CHAPTER 8

Expression of local solutions

Fix $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,\dots,p\\1\le\nu\le n_j}} \in \mathcal{P}_{p+1}$. Suppose $\mathbf{m}$ is monotone and irreducibly realizable. Let $P_{\mathbf{m}}$ be the universal operator with the Riemann scheme (4.15), which is given in Theorem 6.14. Suppose $c_1 = 0$ and $m_{1,n_1} = 1$. We give expressions of the local solution of $P_{\mathbf{m}}u = 0$ at $x = 0$ corresponding to the characteristic exponent λ_{1,n_1}.

THEOREM 8.1. *Retain the notation above and in* Definition 5.12. *Suppose $\lambda_{j,\nu}$ are generic. Let*

$$v(x) = \sum_{\nu=0}^{\infty} C_\nu x^{\lambda(K)_{1,n_1}+\nu} \tag{8.1}$$

be the local solution of $\big(\partial_{\max}^K P_{\mathbf{m}}\big)v = 0$ at $x = 0$ with the condition $C_0 = 1$. Put

$$\lambda(k)_{j,max} = \lambda(k)_{j,\ell(k)_j}. \tag{8.2}$$

Note that if $\mathbf{m}$ is rigid, then

$$v(x) = x^{\lambda(K)_{1,n_1}} \prod_{j=2}^{p} \Big(1 - \frac{x}{c_j}\Big)^{\lambda(K)_{j,max}}. \tag{8.3}$$

The function

$$\begin{aligned} u(x) := \prod_{k=0}^{K-1} &\frac{\Gamma\big(\lambda(k)_{1,n_1} - \lambda(k)_{1,max} + 1\big)}{\Gamma\big(\lambda(k)_{1,n_1} - \lambda(k)_{1,max} + \mu(k) + 1\big)\Gamma\big(-\mu(k)\big)} \\ &\int_0^{s_0} \cdots \int_0^{s_{K-1}} \prod_{k=0}^{K-1} (s_k - s_{k+1})^{-\mu(k)-1} \\ &\cdot \prod_{k=0}^{K-1} \Big(\Big(\frac{s_k}{s_{k+1}}\Big)^{\lambda(k)_{1,max}} \prod_{j=2}^{p} \Big(\frac{1 - c_j^{-1}s_k}{1 - c_j^{-1}s_{k+1}}\Big)^{\lambda(k)_{j,max}}\Big) \\ &\cdot v(s_K)ds_K \cdots ds_1\Big|_{s_0=x} \end{aligned} \tag{8.4}$$

is the solution of $P_{\mathbf{m}}u = 0$ so normalized that $u(x) \equiv x^{\lambda_{1,n_1}} \mod x^{\lambda_{1,n_1}+1}\mathcal{O}_0$.

Here we note that

$$\begin{aligned} &\prod_{k=0}^{K-1} \Big(\Big(\frac{s_k}{s_{k+1}}\Big)^{\lambda(k)_{1,max}} \prod_{j=2}^{p} \Big(\frac{1 - c_j^{-1}s_k}{1 - c_j^{-1}s_{k+1}}\Big)^{\lambda(k)_{j,max}}\Big) \\ &\quad = \frac{s_0^{\lambda(0)_{1,max}}}{s_K^{\lambda(K-1)_{1,max}}} \prod_{j=1}^{p} \frac{(1 - c_j^{-1}s_0)^{\lambda(0)_{j,max}}}{(1 - c_j^{-1}s_K)^{\lambda(K-1)_{j,max}}} \\ &\qquad \cdot \prod_{k=1}^{K-1} \Big(s_k^{\lambda(k)_{1,max}-\lambda(k-1)_{1,max}} \prod_{j=2}^{p} (1 - c_j^{-1}s_k)^{\lambda(k)_{j,max}-\lambda(k-1)_{j,max}}\Big). \end{aligned} \tag{8.5}$$

