APPENDIX A

Rapid Course in p-adic Analysis.

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1. Introduction.

In this appendix, K always denotes a complete field with respect to a non-archimedean valuation $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$. The norm $|\cdot|$ is almost always assumed to be non-trivial, unless otherwise stated.

Let us assume for a while that K is algebraically closed. We view the field K as an affine space, as we do in complex analysis; K is a metrized space with the metric induced from the valuation $|\cdot|$. We can then follow the lines of classical complex analysis and define convergence of power series, Taylor and Laurent expansions, etc...

For instance let us define "holomorphic" functions to be K-valued functions which are locally expressed by convergent power series[(1)] (cf. [**Gou97**, Chap. 4]). This approach drives us however to several problems, which come mainly from the fact that K with the metric topology is totally disconnected; for example:

(i) for an open set $U \subseteq K$ the ring of all such holomorphic functions on U is huge. So is already the subring of all locally constant functions.

(ii) It can be shown that the sheaf of germs of such functions satisfies the principle of unique continuation; but not in a satisfactory way. In particular, Lemma 1.1.2 of Chapter I holds but is of no use, because there is no non-empty connected open subset.

The trouble becomes more apparent when we think the *local* representability by convergent power series in terms of *coverings*. In the complex analytic situation, when we speak of the holomorphy of a function f defined over a connected open set U, we tacitly take a open covering $\{U_i\}_{i\in I}$ of U consisting of sufficiently small open neighborhoods such that f restricted on each member U_i can be seen as evaluation of a power series convergent in U_i. The point is that, since U is connected, members of $\{U_i\}$ must have overlaps so that the local properties can be transmitted to whole U. But, in the non-archimedean situation, such coverings may be refined in such a way that there are no overlap. This is why the analytic continuation does

[(1)] The reader may wonder whether there is another approach by means of Cauchy-Riemann type differential equations. This is unlikely because the differential calculus does not go well, see [**Gou97**, Chap. 4].

not work well. So, for the sake of a reasonable analytic theory, we have to "limit" coverings so that we cannot take arbitrarily fine refinements.

A first method is due to Krasner. He introduced the so-called *quasi-connected sets* and, on such subsets, defined analytic functions as uniform convergent limits of rational functions.

A more modern and systematic treatment was introduced by Tate, with the so-called *Rigid Analysis*. Let us briefly view the main idea: let K be as above, not necessarily algebraically closed. We first set

$$K\{t_1,\ldots,t_n\} = \left\{ \sum_{\nu_1,\ldots,\nu_n\geq 0} a_{\nu_1,\ldots,\nu_n} t_1^{\nu_1}\cdots t_n^{\nu_n} \in K[[t_1,\ldots,t_n]] \;\middle|\; \begin{array}{c} a_{\nu_1,\ldots,\nu_n}\to 0 \text{ for} \\ \nu_1+\cdots+\nu_n\to\infty \end{array} \right\},$$

which is now called the *Tate algebra* over K. This is the ring of all functions expressed by power series convergent on the closed disk $\mathrm{D}(0,1^+) = \{z \in K^n \mid |z_i| \leq 1\}$. The algebra $K\{t_1,\ldots,t_n\}$ is endowed with the sup-norm $\|\cdot\|$, also called the *Gauss norm*. We list up some known properties (cf. [**BGR84**, 5.2], [**FvdP81**, II.3]):

(i) The ring $K\{t_1,\ldots,t_n\}$ satisfies the Weierstrass Preparation Theorem.
(ii) The ring $K\{t_1,\ldots,t_n\}$ is Noetherian and factorial.
(iii) Every ideal of $K\{t_1,\ldots,t_n\}$ is closed.
(iv) For any maximal ideal $\mathfrak{m}$ of $K\{t_1,\ldots,t_n\}$, the field $K\{t_1,\ldots,t_n\}/\mathfrak{m}$ is a finite extension of K.

If, moreover, K is algebraically closed, the set of all maximal ideals, endowed with the natural topology, coincides with $\mathrm{D}(0,1^+)$. So the situation is analogous to classical algebraic geometry; $\mathrm{D}(0,1^+)$ to the affine n-space, and $K\{t_1,\ldots,t_n\}$ to the coordinate ring. In fact, quotients $\mathsf{A} = K\{t_1,\ldots,t_n\}/I$ by ideals I (called *affinoid algebras*), and their associated maximal spectra, $\mathrm{Spm}(\mathsf{A})$, called *affinoids*, forms the fundamental patches from which the non-archimedean function theory will be developed. Although the natural metric topology on $\mathrm{Spm}(\mathsf{A})$ is terrible for the same reason as above, the situation becomes much better as far as we deal only with those functions coming from A. The trick is that, if the algebra A is integral, then we can pretend that the space $\mathrm{Spm}(\mathsf{A})$ is "connected"; for example, the closed disk $\mathrm{D}(0,1^+)$ is "connected" in this sense. So, basically according to this point of view, we can globalize the situation by gluing affinoid patches to obtain a reasonable theory of analytic functions. The actual recipe to do it is furnished by the notion of Grothendieck topology, which censors the plethora of coverings. We will see this a little more precisely in what follows.

Remark 1.1. Although the rigid analysis seems to provide the reasonable topological and analytical framework, one must not expect it to be as nice as the complex case. Firstly, in classical complex analysis, when we expand a holomorphic function centered at a point inside its region of convergence, the resulting Taylor expansion may have a different region of convergence than the original one. This is a important method of analytic continuation. But in the non-archimedean case, this never happens (cf. [**Gou97**, 4.4]).

