: Hörmander's Volume III of *The Analysis of Linear Partial Differential Equations*, Taylor's *Pseudodifferential Operators*, and Treves's Volume 1 of *Introduction to Pseudodifferential and Fourier Integral Operators*.[19]

All three books use the definition $\widehat{f}(\xi) = c\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx$ for the Fourier transform, where $c = 1$ for Hörmander and Treves and $c = (2\pi)^{-n/2}$ for Taylor. The definition here is $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx$; this change forces small differences in the constants involved in the definition of pseudodifferential operators and results like Theorems 7.22 and 8.17. Another difference in notation is that these books include a power of $i = \sqrt{-1}$ in the definition of D^α, and this text does not; inclusion of the power of i follows a tradition dating back to the work of Hermann Weyl and seems an unnecessary encumbrance at this level.

The books by Hörmander and Treves assume extensive knowledge of material in separate books by the authors concerning distributions; Taylor makes extensive use of distributions and includes a very brief summary of them in Chapter I. Treves

[18] Not the standard proof.

[19] Full references for these books and other sources may be found in the section References at the end of the book.

uses a smooth measure on a manifold in order to identify smooth functions with distributions,[20] but Hörmander does not.

The relevant sections of those books for the material in Sections VII.6, VIII.6, and VIII.7 are as follows: Section 18.1 of Hörmander's book, Sections II.1–II.5 and III.1 of Taylor's book, and Sections I.1–I.5 of the Treves book.

The relevant portions of the three books for the mathematics in Section VIII.8 include the following: (1) Hörmander, pp. 90–91, Taylor, Section II.6; Treves, pp. 16–18 and 47. (2) Taylor, Section VI.3; Treves, pp. 92–93. (3) Hörmander, pp. 91–92; Treves, Section I.7. (4) Hörmander, Chapter XIX; Treves, Section II.2.

A larger number of books use pseudodifferential operators for some particular kind of application, sometimes developing a certain amount of the abstract theory of pseudodifferential operators. Among these are Wells, *Differential Analysis on Complex Manifolds*, which addresses applications (5a), (5b), and (5c) above; Lawson–Michelsohn, *Spin Geometry*, which addresses application (5d) above; and Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, which uses pseudodifferential operators to study the behavior of holomorphic functions on the boundaries of domains in $\mathbb{C}^n$, as well as related topics. Hörmander's book is another one that addresses application (5d), but it does so less completely than Lawson–Michelsohn.

For a brief history of pseudodifferential operators and the relationship of the theory to results like the Calderón–Zygmund Theorem, see Hörmander, pp. 178–179. For more detail about how pseudodifferential operators capture the idea of a freezing principle, see Stein, pp. 230–231.

9. Problems

1. Verify that the unit sphere $M = S^n$ in $\mathbb{R}^{n+1}$, the set of vectors of norm 1, can be made into a smooth manifold of dimension n by using two charts defined as follows. One of these charts is

$$\kappa_1(x_1, \dots, x_{n+1}) = \big(\tfrac{x_1}{1-x_{n+1}}, \dots, \tfrac{x_n}{1-x_{n+1}}\big)$$

with domain $M_{\kappa_1} = S^n - \{(0, \dots, 0, 1)\}$, and the other is

$$\kappa_2(x_1, \dots, x_{n+1}) = \big(\tfrac{x_1}{1+x_{n+1}}, \dots, \tfrac{x_n}{1+x_{n+1}}\big)$$

with domain $M_{\kappa_2} = S^n - \{(0, \dots, 0, -1)\}$.

[20] For a while, anyway.

