11. The First Formula for Transforming Inclusions into Equalities.—We can now demonstrate an important formula by which an inclusion may be transformed into an equality, or *vice versa:*

$$(a < b) = (a = ab) \quad \Big| \quad (a < b) = (a + b = b)$$

Demonstration:

1. $(a < b) < (a = ab), \qquad (a < b) < (a + b = b).$

For

(Comp.) $\qquad (a < a)\ (a < b) < (a < ab),$
$\qquad (a < b)\ (b < b) < (a + b < b).$

On the other hand, we have

(Simpl.) $\qquad ab < a, \qquad b < a + b,$

(Def. =) $\qquad (a < ab)\ (ab < a) = (a = ab),$
$\qquad (a + b < b)\ (b < a + b) = (a + b = b);$

2. $\qquad (a = ab) < (a < b), \qquad (a + b = b) < (a < b).$

For

$$(a = ab)\ (ab < b) < (a < b),$$
$$(a < a + b)\ (a + b = b) < (a < b).$$

Remark.—If we take the relation of equality as a primitive idea (one not defined) we shall be able to define the relation of inclusion by means of one of the two preceding formulas.[1] We shall then be able to demonstrate the principle of the syllogism.[2]

From the preceding formulas may be derived an interesting result:

$$(a = b) = (ab = a + b).$$

For

1. $$(a = b) = (a < b)\ (b < a),$$
$$(a < b) = (a = ab), \quad (b < a) = (a + b = a),$$

(Syll.) $\qquad (a = ab)\ (a + b = a) < (ab = a + b).$

---

[1] See HUNTINGTON, *op. cit.*, § 1.

[2] This can be demonstrated as follows: By definition we have $(a < b) = (a = ab)$, and $(b < c) = (b = bc)$. If in the first equality we substitute for b its value derived from the second equality, then $a = abc$. Substitute for a its equivalent ab, then $ab = abc$. This equality is equivalent to the inclusion, $ab < c$. Conversely substitute a for ab; whence we have $a < c$. Q. E. D.

2. $$(ab = a + b) < (a + b < ab),$$

(Comp.) $$(a + b < ab) = (a < ab)\ (b < ab),$$
$$(a < ab)\ (ab < a) = (a = ab) = (a < b),$$
$$(b < ab)\ (ab < b) = (b = ab) = (b < a).$$

Hence
$$(ab = a + b) < (a < b)\ (b < a) = (a = b).$$

12. **The Distributive Law.**—The principles previously stated make it possible to demonstrate the *converse distributive law,* both of multiplication with respect to addition, and of addition with respect to multiplication,
$$ac + bc < (a + b)c, \qquad ab + c < (a + c)\ (b + c).$$

Demonstration:
$$(a < a + b) < [ac < (a + b)c],$$
$$(b < a + b) < [bc < (a + b)c];$$
whence, by composition,
$$[ac < (a + b)c]\ [bc < (a + b)c] < [ac + bc < (a + b)c].$$

2. $$(ab < a) < (ab + c < a + c),$$
$$(ab < b) < (ab + c < b + c),$$
whence, by composition,
$$(ab + c < a + c)\ (ab + c < b + c) < [ab + c < (a + c)\ (b + c)].$$

But these principles are not sufficient to demonstrate the *direct distributive law*
$$(a + b)\,c < ac + bc, \qquad (a + c)\ (b + c) < ab + c,$$
and we are obliged to postulate one of these formulas or some simpler one from which they can be derived. For greater convenience we shall postulate the formula

(Ax. V). $$(a + b)\,c < ac + bc.$$

This, combined with the converse formula, produces the equality
$$(a + b)\,c = ac + bc,$$
which we shall call briefly the *distributive law.*

From this may be directly deduced the formula
$$(a + b)\ (c + d) = ac + bc + ad + bd,$$