Secondly, the topological framework given as above is still not powerful enough to handle paths. So it is hopeless to treat, for example, fundamental groups and monodromy in an intuitive way as in the classical complex case. This difficulty is, in fact, remedied by Berkovich' theory which we shall see later on.

2. Rigid analytic spaces.

Let K be as in the previous section; we do not assume in general that K is algebraically closed.

As mentioned above, an *affinoid algebra* A over K is a quotient $K\{t_1,\ldots,t_n\}/I$ for some n by an ideal I. Since I is closed, A has a norm $|\cdot|$ induced from the sup-norm $\|\cdot\|$ of $K\{t_1,\ldots,t_n\}$ (*"residue norm"*). With the residue norm, A is a Banach K-algebra. The norm $|\cdot|$ itself depends on the presentation $\alpha\colon K\{t_1,\ldots,t_n\}/I \xrightarrow{\sim} \mathsf{A}$, while the induced topology does not. So, strictly speaking, we should write it like $|\cdot|_\alpha$. Clearly, A is Noetherian and, for any maximal ideal $\mathfrak{m}$ of A, the residue field $\mathsf{A}/\mathfrak{m}$ is a finite extension of K, and hence, the valuation $|\cdot|$ of K naturally extends to that of it, denoted again by $|\cdot|$.

The associated *affinoid* $\mathrm{Spm}(\mathsf{A})$ is the set of all maximal ideals of A. For $x \in \mathrm{Spm}(\mathsf{A})$ and $f \in \mathsf{A}$ the value of f at x, denoted by $f(x)$, is the class of f in A/x. The set $\mathrm{Spm}(\mathsf{A})$ has the topology generated by the subsets of form $\{x \in \mathrm{Spm}(\mathsf{A}) \mid |f(x)| \leq 1\}$ for $f \in \mathsf{A}$. But, as we pointed out above, this topology is not very interesting since it makes $\mathrm{Spm}(\mathsf{A})$ totally disconnected. So we should specify the reasonable family of "admissible" open sets (and coverings), on which the function theory will be built.

Let A be an affinoid algebra over K, and f_i $(i = 0,\ldots,n)$ a collection of elements in A which have no common zeros on $\mathrm{Spm}(\mathsf{A})$. The subspace

$$R = \{x \in \mathrm{Spm}(\mathsf{A}) \mid |f_i(x)| \leq |f_0(x)|,\ i = 1,\ldots,n\}$$

can be identified with the affinoid $\mathrm{Spm}(\mathsf{A}_R)$, where

$$\mathsf{A}_R = \mathsf{A}\widehat{\otimes}_K K\{t_1,\ldots,t_n\}/(f_1 - t_1 f_0,\ldots,f_n - t_n f_0).$$

A subset of this form is called a *rational subdomain.* The identification comes as follows: We first note that the morphism of affinoids $\mathrm{Spm}(\mathsf{A}_R) \to \mathrm{Spm}(\mathsf{A})$ induced by $\mathsf{A} \to \mathsf{A}_R$ maps $\mathrm{Spm}(\mathsf{A}_R)$ to R. Then A_R is the unique solution of the following universal property (hence, it can be determined up to canonical isomorphism): for any morphism of affinoids $\phi\colon \mathrm{Spm}(\mathsf{B}) \to \mathrm{Spm}(\mathsf{A})$ such that $\phi(\mathrm{Spm}(\mathsf{B})) \subset R$, there exists a unique K-homomorphism $\mathsf{A}_R \to \mathsf{B}$ such that the resulting diagram

$$\begin{tikzcd} \mathrm{Spm}(B) \arrow[r, "\phi"] \arrow[dr, dotted] & \mathrm{Spm}(A) \\ & \mathrm{Spm}(A_R) \arrow[u] \end{tikzcd}$$

is commutative.

Lemma 2.1. (1) If R and S are rational subdomains in Spm(A), then so is $R \cap S$.

(2) If R_1 is a rational subdomain in Spm(A), and R_2 is a rational subdomain in R_1, then R_2 is a rational subdomain in Spm(A).

PROOF. (1) If $R = \{|f_i(x)| \leq |f_0(x)|,\ i = 1, \dots, n\}$ and $S = \{|g_j(x)| \leq |g_0(x)|,\ j = 1, \dots, m\}$, then one sees easily that $R \cap S = \{|f_i(x)g_j(x)| \leq |f_0(x)g_0(x)|,\ i = 1, \dots, n,\ j = 1, \dots, m\}$.

(2) Exercise. □

Corollary 2.2. Any subspace of the form

$$\{|f_i(x)| \leq 1, |g_j(x)| \geq 1,\ i = 1, \dots, n,\ j = 1, \dots, m\}$$

is a rational subdomain.

PROOF. It is the intersection of rational subdomains $\{|f_i(x)| \leq 1\}$ for $i = 1, \dots, n$ and $\{1 \leq |g_j(x)|\}$ for $j = 1, \dots, m$. □

Example 2.3 (Rational subdomains of the unit polydisk). Let us assume that K is algebraically closed. We consider rational subdomains of the unit polydisk $\mathrm{D}(0, 1^+) = \mathrm{Spm}(K\{t_1, \dots, t_n\})$.