2. Set-theoretically, the real n-dimensional projective space $M = \mathbb{R}P^n$ can be defined as the result of identifying each member x of S^n in the previous problem with its antipodal point $-x$. Let $[x] \in \mathbb{R}P^n$ denote the class of $x \in S^n$.
   (a) Show that $d([x],[y]) = \min\{|x-y|, |x+y|\}$ is well defined and makes $\mathbb{R}P^n$ into metric space such that the function $x \mapsto [x]$ is continuous and carries open sets to open sets.
   (b) For each j with $1 \le j \le n+1$, define

$$\kappa_j[(x_1, \ldots, x_{n+1})] = \Big(\frac{x_1}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_{n+1}}{x_j}\Big)$$

   on the domain $M_{\kappa_j} = \big\{[(x_1, \ldots, x_{n+1})] \bigm| x_j \ne 0\big\}$. Show that the system $\big\{\kappa_j \bigm| 1 \le j \le n+1\big\}$ is an atlas for $\mathbb{R}P^n$ and that the function $x \mapsto [x]$ from S^n to $\mathbb{R}P^n$ is smooth.

3. Let X be a smooth manifold.
   (a) Prove that if X is Lindelöf, or is σ-compact, or has a countable dense set, then X has an atlas with countably many charts.
   (b) Prove that if X has an atlas with countably many charts, then X is separable.

4. The **real general linear group** $G = GL(n, \mathbb{R})$ is the group of invertible n-by-n matrices with entries in $\mathbb{R}$, the group operation being matrix multiplication. The space of *all* n-by-n real matrices A may be identified with $\mathbb{R}^{n^2}$, and $GL(n, \mathbb{R})$ is then the open set where $\det A \ne 0$. As an open subset of $\mathbb{R}^{n^2}$, it is a smooth manifold with an atlas consisting of one chart. The coordinate functions $x_{ij}(g)$ yield the entries g_{ij} of g.
   (a) Prove that matrix multiplication, as a mapping of $G \times G$ into G, is a smooth mapping. Prove that matrix inversion, as a mapping from G into G, is smooth.
   (b) If A is a matrix with entries A_{ij}, identify A as a member of $T_g(G)$ by $A \leftrightarrow \sum_{i,j} A_{ij}\big[\frac{\partial}{\partial x_{ij}}\big]_g$. Let l_g be the diffeomorphism of G given by $l_g(h) = gh$. Define a vector field $\widetilde{A}$ by $\widetilde{A}_g f = (dl_g)_1(A)(f)$ if f is defined near g. Prove that $\widetilde{A}_g f = \sum_{i,j}(gA)_{ij}\frac{\partial f}{\partial x_{ij}}(g)$.
   (c) Prove that $\widetilde{A}$ is smooth and is **left invariant** in the sense of being carried to itself by all l_g's.
   (d) Show that $c(t) = g_0 \exp tA$ is the integral curve for $\widetilde{A}$ such that $c(0) = g_0$.
   (e) Prove that if f is in $C^\infty(G)$, then $\widetilde{A}f(g) = \frac{d}{dt} f(g \exp tX)\big|_{t=0}$.

5. This problem concerns the existence of smooth partitions of unity on a separable smooth manifold M. Let $\{K_l\}_{l \ge 1}$ be an exhausting sequence for M. For $l = 0$, put $L_0 = K_2$ and $U_0 = K_3^o$. For $l \ge 1$, put $L_l = L_{l+2} - K_{l+1}^o$ and $U_l = K_{l+3}^o - K_l$. Each point of M lies in some L_l and has a neighborhood lying in only finitely many U_l's.

(a) Using the exhausting sequence, find an atlas $\{\kappa_\alpha\}$ of compatible charts such that each point of M has a neighborhood lying in only finitely many M_{κ_α}'s.

(b) By applying Proposition 8.2 within each member of a suitable atlas as in (a), show that there exists $\eta_\alpha \in C^\infty_{\text{com}}(M_{\kappa_\alpha})$ for each α with values in $[0, 1]$ such that $\sum \eta_\alpha$ is everywhere > 0. Normalizing, conclude that there exists $\varphi_\alpha \in C^\infty_{\text{com}}(M_{\kappa_\alpha})$ for each α with values in $[0, 1]$ such that $\sum \varphi_\alpha$ is 1 identically on M.

(c) Prove that if K is compact in M and U is open with $K \subseteq U$, then there exists φ in $C^\infty_{\text{com}}(U)$ with values in $[0, 1]$ such that φ is 1 everywhere on K.

