3

DPN surfaces of elliptic type

3.1. Fundamental chambers of $W^{(2,4)}(S)$ for elliptic type

The most important property of lattices S of elliptic type is that the subgroup $W^{(2)}(S) \subset O(S)$ has finite index. We remark that this is parallel to Lemma 1.4, and is an important step to prove that log del Pezzo surfaces of index ≤ 2 are equivalent to DPN surfaces of elliptic type.

This finiteness was first observed and used for classification of hyperbolic lattices M with finite index $[O(M) : W^{(2)}(M)]$ in [Nik79], [Nik83]. We repeat arguments of [Nik79], [Nik83]. Let us take a general pair (X, θ) with $(S_X)_+ = S$. Then $S_X = S$, and the involution θ of X is unique by the condition that it is identical on $S_X = S$ and is -1 on the orthogonal complement to S_X in $H^2(X, \mathbb{Z})$. Thus, $\operatorname{Aut} X = \operatorname{Aut}(X, \theta)$. By Global Torelli Theorem for K3 (see [PS-Sh71]), the action of $\operatorname{Aut} X$ on S_X gives that $\operatorname{Aut} X$ and $O(S_X)/W^{(2)}(S_X)$ are isomorphic up to finite groups. In particular, they are finite simultaneously. Thus, $[O(S) : W^{(2)}(S)]$ is finite, if and only if $\operatorname{Aut}(X, \theta)$ is finite. If (X, θ) has elliptic type, then $\operatorname{Aut}(X, \theta)$ preserves X^θ and its component C_g with $(C_g)^2 > 0$. Since S_X is hyperbolic, it follows that the action of $\operatorname{Aut}(X, \theta)$ in S_X is finite. But it is known for K3 (see [PS-Sh71]) that the kernel of this action is also finite. It follows that $\operatorname{Aut}(X, \theta)$ and $[O(S) : W^{(2)}(S)]$ are finite. See more details on the results we used about K3 in Section 2.2.

Since $O(S)$ is arithmetic, $W^{(2)}(S)$ has a fundamental chamber $\mathcal{M}^{(2)}$ in $\mathcal{L}(S)$ of finite volume and with a finite number of faces (e.g. see [Vin85]). Since $W^{(2)}(S) \subset W^{(2,4)}(S) \subset O(S)$, the same is valid for $W^{(2,4)}(S)$.

Let $\mathcal{M}^{(2,4)} \subset \mathcal{L}(S)$ be a fundamental chamber of $W^{(2,4)}(S)$, and $\Gamma(P(\mathcal{M}^{(2,4)}))$ its Dynkin diagram (see [Vin85]). Vertices corresponding to different elements $f_1,\ f_2 \in P(\mathcal{M}^{(2,4)})$ are **not connected** by any edge, if $f_1 \cdot f_2 = 0$. They are connected by a **simple edge of the weight** m

(equivalently, by $m-2$ **simple edges,** if $m>2$ is small), if

$$\frac{2 f_1 \cdot f_2}{\sqrt{f_1^2 f_2^2}} = 2\cos\frac{\pi}{m}, \quad m \in \mathbb{N}.$$

They are connected by a **thick edge,** if

$$\frac{2 f_1 \cdot f_2}{\sqrt{f_1^2 f_2^2}} = 2.$$

They are connected by a **broken edge** of the weight t, if

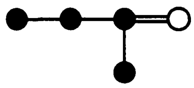
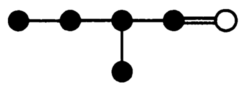
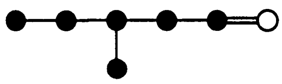
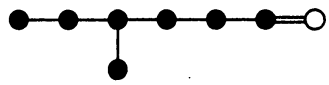
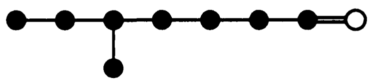
$$\frac{2 f_1 \cdot f_2}{\sqrt{f_1^2 f_2^2}} = t > 2.$$

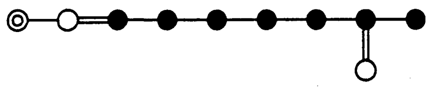
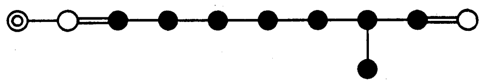
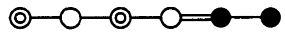
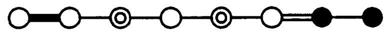
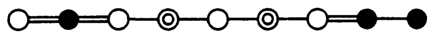
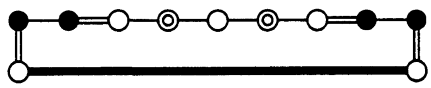
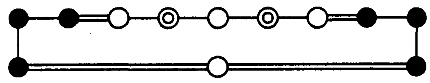
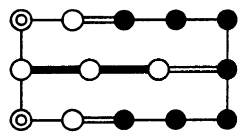
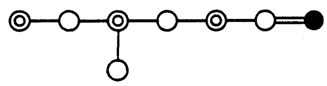
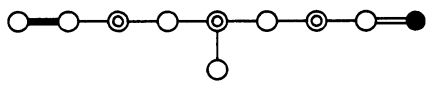
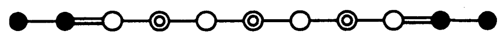
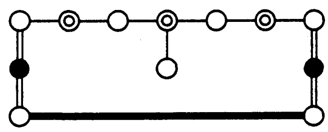
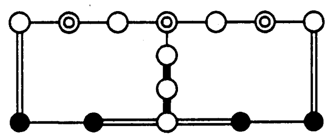
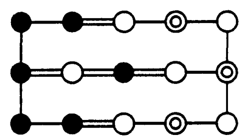
Moreover, a vertex corresponding to $f \in P^{(4)}(\mathcal{M}^{(2,4)})$ is **black.** It is **transparent,** if $f \in P^{(2)}(\mathcal{M}^{(2,4)})$. It is **double transparent,** if $f \in P(X)_{+I}$ (i. e. it corresponds to the class of a rational component of X^θ), otherwise, it is **simple transparent.** Of course, here we assume that $\mathcal{M}^{(2,4)} \subset \mathcal{M}(X)_+$ for a K3 surface with involution (X,θ) and $(S_X)_+ = S$.

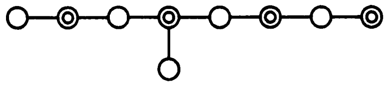
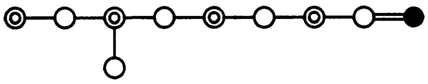
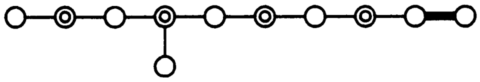
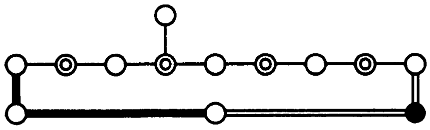
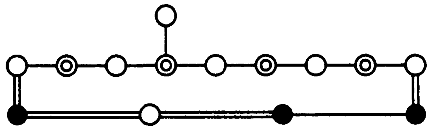
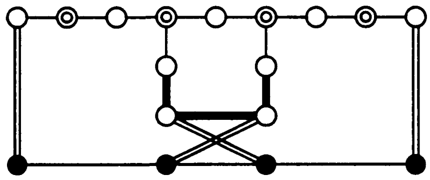
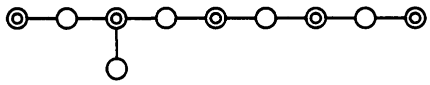
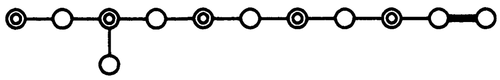
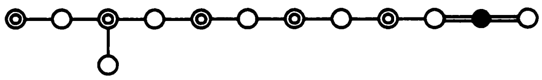
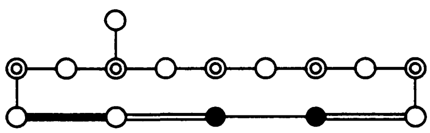
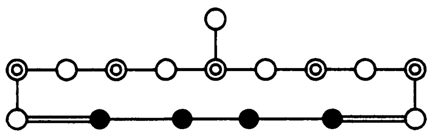
Classification of DPN surfaces of elliptic type is based on the purely arithmetic calculations of the fundamental chambers $\mathcal{M}^{(2,4)}$ (equivalently, of the graphs $\Gamma(P(\mathcal{M}^{(2,4)}))$ of the reflection groups $W^{(2,4)}(S)$ of the lattices S of elliptic type. Since S is 2-elementary and even, $W^{(2,4)}(S) = W(S)$ is the full reflection group of the lattice S, and any root $f \in S$ has $f^2 = -2$ or -4. We have

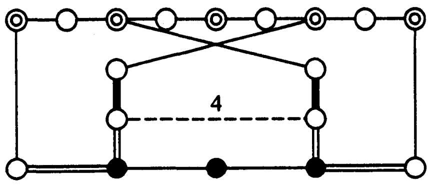
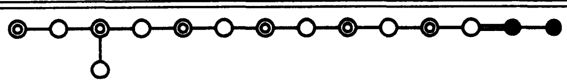
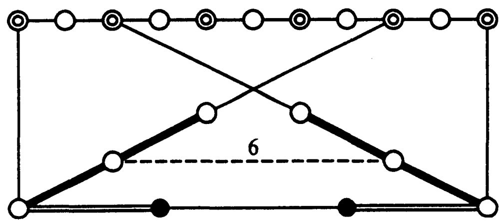
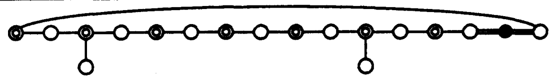
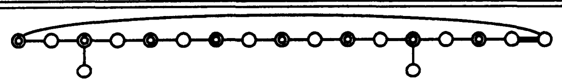
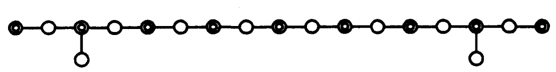
Theorem 3.1. *2-elementary even hyperbolic lattices S of elliptic type have fundamental chambers $\mathcal{M}^{(2,4)}$ for their reflection groups $W^{(2,4)}(S)$ (it is the full reflection group of S), equivalently the corresponding Dynkin diagrams $\Gamma(P(\mathcal{M}^{(2,4)}))$, which are given in Table 1 below, where the lattice S is defined by its invariants (r,a,δ) (equivalently, (k,g,δ)), see Section 2.3.*

TABLE 1. Fundamental chambers $\mathcal{M}^{(2,4)}$ of reflection groups $W^{(2,4)}(S)$ for 2-elementary even hyperbolic lattices S of elliptic type.

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(\mathcal{M}^{(2,4)}))$
1	1	1	1	0	10	0	$\Gamma=\varnothing$
2	2	2	0	0	9	0	
3	2	2	1	0	9	0	
4	3	3	1	0	8	0	
5	4	4	1	0	7	0	
6	5	5	1	0	6	0	
7	6	6	1	0	5	0	
8	7	7	1	0	4	0	
9	8	8	1	0	3	0	
10	9	9	1	0	2	0	
11	2	0	0	1	10	0	
12	3	1	1	1	9	0	
13	4	2	1	1	8	0	
14	5	3	1	1	7	0	
15	6	4	0	1	6	0	
16	6	4	1	1	6	0	
17	7	5	1	1	5	0	
18	8	6	1	1	4	1	

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(\mathcal{M}^{(2,4)}))$
19	9	7	1	1	3	1	
20	10	8	1	1	2	1	
21	6	2	0	2	7	0	
22	7	3	1	2	6	0	
23	8	4	1	2	5	0	
24	9	5	1	2	4	0	
25	10	6	0	2	3	1	
26	10	6	1	2	3	1	
27	11	7	1	2	2	1	
28	8	2	1	3	6	0	
29	9	3	1	3	5	0	
30	10	4	0	3	4	0	
31	10	4	1	3	4	0	
32	11	5	1	3	3	0	
33	12	6	1	3	2	1	

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(\mathcal{M}^{(2,4)}))$
34	9	1	1	4	6	0	
35	10	2	0	4	5	0	
36	10	2	1	4	5	0	
37	11	3	1	4	4	0	
38	12	4	1	4	3	0	
39	13	5	1	4	2	0	
40	10	0	0	5	6	0	
41	11	1	1	5	5	0	
42	12	2	1	5	4	0	
43	13	3	1	5	3	0	
44	14	4	0	5	2	0	

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(\mathcal{M}^{(2,4)}))$
45	14	4	1	5	2	0	
46	14	2	0	6	3	0	
47	15	3	1	6	2	0	
48	16	2	1	7	2	0	
49	17	1	1	8	2	0	
50	18	0	0	9	2	0	

Proof. When S is unimodular (i.e. $a = 0$) or $r = a$ (then $S(1/2)$ is unimodular), i. e. for cases 1—11, 40, 50, these calculations were done by Vinberg [Vin72]. In all other cases they can be done using Vinberg's algorithm for calculation of the fundamental chamber of a hyperbolic reflection group. See [Vin72] and also [Vin85]. These technical calculations take too much space and will be presented in Appendix, Section A.4.1.

To describe elements of $P(X)_{+I}$ (i. e. double transparent vertices), we use the results of Section 2.6 and the fact that their number k is known by Section 2.3. $\square$

Remark 3.2. Using diagrams of Theorem 3.1, one can easily find the class in S of the component C_g of X^θ as an element $C_g \in S$ such that $C_g \cdot x = 0$, if x corresponds to a black or a double transparent vertex, and $C_g \cdot x = 2 - s$ if x corresponds to a simple transparent vertex which has s edges to double transparent vertices.

3.2. Root invariants, and subsystems of roots in $\Delta^{(4)}(\mathcal{M}^{(2)})$ for elliptic case

We use the notation and results of Section 2.4.1. Let $\mathcal{M}^{(2)} \supset \mathcal{M}^{(2,4)}$ be the fundamental chamber of $W^{(2)}(S)$ containing $\mathcal{M}^{(2,4)}$. Dynkin diagram of $P^{(4)}(\mathcal{M}^{(2,4)})$ (i. e. black vertices) consists of components of types A, D or E (see Table 1). Thus, the group $W^{(4)}(\mathcal{M}^{(2)})$ generated by reflections in all elements of $P^{(4)}(\mathcal{M}^{(2,4)})$ is a finite Weyl group. It has to be finite because $W^{(4)}(\mathcal{M}^{(2)})(\mathcal{M}^{(2,4)}) = \mathcal{M}^{(2)}$ has finite volume, and $\mathcal{M}^{(2,4)}$ is the fundamental chamber for the action of $W^{(4)}(\mathcal{M}^{(2)})$ in $\mathcal{M}^{(2)}$. Thus,

$$\Delta^{(4)}(\mathcal{M}^{(2)}) = W^{(4)}(\mathcal{M}^{(2)})P^{(4)}(\mathcal{M}^{(2,4)})$$

is a finite root system of the corresponding type with the negative definite root sublattice

$$R(2) = [P^{(4)}(\mathcal{M}^{(2,4)})] \subset S.$$

Let (X, θ) be a K3 surface with a non-symplectic involution, and $(S_X)_+ = S$. Let $\Delta_+^{(4)} \subset \Delta^{(4)}(S)$ be the subset defined by (X, θ) which is invariant with respect to $W_+^{(2,4)}$ (we remind that it is generated by reflections in $\Delta^{(2)}(S)$ and $\Delta_+^{(4)}$). By Theorem 2.4, $\Delta_+^{(4)} = W^{(2)}(S)\Delta_+^{(4)}(\mathcal{M}^{(2)})$ where $\Delta_+^{(4)}(\mathcal{M}^{(2)}) = \Delta_+^{(4)} \cap \Delta^{(4)}(\mathcal{M}^{(2)})$ is a root subsystem in $\Delta^{(4)}(\mathcal{M}^{(2)})$. Let

$$K^+(2) = [\Delta_+^{(4)}(\mathcal{M}^{(2)})] \subset R(2) \subset S \tag{64}$$

be its negative definite root sublattice in S, and

$$Q = \frac{1}{2}K^+(2)/K^+(2), \quad \xi^+ : q_{K^+(2)}|Q \to q_S \tag{65}$$

a homomorphism such that $\xi^+(x/2 + K^+(2)) = x/2 + S$, $x \in K^+(2)$. We obtain a pair $(K^+(2), \xi^+)$ which is similar to a root invariant, and it is equivalent to the root invariant for elliptic type.

Proposition 3.3. *Let (X, θ) be a K3 surface with a non-symplectic involution of elliptic type, and $S = (S_X)_+$.*

In this case, the root invariant $R(X, \theta)$ is equivalent to the root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$, considered up to the action of $O(S)$ (i. e. two root subsystems $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ and $\Delta_+^{(4)}(\mathcal{M}^{(2)})' \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ are equivalent, if $\Delta_+^{(4)}(\mathcal{M}^{(2)})' = \phi(\Delta_+^{(4)}(\mathcal{M}^{(2)}))$ for some $\phi \in O(S)$):

The root invariant $R(X, \theta) \cong (K^+(2), \xi^+)$ is defined by (64) *and* (65).

The fundamental chamber $\mathcal{M}(X)_+$ is defined by the root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ (up to above equivalence), by Theorem 2.4.

Moreover, $P^{(4)}(\mathcal{M}(X)_+)$ *coincides with a basis of the root subsystem* $\Delta_+^{(4)}(\mathcal{M}^{(2)})$.

Proof. Let E_i, $i \in I$, be all non-singular rational curves on X such that $E_i \cdot \theta(E_i) = 0$, i. e.

$$\mathrm{cl}(E) + \mathrm{cl}(\theta(E)) = \delta \in P^{(4)}(\mathcal{M}(X)_+) = P^{(4)}(X)_+ = P(X)_{+III}.$$

Since $E_i \cdot C_g = 0$ and $C_g^2 = 2g - 2 > 0$, the curves E_i, $i \in I$, generate in S_X a negative definite sublattice. Thus, their components define a Dynkin diagram Γ which consists of several connected components A_n, D_m or E_k. The involution θ acts on these diagrams and corresponding curves without fixed points. Thus it necessarily changes connected components of Γ. Let $\Gamma = \Gamma_1 \bigsqcup \Gamma_2$ where $\theta(\Gamma_1) = \Gamma_2$, and $I = I_1 \bigsqcup I_2$ the corresponding subdivision of vertices of Γ. Then

$$\delta_i^+ = \mathrm{cl}(E_i) + \mathrm{cl}(\theta(E_i)), \quad i \in I_1,$$

and

$$\delta_i^- = \mathrm{cl}(E_i) - \mathrm{cl}(\theta(E_i)), \quad i \in I_1$$

give bases of root systems $\Delta_+^{(4)}(\mathcal{M}^{(2)})$ and $\Delta_-^{(4)} = \Delta^{(4)}(K(2))$ respectively. The map

$$\delta_i^- = \mathrm{cl}(E_i) - \mathrm{cl}(\theta(E_i)) \mapsto \delta_i^+ = \mathrm{cl}(E_i) + \mathrm{cl}(\theta(E_i)),\ i \in I_1,$$

defines an isomorphism $\Delta_-^{(4)} \cong \Delta_+^{(4)}(\mathcal{M}^{(2)})$ of root systems, since it evidently preserves the intersection pairing. The homomorphism ξ of the root invariant $R(X, \theta) = (K(2), \xi)$ of the pair (X, θ) then goes to $(K^+(2), \xi^+)$.