(1) *Closed polydisk*: $\mathrm{D}(0, |\pi|^+) = \{x \in K^n \mid |x_i| \leq |\pi_i|,\ i = 1, \dots, n\}$, for $0 \neq \pi_i \in K$, $|\pi_i| \leq 1$, is a rational subdomain. The corresponding affinoid algebra is

$$K\left\{\frac{t_1}{\pi_1}, \dots, \frac{t_n}{\pi_n}\right\}.$$

(2) *Annulus*: $\mathrm{C}(0, |\pi^{(1)}|^+, |\pi^{(2)}|^+) = \{x \in K^n \mid |\pi_i^{(1)}| \leq |x_i| \leq |\pi_i^{(2)}|,\ i = 1, \dots, n\}$, for $0 \neq \pi_i^{(j)} \in K$, $|\pi_i^{(j)}| \leq 1$, is a rational subdomain. The corresponding affinoid algebra is

$$K\left\{\frac{\pi_1^{(1)}}{t_1}, \dots, \frac{\pi_n^{(1)}}{t_n}, \frac{t_1}{\pi_1^{(2)}}, \dots, \frac{t_n}{\pi_n^{(2)}}\right\}.$$

Let us next review the definition of Grothendieck topology (in a narrow sense). Let X be a topological space. A *Grothendieck topology* (G-topology, in short) on X is a pair $(\mathcal{T}, \mathfrak{Cov})$ consisting of

- a collection $\mathcal{T}$ of open subsets in X,
- an assignment $U \mapsto \mathfrak{Cov}(U)$ for any $U \in \mathcal{T}$, where $\mathfrak{Cov}(U)$ is a collection of coverings by elements in $\mathcal{T}$,

such that the following conditions are satisfied:

(1) $\emptyset \in \mathfrak{T}$; $U, V \in \mathfrak{T} \Rightarrow U \cap V \in \mathfrak{T}$.
(2) $U \in \mathfrak{T} \Rightarrow \{U\} \in \mathfrak{Cov}(U)$.
(3) $\{U_i\}_{i\in I} \in \mathfrak{Cov}(U)$, $V \subseteq U$, $V \in \mathfrak{T} \Rightarrow \{U_i \cap V\}_{i\in I} \in \mathfrak{Cov}(V)$.
(4) $\{U_i\}_{i\in I} \in \mathfrak{Cov}(U)$, $\{U_{i,j}\}_{j\in J_i} \in \mathfrak{Cov}(U_i) \Rightarrow \{U_{i,j}\}_{i\in I, j\in J} \in \mathfrak{Cov}(U)$.

Elements in $\mathfrak{T}$ are called *admissible open sets*, and elements in $\mathfrak{Cov}(U)$ are called *admissible coverings* of U.

Let $X = \mathrm{Spm}(\mathsf{A})$ be an affinoid over K. We introduce a Grothendieck topology to X by the following recipe:

- An admissible open set is a rational subdomain.
- For a rational subdomain, an admissible covering is a finite covering consisting of rational subdomains.

Due to Lemma 2.1 this indeed gives a Grothendieck topology. This topology is equivalent to the so-called weak G-topology in the standard literature (e.g. [**BGR84**, 9.1.4]) due to Gerritzen-Grauert theorem [**BGR84**, 7.3.5].

We define the structure presheaf $\mathcal{O}_X$, with respect to the Grothendieck topology, by assigning for each rational subdomain R the corresponding affinoid algebra A_R. Due to the following weak form of *Tate's acyclicity theorem*, the presheaf $\mathcal{O}_X$ is, in fact, a sheaf (a sheaf of local rings).

Theorem 2.4. Let $U_1, \ldots, U_m$ be rational subdomains of $X = \mathrm{Spm}(\mathsf{A})$. Set $U = U_1 \cup \cdots \cup U_m$. Then the sequence

$$0 \longrightarrow \mathcal{O}_X(U) \longrightarrow \prod_{i=1}^{m} \mathcal{O}_X(U_i) \longrightarrow \prod_{i,j=1}^{m} \mathcal{O}_X(U_i \cap U_j)$$

is exact, where the last arrow is the difference of the two possible restriction morphisms.

For the proof, see, for example, [**BGR84**, 8.2] or [**FvdP81**, III.2.2].

Definition 2.5 (Rigid Analytic Space). A *rigid analytic space* is a locally ringed space $(X, \mathcal{O}_X)$, with a Grothendieck topology in the above sense, locally isomorphic to an affinoid; more precisely, there exists a covering $\{X_i\}_{i\in I}$ (possibly infinite) of X by admissible open sets such that $(X_i, \mathcal{O}_X|_{X_i})$ is isomorphic to a certain affinoid for each i.

Let us see some examples of rigid analytic spaces:

Example 2.6 (Projective Space). The projective space $\mathbb{P}^n(K)$ has the natural rigid analytic structure. Here we limit ourselves to demonstrate it only in the case $n = 1$, and leave the general case to the reader. Let $(X : Y)$ be the homogeneous coordinate in $\mathbb{P}^1(K)$ and set $z = X/Y$. Set

$$\mathsf{U}^+ = \{(X : Y) \mid |X| \leq |Y|\} \text{ and } \mathsf{U}^- = \{(X : Y) \mid |X| \geq |Y|\}.$$

Then U^+ is isomorphic to the closed disk $\{z \in K \mid |z| \leq 1\}$ (the Southern Hemisphere with the equator) , and U^- to $\{z \in K \cup \{\infty\} \mid |z| \geq 1\}$ (the Northern Hemisphere with the equator). The intersection $\mathsf{U}^+ \cap \mathsf{U}^-$ is thus isomorphic to the circle $\{z \in K \mid |z| = 1\}$. Hence $\{\mathsf{U}^+, \mathsf{U}^-\}$ gives the admissible covering of $\mathbb{P}^1(K)$ which induces, in the obvious way, the structure of a rigid analytic space.

More generally, an algebraic variety, separated and of finite type over K, carries the canonical structure of a rigid analytic space.

Any open set (not necessarily admissible) U in a rigid analytic space X will be again a rigid analytic space (*"open subanalytic space"*); more precisely, U has the induced analytic structure from that of X. The admissible open sets of U, for example, are those of X contained in U.