When $\mathbf{m}$ *is rigid,*

$$
\begin{aligned}
u(x) = x^{\lambda_{1,n_1}} \Bigl(\prod_{j=2}^{p}\Bigl(1-\frac{x}{c_j}\Bigr)^{\lambda(0)_{j,max}}\Bigr) &\sum_{\left(\nu_{j,k}\right)_{\substack{2\le j\le p\\ 1\le k\le K}}\in\mathbb{Z}_{\ge 0}^{(p-1)K}} \\
\prod_{i=0}^{K-1} &\frac{\bigl(\lambda(i)_{1,n_1}-\lambda(i)_{1,max}+1\bigr)_{\sum_{s=2}^{p}\sum_{t=i+1}^{K}\nu_{s,t}}}{\bigl(\lambda(i)_{1,n_1}-\lambda(i)_{1,max}+\mu(i)+1\bigr)_{\sum_{s=2}^{p}\sum_{t=i+1}^{K}\nu_{s,t}}} \\
\cdot &\prod_{i=1}^{K}\prod_{s=2}^{p}\frac{\bigl(\lambda(i-1)_{s,max}-\lambda(i)_{s,max}\bigr)_{\nu_{s,i}}}{\nu_{s,i}!}\cdot\prod_{s=2}^{p}\Bigl(\frac{x}{c_s}\Bigr)^{\sum_{i=1}^{K}\nu_{s,i}}.
\end{aligned}
\tag{8.6}
$$

When $\mathbf{m}$ *is not rigid*

$$
\begin{aligned}
u(x) = x^{\lambda_{1,n_1}} \Bigl(\prod_{j=2}^{p}\Bigl(1-\frac{x}{c_j}\Bigr)^{\lambda(0)_{j,max}}\Bigr) &\sum_{\nu_0=0}^{\infty}\sum_{\left(\nu_{j,k}\right)_{\substack{2\le j\le p\\ 1\le k\le K}}\in\mathbb{Z}_{\ge 0}^{(p-1)K}} \\
\prod_{i=0}^{K-1} &\frac{\bigl(\lambda(i)_{1,n_1}-\lambda(i)_{1,max}+1\bigr)_{\nu_0+\sum_{s=2}^{p}\sum_{t=i+1}^{K}\nu_{s,t}}}{\bigl(\lambda(i)_{1,n_1}-\lambda(i)_{1,max}+\mu(i)+1\bigr)_{\nu_0+\sum_{s=2}^{p}\sum_{t=i+1}^{K}\nu_{s,t}}} \\
\cdot &\prod_{s=2}^{p}\frac{\bigl(\lambda(K-1)_{s,max}\bigr)_{\nu_{s,K}}}{\nu_{s,K}!}\cdot\prod_{i=1}^{K-1}\prod_{s=2}^{p}\frac{\bigl(\lambda(i-1)_{s,max}-\lambda(i)_{s,max}\bigr)_{\nu_{s,i}}}{\nu_{s,i}!} \\
\cdot &C_{\nu_0}x^{\nu_0}\prod_{s=2}^{p}\Bigl(\frac{x}{c_s}\Bigr)^{\sum_{i=1}^{K}\nu_{s,i}}.
\end{aligned}
\tag{8.7}
$$

Fix j *and* k *and suppose*

$$
\begin{cases}
\ell(k-1)_j = \ell(k)_\nu & \text{when } \mathbf{m} \text{ is rigid or } k<K,\\
\ell(k-1)_j = 0 & \text{when } \mathbf{m} \text{ is not rigid and } k=K.
\end{cases}
\tag{8.8}
$$

Then the terms satisfying $\nu_{j,k}>0$ *vanish because* $(0)_{\nu_{j,k}}=\delta_{0,\nu_{j,k}}$ *for* $\nu_{j,k}=0,1,2,\ldots$.