In the opposite direction, the root invariant $R(X, \theta)$ defines $\Delta_+^{(4)}$ and $\Delta_+^{(4)}(\mathcal{M}^{(2)}) = \Delta^{(4)}(\mathcal{M}^{(2)}) \cap \Delta_+^{(4)}$.

The last statement follows from Section 2.4.1. $\square$

By Proposition 3.3, in the elliptic case instead of root invariants one can consider root subsystems $\Delta_+^{(4)}(\mathcal{M}^{(2)})$ (in $\Delta^{(4)}(\mathcal{M}^{(2)})$). We say that a **root subsystem** $\Delta_+^{(4)}(\mathcal{M}^{(2)})$ **"is contained"** (respectively "is **primitively contained") in a root subsystem** $\Delta_+^{(4)}(\mathcal{M}^{(2)})'$, if $\phi(\Delta_+^{(4)}(\mathcal{M}^{(2)})) \subset \Delta_+^{(4)}(\mathcal{M}^{(2)})'$ (respectively $[\phi(\Delta_+^{(4)}(\mathcal{M}^{(2)}))] \subset [\Delta_+^{(4)}(\mathcal{M}^{(2)})']$ is a primitive embedding of lattices) for some $\phi \in O(S)$. By Corollary 2.11, we obtain

Proposition 3.4. *If a root subsystem* $\Delta_+^{(4)}(\mathcal{M}^{(2)})$ *in* $\Delta^{(4)}(\mathcal{M}^{(2)})$ *corresponds to a K3 surface with non-symplectic involution* (X, θ)*, then any primitive root subsystem in* $\Delta_+^{(4)}(\mathcal{M}^{(2)})$ *corresponds to a K3 surface with non-symplectic involution.*

Thus, it is enough to describe **extremal pairs** (X, θ) *such that their root subsystems* $\Delta_+^{(4)}(\mathcal{M}^{(2)})$ *in* $\Delta^{(4)}(\mathcal{M}^{(2)})$ *are not contained as primitive* *root subsystems of strictly smaller rank in a root subsystem* $\Delta_+^{(4)}(\mathcal{M}^{(2)})'$ *in* $\Delta^{(4)}(\mathcal{M}^{(2)})$ *corresponding to another pair* (X', θ').

3.3. Classification of non-symplectic involutions (X, θ) of elliptic type of K3 surfaces

We have

Theorem 3.5. *Let* (X, θ) *and* (X', θ') *be two non-symplectic involutions of elliptic type of K3 surfaces.*

Then the following three conditions are equivalent:

(i) Their main invariants (r, a, δ) *(equivalently,* (k, g, δ)*) coincide, and their root invariants are isomorphic.*

(ii) Their main invariants (r, a, δ) *coincide, and the root subsystems* $\Delta_+^{(4)}(\mathcal{M}^{(2)})$ *are equivalent.*

(iii) Dynkin diagrams $\Gamma(P(X)_+)$ *and* $\Gamma(P(X')_+)$ *of their exceptional curves are isomorphic, and additionally the genera* g *are equal, if these diagrams are empty. The diagram* $\Gamma(P(X)_+)$ *is empty if and only if either* $(r, a, \delta) = (1, 1, 1)$ *(then* $g = 10$*), or* $(r, a, \delta) = (2, 2, 0)$ *(then* $g = 9$*) and the root invariant is zero. The corresponding DPN surfaces are* $\mathbb{P}^2$ *or* $\mathbb{F}_0$ *respectively.*

Proof. By Sections 3.2 and 2.5, the conditions (i) and (ii) are equivalent, and they imply (iii).

Let us show that (iii) implies (i).

Assume that $r = \operatorname{rk} S \geq 3$.

First, let us show that S is generated by $\Delta^{(2)}(S)$, if $r = \operatorname{rk} S \geq 3$. If $r \geq a+2$, then it is easy to see that either $S \cong U \oplus T$ or $S \cong U(2) \oplus T$ where T is orthogonal sum of A_1, D_{2m}, E_7, E_8 (one can get all possible invariants (r, a, δ) of S taking these orthogonal sums). We have $U = [c_1, c_2]$ where $c_1^2 = c_2^2 = 0$ and $c_1 \cdot c_2 = 1$ (the same for $U(2)$, only $c_1 \cdot c_2 = 2$). Then S is generated by elements with square -2 which are

$$\Delta^{(2)}(T) \cup (c_1 \oplus \Delta^{(2)}(T)) \cup (c_2 \oplus \Delta^{(2)}(T)).$$

If $r = a$ then $S \cong \langle 2 \rangle \oplus tA_1$. Let $h, e_1, \ldots, e_t$ be the corresponding orthogonal basis of S where $h^2 = 2$ and $e_i^2 = -2$, $i = 1, \ldots, t$. Then S is generated by elements with square (-2) which are $e_1, \ldots, e_t$ and $h - e_1 - e_2$.

Now, let us show that $P(X)_+$ generates S. Indeed, every element of $\Delta^{(2)}(S) \cup \Delta_+^{(4)}$ can be obtained by composition of reflections in elements of $P(X)_+$ from some element of $P(X)_+$. It follows, that it is an integral linear combination of elements of $P(X)_+$. Since we can get in this way all elements of $\Delta^{(2)}(S)$ and they generate S, it follows that $P(X)_+$ generates S.

It follows that the lattice S with its elements $P(X)_+$ is defined by the Dynkin diagram $\Gamma(P(X)_+)$. From S, we can find invariants (r, a, δ) of S, and they define invariants (k, g, δ).

Let $K^+(2) \subset S$ be a sublattice generated by $P^{(4)}(X)_+$ (i. e. by the black vertices), and $\xi^+ : Q = (1/2)K^+(2)/K^+(2) \to q_S$ the homomorphism with $\xi^+(x/2 + K^+(2)) = x/2 + S$. By Proposition 3.3, the pair $(K^+(2), \xi^+)$ coincides with the root invariant $R(X, \theta)$.

Now assume that $r = \operatorname{rk} S = 1, 2$ for the pair (X, θ). Then $S \cong \langle 2 \rangle$, $U(2)$, U or $\langle 2 \rangle \oplus \langle -2 \rangle$.

In the first two cases $\Delta^{(2)}(S) = \emptyset$ and then $P^{(2)}(X)_+ = \emptyset$. In the last two cases $\Delta^{(2)}(S)$ and $P^{(2)}(X)_+$ are not empty.

Thus, only the first two cases give an empty diagram $P^{(2)}(X)_+$. This distinguishes these two cases from all others. In the case $S = \langle 2 \rangle$, the invariant $g = 10$, and the root invariant is always zero because S has no elements with square -4. Thus, in this case, the diagram $P(X)_+$ is always empty. This case gives $Y = X/\{1, \theta\} \cong \mathbb{P}^2$. In the case $S = U(2)$, the diagram $P^{(2)}(X)_+$ is empty, but $P^{(4)}(X)_+ = \emptyset$, if the root invariant is zero, and $P^{(4)}(X)_+$ consists of one black vertex, if the root invariant is not zero (see Table 1 for this case). First case gives $Y = \mathbb{F}_0$. Second case gives $Y = \mathbb{F}_2$. In both these cases $g = 9$. Thus difference between two cases when the diagram is empty ($\mathbb{P}^2$ or $\mathbb{F}_1$) is in genus: $g = 10$ for the first case, and $g = 9$ for the second.

The difference of $S = U(2)$ with a non-empty diagram $\Gamma(P(X)_+)$ from all other cases is that this diagram consists of only one black vertex. All cases with $\operatorname{rk} S \geq 3$ must have at least 3 different vertices to generate S. In cases $S = U$ and $S = \langle 2 \rangle \oplus \langle -2 \rangle$, the diagram $\Gamma(P(X)_+)$ also consists of one vertex, but it is respectively double transparent and simple transparent (see Table 1). Moreover, this consideration also shows the difference between cases $S = U$ and $S = \langle 2 \rangle \oplus \langle -2 \rangle$ and with all other cases. $\square$

Theorem 3.5 shows that to classify pairs (X, θ) of elliptic type, we can use any of the following invariants: either the root invariant, or the root subsystem (together with the main invariants (k, g, δ) or (r, a, δ)), or the Dynkin diagram of exceptional curves.

It seems that the most natural and geometric is the classification by the Dynkin diagram. Using this diagram, on the one hand, it easy to calculate all other invariants. On the other hand, considering the corresponding DPN surface, we get the Gram diagram of all exceptional curves on it and all possibilities to get the DPN surface by blow-ups from relatively minimal rational surfaces.

However, the statements (i) and (ii) of Theorem 3.5 are also very important since they give a simple way to find out if two pairs (X, θ) and (X', θ') (equivalently, the corresponding DPN surfaces) have isomorphic Dynkin diagrams of exceptional curves. Moreover, the classification in terms of root invariants and root subsystems is much more compact, since the full Gram diagram of exceptional curves can be very large (e.g. recall the classical non-singular del Pezzo surface corresponding to E_8).

We have the following

Theorem 3.6 (Classification Theorem in the extremal case of elliptic type). *A K3 surface with a non-symplectic involution (X, θ) of elliptic type is* **extremal,** *if and only if the number of its exceptional curves with the square (-4), i. e. $\#P^{(4)}(X)_+$, is equal to $\#P^{(4)}(\mathcal{M}^{(2,4)})$ (see Theorem 3.1) where $\mathcal{M}^{(2,4)}$ is a fundamental chamber of $W^{(2,4)}(S)$, $S = (S_X)_+$. Equivalently, numbers of black vertices of Dynkin diagrams $\Gamma(P(X)_+)$ and $\Gamma(P(\mathcal{M}^{(2,4)}))$ with the same invariants (r, a, δ) are equal.*

Moreover, the diagram $\Gamma(P(X)_+)$ is isomorphic to (i. e. coincides with) $\Gamma(P(\mathcal{M}^{(2,4)}))$ (see Table 1) in all cases of Theorem 3.1 except cases 7, 8, 9, 10 and 20 of Table 1. In the last five cases, all possible diagrams $\Gamma(P(X)_+)$ are given in Table 2. All diagrams of Tables 1 and 2 correspond to some extremal standard K3 pairs (X, θ).

Proof. It requires long considerations and calculations and will be given in Section 3.4 below. □

Now let us consider a description of non-extremal pairs (X, θ). The worst way to describe them is using full diagrams $\Gamma(P(X)_+)$, since the number of non-extremal pairs (X, θ) is very large and diagrams $\Gamma(P(X)_+)$ can be huge. It is better to describe them using Proposition 3.4 and Theorem 3.5, by primitive root subsystems $\Delta^{(4)}_+(\mathcal{M}^{(2)})'$ in the root subsystems $\Delta^{(4)}_+(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ of extremal pairs $(\widetilde{X}, \widetilde{\theta})$.

Let us choose $\mathcal{M}^{(2)}$ in such a way that $\mathcal{M}^{(2)} \supset \mathcal{M}(\widetilde{X})_+$. By Section 2.4.1, then $\Delta^{(4)}_+(\mathcal{M}^{(2)}) = \Delta^{(4)}([P^{(4)}(\widetilde{X})_+])$ is the subsystem of roots with the basis $P^{(4)}(\widetilde{X})_+$, i. e. $\Delta^{(4)}_+(\mathcal{M}^{(2)}) = \Delta^{(4)}([P^{(4)}(\widetilde{X})_+])$ is the set of all elements with the square (-4) in the sublattice $[P^{(4)}(\widetilde{X})_+]$ generated by $P^{(4)}(\widetilde{X})_+$ in $S = (S_{\widetilde{X}})_+$. Equivalently, $\Delta^{(4)}([P^{(4)}(\widetilde{X})_+]) = W^{(4)}_+(\widetilde{X})(P^{(4)}(\widetilde{X})_+)$, where $W^{(4)}_+(\widetilde{X})$ is the finite Weyl group generated by reflections in all elements of $P^{(4)}(\widetilde{X})_+$.

Replacing a primitive root subsystem $\Delta^{(4)}_+(\mathcal{M}^{(2)})' \subset \Delta^{(4)}([P^{(4)}(\widetilde{X})_+])$ for a non-extremal pair (X, θ) by an equivalent root subsystem $\phi(\Delta^{(4)}_+(\mathcal{M}^{(2)})')$, $\phi \in W^{(4)}_+(\widetilde{X})$, we can assume (by primitivity) that a basis of $\Delta_+^{(4)}(\mathcal{M}^{(2)})'$ is a part of the basis $P^{(4)}(\widetilde{X})_+$ of the root system $\Delta^{(4)}([P^{(4)}(\widetilde{X})_+])$. Thus, we can assume that the root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)})'$ is defined by a subdiagram

$$D \subset \Gamma(P^{(4)}(\widetilde{X})_+)$$

where $\Gamma(P^{(4)}(\widetilde{X})_+)$ is the subdiagram of the full diagram $\Gamma(P(\widetilde{X})_+)$ generated by all its black vertices. The D is a basis of $\Delta_+^{(4)}(\mathcal{M}^{(2)})'$.

By Propositions 2.2, 2.3 and Theorem 2.4, the subdiagram $D \subset \Gamma(P^{(4)}(\widetilde{X})_+)$ defines the full Dynkin diagram $\Gamma(P(X)_+)$ of the pair (X,θ) with the root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)})'$: We have

$$P^{(2)}(X)_+ = \{f \in W_+^{(4)}(\widetilde{X})(P^{(2)}(\widetilde{X})_+) \mid f\cdot D \geq 0\}. \tag{66}$$

The subdiagram of $\Gamma(P(X)_+)$ defined by all its black vertices coincides with D. It is called **Du Val part** of $\Gamma(P(X)_+)$, and it is denoted by $\operatorname{Duv}\Gamma(P(X)_+)$. Thus,

$$\operatorname{Duv}\Gamma(P(X)_+) = D \subset \operatorname{Duv}\Gamma(P(\widetilde{X})_+).$$

Double transparent vertices of $\Gamma(P(X)_+)$ are identified with double transparent vertices of $\Gamma(P(\widetilde{X})_+)$ (see Section 2.6), and single transparent vertices of $P(X)_+$ which are connected by two edges with double transparent vertices of $\Gamma(P(X)_+)$ are identified with such vertices of $\Gamma(P(\widetilde{X})_+)$. Indeed, they are orthogonal to the set $P^{(4)}(\widetilde{X})_+$ which defines the reflection group $W_+^{(4)}(\widetilde{X})$ as the group generated by reflections in all elements of $P^{(4)}(\widetilde{X})_+$. Thus, the group $W_+^{(4)}(\widetilde{X})$ acts identically on all these vertices, and all of them satisfy (66). All double transparent vertices and all single transparent vertices connected by two edges with double transparent vertices of $\Gamma(P(X)_+)$ define the **logarithmic part** of $\Gamma(P(X)_+)$, and it is denoted by $\operatorname{Log}\Gamma(P(X)_+)$.

Thus, we have

$$\operatorname{Log}\Gamma(P(X)_+) = \operatorname{Log}\Gamma(P(\widetilde{X})_+),$$

logarithmic parts of X and $\widetilde{X}$ are identified. Moreover, the Du Val part $\operatorname{Duv}\Gamma(P(X)_+)$ and the logarithmic part $\operatorname{Log}\Gamma(P(X)_+)$ are *disjoint in* $\Gamma(P(X)_+)$ because they are orthogonal to each other. Thus, the logarithmic part of $\Gamma(P(X)_+)$ is stable, it is the same for all pairs (X,θ) with the same main invariants (r,a,δ). On the Du Val part of $\Gamma(P(X)_+)$ we have only a restriction: it is a subdiagram of Du Val part of one of extremal pairs $(\widetilde{X},\widetilde{\theta})$ described in Theorems 3.1 and 3.6 (with the same main invariants (r,a,δ)).

All vertices of $\Gamma(P(X)_+)$ which do not belong to $\operatorname{Duv}\Gamma(P(X)_+) \cup \operatorname{Log}\Gamma(P(X)_+)$ define a subdiagram $\operatorname{Var}\Gamma(P(X)_+)$ which is called the

varying part of $\Gamma(P(X)_+)$. By (66), we have

$$\operatorname{Var} P(X)_+ = \{f \in W_+^{(4)}(\widetilde{X})(\operatorname{Var} P(\widetilde{X})_+) \mid f \cdot D \geq 0\}$$

(we skip Γ when we consider only vertices). It describes $\operatorname{Var}\Gamma(P(X)_+)$ by the intersection pairing in S.

Of course, two Dynkin subdiagrams $D \subset \Gamma(P^{(4)}(\widetilde{X})_+)$ and $D' \subset \Gamma(P^{(4)}(\widetilde{X}')_+)$, with isomorphic Dynkin diagrams $D \cong D'$, of two extremal pairs $(\widetilde{X}, \widetilde{\theta})$ and $(\widetilde{X}', \widetilde{\theta}')$ with the same main invariants can give isomorphic Dynkin diagrams $\Gamma(P(X)_+)$ and $\Gamma(P(X')_+)$ for defining by them K3 pairs (X, θ) and (X', θ'). To have that, it is necessary and sufficient that root invariants $([D], \xi^+)$ and $([D'], (\xi')^+)$ defined by them are isomorphic. We remind that they can be obtained by restriction on $[D]$ and $[D']$ of the root invariants of pairs $(\widetilde{X}, \widetilde{\theta})$ and $(\widetilde{X}', \widetilde{\theta}')$ respectively, and they can be easily computed. We remind that to have $([D], \xi^+)$ and $([D'], (\xi')^+)$ isomorphic, there must exist an isomorphism $\gamma : [D] \to [D']$ of the root lattices and an automorphism $\overline{\phi} \in O(q_S)$ of the discriminant quadratic form of the lattice S which send ξ^+ for $(\xi')^+$. Section 2.5 gives the very simple and effective method for that. Thus, we have a very simple and effective method to find out when different subdiagrams D above give K3 pairs with isomorphic diagrams.