3. Relation with Formal Geometry.

In this section we need to consider the valuation ring R of K. Let us denote by $\mathfrak{m}$ the maximal ideal of R, and fix a topological generator $0 \neq \pi \in \mathfrak{m}$. The residue field $R/\mathfrak{m}$ is denoted by k.

Let A be an affinoid algebra over K. The *spectral semi-norm* on A is the function $|\cdot|_{\sup}\colon \mathsf{A} \to \mathbb{R}_{\geq 0}$ defined, for any $f \in \mathsf{A}$, by

$$|f|_{\sup} = \sup_{x \in \mathrm{Spm}(\mathsf{A})} |f(x)|.$$

It is known that for any representation $\alpha\colon K\{t_1, \ldots, t_n\}/I \xrightarrow{\sim} \mathsf{A}$, we have $|f|_{\sup} \leq |f|_\alpha$ for every $f \in \mathsf{A}$ (cf. [**BGR84**, 6.2.1]).

Theorem 3.1 (Maximal modulus principle). For an affinoid algebra A and an element $f \in \mathsf{A}$, there exists a point $x \in \mathrm{Spm}(\mathsf{A})$ such that $|f(x)| = |f|_{\sup}$.

We refer [**BGR84**, 6.2.1] for the proof.

Given an affinoid algebra A over K, we introduce the following notation:

$$\begin{array}{rcl} \mathsf{A}^\circ & = & \{f \in \mathsf{A} \mid |f|_{\sup} \leq 1\}, \\ \mathsf{A}^{\circ\circ} & = & \{f \in \mathsf{A} \mid |f|_{\sup} < 1\}, \\ \overline{\mathsf{A}} & = & \mathsf{A}^\circ/\mathsf{A}^{\circ\circ}; \end{array}$$

A° is a R-subalgebra of A, and $\mathsf{A}^{\circ\circ}$ is an ideal of it. For instance, $K^\circ = R$, $K^{\circ\circ} = \mathfrak{m}$, and $\overline{K} = k$.

Example 3.2. We can immediately calculate:

$$\begin{array}{rcl} \mathsf{T}_K^{n\,\circ} & = & R\{t_1, \ldots, t_n\} \\ & := & \left\{ \begin{array}{r|l} \sum_{\nu_1,\ldots,\nu_n \geq 0} a_{\nu_1,\ldots,\nu_n} t_1^{\nu_1} \cdots t_n^{\nu_n} & |a_{\nu_1,\ldots,\nu_n}| \to 0 \text{ for} \\ \in R[[t_1, \ldots, t_n]] & \nu_1 + \cdots + \nu_n \to \infty \end{array} \right\} \\ \overline{\mathsf{T}_K^n} & = & k[t_1, \ldots, t_n]. \end{array}$$

Note that $\mathsf{T}_K^{n\,\circ}$ is the (π)-adic completion of the polynomial ring $R[t_1, \ldots, t_n]$ over R. It should be noticed that $\overline{\mathsf{T}_K^n}$ is a *polynomial ring* over k; indeed, each element in $R\{t_1, \ldots, t_n\}$ lies, by the convergence condition, in $\mathsf{T}_K^{n\,\circ\circ}$ modulo finitely many terms.

Proposition 3.3. Let A be an affinoid algebra over K. Then,

(i) A° is a model of A; i.e. $\mathsf{A}^\circ \otimes_R K \cong \mathsf{A}$,

(ii) A° is (π)-adically completed,
(iii) A° is topologically of finite type over R; i.e. $R\{t_1,\dots,t_n\}/\mathfrak{a} \cong \mathsf{A}$ for some ideal $\mathfrak{a}$ in $R\{t_1,\dots,t_n\}$,
(iv) A° is flat over R,
(v) $\overline{\mathsf{A}}$ is a k-algebra of finite type.

PROOF. (i) and (ii) are easy to see. The flatness of R is now equivalent to the lack of R-torsion. So (iv) can be seen immediately.

Before proving (iii) and (v), we need some terminology: let $\|\cdot\|_\alpha$ be a Banach norm of A ($\alpha\colon \mathsf{T}^n_K \to \mathsf{A}$: surjective). We say an element f of A is *power-bounded* if $\{\|f^n\|_\alpha \mid n \in \mathbb{N}\}$ is bounded. This does not depend on the choice of the presentation α since the equivalence class of the resulting norms do not change. An element f of A is said to be *topologically nilpotent* if $\lim f^n = 0$; here the limit is taken with respect to the Banach norm $\|\cdot\|_\alpha$, and is not dependent of the choice.

Let us prove (iii). By [**BGR84**, 6.2.3], A° is the set of all power-bounded elements of A, and $\mathsf{A}^{\circ\circ}$ is the set of all topologically nilpotent elements. Hence a choice of a presentation $\alpha\colon \mathsf{T}^n_K \to \mathsf{A}$ induces surjections

$$\begin{array}{ccc} \mathsf{T}^{n\,\circ}_K & \longrightarrow & \mathsf{A}^\circ \\ \mathsf{T}^{n\,\circ\circ}_K & \longrightarrow & \mathsf{A}^{\circ\circ}. \end{array}$$

(iii) is due to the surjectivity of the first arrow. By the surjectivity of these arrows and 3.2, we have (v). $\square$

Definition 3.4. An R-algebra A is said to be *admissible* if A is (π)-adically completed, flat over R, and topologically of finite type over R.