Proof. The theorem follows from (5.26), (5.27), (5.28), (3.2) and (3.6) by the induction on K. Note that the integral representation of the normalized solution of $\bigl(\partial_{max}P\bigr)v=0$ corresponding to the exponent $\lambda(1)_{n_1}$ equals

$$
\begin{aligned}
v(x) := &\prod_{k=1}^{K-1}\frac{\Gamma\bigl(\lambda(k)_{1,n_1}-\lambda(k)_{1,max}+1\bigr)}{\Gamma\bigl(\lambda(k)_{1,n_1}-\lambda(k)_{1,max}+\mu(k)+1\bigr)\Gamma\bigl(-\mu(k)\bigr)} \\
&\cdot\int_0^{s_1}\cdots\int_0^{s_{K-1}}\prod_{k=0}^{K-1}(s_k-s_{k+1})^{-\mu(k)-1} \\
&\cdot\prod_{k=0}^{K-1}\Bigl(\Bigl(\frac{s_k}{s_{k+1}}\Bigr)^{\lambda(k)_{1,max}}\prod_{j=2}^{p}\Bigl(\frac{1-c_j^{-1}s_k}{1-c_j^{-1}s_{k+1}}\Bigr)^{\lambda(k)_{j,max}}\Bigr) \\
&\cdot v(s_K)ds_K\cdots ds_1\Big|_{s_1=x} \\
\equiv\ & x^{\lambda(1)_{1,n_1}} \mod x^{\lambda(1)_{1,n_1}+1}\mathcal{O}_0
\end{aligned}
$$

by the induction hypothesis and the normalized solution of $Pu = 0$ corresponding to the exponent λ_{1,n_1} equals

$$\frac{\Gamma\bigl(\lambda(0)_{1,n_1} - \lambda(0)_{1,max} + 1\bigr)}{\Gamma\bigl(\lambda(0)_{1,n_1} - \lambda(0)_{1,max} + \mu(0) + 1\bigr)\Gamma\bigl(-\mu(0)\bigr)} \cdot \int_0^x (x - s_0)^{-\mu(0)-1} \frac{x^{-\lambda(0)_{1,\max}}}{s_0^{-\lambda(0)_{1,\max}}} \prod_{j=2}^{p} \Bigl(\frac{1 - c_j^{-1}x}{1 - c_j^{-1}s_0}\Bigr)^{-\lambda(0)_{j,\max}} v(s_0)ds_0$$

and hence we have (8.4). Then the integral expression (8.4) with (8.5), (3.2) and (3.6) inductively proves (8.6) and (8.7). □

Example 8.2 (Gauss hypergeometric equation). The reduction (10.54) shows

$$\begin{aligned}&\lambda(0)_{j,\nu} = \lambda_{j,\nu},\ m(0)_{j,\nu} = 1 \quad (0 \le j \le 2,\ 1 \le \nu \le 2),\ \mu(0) = -\lambda_{0,2} - \lambda_{1,2} - \lambda_{2,2},\\ &m(1)_{j,1} = 0,\ m(1)_{j,2} = 1 \quad (j = 0, 1, 2),\\ &\lambda(1)_{0,1} = \lambda_{0,1} + 2\lambda_{0,2} + 2\lambda_{1,2} + 2\lambda_{2,2},\ \lambda(1)_{1,1} = \lambda_{1,1},\ \lambda(1)_{2,1} = \lambda_{2,1},\\ &\lambda(1)_{0,2} = 2\lambda_{0,2} + \lambda_{1,2} + \lambda_{2,2},\ \lambda(1)_{1,2} = -\lambda_{0,2} - \lambda_{2,2},\ \lambda(1)_{2,2} = -\lambda_{0,2} - \lambda_{1,2}\end{aligned}$$

and therefore

$$\begin{aligned}\lambda(0)_{1,n_1} - \lambda(0)_{1,max} + \mu(0) + 1 &= \lambda_{1,2} - \lambda_{1,1} - (\lambda_{0,2} + \lambda_{1,2} + \lambda_{2,2}) + 1\\ &= \lambda_{0,1} + \lambda_{1,2} + \lambda_{2,1},\\ \lambda(0)_{2,max} - \lambda(1)_{2,max} &= \lambda(0)_{2,1} - \lambda(1)_{2,2} = \lambda_{2,1} + \lambda_{0,2} + \lambda_{1,2}.\end{aligned}$$