(d) Prove that if K is compact in M and $\{U_1, \ldots, U_r\}$ is a finite open cover of K, then there exist φ_j in $C^\infty_{\text{com}}(U_j)$ for $1 \leq j \leq r$ with values in $[0, 1]$ such that $\sum_{j=1}^r \varphi_j$ is 1 on K.

Problems 6–7 concern local coordinate systems on smooth manifolds.

6. Let M and N be smooth manifolds of dimensions n and k, let p be in M, suppose that $F : M \to N$ is a smooth function such that dF_p carries $T_p(M)$ onto $T_{F(p)}(N)$, and suppose that λ is a compatible chart for N about $F(p)$ such that $\lambda = (y_1, \ldots, y_k)$. Prove that the functions $y_1 \circ F, \ldots, y_k \circ F$ can be taken as the first k of n functions that generate a system of local coordinates near p in the sense of Proposition 8.4.

7. Let M and N be smooth manifolds of dimensions n and k, let p be in M, suppose that $F : M \to N$ is a smooth function such that dF_p is one-one, and suppose that $\psi = (y_1, \ldots, y_k)$ is a compatible chart for N about $F(p)$.
   (a) Prove that it is possible to select from the set of functions $y_1 \circ F, \ldots, y_k \circ F$ a subset of n of them that generate a system of local coordinates near $F(p)$ in the sense of Proposition 8.4.
   (b) Let $\varphi = (x_1, \ldots, x_n)$ be a compatible chart for M about p. Prove that there exists a system of local coordinates $(z_1, \ldots, z_k)$ near $F(p)$ such that x_j coincides in a neighborhood of p with $z_j \circ F$ for $1 \leq j \leq n$.

Problems 8–9 concern extending Sard's Theorem (Theorem 6.35 of *Basic*) to separable smooth manifolds. Let M be an n-dimensional separable smooth manifold, and let $\{\kappa_\alpha\}$ be an atlas of charts. A subset S of M has **measure 0** if $\kappa_\alpha(S \cap M_\alpha)$ has n-dimensional Lebesgue measure 0 for all α. If $F : M \to N$ is a smooth map between smooth n-dimensional manifolds M and N, a **critical point** p of F is a point where the differential $(dF)_p$ has rank $< n$. In this case, $F(p)$ is called a **critical value**.

8. Prove that if $F : M \to N$ is a smooth map between two smooth separable n-dimensional manifolds M and N, then the set of critical values of F has measure 0 in N.

9. Prove that if $F : M \to N$ is a smooth map between two separable smooth manifolds and if $\dim M < \dim N$, then the image of F has measure 0 in N.

Problems 10–13 introduce equivalence of vector bundles, which is the customary notion of isomorphism for vector bundles with the same base space. Let $\pi : B \to M$ and $\pi' : B' \to M$ be two smooth coordinate vector bundles of the same rank n with the same field of scalars and same base space M, but with distinct bundle spaces, distinct projections, possibly distinct atlases $\mathcal{A} = \{\kappa_j\}$ and $\mathcal{A}' = \{\kappa'_k\}$ for M, distinct coordinate functions ϕ_j and ϕ'_k, and distinct transition functions $g_{jk}(x)$ and $g'_{kl}(x)$. Let $h : B \to B'$ be a fiber-preserving smooth map covering the identity map of M, i.e., a smooth map such that $h(\pi^{-1}(x)) = \pi'^{-1}(x)$ for all x in M. For each x in M, define h_x to be the smooth map obtained by restriction $h_x = h\big|_{\pi^{-1}(x)}$; this carries $\pi^{-1}(x)$ to $\pi'^{-1}(x)$. Say that h exhibits $\pi : B \to M$ and $\pi' : B' \to M$ as **equivalent** coordinate vector bundles if the following two conditions are satisfied:

- whenever κ_j and κ'_k are charts in $\mathcal{A}$ and $\mathcal{A}'$ about a point x of M, then the map
$$\bar{g}_{kj}(x) = {\phi'_{k,x}}^{-1} \circ h_x \circ \phi_{j,x}$$
of $\mathbb{F}^n$ into itself coincides with the operation of a member of $GL(n, \mathbb{F})$,
- the map $\bar{g}_{kj} : M_{\kappa_j} \cap M_{\kappa'_k} \to GL(n, \mathbb{F})$ is smooth.