Remark 3.5. By [**BL93**, 1.1 (c)], any admissible R-algebra A is topologically of finite presentation, i.e., there exists a finitely generated ideal $\mathfrak{a}$ in $R\{t_1,\dots,t_n\}$ such that $R\{t_1,\dots,t_n\}/\mathfrak{a} \cong \mathsf{A}$.

Definition 3.6 (Formal model and Analytic reduction)**.** Let $X = \mathrm{Spm}\mathsf{A}$ be an affinoid over K. Then the formal scheme $\mathfrak{X} = \mathrm{Spf}\,\mathsf{A}^\circ$ over R is called the *formal model* of X, and the algebraic scheme $\overline{X} = \mathrm{Spec}\,\overline{\mathsf{A}}$ over k is called the *analytic reduction* of X.

We define the so-called *reduction map*

$$\mathrm{Red}_X\colon X = \mathrm{Spm}\mathsf{A} \longrightarrow \overline{X} = \mathrm{Spec}\,\overline{\mathsf{A}}$$

by $\mathfrak{a} \mapsto \mathfrak{a} \cap \mathsf{A}^\circ/\mathfrak{a} \cap \mathsf{A}^{\circ\circ}$.

Proposition 3.7. The map Red_X is continuous with respect to the Zariski topology of $\overline{X}$ and the G-topology of X; it maps X surjectively onto the set of all closed points of $\overline{X}$.

PROOF. Let us consider the affine open set $U_{\overline{f}} = \mathrm{Spec}\,\overline{\mathsf{A}}_{\overline{f}}$ of $\mathrm{Spec}\,\overline{\mathsf{A}}$ relevant to $\overline{f} \in \overline{\mathsf{A}}$. Take $f \in \mathsf{A}^\circ$ from the residue class $\overline{f}$. The function f takes values less than equal to 1 on $\mathrm{Spm}\mathsf{A}$. So $U_f = \{x \in \mathrm{Spm}\mathsf{A} \mid |f(x)| = 1\}$ is a rational subdomain whose associated affinoid is $\mathrm{Spm}A\{f, f^{-1}\}$. Since the condition

$|f(x)| = 1$ is equivalent to $\overline{f}(\mathrm{Red}_X(x)) \neq 0$, we have $\mathrm{Red}_X^{-1}(U_{\overline{f}}) = U_f$. Then the first assertion follows. The last one is almost trivial. $\square$

Example 3.8 (Analytic reduction of the unit disk). Let us consider the analytic reduction of the unit disk $\mathsf{D}^1 = \mathsf{D}(0, 1^+) = \mathrm{Spm}K\{T\}$ (here we assume that K is algebraically closed for simplicity). The analytic reduction is given by the affine line $\mathbb{A}^1_k = \mathrm{Spec}\, k[T]$. The reduction map $\mathrm{Red}_{\mathsf{D}^1}$ maps all "interior" points, i.e. those points lying in $\mathsf{D}(0, 1^-)$ to the origin in $\mathbb{A}^1_k$, and the other points to the non-zero points. If we identify D^1 and R, $\mathbb{A}^1_k$and k, then this is nothing but the mapping obtained by the reduction modulo $\mathfrak{m}$.

It is very important to see how the formal models behave under localization of affinoids. Let $X = \mathrm{Spm}\mathsf{A}$ be an affinoid over K and $f_0, \ldots, f_n \in \mathsf{A}$ a sequence of elements which does not have common zeros on $\mathrm{Spm}\mathsf{A}$. We consider the admissible covering $X = \bigcup_i U_i$ consisting of rational subdomains

$$U_i = \{x \in X \mid |f_j(x)| \leq |f_i(x)| \text{ for } j \neq i\};$$

by multiplying a suitable power of π, we may assume that each f_i belongs to A°. For each i,

$$U_i = \mathrm{Spm}\mathsf{A}\left\{\frac{f_0}{f_i}, \ldots, \frac{f_n}{f_i}\right\},$$

and then the corresponding formal model is given by

$$\mathfrak{U}_i = \mathrm{Spf}\,\mathsf{A}^\circ\left\{\frac{f_0}{f_i}, \ldots, \frac{f_n}{f_i}\right\} / \left((\pi)\text{–torsions}\right).$$

These formal schemes glue and the resulting morphism

$$\bigcup_{0 \leq i \leq n} \mathfrak{U}_i \longrightarrow \mathfrak{X} = \mathrm{Spf}\,\mathsf{A}^\circ$$

is the formal blow-up along the ideal $\mathfrak{a} = (f_0, \ldots, f_n) \subset \mathsf{A}^\circ$. We note here that the ideal $\mathfrak{a}$ contains a power of π, or what amounts to the same, $\mathfrak{a}$ is an open ideal, for $\mathfrak{a}\mathsf{A} = \mathsf{A}$.

This simple observation indicates:

Slogan (vague): *Refinements of admissible coverings corresponds to formal blow-up's along open coherent ideals.*

Example 3.9. Let us assume in this example that K is algebraically closed. We consider the admissible covering $\mathsf{D}^1 = U_1 \cup U_2$ of the unit disk $\mathsf{D}^1 = \mathrm{Spm}K\{T\}$ given by

$$\begin{aligned} U_1 &= \{z \in K \mid |z| \leq |\pi|\} &&= \mathrm{Spm}K\left\{\tfrac{T}{\pi}\right\}, \\ U_2 &= \{z \in K \mid |\pi| \leq |z| \leq 1\} &&= \mathrm{Spm}K\left\{T, \tfrac{\pi}{T}\right\}. \end{aligned}$$