Hence (8.4) says that the normalized local solution corresponding to the characteristic exponent $\lambda_{1,2}$ with $c_1 = 0$ and $c_2 = 1$ equals

$$\begin{aligned}u(x) &= \frac{\Gamma\bigl(\lambda_{1,2} - \lambda_{1,1} + 1\bigr)x^{\lambda_{1,1}}(1 - x)^{\lambda_{2,1}}}{\Gamma\bigl(\lambda_{0,1} + \lambda_{1,2} + \lambda_{2,1}\bigr)\Gamma\bigl(\lambda_{0,2} + \lambda_{1,2} + \lambda_{2,2}\bigr)}\\ &\qquad \int_0^x (x - s)^{\lambda_{0,2}+\lambda_{1,2}+\lambda_{2,2}-1} s^{-\lambda_{0,2}-\lambda_{1,1}-\lambda_{2,2}} (1 - s)^{-\lambda_{0,2}-\lambda_{1,2}-\lambda_{2,1}} ds\end{aligned} \tag{8.9}$$

and moreover (8.6) says

$$u(x) = x^{\lambda_{1,2}}(1 - x)^{\lambda_{2,1}} \sum_{\nu=0}^{\infty} \frac{(\lambda_{0,1} + \lambda_{1,2} + \lambda_{2,1})_\nu (\lambda_{0,2} + \lambda_{1,2} + \lambda_{2,1})_\nu}{(\lambda_{1,2} - \lambda_{1,1} + 1)_\nu \nu!} x^\nu. \tag{8.10}$$

Note that $u(x) = F(a, b, c; x)$ when

$$\begin{Bmatrix} x = \infty & 0 & 1\\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1}\\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2} \end{Bmatrix} = \begin{Bmatrix} x = \infty & 0 & 1\\ a & 1 - c & 0\\ b & 0 & c - a - b \end{Bmatrix}. \tag{8.11}$$

The integral expression (8.9) is based on the minimal expression $w = s_{0,1}s_{1,1}s_{1,2}s_0$ satisfying $w\alpha_{\mathbf{m}} = \alpha_0$. Here $\alpha_{\mathbf{m}} = 2\alpha_0 + \sum_{j=0}^{2} \alpha_{j,1}$. When we replace w and its minimal expression by $w' = s_{0,1}s_{1,1}s_{1,2}s_0s_{0,1}$ or $w'' = s_{0,1}s_{1,1}s_{1,2}s_0s_{2,1}$, we get the different integral expressions

$$\begin{aligned}u(x) &= \frac{\Gamma\bigl(\lambda_{1,2} - \lambda_{1,1} + 1\bigr)x^{\lambda_{1,1}}(1 - x)^{\lambda_{2,1}}}{\Gamma\bigl(\lambda_{0,2} + \lambda_{1,2} + \lambda_{2,1}\bigr)\Gamma\bigl(\lambda_{0,1} + \lambda_{1,2} + \lambda_{2,2}\bigr)}\\ &\qquad \int_0^x (x - s)^{\lambda_{0,1}+\lambda_{1,2}+\lambda_{2,2}-1} s^{-\lambda_{0,1}-\lambda_{1,1}-\lambda_{2,2}} (1 - s)^{-\lambda_{0,1}-\lambda_{1,2}-\lambda_{2,1}} ds\\ &= \frac{\Gamma\bigl(\lambda_{1,2} - \lambda_{1,1} + 1\bigr)x^{\lambda_{1,1}}(1 - x)^{\lambda_{2,2}}}{\Gamma\bigl(\lambda_{0,1} + \lambda_{1,2} + \lambda_{2,2}\bigr)\Gamma\bigl(\lambda_{0,2} + \lambda_{1,2} + \lambda_{2,1}\bigr)}\\ &\qquad \int_0^x (x - s)^{\lambda_{0,2}+\lambda_{1,2}+\lambda_{2,1}-1} s^{-\lambda_{0,2}-\lambda_{1,1}-\lambda_{2,1}} (1 - s)^{-\lambda_{0,2}-\lambda_{1,2}-\lambda_{2,2}} ds.\end{aligned} \tag{8.12}$$

These give different integral expressions of $F(a,b,c;x)$ under (8.11).