Note that we have used all equivalent conditions (i), (ii) and (iii) of Theorem 3.5 which shows their importance. Finally, we get

Theorem 3.7 (Classification Theorem in the non-extremal, i. e. arbitrary, case of elliptic type). *Dynkin diagrams $\Gamma(P(X)_+)$ of exceptional curves of non-extremal (i. e. arbitrary) non-symplectic involutions (X, θ) of elliptic type of K3 surfaces are described by arbitrary (without restrictions) Dynkin subdiagrams $D \subset \operatorname{Duv}\Gamma(P(\widetilde{X})_+)$ of extremal pairs $(\widetilde{X}, \widetilde{\theta})$ (see Theorem 3.6) with the same main invariants (r, a, δ) (equivalently (k, g, δ)). Moreover,*

$$\operatorname{Duv}\Gamma(P(X)_+) = D, \quad \operatorname{Log}\Gamma(P(X)_+) = \operatorname{Log}\Gamma(P(\widetilde{X})_+),$$

and they are disjoint to each other,

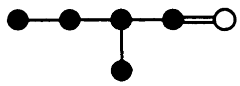
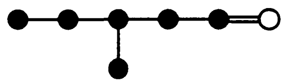
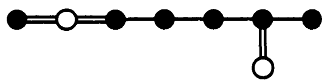
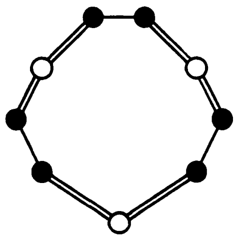
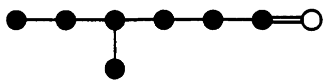
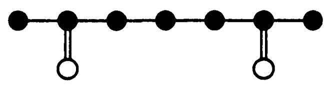
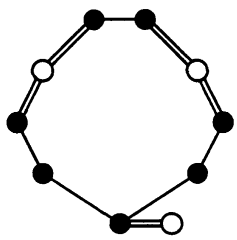
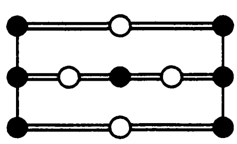
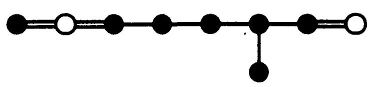
$$\operatorname{Var} P(X)_+ = \{f \in W_+^{(4)}(\widetilde{X})(\operatorname{Var} P(\widetilde{X})_+) \mid f \cdot D \geq 0\}$$

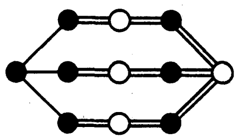
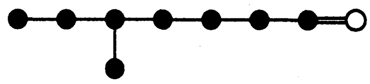
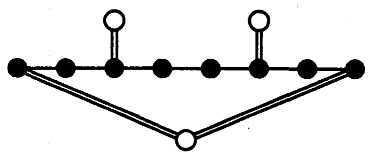
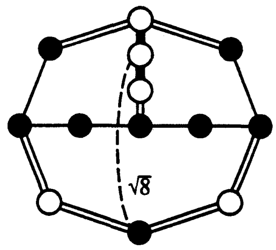
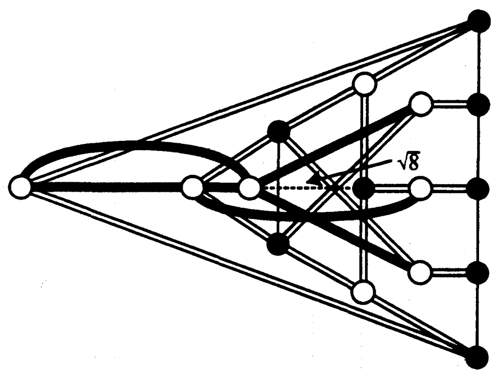
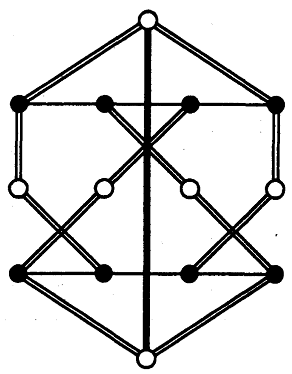
where the group $W_+^{(4)}(\widetilde{X})$ is generated by reflections in all elements of $\operatorname{Duv}\Gamma(P(\widetilde{X})_+) = P^{(4)}(\widetilde{X})_+$.

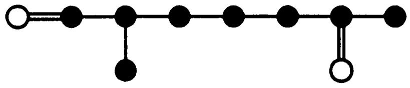
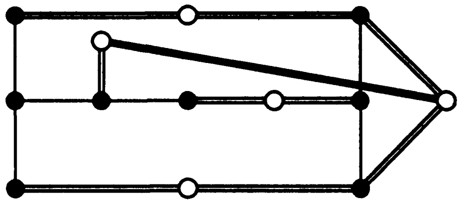
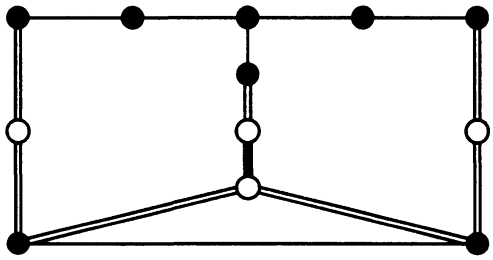
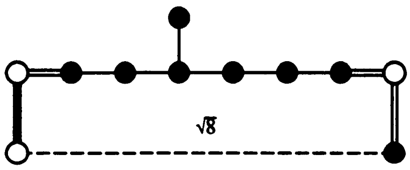
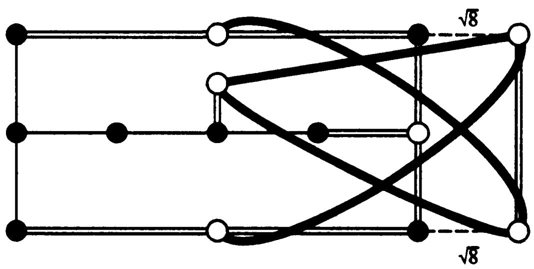
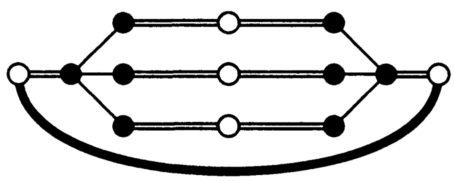
Dynkin subdiagrams $D \subset \operatorname{Duv}\Gamma(P(\widetilde{X})_+)$, $D' \subset \operatorname{Duv}\Gamma(P(\widetilde{X'})_+)$ (with the same main invariants) give K3 pairs (X, θ), (X', θ') with isomorphic Dynkin diagrams $\Gamma(P(X)_+) \cong \Gamma(P(X')_+)$, if and only if the *root invariants* $([D], \xi^+)$, $([D'], (\xi')^+)$ *defined by* $D \subset \mathrm{Duv}\,\Gamma(P(\widetilde{X})_+)$, $D' \subset \mathrm{Duv}\,\Gamma(P(\widetilde{X'})_+)$ *are isomorphic.*

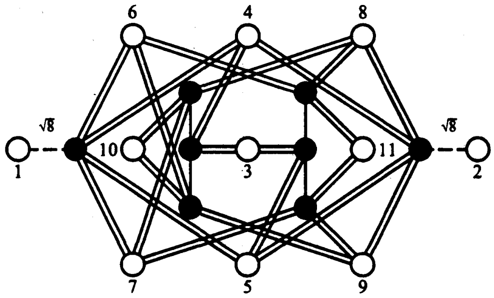
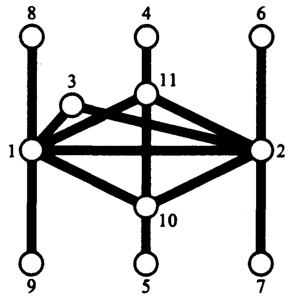
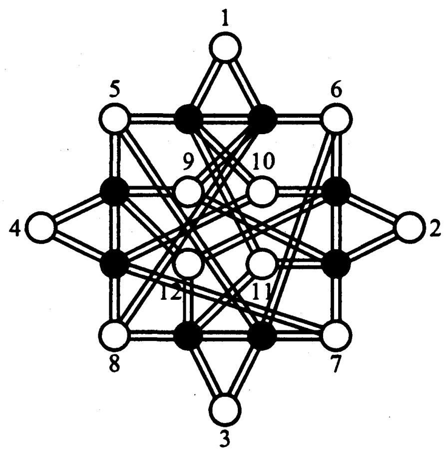
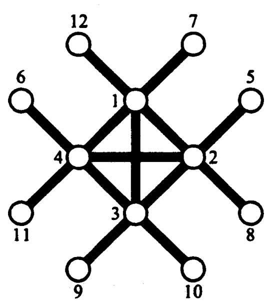
TABLE 2. Diagrams $\Gamma(P(X)_+)$ of extremal K3 surfaces (X,θ) of elliptic type which are different from Table 1

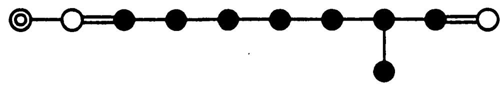
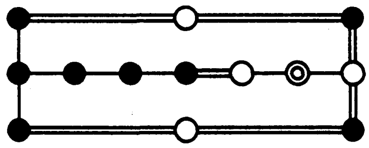
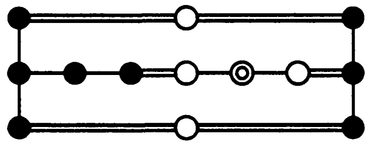
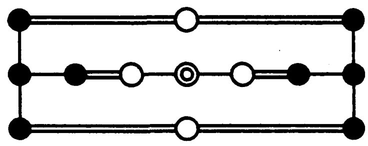
(In (a) we repeat the corresponding case of Table 1)

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(X)_+)$
7	6	6	1	0	5		
a						0	
b						1	
8	7	7	1	0	4		
a						0	
b						1	
c						0	
9	8	8	1	0	3		
a						0	
b						1	
c						0	
d						1	
e						1	

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(X)_+)$
9	8	8	1	0	3		
f						2	
10	9	9	1	0	2		
a						0	
b						0	
c						1	
d						1	
e						0	

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(X)_+)$
10	9	9	1	0	2		
f						1	
g						1	
h						0	
i						1	
j						2	
k						2	

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(X)_+)$
10	9	9	1	0	2		
1						2	  
m						0	

N	r	a	δ	k	g	$l(H)$	$\Gamma(P(X)_+)$
20	10	8	1	1	2		
a						1	
b						2	
c						1	
d						2	

3.4. Proof of Classification Theorem 3.6

Let (X, θ) be a non-symplectic involution of elliptic type of a K3 surface, with the main invariants (r, a, δ), and (X, θ) is an extremal pair.

By Theorem 3.5, the $\Gamma(P(X)_+)$ is defined by the root subsystem $\Delta^{(4)}_+(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ corresponding to (X, θ) where $\mathcal{M}^{(2)}$ is a fundamental chamber of $W^{(2)}(S)$, and $S = (S_X)_+$ has the invariants (r, a, δ). We can assume that $\mathcal{M}^{(2)} \supset \mathcal{M}(X)_+ \supset \mathcal{M}^{(2,4)}$ where $\mathcal{M}^{(2,4)}$ is a fundamental chamber of $W^{(2,4)}(S)$ defined by a choice of a basis $P^{(4)}(\mathcal{M}^{(2,4)})$ of the root system $\Delta^{(4)}(\mathcal{M}^{(2)})$ (see Section 2.4.1).

Let $\Gamma(P^{(4)}(\mathcal{M}^{(2,4)}))$ be the Dynkin diagram of the root system $\Delta^{(4)}(\mathcal{M}^{(2)})$ and $W^{(4)}(\mathcal{M}^{(2)})$ the Weyl group of the root system $\Delta^{(4)}(\mathcal{M}^{(2)})$. We use the description of root subsystems $\Delta^{(4)}_+(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ given below.

3.4.1

Let $T \subset R$ be a root subsystem of a root system R and all components of R have types A, D or E. We consider two particular cases of root subsystems.

Let B be a basis of R. Let $T \subset R$ be a primitive root subsystem. Then T can be replaced by an equivalent root subsystem $\phi(T)$, $\phi \in W(R)$, such that a part of the basis B gives a basis of T (see [Bou68]). Thus (up to equivalence defined by the Weyl group $W(R)$), primitive root subsystems $T \subset R$ can be described by Dynkin subdiagrams $\Gamma \subset \Gamma(B)$.

Now let $T \subset R$ be a root subsystem of a finite index. Let R_i be a component of R. Let r_j, $j \in J$, be a basis of R_i. Let $r_{\max} = \sum_{j \in J} k_j r_j$ be the maximal root of R_i corresponding to this basis. Dynkin diagram of the set of roots

$$\{r_j \mid j \in J\} \cup \{-r_{\max}\}$$

is an extended Dynkin diagram of the Dynkin diagram $\Gamma(\{r_j \mid j \in J\})$. Let us replace the component R_i of the root system R by the root subsystem $R_i' \subset R_i$ having by its basis the set $(\{r_j \mid j \in J\} \cup \{-r_{\max}\}) - \{r_t\}$ where $t \in J$ is some fixed element. We get a root subsystem $R' \subset R$ of finite index k_t. It can be shown [Dyn57] that iterations of this procedure give any root subsystem of finite index of R up to the action of $W(R)$.

Description of an arbitrary root subsystem $T \subset R$ can be reduced to these two particular cases, moreover it can be done in two ways.

Firstly, any root subsystem $T \subset R$ is a subsystem of finite index $T \subset T_{\text{pr}}$ where $T_{\text{pr}} \subset R$ is a primitive root subsystem generated by T.

Secondly, any root subsystem $T \subset R$ can be considered as a primitive root subsystem $T \subset R_1$ where $R_1 \subset R$ is root subsystem of finite index. One can take R_1 generated by T and by any $u = \operatorname{rk} R - \operatorname{rk} T$ roots $r_1, \dots, r_u$ such that $\operatorname{rk}[T, r_1, \dots, r_u] = \operatorname{rk} R$.

3.4.2

Here we show that the root subsystems $\Delta_+(\mathcal{M}^{(2)})$ which coincide with the full root systems $\Delta^{(4)}(\mathcal{M}^{(2)})$ can be realized by K3 pairs (X, θ). Obviously, they are extremal. For them $\mathcal{M}(X)_+ = \mathcal{M}^{(2,4)}$, and the Dynkin diagrams $\Gamma(P(X)_+) = \Gamma(P(\mathcal{M}^{(2,4)}))$ coincide. All these diagrams are described in Table 1 of Theorem 3.1. It is natural to call such pairs (X, θ) super-extremal. Thus, a non-symplectic involution (X, θ) of elliptic type of K3 (equivalently, the corresponding DPN pair (Y, C) or DPN surface) is called **super-extremal**, if for the corresponding root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ we have $\Delta_+^{(4)}(\mathcal{M}^{(2)}) = \Delta^{(4)}(\mathcal{M}^{(2)})$ (equivalently, $\Delta_+^{(4)} = \Delta^{(4)}(S)$). We have

Proposition 3.8. *For any possible elliptic triplet of main invariants (r, a, δ) there exists a super-extremal, i. e.*

$$\Gamma(P(X)_+) = \Gamma(P(\mathcal{M}^{(2,4)})),$$

and standard (see Section 2.7) K3 pair (X, θ).

See the description of their graphs $\Gamma(P(X)_+) = \Gamma(P(\mathcal{M}^{(2,4)}))$ *in Table 1 of Theorem 3.1.*

Proof. Let us consider an elliptic triplet of main invariants (r, a, δ) and the corresponding Dynkin diagram $\Gamma(P(\mathcal{M}^{(2,4)}))$ which is described in Theorem 3.1. Denote $K^+(2) = [P^{(4)}(\mathcal{M}^{(2,4)})]$, i. e. it is the sublattice generated by all black vertices of $\Gamma(P(\mathcal{M}^{(2,4)}))$. Consider the corresponding root invariant $(K^+(2), \xi^+)$, see (64) and (65). Consider $H = \operatorname{Ker} \xi^+$. By Propositions 3.3 and 2.9, there exists a super-extremal standard pair (X, θ), if the inequalities

$$r + \operatorname{rk} K^+ + l(\mathfrak{A}_{(K^+)_p}) < 22 \text{ for all prime } p > 2,$$

$$r + a + 2l(H) < 22$$

are valid together with Conditions 1 and 2 from Section 2.7.

By trivial inspection of all cases in Table 1, we can see that first inequality is valid. To prove second inequality, it is enough to show that $l(H) \leq 1$ since $r + a \leq 18$ in elliptic case. The inequality $l(H) \leq 1$ can be proved by direct calculation of $l(H)$ in all cases of Table 1 of Theorem 3.1.1. These calculations are simplified by the general statement.

Lemma 3.9. *In elliptic super-extremal case,*

$$l(H) = \#P^{(4)}(\mathcal{M}^{(2,4)}) - l(\mathfrak{A}_S^{(1)})$$

where $\mathfrak{A}_S^{(1)} \subset \mathfrak{A}_S$ *is the subgroup generated by all elements* $x \in \mathfrak{A}_S$ *such that* $q_S(x) = 1 \mod 2$. *Moreover, we have:*

If $\delta = 0$ *then* $l(\mathfrak{A}_S^{(1)}) = a$ *except* ($a = 2$ *and* $\operatorname{sign} S = 2 - r \equiv 0 \mod 8$). *In the last case* $l(\mathfrak{A}_S^{(1)}) = a - 1$.

If $\delta = 1$, *then* $l(\mathfrak{A}_S^{(1)}) = a - 1$ *except cases* ($a = 2$ *and* $\operatorname{sign} S \equiv 0 \mod 8$), ($a = 3$ *and* $\operatorname{sign} S \equiv \pm 1 \mod 8$), *and* ($a = 4$ *and* $\operatorname{sign} S \equiv 0 \mod 8$). *In these cases* $l(\mathfrak{A}_S^{(1)}) = a - 2$.

Proof. We know (see Section 2.4.1) that $\Delta^{(4)}(S) = W^{(2)}(S)(\Delta^{(4)}(\mathcal{M}^{(2,4)}))$. The group $W^{(2)}(S)$ acts identically on $\mathfrak{A}_S$. Therefore,

$$\operatorname{Im} \xi^+ = [\{\xi^+(f/2 + K^+(2)) \mid f \in \Delta^{(4)}(\mathcal{M}^{(2,4)})\}] =$$

$$[\{f/2 + S \mid f \in \Delta^{(4)}(S)\} = \mathfrak{A}_S^{(1)}.$$

In the last equality, we use Lemma 2.6. For $Q = (K^+(2)/2)/K^+(2)$, we have $l(Q) = \operatorname{rk} K^+ = \#P^{(4)}(\mathcal{M}^{(2,4)})$. Thus, $l(H) = l(Q) - l(\mathfrak{A}_S^{(1)}) = \#P^{(4)}(\mathcal{M}^{(2,4)}) - l(\mathfrak{A}_S^{(1)})$.