The corresponding formal models are

$$\begin{aligned} \mathfrak{U}_1 &= \mathrm{Spf}\,R\left\{\tfrac{T}{\pi}\right\} &&= \mathrm{Spf}\,R\{T, U\}/(\pi U - T), \\ \mathfrak{U}_2 &= \mathrm{Spf}\,R\left\{T, \tfrac{\pi}{T}\right\} &&= \mathrm{Spf}\,R\{T, V\}/(TV - \pi) \end{aligned}$$

respectively. We glue them as $U = V^{-1}$ and get $\mathcal{U}_1 \cup \mathcal{U}_2 \to \mathcal{D}^1 = \operatorname{Spf} R\{T\}$, which is the formal blow-up along the ideal (π, T). Meanwhile, the analytic reduction is obtained by gluing the following two affine sets:

$$\begin{aligned} \overline{U}_1 &= \operatorname{Spec} k\,[T]\,, \\ \overline{U}_2 &= \operatorname{Spec} k\,[T, V]\,/(TV), \end{aligned}$$

this is the normal crossing of the affine line with coordinate T and the projective line with the inhomogeneous coordinate V.

We are going to state the Raynaud's theorem, which provides a close relationship between rigid geometry and formal geometry; we need some terminology:

Definition 3.10 (Admissible formal scheme). An R-formal scheme[2] $\mathcal{X}$ is said to be *admissible* if it is Zariski locally isomorphic to admissible $\operatorname{Spf} A$ with A admissible in the sense of Definition 3.4.

Definition 3.11 (Admissible formal blow-up). Let $\mathcal{X}$ be an admissible formal scheme. An *admissible formal blow-up* of $\mathcal{X}$ is the formal blow-up along a coherent open ideal $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$, i.e.,

$$\mathcal{X}' = \varinjlim_{\lambda} \operatorname{Proj} \bigoplus_{n=0}^{\infty} \left(\mathcal{I}^n \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}/\pi^{\lambda} \right) \longrightarrow \mathcal{X}.$$

We are going to construct the functor from the category of admissible formal schemes to the category of rigid analytic spaces; let us start by observing the affine case. From an affine admissible formal scheme $X = \operatorname{Spf} A$, it is easy to get the corresponding rigid analytic space: we just set

$$A_{\text{rig}} := A \otimes_R K$$

which is an affinoid algebra over K. If A is presented as $A = R\{t_1, \ldots, t_n\}/\mathfrak{a}$, then $K\{t_1, \ldots, t_n\}/\mathfrak{a}K\{t_1, \ldots, t_n\}$ gives a presentation of A_{rig}.

To globalize, we need to see the compatibility with localization. Let us consider the completed localization $A\{f^{-1}\}$ of A with respect to $f \in A$. The corresponding affinoid algebra is

$$\begin{aligned} A\{f^{-1}\} \otimes_R K &= A\{T\}/(1 - Tf) \otimes_R K \\ &= A_{\text{rig}}\{T\}/(1 - Tf) = A_{\text{rig}}\{f^{-1}\}, \end{aligned}$$

and the latter is the affinoid algebra attached to the rational subdomain

$$\{x \in \operatorname{Spm} A_{\text{rig}} \mid |f| \geq 1\};$$

this means that the completed localization of formal schemes just corresponds to the localization of rigid spaces with respect to the G-topology. So we globalize the recipe as above to obtain the functor

$$\operatorname{Rig}\colon \mathcal{X} \mapsto \mathcal{X}_{\text{rig}}$$

from admissible formal R-schemes to K-rigid analytic spaces. The rigid analytic space $\mathcal{X}_{\text{rig}}$ is called the *Raynaud generic fiber* of $\mathcal{X}$.

[2] We follow [**EGA I**, §10] for generalities of formal schemes.

Proposition 3.12. The functor Rig maps admissible formal blow-up's of admissible formal schemes to isomorphisms of rigid analytic spaces.

PROOF. This has already been essentially shown just before the Slogan. Indeed, we may limit ourselves to the affine case, for the centers of blow-up's are coherent. We have seen in this case that the Raynaud generic fiber of admissible formal blow-up's is the decomposition into rational subdomains, which does not change the analytic structure. $\square$

Now we state the theorem of Raynaud:

Theorem 3.13 (Raynaud 1972). The functor gives the equivalence of the following categories,

(i) the category of quasi-compact admissible formal R-schemes, localized by admissible formal blow-up's,
(ii) the category of quasi-compact and quasi-separated rigid analytic spaces over K.

The reader finds a nice comprehensive account of the proof of this theorem in [**BL93**].

4. Topology of Rigid Analytic Space.

In this section, we assume, for simplicity, that K is algebraically closed.

Definition 4.1. (1) A rigid analytic space X is said to be *quasi-compact* if every admissible covering of X has a finite admissible refinement.
(2) A rigid analytic space X is said to be *connected* if there is no admissible covering $\{U_i\}_{i\in I}$ of X such that

$$\bigcup_{i\in I_1} U_i \cap \bigcup_{i\in I_2} U_i = \emptyset \text{ and } \bigcup_{i\in I_1} U_i \neq \emptyset \neq \bigcup_{i\in I_2} U_i$$

for some non-empty subsets $I_1, I_2 \subseteq I$ with $I = I_1 \sqcup I_2$.

Since admissible coverings of affinoids are fixed as finite, every affinoid is quasi-compact. So, we see that a rigid analytic space is quasi-compact if and only if it is a finite union of affinoids. It is easy to see that a rigid analytic space X is connected if and only if the ring $\Gamma(X, \mathcal{O}_X)$ has no other idempotent than 0 and 1.