Since $s_{\alpha_0+\alpha_{0,1}+\alpha_{0,2}}\alpha_{\mathbf{m}} = \alpha_{\mathbf{m}}$, we have

$$\begin{Bmatrix} x=\infty & 0 & 1 \\ a & 1-c & 0 \\ b & 0 & c-a-b \end{Bmatrix} \xrightarrow{x^{c-1}} \begin{Bmatrix} x=\infty & 0 & 1 \\ a-c+1 & 0 & 0 \\ b-c+1 & c-1 & c-a-b \end{Bmatrix}$$

$$\xrightarrow{\partial^{c-d}} \begin{Bmatrix} x=\infty & 0 & 1 \\ a-d+1 & 0 & 0 \\ b-d+1 & d-1 & d-a-b \end{Bmatrix} \xrightarrow{x^{1-d}} \begin{Bmatrix} x=\infty & 0 & 1 \\ a & 1-d & 0 \\ b & 0 & d-a-b \end{Bmatrix}$$

and hence (cf. (3.6))

$$F(a,b,d;x) = \frac{\Gamma(d)x^{1-d}}{\Gamma(c)\Gamma(d-c)} \int_0^x (x-s)^{d-c-1} s^{c-1} F(a,b,c;s)ds. \tag{8.13}$$

REMARK 8.3. The integral expression of the local solution $u(x)$ as is given in Theorem 8.1 is obtained from the expression of the element w of W_∞ satisfying $w\alpha_{\mathbf{m}} \in B \cup \{\alpha_0\}$ as a product of simple reflections and therefore the integral expression depends on such element w and the expression of w as such product. The dependence on w seems non-trivial as in the preceding example but the dependence on the expression of w as a product of simple reflections is understood as follows.

First note that the integral expression doesn't depend on the coordinate transformations $x \mapsto ax$ and $x \mapsto x+b$ with $a \in \mathbb{C}^\times$ and $b \in \mathbb{C}$. Since

$$\begin{aligned} \int_c^x (x-t)^{\mu-1}\phi(t)dt &= -\int_{\frac{1}{c}}^{\frac{1}{x}} (x-\tfrac{1}{s})^{\mu-1}\phi(\tfrac{1}{s})s^{-2}ds \\ &= -(-1)^{\mu-1}x^{\mu-1}\int_{\frac{1}{c}}^{\frac{1}{x}} (\tfrac{1}{x}-s)^{\mu-1}(\tfrac{1}{s})^{\mu+1}\phi(\tfrac{1}{s})ds, \end{aligned}$$

we have

$$I_c^\mu(\phi) = -(-1)^{\mu-1}x^{\mu-1}\left(I_{\frac{1}{c}}^x \left(x^{\mu+1}\phi(x)\right)\Big|_{x\mapsto\frac{1}{x}}\right)\Big|_{x\to\frac{1}{x}}, \tag{8.14}$$

which corresponds to (5.11). Here the value $(-1)^{\mu-1}$ depends on the branch of the value of $(x-\frac{1}{s})^{\mu-1}$ and that of $x^{\mu-1}x^{1-\mu}(\frac{1}{x}-s)^{\mu-1}$.

Hence the argument as in the proof of Theorem 7.5 shows that the dependence on the expression of w by a product of simple reflections can be understood by the identities (8.14) and $I_c^{\mu_1}I_c^{\mu_2} = I_c^{\mu_1+\mu_2}$ (cf. (3.4)) etc.