The remaining statements of Lemma can be proved by direct calculations using a decomposition of a 2-elementary non-degenerate finite quadratic form as sum of elementary ones: $q_{\pm 1}^{(2)}(2)$, $u_+^{(2)}(2)$ and $v_+^{(2)}(2)$ (in notation of [Nik80b]). See Appendix, Section A.1.3. □

One can easily check Condition 2 of Section 2.7.

To check Condition 1 of Section 2.7, note that if the lattice $K_H^+(2)$ has elements with the square (-2), then the sublattice $[P^{(4)}(\mathcal{M}^{(2,4)})]_{\text{pr}}$ of S also has elements with the square (-2). Let us show that this is not the case.

Let us consider the subspace

$$\gamma = \bigcap_{f \in P^{(4)}(\mathcal{M}^{(2,4)})} \mathcal{H}_f$$

of $\mathcal{L}(S)$ which is orthogonal to $[P^{(4)}(\mathcal{M}^{(2,4)})]$ (equivalently, we consider the corresponding face $\gamma \cap \mathcal{M}^{(2,4)}$ of $\mathcal{M}^{(2,4)}$). If the sublattice $[P^{(4)}(\mathcal{M}^{(2,4)})]_{\text{pr}} \subset S$ has elements with square (-2), then some hyperplanes $\mathcal{H}_e$, $e \in \Delta^{(2)}(S)$, also contain γ and give reflections from $W^{(2,4)}(S)$. On the other hand (e.g. see [Vin85]), all hyperplanes of reflections from $W^{(2,4)}(S)$ containing γ must be obtained from the hyperplanes $\mathcal{H}_f$, $f \in P^{(4)}(\mathcal{M}^{(2,4)})$, by the group generated by reflections in $P^{(4)}(\mathcal{M}^{(2,4)})$. All these hyperplanes are then also orthogonal to elements with square (-4) from S. They cannot be orthogonal to elements with square (-2) from S too.

This finishes the proof of Proposition 3.8.

3.4.3

Let us prove Theorem 3.6 in all cases except 7 — 10 and 20 of Table 1. These cases (i. e. different from 7 — 10 and 20 of Table 1) are characterized by the property that Dynkin diagram $\Gamma(P^{(4)}(\mathcal{M}^{(2,4)}))$ consists of components of type A only. By Section 3.4.1, any root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ is then primitive. In particular, any root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ of finite index is $\Delta_+^{(4)}(\mathcal{M}^{(2)}) = \Delta^{(4)}(\mathcal{M}^{(2)})$. By Proposition 3.8, we then obtain

Proposition 3.10. *For any elliptic triplet (r, a, δ) of main invariants which is different from (6,6,1), (7,7,1), (8,8,1), (9,9,1) and (10,8,1), any extremal K3 pair (X, θ) is super-extremal, i. e. $\Gamma(P(X)_+) = \Gamma(P(\mathcal{M}^{(2,4)}))$ (see their description in Table 1 of Theorem 3.1).*

Above, we have proved that the primitive sublattice $[P^{(4)}(\mathcal{M}^{(2,4)})]_{\text{pr}}$ in S generated by $P^{(4)}(\mathcal{M}^{(2,4)})$ has no elements with square -2. The lattice $[P^{(4)}(\mathcal{M}^{(2,4)})]$ coincides with the root lattice $[\Delta^{(4)}(\mathcal{M}^{(2)})]$. Thus, its primitive sublattice $[\Delta^{(4)}(\mathcal{M}^{(2)})]_{\text{pr}}$ in S also has no elements with square -2.

This fact is very important. Using (64) and (65), we can define the **root invariant** $(K^+(2), \xi^+)$ **for any root subsystem** $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$. Like for root subsystems of K3 pairs (X, θ), we then have

Lemma 3.11. *Root subsystems $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ and $\Delta_+^{(4)}(\mathcal{M}^{(2)})' \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ are $O(S)$ equivalent, if and only if their root invariants are isomorphic.*

Proof. Assume that the root invariants are isomorphic. Since ± 1 and the group $W^{(2)}(S)$ act identically on the discriminant form q_S, there exists an automorphism $\phi \in O(S)$ such that $\phi(\Delta^{(4)}(\mathcal{M}^{(2)})) = \Delta^{(4)}(\mathcal{M}^{(2)})$ and, identifying by ϕ the root subsystem $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ with $\phi\left(\Delta_+^{(4)}(\mathcal{M}^{(2)})\right) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$, we have the following. There exists an isomorphism $\alpha : \Delta_+^{(4)}(\mathcal{M}^{(2)}) \cong \Delta_+^{(4)}(\mathcal{M}^{(2)})'$ of root systems such that $\alpha(f)/2 + S = f/2 + S$ for any $f \in \Delta_+^{(4)}(\mathcal{M}^{(2)})$. Equivalently, $(\alpha(f) + f)/2 \in S$.

Assume that $\alpha(f) \neq \pm f$. Then, since $\alpha(f)$ and f are two elements of a finite root system $\Delta^{(2)}(\mathcal{M}^{(2)})$ which is a sum of A_n, D_m, E_k, it follows that either $\alpha(f) \cdot f = \pm 2$, or $\alpha(f) \cdot f = 0$. First case gives $f \cdot (\alpha(f) + f)/2 \equiv 1 \mod 2$ which is impossible because $f \in S$ is a root. Second case gives that $\beta = (\alpha(f) + f)/2$ has $\beta^2 = -2$ which is impossible because $[\Delta^{(4)}(\mathcal{M}^{(2)})]_{\mathrm{pr}}$ has no elements with square -2. Thus, $\alpha(f) = \pm f$. It follows that $\Delta_+^{(4)}(\mathcal{M}^{(2)}) = \Delta_+^{(4)}(\mathcal{M}^{(2)})'$ are identically the same root subsystems of $\Delta_+^{(4)}(\mathcal{M}^{(2)})$. □

3.4.4

Now let us consider cases 7—10 and 20 of Table 1. In these cases, the root system $R = \Delta^{(4)}(\mathcal{M}^{(2)})$ is D_5 in the case 7, E_6 in the case 8, E_7 in the case 9, E_8 in the case 10, and D_8 in the case 20.

We have

Lemma 3.12. *If R is a root system of one of types D_5, E_6, E_7, E_8 or D_8, then its root subsystem $T \subset R$ of finite index is determined by the isomorphism type of the root system T itself, up to the action of $W(R)$. Moreover, the type of T can be the following and only the following which is given in Table of Lemma 3.12 below (we identify the type with the isomorphism class of the corresponding root lattice).*

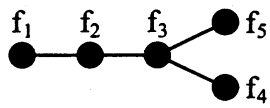
Moreover, in the corresponding cases labelled by $N = 7, 8, 9, 10$ and 20 of Table 1 the above statement is equivalent to the fact that the root invariant of the corresponding root subsystem $T \subset R = \Delta^{(4)}(\mathcal{M}^{(2)})$ of finite index is defined by its type. The root invariants (T, ξ^+) for them are given below by *showing the kernel* $H = \operatorname{Ker} \xi^+$ *and the invariants* α *and* $\overline{a}$, *if* $\alpha = 0$ *(we use Proposition 2.8).*

Table of Lemma 3.12

N	R	T
7	D_5	a) D_5, b) $A_3 \oplus 2A_1$
8	E_6	a) E_6, b) $A_5 \oplus A_1$, c) $3A_2$
9	E_7	a) E_7, b) A_7, c) $A_5 \oplus A_2$, d) $2A_3 \oplus A_1$, e) $D_6 \oplus A_1$, f) $D_4 \oplus 3A_1$, g) $7A_1$
10	E_8	a) E_8, b) A_8, c) $A_7 \oplus A_1$, d) $A_5 \oplus A_2 \oplus A_1$, e) $2A_4$, f) D_8, g) $D_5 \oplus A_3$, h) $E_6 \oplus A_2$, i) $E_7 \oplus A_1$, j) $D_6 \oplus 2A_1$, k) $2D_4$, l) $2A_3 \oplus 2A_1$, m) $4A_2$, n) $D_4 \oplus 4A_1$, o) $8A_1$
20	D_8	a) D_8, b) $D_6 \oplus 2A_1$, c) $D_5 \oplus A_3$, d) $2D_4$, e) $2A_3 \oplus 2A_1$, f) $D_4 \oplus 4A_1$, g) $8A_1$

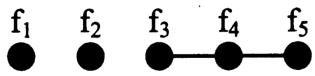
The root invariants for $T \subset R$ *of Lemma 3.12:*

7a, $D_5 \subset D_5$: with the basis in T



$H = 0 \mod T, \overline{a} = (f_4 + f_5)/2 \mod H$ (since $\overline{a}$ is defined, the invariant $\alpha = 0$).

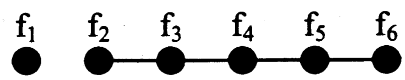
7b, $A_3 \oplus A_1 \subset D_5$: with the basis (in T)



$H = [(f_1 + f_2 + f_3 + f_5)/2] \mod T, \overline{a} = (f_3 + f_5)/2 \mod H$.

8a, $E_6 \subset E_6$: Then $H = 0 \mod T$ and $\alpha = 1$ (it follows that $\alpha = 1$ and $\overline{a}$ is not defined for all cases 8a—c below).

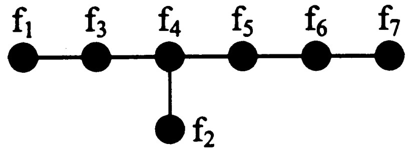
8b, $A_1 \oplus A_5 \subset E_6$: with the basis



$H = [(f_1 + f_2 + f_4 + f_6)/2] \mod T$ and $\alpha = 1$.

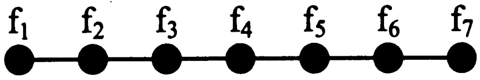
8c, $3A_2 \subset E_6$: Then $H = 0 \mod T$ and $\alpha = 1$.

9a, $E_7 \subset E_7$: with the basis



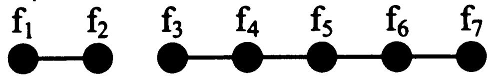
$H = 0 \mod T$ and $\overline{a} = (f_2 + f_5 + f_7)/2 \mod T$.

9b, $A_7 \subset E_7$: with the basis



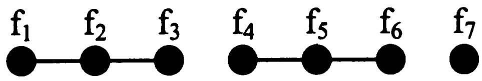
$H = [(f_1 + f_3 + f_5 + f_7)/2] \mod T$ and $\alpha = 1$.

9c, $A_5 \oplus A_2 \subset E_7$: with the basis



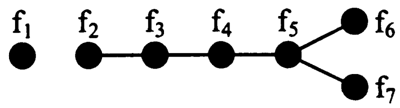
$H = 0 \mod T$ and $\overline{a} = (f_3 + f_5 + f_7)/2 \mod H$.

9d, $2A_3 \oplus A_1 \subset E_7$: with the basis



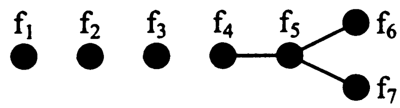
$H = [(f_1 + f_3 + f_4 + f_6)/2] \mod T$ and $\overline{a} = (f_1 + f_3 + f_7)/2 \mod H$.

9e, $D_6 \oplus A_1 \subset E_7$: with the basis



$H = [(f_1 + f_2 + f_4 + f_6)/2] \mod T$ and $\overline{a} = (f_1 + f_6 + f_7)/2 \mod H$.

9f, $D_4 \oplus 3A_1 \subset E_7$: with the basis

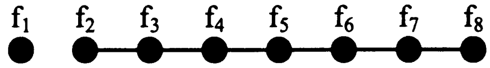


$H = [(f_1 + f_2 + f_4 + f_6)/2, (f_2 + f_3 + f_6 + f_7)/2] \mod T$ and $\overline{a} = (f_1 + f_2 + f_3)/2 \mod H$.

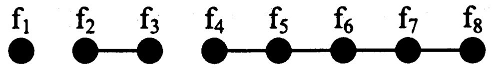
9g, $7A_1 \subset E_7$: with the basis f_v, $v \in \mathbb{P}^2(F_2)$ where $\mathbb{P}^2(F_2)$ is the projective plane over the field F_2 with two elements, the group H is generated by $\left(\sum_{v \in \mathbb{P}^2(F_2) - l} f_v\right)/2$ where l is any line in $\mathbb{P}^2(F_2)$. The element $\overline{a} = \left(\sum_{v \in l} f_v\right)/2$ where l is any line in $\mathbb{P}^2(F_2)$.

10a, $E_8 \subset E_8$: Then $H = 0 \mod T$ and $\alpha = 1$ (it follows that $\alpha = 1$ and the element $\overline{a}$ is not defined for all cases 10a—o).

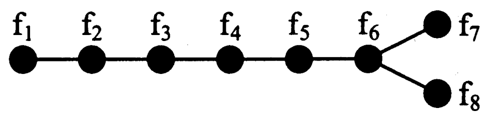
10b, $A_8 \subset E_8$: Then $H = 0 \mod T$ and $\alpha = 1$.
10c, $A_7 \oplus A_1 \subset E_8$: with the basis



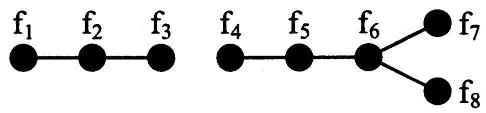
$H = [(f_2 + f_4 + f_6 + f_8)/2] \mod T$ and $\alpha = 1$.
10d, $A_5 \oplus A_2 \oplus A_1 \subset E_8$: with the basis



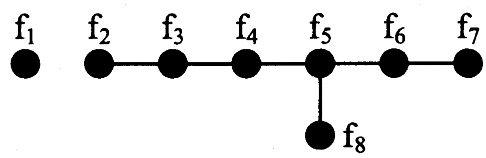
$H = [(f_1 + f_4 + f_6 + f_8)/2] \mod T$ and $\alpha = 1$.
10e, $2A_4 \subset E_8$: Then $H = 0 \mod T$ and $\alpha = 1$.
10f, $D_8 \subset E_8$: with the basis



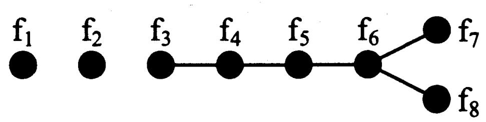
$H = [(f_1 + f_3 + f_5 + f_7)/2] \mod T$ and $\alpha = 1$.
10g, $D_5 \oplus A_3 \subset E_8$: with the basis



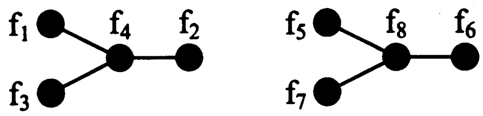
$H = [(f_1 + f_3 + f_7 + f_8)/2] \mod T$ and $\alpha = 1$.
10h, $E_6 \oplus A_2 \subset E_8$: Then $H = 0$ and $\alpha = 1$.
10i, $E_7 \oplus A_1 \subset E_8$: with the basis



$H = [(f_1 + f_2 + f_4 + f_8)/2] \mod T$ and $\alpha = 1$.
10j, $D_6 \oplus 2A_1 \subset E_8$: with the basis

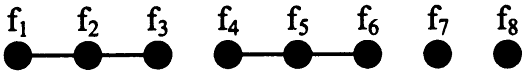


$H = [(f_1 + f_3 + f_5 + f_7)/2,\ (f_2 + f_3 + f_5 + f_8)/2] \mod T$ and $\alpha = 1$.
10k, $2D_4 \subset E_8$: with the basis



$H = [(f_1 + f_2 + f_5 + f_6)/2,\ (f_2 + f_3 + f_6 + f_7)/2] \mod T$ and $\alpha = 1$.

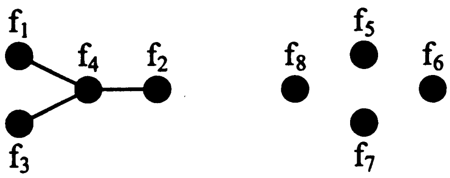
10l, $2A_3 \oplus 2A_1 \subset E_8$: with the basis



$H = [(f_1 + f_3 + f_7 + f_8)/2,\ (f_4 + f_6 + f_7 + f_8)/2] \mod T$ and $\alpha = 1$.

10m, $4A_2 \subset E_8$: Then $H = 0 \mod T$ and $\alpha = 1$.

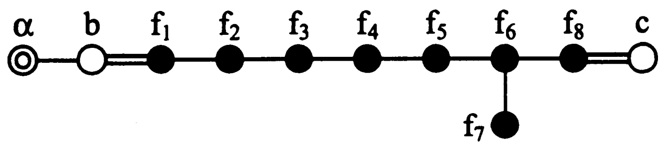
10n, $D_4 \oplus 4A_1 \subset E_8$:



$H = [(f_1 + f_2 + f_5 + f_6)/2,\ (f_2 + f_3 + f_6 + f_7)/2,\ (f_5 + f_6 + f_7 + f_8)/2] \mod T$ and $\alpha = 1$.

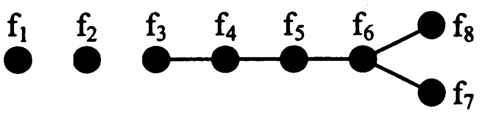
10o, $8A_1 \subset E_8$: with the basis f_v, $v \in V$ and V has the structure of 3-dimensional affine space over F_2, the group H is generated by $\left(\sum_{v\in\pi} f_v\right)/2$ where $\pi \subset V$ is any 2-dimensional affine subspace in V. The invariant $\alpha = 1$.

20a, $D_8 \subset D_8$: with the basis $f_1, \ldots, f_8$ shown below



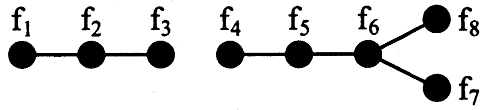
$H = [(f_1 + f_3 + f_5 + f_7)/2] \mod T$, $\bar{a} = (f_7 + f_8)/2 \mod H$.