Definition 4.2. A morphism $\pi\colon Y \to X$ of rigid analytic spaces over K is said to be an *analytic covering* if there exists an admissible covering(3) $\{X_i\}_{i\in I}$ of X such that for each $i \in I$ $\pi^{-1}(X_i)$ is isomorphic to the disjoint union of copies of X_i. A connect rigid analytic space X is said to be *simply connected* if there is no other connected analytic covering over X than the trivial one id: $X \to X$ (cf. [**vdP87**]).

(3)or else a topological covering, referring to the Grothendieck topology; cf. Chapter I, §§1.4

Lemma 4.3. Let X be a connect rigid analytic space X.

(1) The space X is simply connected if and only if every locally constant sheaf of sets, with respect to the Grothendieck topology, is constant.

(2) If there exists an admissible covering $\{X_i\}_{i\in\mathbb{N}}$ of X such that (a) every X_i is simply connected, and (b) $X_i \subseteq X_{i+1}$ for all i, then X is simply connected.

PROOF. Both are straightforward. □

Example 4.4 (Topology of the closed disk). Let $\mathsf{D} = \mathsf{D}(0, 1^+)$ be the 1-dimensional closed disk of radius 1. A rational subdomain in D of form

$$\{x \in \mathsf{D} \mid |x - a| \leq \rho \text{ and } |x - a_i| \geq \rho_i \text{ for } i = 1, \dots, s\},$$

where $\rho, \rho_i \in |K^*|$ and $a, a_i \in \mathsf{D}$, is called a *standard domain*. This is the complement of finite union of open disks. The following properties are easy to verify:

- If S_1 and S_2 are standard domains such that $S_1 \cap S_2 \neq \emptyset$, then so are $S_1 \cup S_2$ and $S_1 \cap S_2$.
- Every finite union of standard domains is uniquely decomposed as a disjoint union of standard domains.

What is more interesting is the following proposition:

Proposition 4.5. Every rational subdomain in D is a finite union of standard domains. (Hence standard domains generate the Grothendieck topology of D.)

SKETCH OF PROOF. (cf. [**GvdP80**, III.1.18]). Let R be the rational subdomain given by $|f_i(x)| \leq |f_0(x)|$ for $i = 1, \dots, n$. Deforming those functions slightly, if necessary, we may assume f_i and f_j have no common zero for any $i \neq j$, and thus, we may concentrate to the single inequality $|f_1(x)| \leq |f_0(x)|$. By Weierstrass Preparation Theorem we may assume f_0 and f_1 are polynomials having all their roots in D. Then the proposition follows from an easy calculation. □

Corollary 4.6. The Grothendieck topology of the projective line $\mathbb{P}^{1,\mathrm{an}}_K$ is generated by standard domains.

Proposition 4.7. Standard domains are connected. Moreover, every connected rational subdomain in $\mathbb{P}^{1,\mathrm{an}}_K$ is simply connected.

PROOF. The first assertion is obvious since the corresponding affinoid algebra

$$K\{t, t_0, \dots, t_s\}/((t - a) - t_0\pi, \pi_i - t_i(t - a_i) \text{ for } i = 1, \dots, s),$$

where $a, a_i \in K$ and $\pi, \pi_i \in K$, is integral. (Note that $K\{t, t_0, \dots, t_s\}$ is Noetherian and factorial.) The second one is proved by using 4.3 (1). Let S be a standard domain and $\mathcal{F}$ a constant sheaf of sets on S. We can take an admissible covering $\{S_i\}$ of S which trivializes $\mathcal{F}$. The point is that any non-disjoint union of standard domains is standard, and hence, connected. So the restriction maps $\mathcal{F}(S_i) \to \mathcal{F}(S_i \cap S_j)$ must be bijective as far as $S_i \cap S_j \neq \emptyset$. It follows that $\mathcal{F}(S_i \cup S_j) \to \mathcal{F}(S_i)$ are bijective, and repeating this argument, we concludes $\mathcal{F}$ is constant. $\square$

The following corollary may seen at first:

Corollary 4.8. Every connected open subanalytic space[(4)] in $\mathbb{P}^{1,\mathrm{an}}_K$ is simply connected.

PROOF. A connected open subanalytic space U has an admissible covering consisting of standard domains. By [**GvdP80**, III,2.6] and the easy fact that a non-disjoint union of connected sets is again connected, we see that a finite non-disjoint union of standard domain is a connected rational subdomain. Hence the corollary follows from Lemma 4.3 (2) and Proposition 4.7. $\square$

Finally we quote, without proof, a theorem by van der Put which indicates a close relation between the topology of a rigid analytic space and that of its analytic reduction:

Theorem 4.9. Let X be a quasi-compact rigid analytic space over K which has the irreducible and smooth analytic reduction. Then X is simply connected.

For the proof, see [**vdP87**].

Remark 4.10. By the corollary, we know, for example, that the space $\mathbb{G}^{\mathrm{an}}_K = K^\times$ is simply connected. But, on the other hand, it is easy to see that the map $K^\times \to K^\times$ by $x \mapsto x^n$ is an analytic morphism. As far as the characteristic of K does not divide n, we are tempted to think of this morphisms also as an analytic covering. But it is not. Certainly, this morphism induces isomorphisms between stalks of the structure sheaf, but there is no *admissible* covering which trivializes this morphism! This kind of morphisms is said to be *étale*; analytic coverings are étale, but not vice versa. Simply connected analytic spaces may have many étale coverings.