The functions $x \mapsto \bar{g}_{kj}(x)$ will be called the **mapping functions** of h.

10. Prove for coordinate vector bundles that "equivalent" is reflexive and transitive and that strictly equivalent implies equivalent.

11. Prove that if h exhibits two coordinate vector bundles $\pi : B \to M$ and $\pi' : B' \to M$ as equivalent, then the mapping functions $x \mapsto \bar{g}_{kj}(x)$ of h satisfy the conditions
$$\bar{g}_{kj}(x) g_{ji}(x) = \bar{g}_{ki}(x) \qquad \text{for } x \in M_{\kappa_i} \cap M_{\kappa_j} \cap M_{\kappa'_k},$$
$$g'_{lk}(x) \bar{g}_{kj}(x) = \bar{g}_{lj}(x) \qquad \text{for } x \in M_{\kappa_j} \cap M_{\kappa'_k} \cap M_{\kappa'_l}.$$

12. Suppose that $\pi : B \to M$ and $\pi' : B' \to M$ are two smooth coordinate vector bundles of the same rank n with the same field of scalars relative to atlases $\mathcal{A} = \{\kappa_j\}$ and $\mathcal{A}' = \{\kappa'_k\}$ of M.
    (a) If smooth functions $x \mapsto \bar{g}_{kj}(x)$ of $M_{\kappa_j} \cap M_{\kappa'_k}$ into $GL(n, \mathbb{F})$ are given that satisfy the displayed conditions in Problem 11, prove that there exists at most one equivalence $h : B \to B'$ of coordinate vector bundles having $\{\bar{g}_{kj}\}$ as mapping functions and that it is given by $h(\phi_{j,x}(y)) = \phi'_{k,x} \bar{g}_{kj}(x)(y)$.
    (b) Prove that "equivalent" for coordinate vector bundles is symmetric, and conclude that "equivalent" is an equivalence relation whose equivalence classes are unions of equivalence classes under strict equivalence. (Educational note: Therefore the notion of **equivalent vector bundles** is well defined.)

13. Suppose that $\pi : B \to M$ and $\pi' : B' \to M$ are two smooth coordinate vector bundles of the same rank n with the same field of scalars relative to atlases $\mathcal{A} = \{\kappa_j\}$ and $\mathcal{A}' = \{\kappa'_k\}$ of M, and suppose that smooth functions $x \mapsto \bar{g}_{kj}(x)$ of $M_{\kappa_j} \cap M_{\kappa'_k}$ into $GL(n, \mathbb{F})$ are given that satisfy the displayed conditions in Problem 11.

(a) Define a smooth mapping h_{kj} from $\pi^{-1}(M_{\kappa_j} \cap M_{\kappa'_k})$ in B to $\pi'^{-1}(M_{\kappa_j} \cap M_{\kappa'_k})$ as follows: If b is in B with $x = \pi(b)$ in $M_{\kappa_j} \cap M_{\kappa'_k}$, let $\pi_j(b) = \phi_{j,x}^{-1}(b) \in \mathbb{F}^n$, and set

$$h_{kj}(b) = \phi'_{k,x}\,\bar{g}_{kj}(x)(p_j(b)).$$

Prove that $\{h_{kj}\}$ is consistently defined as one moves from chart to chart, i.e., that if x lies also in $M_{\kappa_i} \cap M_{\kappa'_l}$, then $h_{kj}(b) = h_{li}(b)$, and conclude that the functions h_{kj} piece together as a single smooth function $h : B \to B'$.

(b) Prove that the functions $x \mapsto \bar{g}_{kj}(x)$ coincide with the mapping functions of h, and conclude that the existence of functions satisfying the displayed conditions in Problem 11 is necessary and sufficient for equivalence.