20b, $D_6 \oplus 2A_1 \subset D_8$: with the basis



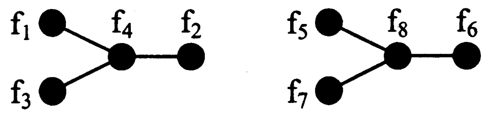
$H = [(f_1{+}f_3{+}f_5{+}f_7)/2,\ (f_2{+}f_3{+}f_5{+}f_8)/2] \mod T$ and $\bar{a} = (f_7{+}f_8)/2 \mod H$.

20c, $D_5 \oplus A_3$: with the basis



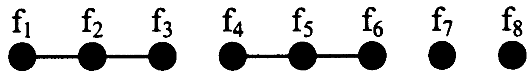
$H = [(f_1 + f_3 + f_7 + f_8)/2] \mod T$ and $\overline{a} = (f_7 + f_8)/2 \mod H$.

20d, $2D_4 \subset D_8$: with the basis



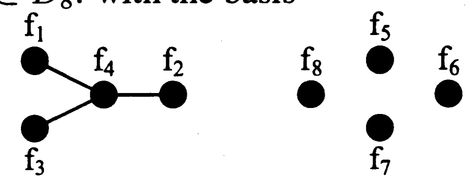
$H = [(f_1+f_2+f_5+f_6)/2,\ (f_2+f_3+f_6+f_7)/2] \mod T$ and $\overline{a} = (f_6+f_7)/2 \mod H$.

20e, $2A_3 \oplus 2A_1 \subset D_8$: with the basis



$H = [(f_1+f_3+f_7+f_8)/2,\ (f_4+f_6+f_7+f_8)/2] \mod T$ and $\overline{a} = (f_7+f_8)/2 \mod H$.

20f, $D_4 \oplus 4A_1 \subset D_8$: with the basis



$H = [(f_1 + f_2 + f_5 + f_6)/2,\ (f_2 + f_3 + f_6 + f_7)/2,\ (f_5 + f_6 + f_7 + f_8)/2] \mod T$ and $\overline{a} = (f_7 + f_8)/2 \mod H$.

20g, $8A_1 \subset D_8$: with the basis f_v, $v \in V$ and V has the structure of 3-dimensional affine space over F_2, the group H is generated by $\left(\sum_{v\in\pi} f_v\right)/2$ where $\pi \subset V$ is any 2-dimensional affine subspace in V. The element $\overline{a} = (f_{v_1} + f_{v_2})/2 \mod H$ where v_1v_2 is a fixed non-zero vector in V. This structure can be seen in Figure 4 below.

Proof of Lemma 3.12. Let us consider cases $N = 7, 8, 9, 10$ and 20 of the main invariants S in Table 1. By Lemma 2.5, the canonical homomorphism $O(S) \to O(q_S)$ is epimorphic. Since ± 1 acts identically on the 2-elementary form q_S, it follows that $O'(S) \to O(q_S)$ is epimorphic. The group $O'(S)$ is the semi-direct product of $W^{(2,4)}(S)$ and the automorphism group of the diagram $\Gamma(P(\mathcal{M}^{(2,4)}))$. The last group is trivial in all these cases. Thus $W^{(2,4)}(S) \to O(q_S)$ is epimorphic. The group $W^{(2,4)}(S)$ is the semi-direct product of $W^{(2)}(S)$ and the symmetry group $W^{(4)}(\mathcal{M}^{(2)})$ of the fundamental chamber $\mathcal{M}^{(2)}$. The group $W^{(2)}(S)$ acts identically on $O(q_S)$. It follows that the corresponding homomorphism $W^{(4)}(\mathcal{M}^{(4)}) \to O(q_S)$ is epimorphic. Here $W^{(4)}(\mathcal{M}^{(2)})$ is exactly the Weyl group of the root system R defined by black vertices of the diagram $\Gamma(P(\mathcal{M}^{(2,4)}))$.

$N = 7$: Then $q_S \cong q_1^{(2)}(2) \oplus q_{-1}^{(2)}(2) \oplus u_+^{(2)}(2) \oplus v_+^{(2)}(2)$ (we use notation of [Nik80b]), and $R = D_5$. By direct calculation (using Lemma 2.7), we get $\#O(q_S) = 5 \cdot 3 \cdot 2^7$. It is known [Bou68], that $\#W(D_5) = 5 \cdot 3 \cdot 2^7$. Thus we get the canonical isomorphism $W(D_5) \cong O(q_S)$. By Lemma 3.11, it follows that any two root subsystems $T_1 \subset D_5$ and $T_2 \subset D_5$ are conjugate by $W(D_5)$, if and only if their root invariants (T_1, ξ_1^+) and (T_2, ξ_2^+) are isomorphic.

In all other cases considerations are the same.

$N = 8$: Then $q_S \cong q_{-1}^{(2)}(2) \oplus v_+^{(2)}(2) \oplus 2u_+^{(2)}(2)$ and $R = E_6$. We have $\#O(q_S) = \#W(E_6) = 5 \cdot 3^4 \cdot 2^7$. It follows, $W(E_6) \cong O(q_S)$.

$N = 9$: Then $q_S \cong 2q_1^{(2)}(2) \oplus 3u_+^{(2)}(2)$ and $R = E_7$. We have $\#O(q_S) = \#W(E_7) = 7 \cdot 5 \cdot 3^4 \cdot 2^{10}$. It follows, $W(E_7) \cong O(q_S)$.

$N = 10$: Then $q_S \cong q_1^{(2)}(2) \oplus 4u_+^{(2)}(2)$ and $R = E_8$. We have $\#O(q_S) = 7 \cdot 5^2 \cdot 3^5 \cdot 2^{13}$ and $\#W(E_8) = 7 \cdot 5^2 \cdot 3^5 \cdot 2^{14}$. It follows that the homomorphism $W(E_8) \to O(q_S)$ is epimorphic and has the kernel ± 1.

$N = 20$: Then $q_S \cong q_1^{(2)}(2) \oplus q_{-1}^{(2)}(2) \oplus 3u_+^{(2)}(2)$ and $R = D_8$. We have $\#O(q_S) = 7 \cdot 5 \cdot 3^2 \cdot 2^{13}$ and $\#W(E_8) = 7 \cdot 5 \cdot 3^2 \cdot 2^{14}$. It follows that the homomorphism $W(D_8) \to O(q_S)$ is epimorphic and has the kernel ± 1.

Any root subsystem $T \subset R$ of finite index can be obtained by the procedure described in Section 3.4.1. In each case $N = 7$, 8, 9, 10 and 20 of R, applying this procedure, it is very easy to find all root subsystems $T \subset R$ of finite index and calculate their root invariants. One can see that it is prescribed by the type of the root system T itself. We leave these routine calculations to a reader. They are presented above and will be also very important for further considerations.

This finishes the proof of Lemma 3.12.

Remark 3.13. As in the proof above, using the homomorphism $W^{(4)}(\mathcal{M}^{(2)}) \to O(q_S)$, one can give the direct proof of the important Lemma 2.5 in all elliptic cases of main invariants. Indeed, it is easy to study its kernel and calculate orders of the groups. This proof uses calculations of $W^{(2,4)}(S)$ and $O(S)$ of Theorem 3.5.

Consider a root subsystem $T \subset R$ of Lemma 3.12. By Theorem 2.4, the root subsystem $T \subset R$ defines a subset $\Delta_+^{(4)}(S) \subset \Delta^{(4)}(S)$, the corresponding reflection group $W_+^{(2,4)}$, and Dynkin diagram $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ of its fundamental chamber $\mathcal{M}_+^{(2,4)}$. Direct calculation of these diagrams using Theorem 2.4 gives diagrams of Table 2 of Theorem 3.6 (where $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ is replaced by $\Gamma(P(X)_+))$ in all cases 7a, b; 8a — c; 9a — f; 10a — m; 20a — d. In the remaining cases 9g; 10n, o; 20e — g we get diagrams

$\Gamma(P(\mathcal{M}_+^{(2,4)}))$ which we describe below. Details of these calculations are presented in Appendix, Sections A.4.2–A.4.6.

In the *Case 9g,* it is better to describe $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ indirectly. Its black vertices correspond to all points of $\mathbb{P}^2(F_2)$ which is the projective plane over the field F_2 with two elements. Its transparent vertices correspond to all lines in $\mathbb{P}^2(F_2)$. Both sets have seven elements. Black vertices are disjoint; transparent vertices are also disjoint; a black vertex is connected with a transparent vertex by the double edge, if the corresponding point belongs to the corresponding line, otherwise, they are disjoint.

In the *Case 10n,* the diagram $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ is given below in Figure 1. Since it is quite complicated, we divide it in three subdiagrams shown. The first one shows all its edges connecting black and transparent vertices. The second one shows the edge connecting the transparent vertices numerated by 1 and 2. The third one shows edges connecting transparent vertices 3 — 12. Each edge of $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ is shown in one of these diagrams. All other our similar descriptions of diagrams as unions of their subdiagrams have the same meaning. In particular, we have used it in some diagrams of Table 2.

In the *Case 10o,* we describe the diagram $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ indirectly. Its black vertices f_v, $v \in V$, correspond to all points of a three-dimensional affine space V over F_2. Its transparent vertices are of two types. Vertices e_v of the first type also correspond to all points $v \in V$. Vertices e_π of the second type correspond to all (affine) planes $\pi \subset V$ (there are 14 of them). Black vertices f_v are disjoint. A black vertex f_v is connected with a transparent vertex $e_{v'}$, if and only if $v = v'$; the edge has the weight $\sqrt{8}$. A black vertex f_v is connected with a transparent vertex e_π, if and only if $v \in \pi$; the edge is double. Transparent vertices e_v, $e_{v'}$ are connected by a thick edge. A transparent vertex e_v is connected with a transparent vertex e_π, if and only if $v \notin \pi$; the edge is thick. Transparent vertices e_π, $e_{\pi'}$ are connected by edge, if and only if $\pi \| \pi'$; the edge is thick.

In *Cases 20e, 20f and 20g* diagrams $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ are shown in figures 2—4 below.

We remark that a calculation of $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ in cases 7a,b, 8a—c, 9a—g and 10a—o can be obtained from results of [BBD84] where (in our notation) the dual diagram of all exceptional curves on the quotient $Y = X/\{1, \theta\}$ is calculated using completely different method (under the assumption that Y does exist). By Section 2.5, both diagrams can be easily obtained from one another (compare with Section 3.5 below). Therefore, we explain our method of calculation of $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ in more details than it has done in Section 2.4.1 only in the Case 20 (i. e. cases 20a—g).

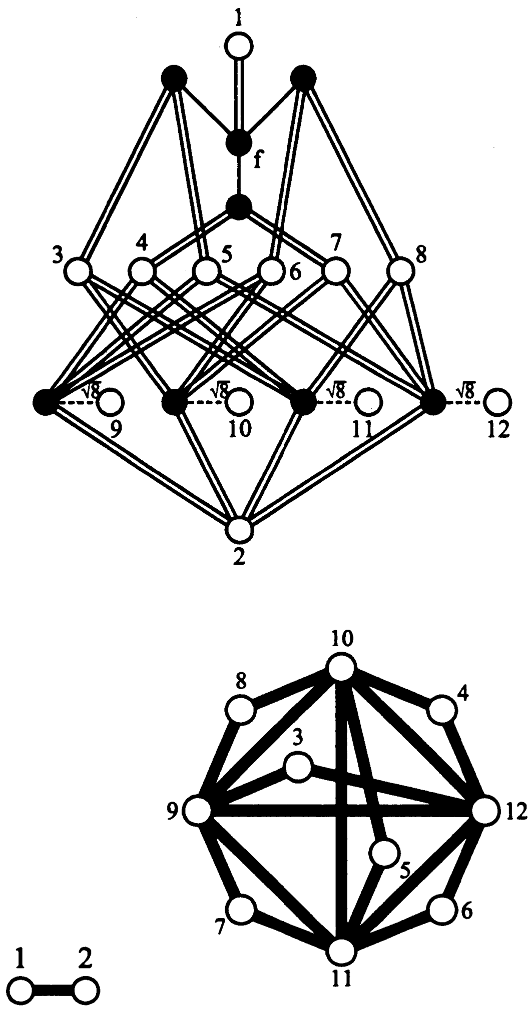


FIGURE 1. The diagram 10n

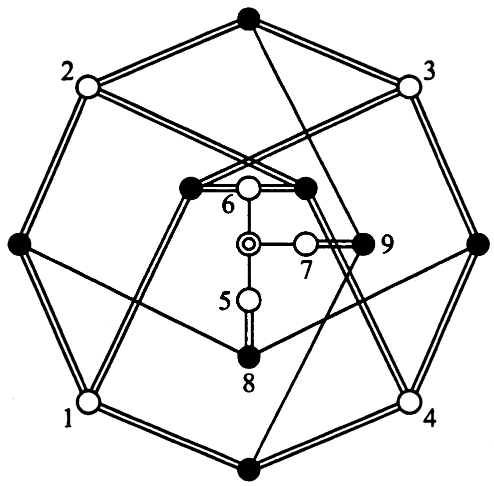


FIGURE 2. The diagram 20e

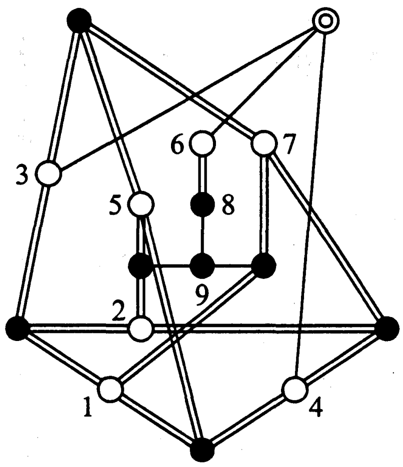


FIGURE 3. The diagram 20f

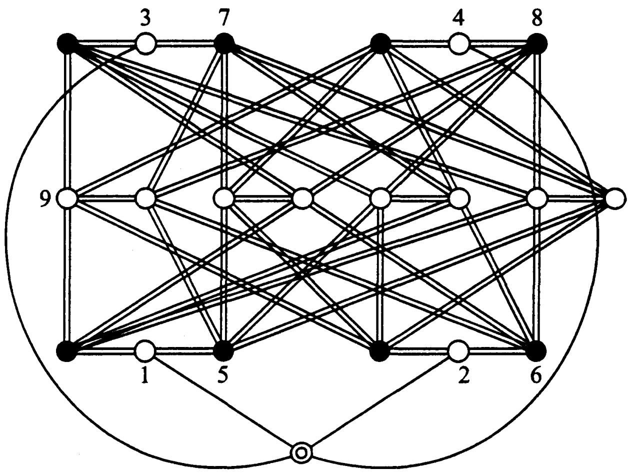


FIGURE 4. The diagram 20g

In the *Case 20,* the lattice S has invariants $(r, a, \delta) = (10, 8, 1)$, and we can take in $S \otimes \mathbb{Q}$ an orthogonal basis h, α, $v_1, \dots, v_8$ with $h^2 = 2$, $\alpha^2 = v_1^2 = \cdots = v_8^2 = -2$. As $P(\mathcal{M}^{(2,4)})$, we can take

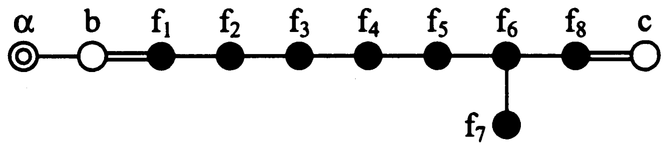
(67)

$$\begin{aligned} P^{(4)}(\mathcal{M}^{(2,4)}) =& \{f_1 = v_1 - v_2, f_2 = v_2 - v_3, f_3 = v_3 - v_4, f_4 = v_4 - v_5, \\ & f_5 = v_5 - v_6, f_6 = v_6 - v_7, f_7 = v_7 - v_8, f_8 = v_7 + v_8\}, \end{aligned}$$

and

$$P^{(2)}(\mathcal{M}^{(2,4)}) = \{\alpha,\, b = \frac{h}{2} - \frac{\alpha}{2} - v_1,\, c = h - \frac{1}{2}(v_1 + v_2 + \cdots + v_8)\}.$$

These elements have Dynkin diagram



of the case 20a, and they generate and define S.

By Section 2.4.1, the set $P^{(2)}(\mathcal{M}^{(2)})$, where $\mathcal{M}^{(2)} \supset \mathcal{M}^{(2,4)}$, is

$$W^{(4)}(\mathcal{M}^{(2)})(\{\alpha, b, c\})$$

where $W^{(4)}(\mathcal{M}^{(2)})$ is generated by reflections in $f_1, \dots, f_8$. It follows that

$$P(\mathcal{M}^{(2)}) = P^{(2)}(\mathcal{M}^{(2)}) = \{\alpha; b_{\pm i}; c_{i_1 \dots i_k}\}$$

where

$$b_{\pm i} = \frac{h}{2} - \frac{\alpha}{2} \pm v_i, \ i = 1, 2, \dots, 8;$$
$$c_{i_1 \dots i_k} = h + \frac{1}{2}(v_1 + v_2 + \dots + v_8) - v_{i_1} - \dots - v_{i_k},$$

where $1 \le i_1 < i_2 < \dots < i_k \le 8$ and $k \equiv 0 \mod 2$. Here all $b_{\pm i}$ give the $W^{(4)}(\mathcal{M}^{(2)})$-orbit of b, and all $c_{i_1 \dots i_k}$ give the $W^{(4)}(\mathcal{M}^{(2)})$-orbit of c.

Elements $f_1, \dots, f_8$ give a basis of the root system R of type D_8. If $T \subset R$ is its subsystem of rank m, and $t_1, \dots, t_m$ a basis of T, then the fundamental chamber $\mathcal{M}_+^{(2,4)} \subset \mathcal{M}^{(2)}$ defined by T and by its basis $t_1, \dots, t_m$ has $P(\mathcal{M}_+^{(2,4)}) = P^{(4)}(\mathcal{M}_+^{(2,4)}) \cup P^{(2)}(\mathcal{M}_+^{(2,4)})$ where

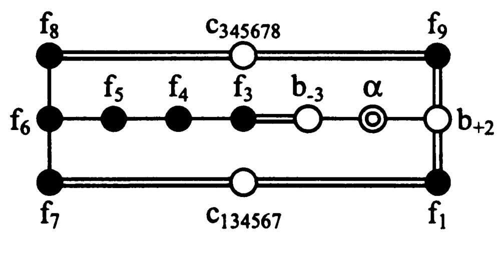
$$\begin{aligned} P^{(4)}(\mathcal{M}_+^{(2,4)}) &= \{t_1, \dots, t_m\}, \\ P^{(2)}(\mathcal{M}_+^{(2,4)}) &= \{\alpha\} \cup \{b_{\pm i} \mid b_{\pm i} \cdot t_s \ge 0, \ 1 \le s \le m\} \\ &\cup \{c_{i_1 \dots i_k} \mid c_{i_1 \dots i_k} \cdot t_s \ge 0, \ 1 \le s \le m\}. \end{aligned} \tag{68}$$

This describes $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ completely.