5. Berkovich' approach to non-archimedean analysis

Berkovich' viewpoint provides an innovative approach to topological problems of p-adic geometry and analysis, which is much closer to the familiar intuition derived from the complex context [**Ber90**][(5)]. His analysis, in fact, generalizes, or properly speaking, "completes" the rigid analysis. This situation is, according to what he says in the introduction of his book, somehow analogous to that of $\mathbb{R}$ completing $\mathbb{Q}$. The standard archimedean metric space $\mathbb{Q}$ is totally disconnected. The rigid analytic viewpoint corresponds, to some extent, to regarding every rational interval $\{r \in \mathbb{Q} \mid a \leq r \leq b\}$ as

[(4)] We refer [**BGR84**, 9.3.1] for the definition of open subanalytic spaces.

[(5)] Already in [**vdP82**] the *esquisse* of Berkovich' idea can be seen.

"connected", and considering only those functions which come from analytic functions on the corresponding real interval. Berkovich' non-archimedean analysis can be compared, in this context, to analysis on $\mathbb{R}$ itself. Namely, his presentation of analytic spaces complements those of rigid analysis by putting some "generic points", and consequently, simplifies their topology.

Let us be a little more precise about his theory. Let A be a Banach ring, i.e., a ring together with a norm $\|\cdot\|$ with respect to which A is complete.

Definition 5.1. (1) A semi-norm $|\cdot|$ on A is said to be *bounded* if there exists $C > 0$ such that $|f| \leq C\|f\|$ for any $f \in \mathsf{A}$.

(2) A semi-norm $|\cdot|$ on A is said to be *multiplicative* if $|fg| = |f||g|$ for all $f, g \in \mathsf{A}$.

Replacing the maximal spectra $\mathrm{Spm}(\mathsf{A})$ of rigid geometry, we have:

Definition 5.2. The Berkovich' *spectrum* $\mathsf{M}(\mathsf{A})$ for a Banach ring A is the set of all bounded multiplicative semi-norms on A, endowed with the weakest topology so that the real valued functions of form $|\cdot| \mapsto |f|$ for $f \in \mathsf{A}$ are continuous.

In case A is an affinoid algebra (a *strict* affinoid algebra, in Berkovich' term) over K, the affinoid space $\mathrm{Spm}(\mathsf{A})$ is naturally viewed as a subspace, with the relative topology, of the Berkovich' spectrum $\mathsf{M}(\mathsf{A})$. This is done by identifying an element $\mathfrak{m}$ in $\mathrm{Spm}(\mathsf{A})$ with the unique semi-norm $|\cdot|$ on A such that $\mathfrak{m} = \mathrm{Ker}\,|\cdot|$. The space $\mathrm{Spm}(\mathsf{A})$ is identified, in terms of this correspondence, with the subspace of $\mathsf{M}(\mathsf{A})$ consisting of semi-norms $|\cdot|$ such that $\dim_K \mathsf{A}/\mathrm{Ker}\,|\cdot|$ is finite (the "classical points").

Example 5.3. Let us consider the usual absolute-value norm $|\cdot|_\infty$ on $\mathbb{Z}$. The spectrum $\mathsf{M}(\mathbb{Z})$ consists of the following points:

(i) $|\cdot|_{\infty,\epsilon} := |\cdot|_\infty^\epsilon$ $(0 < \epsilon \leq 1)$.
(ii) The p-adic norm $|\cdot|_{p,\epsilon}$: with $|p|_{p,\epsilon} = \epsilon$ $(0 < \epsilon \leq 1)$.
(iii) The semi-norm $|\cdot|_p$ induced from the trivial norm on $\mathbb{Z}/p\mathbb{Z}$.
(iv) The trivial norm $|\cdot|_0$.

As a topological space, $\mathsf{M}(\mathbb{Z})$ is a tree with end points $|\cdot|_p$ and $|\cdot|_\infty$. Each of these points is connected by a single edge with $|\cdot|_0$.

On the Berkovich' spectrum, the *affinoid subdomains* (incl. *rational subdomains*) are defined in the similar way to rigid analysis. Thus one can define the structure sheaves on spectra, which one calls affinoid spaces. The local data consisting of Berkovich' affinoids and the structure sheaves can glue to locally ringed spaces; thus we obtain *analytic spaces* in Berkovich' sense.

Example 5.4. (Cf. [**Ber90**, 1.5].) Let K be a non archimedean complete field. An affine space over K is defined by

$$\mathbb{A}^n_K = \{\text{multiplicative seminorm } |\cdot| \text{ on } K[t_1, \ldots, t_n] \mid |\cdot|_{|K} \text{ is bounded}\}$$

together with the weakest topology so that $|\cdot| \mapsto |f|$ ($f \in K[t_1, \ldots, t_n]$) is continuous. This has a structure of analytic space so that the analytic functions over it is characterized by limits of rational functions. When $K = \mathbb{C}$ the space $\mathbb{A}^n_K$ is nothing but the usual affine space in the complex analytic sense. If K is a non-archimedean field, then $\mathbb{A}^n_K$ is a union of balls; i.e., the spectrum of Tate algebra.

Here are some general properties of Berkovich' analytic spaces:

(i) Every connected K-analytic space is arcwise connected.
(ii) If $X = \mathsf{M}(\mathsf{A})$ with A an affinoid algebra over K, then Krull-dim$\mathsf{A} = \dim|X|$, where $|X|$ is the underlying topological space of X.
(iii) If a K-analytic space X is smooth, then it is locally contractible.

Relation between Berkovich' analytic spaces and rigid analytic spaces is as follows. If a Berkovich' K-analytic space X is formed only by spectra of (strict) affinoid algebras, we say X is strict. There exists a functor from the category of separated strict K-analytic spaces to that of separated rigid analytic spaces over K by

$$X \mapsto X_0 = \{x \in X \mid [K(x)\colon K] < \infty\},$$

where $K(x)$ is the residue field at x.

Theorem 5.5. This functor is fully-faithful, and preserves fiber products.