For example, assume that $T \subset R$ has the type $2A_1 \oplus D_6$ with the basis f_1, $f_9 = -v_1 - v_2$, f_3, $\dots$, f_8. Then we get (after simple calculations)

$$P^{(2)}(\mathcal{M}_+^{(2,4)}) = \{\alpha, b_{+2}, b_{-3}, c_{345678}, c_{134567}\},$$

and $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ is



This gives Case 20b of Table 2.

Exactly the same calculations of $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ can be done in all cases 20a—g, and cases 7a,b — 10a—o of Table of Lemma 3.12 as well. See Appendix, Sections A.4.2–A.4.6.

3.4.5

Here we prove

Proposition 3.14. *Cases 9g, 10n,o and 20 e — g of root subsystems $T \subset R$ of Lemma 3.12 do not correspond to non-symplectic involutions (X, θ) of K3 (in characteristic 0 and even in characteristic ≥ 3).*

Proof. Assume that a root subsystem $T \subset R$ corresponds to a K3 pair (X, θ). Then the corresponding Dynkin diagram $\Gamma(P(\mathcal{M}_+^{(2,4)}))$ given in Section 3.4.4 coincides with Dynkin diagram $\Gamma(P(X)_+)$ of exceptional curves of the pair (X, θ). It follows the dual diagram of exceptional curves $\Gamma(P(Y))$ on the corresponding DPN surface $Y = X/\{1, \theta\}$ (see Section 2.4). Using this diagram, it is easy to find a sequence of exceptional curves $E_1, \dots, E_k$ on Y where $k = r - 1$ such that their contraction gives a morphism $\sigma : Y \to \mathbb{P}^2$. Then other (different from $E_1, \dots, E_k$) exceptional curves on Y corresponding to Du Val and logarithmic part of $\Gamma(P(Y))$ give a configuration of rational curves on $\mathbb{P}^2$ which cannot exist in characteristic 0 and even in characteristic ≥ 3 (but it exists in characteristic 2). In cases 9g; 10n,o; 20e,f we get Fano's configuration of seven lines of the finite projective plane over F_2 which can exist only in characteristic 2. In the case 9g one should contract exceptional curves corresponding to all transparent vertices. In the case 10n — corresponding to vertices 1, f, 3 — 8. In the case 10o — corresponding to vertices e_π where π contains a fixed point $0 \in V$ and e_0; then curves corresponding to f_v, $v \neq 0$, give Fano's configuration. In cases 20e,f — corresponding to vertices 1 — 9. In the case 20g — corresponding to vertices 1 — 9, then we get a conic (corresponding to the double transparent vertex) and four its tangent lines (corresponding to black vertices different from 5 — 8) passing through one point. It is possible only in characteristic 2. $\square$

Another purely arithmetic proof of Proposition 3.14 (over $\mathbb{C}$) can be obtained using Proposition 2.10. This proof is more complicated, but it can also be done. Here we preferred shorter and geometric considerations (if diagrams have calculated). $\square$

3.4.6

Here we prove

Proposition 3.15. *Cases 7a,b; 8a—c, 9a—f; 10a—m and 20a—d of Table 2 of Theorem 3.6 correspond to standard extremal non-symplectic K3 involutions* (X, θ).

Proof. Let us calculate root invariants $(K^+(2), \xi^+)$ corresponding to these cases.

Consider the sequence of embeddings of lattices

$$K^+(2) = [T] \subset [R] \subset S.$$

It defines the homomorphism

$$\xi^+ : Q = \frac{1}{2}K^+(2)/K^+(2) \to S^*/S \subset \frac{1}{2}S/S$$

with the kernel H. It can be decomposed as

$$\xi^+ : Q \xrightarrow{\widetilde{\xi}^+} \frac{1}{2}[R]/[R] \xrightarrow{\xi_R^+} S^*/S \subset \frac{1}{2}S/S. \tag{69}$$

Let $H_R = \operatorname{Ker} \xi_R^+$. Then $H = (\widetilde{\xi}^+)^{-1}(H_R)$. As we know (from our considerations in the super-extremal case), $H_R = 0$ in cases 7, 8, 9, 10. In the case 20, the $H_R = \mathbb{Z}/2\mathbb{Z}$ is

$$H_R = [\frac{1}{2}(f_1 + f_3 + f_5 + f_7) + R]/[R]$$

(see Section 3.4.4 about this case). Thus, H can be identified with $H = (\frac{1}{2}[T] \cap [R])/[T]$ in cases 7, 8, 9, 10, and with

$$H = (\frac{1}{2}[T] \cap [\frac{1}{2}(f_1 + f_3 + f_5 + f_7) + R])/[T]$$

in the case 20.

Further details of this calculations in all cases N=7, 8, 9, 10 and 20 are presented in Lemma 3.12.

From these calculations, we get values of $l(H)$ given in Table 2 of Theorem 3.6.

As in Section 3.4.2, using Proposition 2.9, one can prove that all these cases when

$$r + a + 2l(H) < 22 \tag{70}$$

correspond to standard extremal non-symplectic K3 involutions (X, θ). Therefore, we only need to consider cases when the inequality (70) fails. There are exactly five such cases: 10j,k,l and 20b,d. Further we consider these cases only.

Below we use some notations and results from [Nik80b] about lattices and their discriminant forms. They are all presented in Appendix, Sections A.1, A.2.

In cases 10j,k,l the discriminant form of S is $q_S = q_1^{(2)}(2) \oplus 4u_+^{(2)}(2)$. Here, the generator of the first summand $q_1^{(2)}(2)$ gives the characteristic element a_{q_S} of the q_S, and the second summand $4u_+^{(2)}(2)$ gives the image of ξ_R^+ from (69), by Lemma 3.9. Thus, the image of ξ^+ belongs to $4u_+^{(2)}(2)$. The discriminant form of the lattice M (from 2.7) is obtained as follows. Let

$$\Gamma_{\xi^+} \subset Q \oplus \mathfrak{A}_S \subset \mathfrak{A}_{K^+(2)} \oplus \mathfrak{A}_S$$

be the graph of the homomorphism ξ^+ in $\mathfrak{A}_{K^+(2)} \oplus \mathfrak{A}_S$. Then

$$q_M = (q_{K^+(2)} \oplus q_S \mid (\Gamma_{\xi^+})^{\perp}_{q_{K^+(2)} \oplus q_S})/\Gamma_{\xi^+} \tag{71}$$

(here Γ_{ξ^+} is an isotropic subgroup).Therefore, $q_M \cong q_1^{(2)}(2) \oplus q'$ since the image of ξ^+ belongs to the orthogonal complement of the summand $q_1^{(2)}(2)$. Considerations in the proof of Proposition 2.9 show that

$$\operatorname{rk} M + l(\mathfrak{A}_{M_2}) \leq 22 \tag{72}$$

since $r + a + 2l(H) = 22$ in cases 10j,k,l. It is easy to see that

$$\operatorname{rk} M + l(\mathfrak{A}_{M_p}) < 22$$

for all prime $p > 2$. Then, by Theorem 1.12.2 in [Nik80b] (see Appendix, Theorem A.5), there exists a primitive embedding $M \subset L_{K3}$ when either the inequality (72) is strict or $q_{M_2} \cong q_{\pm 1}^{(2)}(2) \oplus q'$, if it gives the equality. Thus, it always does exist. It follows that all cases 10j,k,l correspond to standard extremal non-symplectic K3 involutions (X, θ) by Proposition 2.10 (where we used fundamental Global Torelli Theorem [PS-Sh71] and surjectivity of Torelli map [Kul77] for K3).

In cases 20b,d, the proof is exactly the same, but it is more difficult to prove that $q_{M_2} \cong q_\theta^{(2)}(2) \oplus q'$ where $\theta = \pm 1$. In these cases

$$q_S = 3u_+^{(2)}(2) \oplus q_1^{(2)}(2) \oplus q_{-1}^{(2)}(2).$$

If α_1 and α_2 are generators of the summands $q_1^{(2)}(2)$ and $q_{-1}^{(2)}(2)$ respectively, then $\alpha_{q_S} = \alpha_1 + \alpha_2$ is the characteristic element of q_S, and the image of ξ^+ belongs to $3u_+^{(2)}(2) \oplus [\alpha_{q_S}]$. In these cases, the lattice $K_H^+(2)$ (see Section 2.7) is isomorphic to $E_8(2)$. For example, this is valid because the subgroups H are the same in cases 10j and 20b, and in cases 10k and 20d, besides, in cases 10j and 10k we have $E_8/K^+ \cong H$. It follows that

$$q_{K_H^+(2)} = (q_{K^+(2)} \mid (H)^{\perp}_{q_{K^+(2)}})/H \cong q_{E_8(2)} \cong 4u_+^{(2)}(2).$$

We set $\overline{\Gamma}_{\xi^+} = \Gamma_{\xi^+}/H$. By (71)

$$q_M = (q_{K_H^+(2)} \oplus q_S \mid (\overline{\Gamma}_{\xi^+})^{\perp}_{q_{K_H^+(2)} \oplus q_S})/\overline{\Gamma}_{\xi^+}.$$

We have $q_{K_H^+(2)} \oplus q_S = 7u_+^{(2)}(2) \oplus q_1^{(2)}(2) \oplus q_{-1}^{(2)}(2)$. Since $u_+^{(2)}(2)$ takes values in $\mathbb{Z}/2\mathbb{Z}$, the element α_{q_S} (more exactly, $0 \oplus \alpha_{q_S}$) is the characteristic element of $q_{K_H^+(2)} \oplus q_S$ again. Moreover, $\alpha_{q_S} \notin \overline{\Gamma}_{\xi^+}$ since Γ_{ξ^+} is the graph of a homomorphism with the kernel H. Therefore $(\overline{\Gamma}_{\xi^+})^{\perp}_{q_{K_H^+(2)} \oplus q_S}$ contains v which is not orthogonal to α_{q_S}. Then

$$(q_{K_H^+(2)} \oplus q_S)(v) = \pm\frac{1}{2} \mod 2$$

and

$$[v \mod \overline{\Gamma}_{\xi^+}] \cong q_\theta^{(2)}(2),\ \theta = \pm 1,$$

is the orthogonal summand of q_{M_2} we were looking for. □

Remark 3.16. We can give another proof of Proposition 3.15 which uses Theorem 1.5 and considerations which are inverse to the proof of the previous Proposition 3.14. Indeed, by Theorem 1.5, it is enough to prove existence of rational surfaces with Picard number r and configuration of rational curves defined by Dynkin diagram of Table 2 of Theorem 3.6 (assuming that these Dynkin diagrams correspond to K3 pairs (X, θ) and considering the quotient by θ). One can prove existence of these rational surfaces considering appropriate sequences of blow-ups of appropriate relatively minimal rational surfaces $\mathbb{P}^2$, $\mathbb{F}_0$, $\mathbb{F}_1$, $\mathbb{F}_2$, $\mathbb{F}_3$ or $\mathbb{F}_4$ with appropriate configurations of rational curves defined by Dynkin diagrams of Table 2 of Theorem 3.6 (see the proof of Proposition 3.14). This proof does not use Global Torelli Theorem and surjectivity of Torelli map for K3. This gives a hope that results of Chapter 2 and Chapter 3 can be generalized to characteristic $p > 0$. Unfortunately, we have proved Theorem 1.5 in characteristic 0 only. Thus, we preferred the proof of Proposition 3.15 which is independent of the results of Chapter 1.

3.4.7

To finish the proof of Theorem 3.6, we need to prove only

Proposition 3.17. *Let (X, θ) be a non-symplectic involution of K3 which corresponds to one of cases 7 — 10 or 20 of Table 1 of Theorem 3.1 and a root subsystem $T \subset R = \Delta^{(4)}(\mathcal{M}^{(2)})$.*

If (X, θ) is extremal, then $\operatorname{rk} T = \operatorname{rk} R$.

Proof. We can assume (see Section 3.4.1) that T has a basis which gives a part of a basis of a root subsystem $\widetilde{T} \subset R = \Delta^{(4)}(\mathcal{M}^{(2)})$ of the same rank $\operatorname{rk} \widetilde{T} = \operatorname{rk} R$. Then $\widetilde{T} \subset R$ is one of root subsystems of Lemma 3.12. If the root subsystem $\widetilde{T} \subset R$ corresponds to a non-symplectic involution of

K3, i. e. $\widetilde{T}$ gives cases 7a—b, 8a—c, 9a—f, 10a—m and 20a—d, then T is extremal, only if $T = \widetilde{T}$ (by definition). Then $\operatorname{rk} T = \operatorname{rk} \widetilde{T} = \operatorname{rk} R$ as we want. Thus, it is enough to consider $\widetilde{T}$ of cases 9g, 10n—o, 20e—g and $T \subset \widetilde{T}$ to be a primitive root subsystem of a strictly smaller rank.

Below we consider all these cases. The following is very important. In Lemma 3.12 we calculated root invariants of root subsystems $\widetilde{T} \subset R$ of finite index. Restricting the root invariant of $\widetilde{T}$ on a root subsystem $T \subset \widetilde{T}$, we get the root invariant of $T \subset R$. In considerations below, we always consider $T \subset R$ together with its root invariant. Two root subsystems of R are considered to be the same, if and only if they are isomorphic root systems together with their root invariants: then they give equivalent root subsystems (even with respect to the finite Weyl group $W^{(4)}(\mathcal{M}^{(2)})$, see the proof of Lemma 3.12) and isomorphic diagrams.

Case 9g. Then $\widetilde{T} = 7A_1$, and $T = kA_1$, $k \le 6$, is its root subsystem (it is always primitive). It is easy to see that the same root subsystem T can be obtained as a primitive root subsystem $T \subset D_4 \oplus 3A_1$. Then T is not extremal because $D_4 \oplus 3A_1$ corresponds to K3.

Case 10n. Then $\widetilde{T} = D_4 \oplus 4A_1$ and $T \subset \widetilde{T}$ is a primitive root subsystem of the rank ≤ 7. It is easy to see that the same root subsystem can be obtained as a primitive root subsystem T of $D_6 \oplus 2A_1$ or $D_4 \oplus D_4$ (then it is not extremal because $D_6 \oplus 2A_1$ and $D_4 \oplus D_4$ correspond to K3) in all cases except when $T = 7A_1$.

Let us consider the last case $T = 7A_1$ and show (as in Section 3.4.5) that it does not correspond to K3. As in Section 3.4.4, one can calculate Dynkin diagram $\Gamma = \Gamma(P(\mathcal{M}_+^{(2,4)}))$. See Appendix, Section A.4.5, Case $7A_1 \subset E_8$. It is similar to the case 10o (see Section 3.4.4), but it is more complicated. We describe it indirectly. One can relate with this diagram a 3-dimensional linear vector space V over F_2.

Black vertices f_v of Γ correspond to $v \in V - \{0\}$ (there are seven of them). Its transparent vertices (all of them are simple) are
e_v, $v \in V - \{0\}$; $e_0^{(+)}$, $e_0^{(-)}$;
e_π, $\pi \subset V$ is any affine hyperspace in V which does not contain 0;
$e_\pi^{(+)}$, $e_\pi^{(-)}$, $\pi \subset V$ is any hyperspace ($0 \in \pi$) of V.

Edges which connect f_v, e_v, $e_0^{(+)}$, e_π, $e_\pi^{(+)}$ are the same as for the diagram 10o (forget about $(+)$). The same is valid for f_v, e_v, $e_0^{(-)}$, e_π, $e_\pi^{(-)}$ (forget about $(-)$). Vertices $e_0^{(+)}$ and $e_0^{(-)}$ are connected by the broken edge of the weight 6. Vertices $e_0^{(+)}$ and $e_\pi^{(-)}$ (and $e_0^{(-)}$, $e_\pi^{(+)}$ as well) are connected by the broken edge of the weight 4. This gives all edges of Γ.

Assume that Γ corresponds to a K3 pair (X, θ). Consider the corresponding DPN surface and contract exceptional curves corresponding to

$e_{\pi}^{(+)}$ and $e_0^{(+)}$. Then exceptional curves of f_v, $v \in V - \{0\}$, give Fano's configuration on $\mathbb{P}^2$ which exists only in characteristic 2. We get a contradiction.

Case 10o. This is similar to the previous case.

Case 20e. Then $\widetilde{T} = 2A_3 \oplus 2A_1$ and T is its primitive root subsystem of the rank ≤ 7. It is easy to see that the same root subsystem can be obtained as a primitive root subsystem of $D_6 \oplus 2A_1$ or $D_5 \oplus A_3$ (and it is not then extremal because $D_6 \oplus 2A_1$ and $D_5 \oplus A_3$ correspond to K3) in all cases except $T = A_3 \oplus 4A_1$.

Let us consider the last case $T = A_3 \oplus 4A_1$ and show (as in Section 3.4.5) that it does not correspond to K3. As in Section 3.4.4, one can calculate Dynkin diagram $\Gamma = \Gamma(P(\mathcal{M}_+^{(2,4)}))$. See Appendix, Section A.4.6, Case $4A_1 \oplus A_3 \subset D_8$. It has exactly one transparent double vertex α and eight simple transparent vertices c_v, $v \in V(K)$, where $V(K)$ is the set of vertices of a 3-dimensional cube K with distinguished two opposite 2-dimensional faces β, $\beta' \in \gamma(K)$ where $\gamma(K)$ is the set of all 2-dimensional faces of K. Black vertices of Γ are f_γ, $\gamma \in \gamma(K)$, and one more black vertex f_0. Simple transparent vertices of Γ which are connected by a simple edge with α are either $b_{\overline{\gamma}}$, $\overline{\gamma} \in \overline{\gamma(K)}$, where $\overline{\gamma(K)}$ is the set of pairs of opposite 2-dimensional faces of K, or b_t, $t \in \overline{V(K)}$. Here $\overline{V(K)}$ consists of two elements corresponding to a choice of one vertex from each pair of opposite vertices of K in such a way that neither of three of them are contained in a 2-dimensional face $\gamma \in \gamma(K)$ (they define a regular tetrahedron with edges which are diagonals of 2-dimensional faces of K).

Let us describe edges of Γ different from above. Thick edges connect c_v corresponding to opposite vertices $v \in V(K)$, vertices b_{t_1} and b_{t_2} where $\{t_1, t_2\} = \overline{V(K)}$, vertices b_t and c_v where $v \in t$. Simple edges connect f_0 with f_β and $f_{\beta'}$. Double simple edges connect c_v with f_γ, if $v \in \gamma$, and $b_{\overline{\gamma}}$ with f_γ, if $\gamma \in \overline{\gamma} - \{\beta, \beta'\}$, and the vertex $b_{\overline{\beta}}$ with f_0.

Assume that Γ corresponds to a K3 pair (X, θ). On its DPN surface, let us contract exceptional curves corresponding to c_v, $v \in t$; $b_{\overline{\gamma}}$, $\overline{\gamma} \in \overline{\gamma(K)}$; f_0 and $b_{t'}$, $t' \neq t$ (here $t \in \overline{V(K)}$ is fixed). Then curves corresponding to f_v, $v \in V(K)$, and the vertex α define Fano's configuration of lines in $\mathbb{P}^2$ which can exist only in characteristic 2.

Cases 20f,g. In these cases, $\widetilde{T} = D_4 \oplus 4A_1$ or $\widetilde{T} = 8A_1$. As for analogous cases 10n,o, everything is reduced to prove that $T = 7A_1$ does not correspond to a K3 pair (X, θ).

In this case, $\Gamma = \Gamma(P(\mathcal{M}_+^{(2,4)}))$ is as follows. See Appendix, Section A.4.6, Case $7A_1 \subset D_8$. Let $I = \{1, 2, 3, 4\}$ and $J = \{1, 2\}$. The Γ has: exactly one double transparent vertex α; black vertices f_{ij}, $i \in I$, $j \in J$, and $(i,j) \neq (4,2)$; simple transparent vertices b_i, $i = 1, 2, 3$, and $b_{4(+)}, b_{4(-)}$ which are connected by a simple edge with α; simple transparent vertices $c_{j_1 j_2 j_3 j_4}$ where $j_1, j_2, j_3 \in J$, $j_4 \in \{1, -2, +2\}$ and $j_1 + j_2 + j_3 + j_4 \equiv 0 \mod 2$ which are disjoint to α.

Edges of Γ which are different from above, are as follows.

Double edges connect b_i with f_{ij}, if $i = 1, 2, 3$, and $b_{4(+)}, b_{4(-)}$ with f_{41}, and $c_{j_1 j_2 j_3 j_4}$ with f_{1j_1}, f_{2j_2}, f_{3j_3}, and $c_{j_1 j_2 j_3 1}$ with f_{41}.

Thick edges connect $b_{4(\pm)}$ with $c_{j_1 j_2 j_3 (\mp 2)}$, and $c_{j_1 j_2 j_3 j_4}$ with $c_{j'_1 j'_2 j'_3 j'_4}$, if $j_1 \neq j'_1$, $j_2 \neq j'_2$, $j_3 \neq j'_3$, $|j_4| \neq |j'_4|$, and $c_{j_1 j_2 j_3 (+2)}$ with $c_{j'_1 j'_2 j'_3 (-2)}$, if $(j_1, j_2, j_3) \neq (j'_1, j'_2, j'_3)$.

Assume that Γ corresponds to a K3 pair (X, θ). On its DPN surface, let us contract exceptional curves corresponding to b_1, b_2, b_3, $b_{4(+)}$, f_{11}, f_{21}, f_{31}, f_{41}, $c_{222(+2)}$. The curve corresponding to α gives a conic in $\mathbb{P}^2$. Curves corresponding to f_{12}, f_{22}, f_{32} give lines touching to the conic and having a common point. This is possible in characteristic 2 only.

This finishes the proof of Theorem 3.6 □

3.5. Classification of DPN surfaces of elliptic type

Each non-symplectic involution of elliptic type (X, θ) of K3 gives rise to the right DPN pair (Y, C) where

$$Y = X/\{1, \theta\},\ C = \pi(X^\theta) \in |-2K_Y|, \tag{73}$$

$\pi : X \to Y$ the quotient morphism; and vice versa. From Theorems 3.6, 3.7, we then get classification of right DPN pairs (Y, C) and DPN surfaces Y of elliptic type. See Chapter 2 and especially Sections 2.1 and 2.8. It is obtained by the reformulation of Theorems 3.6 and 3.7 and by redrawing of the diagrams. But, for readers' convenience, we do it below.

Theorem 3.18 (Classification Theorem for right DPN surfaces of elliptic type in the extremal case). *A right DPN surface Y of elliptic type is* **extremal** *if and only if the number of its exceptional curves with the square (-2) is maximal for the fixed main invariants (r, a, δ) (equivalently, (k, g, δ)). (It is equal to the number of black vertices in the diagram Γ of Table 3 below.)*

Moreover, the dual diagram $\Gamma(Y)$ of all exceptional curves on extremal Y is isomorphic to one of diagrams Γ given in Table 3. Vice versa any diagram Γ of Table 3 corresponds to some of the Y (the Y can be even taken standard).

In the diagrams Γ, simple transparent vertices correspond to curves of the 1st kind (i. e. to non-singular rational irreducible curves with the *square* (-1)*), double transparent vertices correspond to non-singular rational irreducible curves with the square* (-4)*, black vertices correspond to non-singular rational irreducible curves with the square* (-2)*, a* m*-multiple edge (or an edge with the weight* m *when* m *is large) means the intersection index* m *for the corresponding curves. Any exceptional curve on* Y *is one of these curves.*

For a not necessarily extremal right DPN surface Y of elliptic type the dual diagram $\Gamma(Y)$ of all exceptional curves on Y also consists of simple transparent, double transparent and black vertices which have exactly the same meaning as in Theorem 3.18 above. All black vertices of $\Gamma(Y)$ define the *Du Val part* $\operatorname{Duv}\Gamma(Y)$ of $\Gamma(Y)$. All double transparent vertices of $\Gamma(Y)$, and all simple transparent vertices of $\Gamma(Y)$ which are connected by two edges with double transparent vertices of $\Gamma(Y)$ (there are always two of these double transparent vertices) define the *logarithmic part* $\operatorname{Log}\Gamma(Y)$ of $\Gamma(Y)$. The rest of vertices (different from vertices of $\operatorname{Duv}\Gamma(Y)$ and $\operatorname{Log}\Gamma(Y)$) define the *varying part* $\operatorname{Var}\Gamma(Y)$ of $\Gamma(Y)$. In Theorem below we identify vertices of $\Gamma(Y)$ with elements of Picard lattice $\operatorname{Pic} Y$, then weights of edges are equal to the corresponding intersection pairing in this lattice which makes sense to the descriptions of the graphs $\operatorname{Var}\Gamma(Y)$ and $\Gamma(Y)$.

Theorem 3.19 (Classification Theorem for right DPN surfaces in the non-extremal, i. e. arbitrary, case of elliptic type). *Dual diagrams* $\Gamma(Y)$ *of all exceptional curves of not necessarily extremal right DPN surfaces* Y *of elliptic type are described by arbitrary (without any restrictions) subdiagrams* $D \subset \operatorname{Duv}\Gamma$ *of extremal DPN surfaces described in Theorem 3.18 above with the same main invariants* (r, a, δ) *(equivalently* (k, g, δ)*).*

Moreover, $\operatorname{Duv}\Gamma(Y) = D$*,* $\operatorname{Log}\Gamma(Y) = \operatorname{Log}\Gamma$*, and these subdiagrams are disjoint to each other;*

$$\operatorname{Var}\Gamma(Y) = \{f \in W(\operatorname{Var}\Gamma) \mid f \cdot D \geq 0\}$$

where W *is the subgroup of automorphisms of the Picard lattice of the extremal DPN surface (the Picard lattice is defined by the diagram* Γ*), generated by reflections in elements with square* -2 *corresponding to all vertices of* $\operatorname{Duv}\Gamma$.

Two such subdiagrams $D \subset \operatorname{Duv}\Gamma$ *and* $D' \subset \operatorname{Duv}\Gamma'$ *(with the same main invariants) give DPN surfaces* Y *and* Y' *with isomorphic diagrams* $\Gamma(Y) \cong \Gamma(Y')$*, if and only if they have isomorphic root invariants* $([D], \xi^+)$ *and* $([D'], (\xi')^+)$ *(see Theorem 3.7).*

To calculate the root invariant $([D], \xi^+)$ *of a DPN surface, one has to go back from the graph* Γ *of Table 3 to the corresponding graph of Tables 1 or 2.*

From our point of view, classification above by graphs of exceptional curves is the best classification of DPN surfaces Y. It shows a sequence (actually all sequences) of -1 curves which should be contracted to get the corresponding relatively minimal rational surface $\overline{Y}$ isomorphic to $\mathbb{P}^2$ or $\mathbb{F}_n$, $n \leq 4$ (see Section 3.6 and Table 4 below). Images of exceptional curves on Y which are not contracted then give some configuration of rational curves on $\overline{Y}$ which should exist to get the DPN surface Y back from $\overline{Y}$ by the corresponding sequence of blow ups. Here the following inverse statement is very important because it shows that any surface Y' obtained by a "similar" sequence of blow ups of $\overline{Y}$ which are related with a "similar" configuration of rational curves on $\overline{Y}$ will be also a DPN surface with the graph $\Gamma(Y')$ of exceptional curves which is isomorphic to $\Gamma(Y)$. Here is the exact statement.

Theorem 3.20. *Let Y be a right DPN surface of elliptic type, and the set of exceptional curves on Y is not empty (i. e. Y is different from $\mathbb{P}^2$ and $\mathbb{F}_0$). Let Y' be a non-singular rational surface such that*

1) the Picard number of Y' is equal to the Picard number of Y.

2) there exists a set $E_1, \dots, E_m$ of non-singular irreducible rational exceptional curves on Y' such that their dual graph is isomorphic to the dual graph $\Gamma(Y)$ of exceptional curves on Y.

Then Y' is also a DPN surface and $E_1, \dots, E_m$ are all exceptional curves on Y' (of course, then $\Gamma(Y') \cong \Gamma(Y)$).

Proof. Let r be the Picard number of Y and Y'. If $r = 2$, then obviously $Y \cong Y' \cong \mathbb{F}_n$, $n > 0$. Further we assume that $r \geq 3$. We denote by S_Y and $S_{Y'}$ the Picard lattices of Y and Y' respectively. Like for K3 surfaces we shall consider the light cones $V(Y) \subset S_Y \otimes \mathbb{R}$, $V(Y') \subset S_Y' \otimes \mathbb{R}$ (of elements with positive square) and their halves $V^+(Y)$ and $V^+(Y')$ containing polarizations.

Let $D_1, \dots, D_m$ are all exceptional curves on Y (corresponding to vertices of $\Gamma(Y)$). Their number is finite and they generate S_Y since $r \geq 3$. We claim that Kleiman–Mori cone $\overline{\mathrm{NE}}(Y) = \mathbb{R}^+D_1 + \dots + \mathbb{R}^+D_m$ is generated by $D_1, \dots D_m$. This is equivalent to

$$\overline{V^+(Y)} \subset \mathbb{R}^+D_1 + \dots + \mathbb{R}^+D_m \tag{74}$$

since D_j are all exceptional curves on Y and $V^+(Y) \subset \overline{\mathrm{NE}}(Y)$ by Riemann-Roch Theorem on Y. The condition (74) is equivalent to the embedding of dual cones

$$(\mathbb{R}^+D_1 + \dots + \mathbb{R}^+D_m)^* \subset \overline{V^+(Y)} \tag{75}$$

because the light cone $V^+(Y)$ is self-dual. By considering the corresponding K3 double cover $\pi : X \to Y$, the embedding (75) is equivalent to the embedding

$$(\mathbb{R}^+\pi^*(D_1) + \cdots + \mathbb{R}^+\pi^*(D_m))^* \subset \overline{V^+(S)} \tag{76}$$

which is equivalent to finiteness of volume of $\mathcal{M}(X)_+ \subset \mathcal{L}(S)$ which we know.

The equivalent conditions (74) and (75) are numerical. Thus, similar conditions

$$\overline{V^+(Y')} \subset \mathbb{R}^+E_1 + \cdots + \mathbb{R}^+E_m \tag{77}$$

and

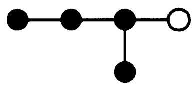
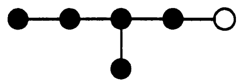
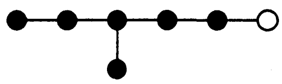
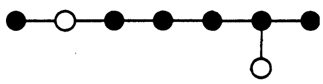
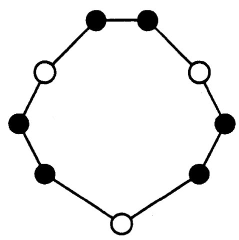
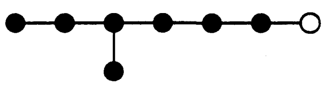
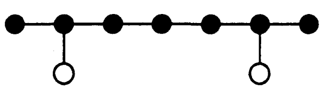
$$(\mathbb{R}^+E_1 + \cdots + \mathbb{R}^+E_m)^* \subset \overline{V^+(Y')} \tag{78}$$

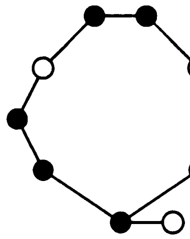
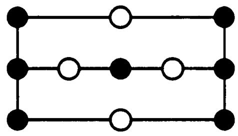
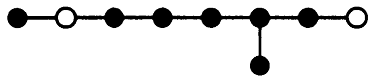
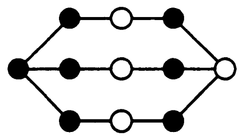
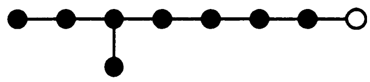
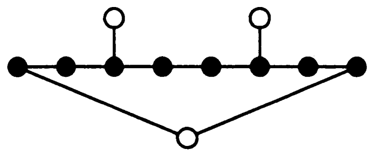
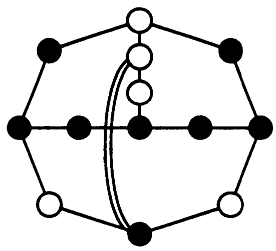
are valid for Y'. This shows that $E_1, \dots E_m$ are the only exceptional curves on Y'. Indeed, if E is any other irreducible curve E on Y' satisfying $E \cdot E_i \geq 0$, then $E^2 \geq 0$ by (78) and the curve E is not exceptional. Thus, $\Gamma(Y')$ and $\Gamma(Y)$ are isomorphic. In the same way as for Y above, we then get from (77) or (78) that the Kleiman–Mori cone $\overline{\mathrm{NE}}(Y') = \mathbb{R}^+E_1 + \cdots + \mathbb{R}^+E_m$ is generated by $E_1, \dots, E_m$.

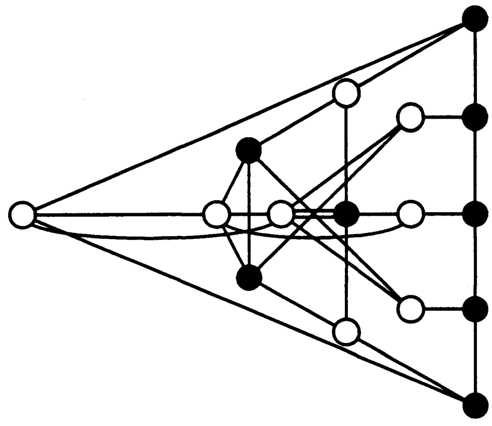
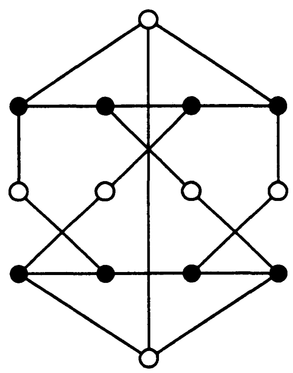
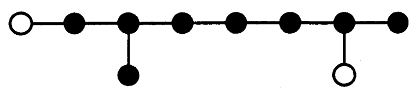
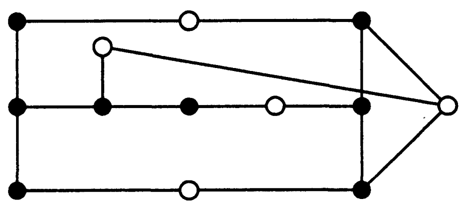
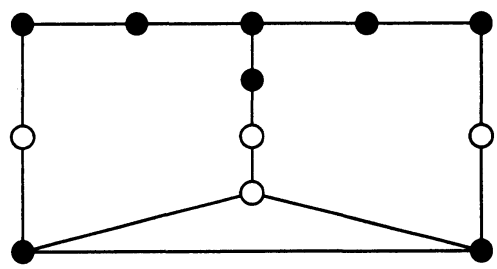
Let us show that Y' is a DPN surface. Definitions of Du Val, logarithmic parts of $\Gamma(Y)$ were purely numerical. Since $\Gamma(Y')$ and $\Gamma(Y)$ are isomorphic, we can use similar notions for Y'.

In Section 4.1 we shall prove (without using Theorem 3.20) that there exists a contraction $p : Y \to Z$ of Du Val and logarithmic parts of exceptional curves of Y which gives the right resolution of singularities of a log del Pezzo surface Z of index ≤ 2. (Remark that by Lemma 1.4 it also gives another proof of the above statements about Kleiman–Mori cone and exceptional curves on Y and Y'.) Thus, the element $p^*(-2K_Z) \in S_Y$ is defined. It equals to $-2K_Y$ minus sum of all exceptional curves on Y with square -4. Thus, similar element can be defined for Y'. Let us denote it by $R \in S_{Y'}$. In Section 1.4, we had proved (for any log del Pezzo surface Z of index ≤ 2) that the linear system $p^*(-2K_Z)$ contains a non-singular curve. The proof was purely numerical and only used the fact that $-2K_Y - \sum E_i$ is big and nef. The same proof for Y' gives that R contains a non-singular curve. It follows that Y' is a right DPN surface of elliptic type. □

TABLE 3. Dual diagrams Γ of all exceptional curves of extremal right DPN surfaces of elliptic type

N	r	a	δ	k	g	$\widetilde{r}$	Γ
1	1	1	1	0	10	1	$\Gamma = \emptyset$, $\mathbb{P}^2$
2	2	2	0	0	9	1	 $\mathbb{F}_0$ or $\mathbb{F}_2$
3	2	2	1	0	9	2	 $\mathbb{F}_1$
4	3	3	1	0	8	2	
5	4	4	1	0	7	1	
6	5	5	1	0	6	1	
7	6	6	1	0	5	1	
a							
b							
8	7	7	1	0	4	1	
a							
b							
c							
9	8	8	1	0	3	1	
a							
b							

N	r	a	δ	k	g	$\widetilde{r}$	Γ
9	8	8	1	0	3	1	
c							
d							
e							
f							
10	9	9	1	0	2	1	
a							
b							
c							

N	r	a	δ	k	g	$\widetilde{r}$	Γ
10	9	9	1	0	2	1	
d							
e							
f							
g							
h							

N	r	a	δ	k	g	$\widetilde{r}$	Γ
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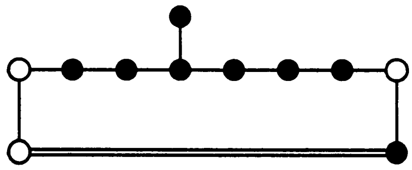
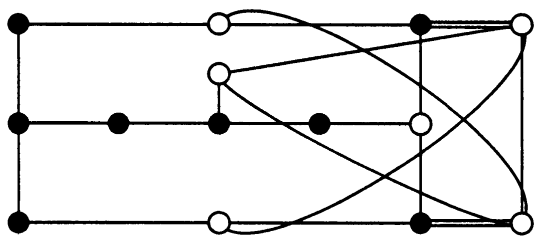
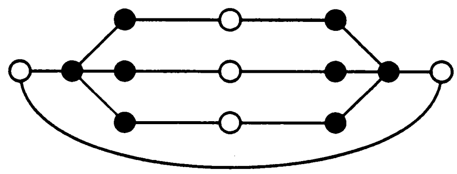
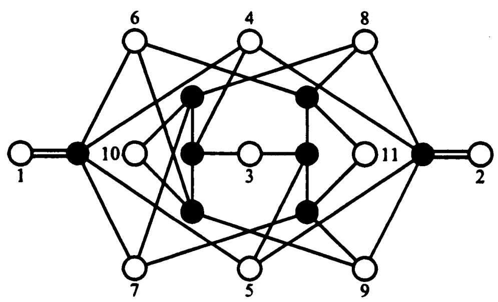
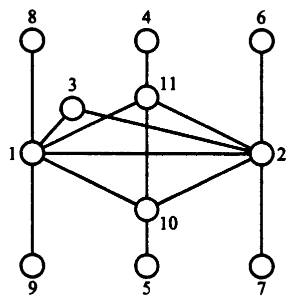
i							
j							
k							
l							 

N	r	a	δ	k	g	$\widetilde{r}$	Γ
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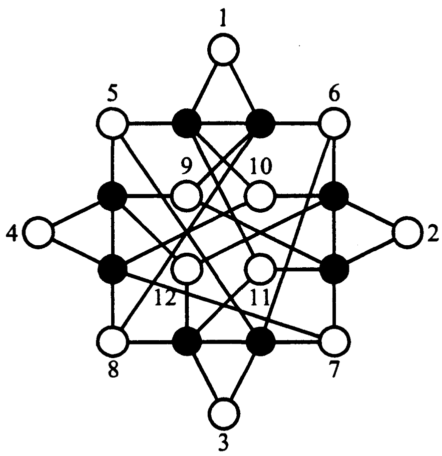
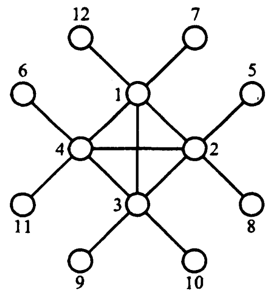
m							 
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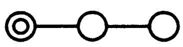
12	3	1	1	1	9	2	
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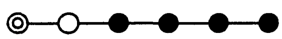
14	5	3	1	1	7	2	
15	6	4	0	1	6	1	
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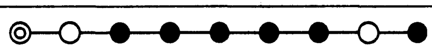
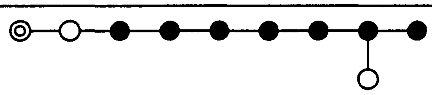
17	7	5	1	1	5	2	
18	8	6	1	1	4	1	
19	9	7	1	1	3	1	

N	r	a	δ	k	g	$\widetilde{r}$	Γ
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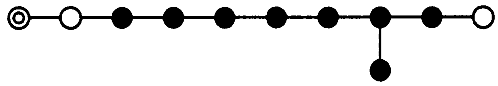
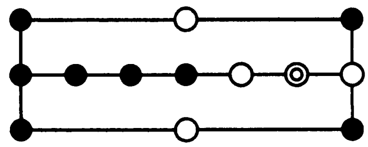
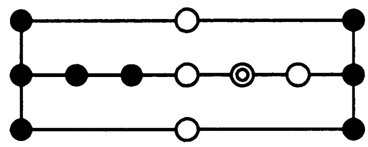
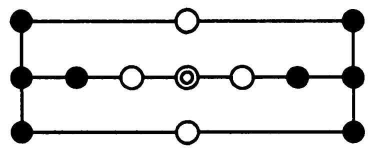
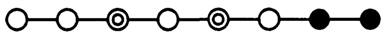
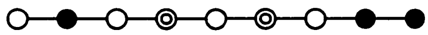
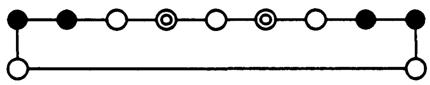
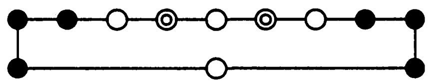
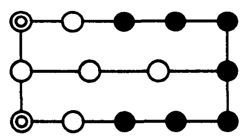
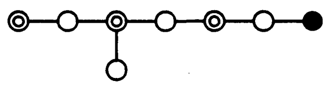
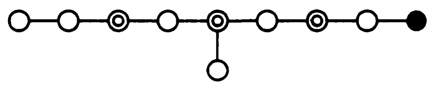
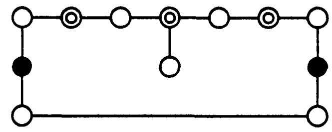
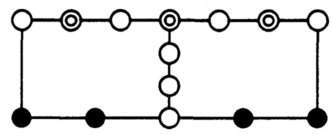
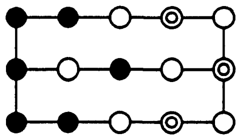
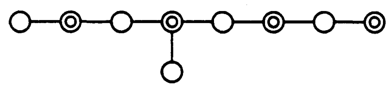
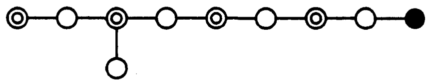
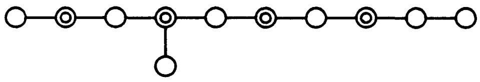
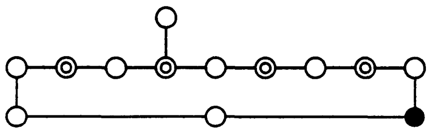
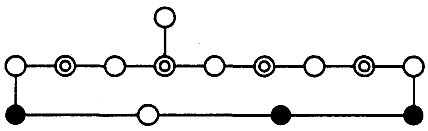
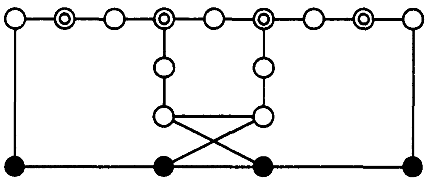
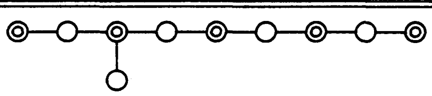
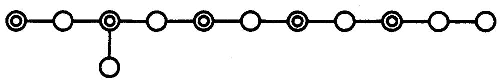
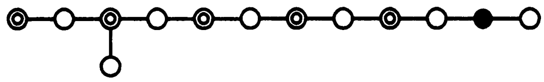
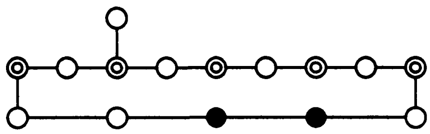
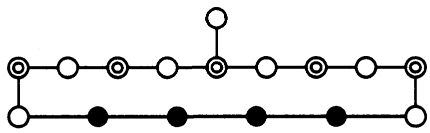
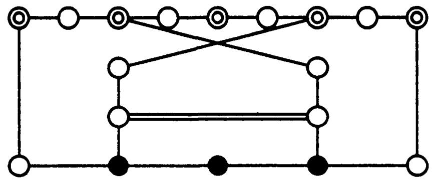
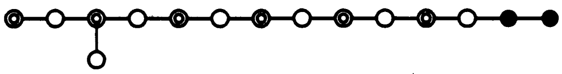
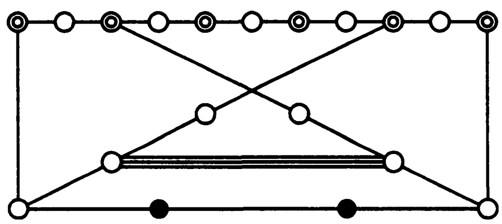
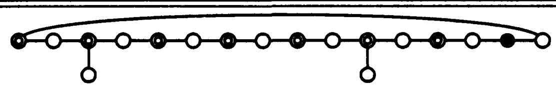
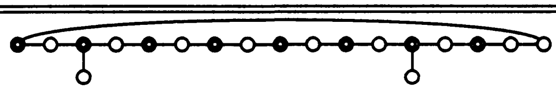
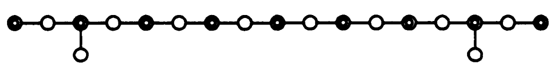
a							
b							
c							
d							
21	6	2	0	2	7	1	
22	7	3	1	2	6	2	
23	8	4	1	2	5	2	
24	9	5	1	2	4	2	
25	10	6	0	2	3	1	
26	10	6	1	2	3	1	
27	11	7	1	2	2	1	
28	8	2	1	3	6	2	
29	9	3	1	3	5	3	
30	10	4	0	3	4	1	

N	r	a	δ	k	g	$\widetilde{r}$	Γ
31	10	4	1	3	4	3	
32	11	5	1	3	3	2	
33	12	6	1	3	2	1	
34	9	1	1	4	6	2	
35	10	2	0	4	5	2	
36	10	2	1	4	5	3	
37	11	3	1	4	4	3	
38	12	4	1	4	3	2	
39	13	5	1	4	2	2	
40	10	0	0	5	6	1	
41	11	1	1	5	5	2	

N	r	a	δ	k	g	$\widetilde{r}$	Γ
42	12	2	1	5	4	2	
43	13	3	1	5	3	2	
44	14	4	0	5	2	1	
45	14	4	1	5	2	2	
46	14	2	0	6	3	1	
47	15	3	1	6	2	2	
48	16	2	1	7	2	2	
49	17	1	1	8	2	2	
50	18	0	0	9	2	1	

3.6. Application: On classification of plane sextics with simple singularities

Let Y be a right DPN surface of elliptic type which were classified in Theorems 3.18, 3.19 and 3.20. Let $\Gamma(Y)$ be the dual diagram of all exceptional curves on Y. By definition of right DPN surfaces, there exists a non-singular curve

$$C = C_g + E_{a_1} + \cdots + E_{a_k} \in |-2K_Y| \tag{79}$$

where $E_{a_1}, \dots, E_{a_k}$ are exceptional curves with square (-4) corresponding to all double transparent vertices $a_1, \dots, a_k$ of $\Gamma(Y)$ and $g > 1$ the genus of the irreducible non-singular curve C_g. Here (k, g, δ) (equivalent to (r, a, δ)) are the main invariants of Y.

We denote by E_v the exceptional curve on Y corresponding to a vertex $v \in V(\Gamma(Y))$. If v is black, then $C \cdot E_v = C_g \cdot E_v = 0$. If v is simple transparent, then $C \cdot E_v = 2$.

If v is simple transparent and v is not connected by any edge with double transparent vertices of $\Gamma(Y)$ (i. e. $E_v \cdot E_{a_i} = 0, i = 1, \dots, k$) then $C_g \cdot E_v = 2$. This intersection index can be obtained in two ways:

(80) C_g intersects E_v transversally in two points;

(81) C_g simply touches E_v in one point.

(For example, in Case 47 of Table 3 we have two such vertices v.)

Up to this ambiguity, we know (from the diagram $\Gamma(Y)$) how components of C intersect exceptional curves. Which of possibilities (80) or (81) does take place is defined by the generalized root invariant which we don't consider in this work.

Let $t_1, \dots, t_{r-1} \in V(\Gamma(Y))$ be a sequence of vertices such that the contraction of exceptional curves $E_{t_1}, \dots, E_{t_{r-1}}$ gives a morphism $\sigma : Y \to \mathbb{P}^2$ which is a sequence of contractions of curves of the 1st kind. By Section 2.1, the image $D = \sigma(C) \subset \mathbb{P}^2$ is then a sextic (it belongs to $|-2K_{\mathbb{P}^2}|$) with simple singularities. What components and what singularities the curve D does have is defined by the subgraph $\Gamma(t_1, \dots, t_{r-1})$ generated by vertices $t_1, \dots, t_{r-1}$ in $\Gamma(Y)$. We formalize that below.

Let

$$\widetilde{D} = C_g + \sum_{v_i \in \{a_1, \dots, a_k\} - \{t_1, \dots, t_{r-1}\}} E_{v_i}$$

be the curve of components of C which are not contracted by σ. Then $\sigma : \widetilde{D} \to D$ is the normalization of D. In pictures, we denote $\widetilde{D}$ (or D) by the symbol $\otimes$ and evidently denote the intersection of this curve and its local branches at the corresponding singular point with the components E_{t_j} which are contracted to this point. For connected components of $\Gamma(t_1, \dots, t_{r-1})$ we then have possibilities presented in Table 4 below depending on types of the corresponding singular points of D.

By Table 4, the ambiguity (80) or (81) takes place only for singularities of the types A_{2k-1} or A_{2k}. Thus, we have to introduce the notation $\mathfrak{A}_{2k-1}$ for the singularity of the type A_{2k-1} or A_{2k} of the component $\sigma(C_g)$ of D of the geometric genus $g > 1$.

In the right column of Table 4, we denote by $\mathcal{A}_n$, $\mathcal{D}_n$, $\mathcal{E}_n$ connected components of graphs $\Gamma(t_1, \dots, t_{r-1})$ corresponding to singularities A_n, D_n and

E_n of the curve D respectively. Obviously, finding of all possible contractions $\sigma : Y \to \mathbb{P}^2$ reduces to enumeration of all subgraphs $\Gamma \subset \Gamma(Y)$ with the connected components $\mathcal{A}_n$, $\mathcal{D}_n$, $\mathcal{E}_n$ and with the common number $r-1$ of vertices. A choice of such a subgraph $\Gamma \subset \Gamma(Y)$ defines the sextic D with the corresponding irreducible components and simple singularities, and the related configuration of rational curves

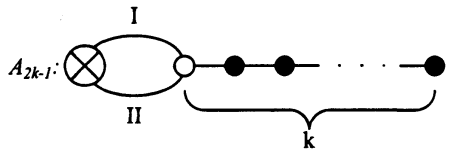
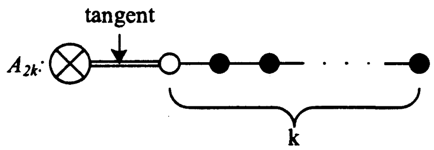
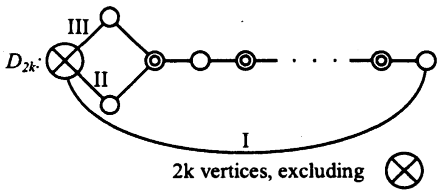
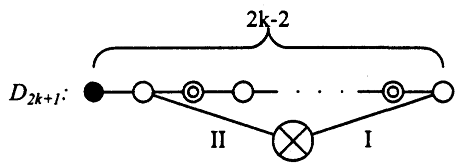
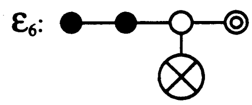
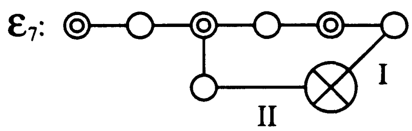
$$\sigma(E_v),\ v \in V(\Gamma(Y)) - (\{a_1, \dots, a_k\} \cup \{t_1, \dots, t_{r-1}\}), \tag{82}$$

which one can call the **exceptional curves of a sextic** D with simple singularities.

Thus, the classification in Theorems 3.18 and 3.19 of DPN surfaces of elliptic type implies a quite delicate classification of sextics D having an irreducible component of the geometric genus $g \geq 2$. For this classification, we correspond to a sextic $D \subset \mathbb{P}^2$ a subgraph $\Gamma \subset \Gamma(Y)$ up to isomorphisms of graphs $\Gamma(Y)$ which send the subgraphs Γ to one another. The analogous classification can be repeated to classify curves with simple singularities in $|-2K_{\mathbb{F}_n}|$, $n = 0, \dots, 4$. One should only replace $r-1$ by $r-2$. We also note that a choice of different subgraphs $\Gamma \subset \Gamma(Y)$ for the same curve C defines birational transformations of the corresponding rational surfaces ($\mathbb{P}^2$ or $\mathbb{F}_n$) which transform the curves D to one another. Thus, the graph $\Gamma(Y)$ itself classifies the corresponding curves D up to some their birational equivalence.

A complete enumeration of all cases has no principal difficulties, and it is only related to a long enumeration using Theorem 3.19 of all possible diagrams $\Gamma(Y)$ and their subdiagrams Γ. Unfortunately, it seems, number of cases is enormous. But the complete enumeration can be important in some problems of real algebraic geometry and singularity theory. For example, it could be important for classification of irreducible quartics in $\mathbb{P}^3$ with double rational singularities by the method of projection from a singular point. To remove the ambiguity (80) or (81), one has to perform similar (to ours) classification of generalized root invariants.

TABLE 4. Correspondence between connected components of Γ and singularities of $D = \sigma(\widetilde{D})$.

Type of Singular point of D	Equations of the Singularity and its Branches	Connected Components of Γ, and Curve $\widetilde{D}$ (denoted by $\otimes$)
$A_{2k-1} = \mathfrak{A}_{2k-1}$	$y^2 - x^{2k} = 0$ I: $y - x^k = 0$ II: $y + x^k = 0$	
$A_{2k} = \mathfrak{A}_{2k-1}$	$y^2 - x^{2k+1} = 0$	
D_{2k}	$xy^2 - x^{2k-1} = 0$ I: $x = 0$ II: $y - x^{k-1} = 0$ III: $y + x^{k-1} = 0$	
D_{2k+1}	$xy^2 - x^{2k} = 0$ I: $x = 0$ II: $y^2 - x^{2k-1} = 0$	
E_6	$y^3 - x^4 = 0$	
E_7	$y^3 - yx^3 = 0$ I: $y = 0$ II: $y^2 - x^3 = 0$	
E_8	$y^3 - x^5 = 0$