Part I

Two-dimensional chiral quantum fields

In Section 1, the notion of fields and the residue products are introduced algebraically and some elementary properties are summarized. Section 2 is devoted to the study of the mutual locality of fields. The notion of operator product expansion is explained and some examples are given. In Section 3, we will derive our identity (3) in the introduction.

1 Fields and their residue products

We describe the definitions and basic properties concerning two-dimensional chiral quantum fields in the language of formal Laurent series.

1.1 Preliminaries

Throughout the paper, we always work over a field $\mathbf{k}$ of characteristic zero. We denote by $V[[z, z^{-1}]]$ the set of all formal Laurent series in the variable z with coefficients in a vector space V possibly having infinitely many terms both of positive and of negative degree:

$$V[[z, z^{-1}]] = \left\{ \sum_{n=-\infty}^{\infty} v_n z^{-n-1} \,\middle|\, v_n \in V \right\}.$$

The subset consisting of all series with only finitely many terms of negative degree is denoted by

$$V((z)) = \left\{ \sum_{n=n_0}^{-\infty} v_n z^{-n-1} \,\middle|\, v_n \in V, n_0 \in \mathbb{Z} \right\}.$$

Similarly we write

$$V[[y, y^{-1}, z, z^{-1}]] = \left\{ \sum_{m,n=-\infty}^{\infty} v_{m,n} y^{-m-1} z^{-n-1} \,\middle|\, v_{m,n} \in V \right\},$$

$$V((y, z)) = \left\{ \sum_{m=m_0}^{-\infty} \sum_{n=n_0}^{-\infty} v_{m,n} y^{-m-1} z^{-n-1} \,\middle|\, v_{m,n} \in V, m_0, n_0 \in \mathbb{Z} \right\}.$$

Let the symbols $(y-z)^n|_{|y|>|z|}$ and $(y-z)^n|_{|y|<|z|}$, $(n \in \mathbb{Z})$ denote the elements of $\mathbf{k}[[y, y^{-1}, z, z^{-1}]]$ obtained by expanding the rational function $(y-z)^n$ into series convergent in the regions $|y| > |z|$ and $|y| < |z|$ respectively:

$$(y-z)^n|_{|y|>|z|} = \sum_{i=0}^{\infty}(-1)^i \binom{n}{i} y^{n-i} z^i, \quad (y-z)^n|_{|y|<|z|} = \sum_{i=0}^{\infty}(-1)^{n+i}\binom{n}{i} y^i z^{n-i}.$$

We will often neglect writing the region of expansion in case it is obvious from the context (see Notation 1.3.3).

The following lemma will be frequently used later.

Lemma 1.1.1. *Let $a(y,z)$ be a series with only finitely many terms of negative or positive degree in y or z. If $a(y,z)$ satisfies $(y-z)^m a(y,z) = 0$ for some nonnegative integer m, then $a(y,z) = 0$.*

Proof. Consider the series $b(y,z) = (y-z)^{m-1}a(y,z)$ and let the coefficients $b_{k,\ell} \in V$ be defined by $b(y,z) = \sum_{k,\,\ell \in \mathbb{Z}} b_{k,\,\ell}\, y^{-k-1} z^{-\ell-1}$. It follows from $(y-z)b(y,z) = (y-z)^n a(y,z) = 0$ that

$$b_{k+1,\ell} = b_{k,\ell+1} \quad \text{for all} \quad k, \ell \in \mathbb{Z}. \tag{1.1.1}$$

Since $a(y,z)$ has only finitely many terms of negative or positive degree in y or z, so does $b(y,z)$. Thus (1.1.1) implies $b(y,z) = 0$. Repeating this procedure, we arrive at $a(y,z) = 0$. □

1.2 Series on a vector space

Let M be a vector space and consider the space $(\operatorname{End} M)[[z, z^{-1}]]$ of all formal Laurent series with coefficients being endomorphisms (operators) on M. We simply call such series[1] $A(z)$ a *series* on M. For a series $A(z)$ on M, we set

$$A_n = \operatorname*{Res}_{z=0} A(z) z^n = \text{ coefficient of } z^{-n-1} \text{ in } A(z)$$

so that the expansion of $A(z)$ is

$$A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1}, \quad A_n \in \operatorname{End} M.$$

The A_n is called a *Fourier mode* of $A(z)$. We write

$$A(z)v = \sum_{n \in \mathbb{Z}} A_n v z^{-n-1}$$

[1] It is called a formal distribution in [K].

for a vector $v \in M$. Given two series $A(z)$ and $B(z)$ on M, we set

$$A(y)B(z) = \sum_{m,n\in\mathbb{Z}} A_m B_n y^{-m-1} z^{-n-1},$$

where $A_m B_n$ denotes the composition of endomorphisms.

The derivative of a series $A(z)$ is defined by

$$\partial A(z) = \sum_{n\in\mathbb{Z}} (-n-1) A_n z^{-n-2} = -\sum_{n\in\mathbb{Z}} n A_{n-1} z^{-n-1}.$$

More generally, for a nonnegative integer $k \in \mathbb{N}$, we define

$$\partial^{(k)} A(z) = \sum_{n\in\mathbb{Z}} \binom{-n-1}{k} A_n z^{-n-k-1} = \sum_{n\in\mathbb{Z}} (-1)^k \binom{n}{k} A_{n-k} z^{-n-1}.$$

Finally, we set

$$A(z)_+ = \sum_{n\geq 0} A_n z^{-n-1} \quad \text{and} \quad A(z)_- = \sum_{n<0} A_n z^{-n-1}$$

respectively[2]. Then we have $\partial(A(z)_\pm) = (\partial A(z))_\pm$. For two series $A(z)$ and $B(z)$, we set

$$\,^\circ_\circ\, A(y)B(z) \,^\circ_\circ\, = A(y)_- B(z) + B(z) A(y)_+,$$

which is an element of $(\operatorname{End} M)[[y, y^{-1}, z, z^{-1}]]$.

1.3 Fields on a vector space

Next we consider the notion of two-dimensional chiral[3] quantum fields.

Definition 1.3.1. A series $A(z) \in (\operatorname{End} M)[[z, z^{-1}]]$ is called a *field*[4] on M if $A(z)v \in M((z))$ for any $v \in M$.

In other words, $A(z)$ is a field if and only if, for any $v \in M$, there exists an integer n_0 depending on v such that $A_n v = 0$ holds for all $n \geq n_0$. In this case, we will briefly say that $A_n v = 0$ for $n \gg 0$. Note that if $A(z)$ is a field, then so is $\partial A(z)$.

The following simplest example of a field will play an important role later:

[2]The parts $A(z)_+$ and $A(z)_-$ respectively appears as the annihilation and the creation parts in the context of vertex algebra, see the axiom (B2) in Subsection 4.1

[3]Here *chiral* means the "holomorphic part" in the sense of conformal field theory.

[4]We followed the terminology in [K]. It is also called a weak vertex operator in the literature of vertex operator algebras (cf. [Li2]).

Definition 1.3.2. The *identity field* on M is the field

$$I(z) = \mathrm{id}_M$$

of which the only nonzero term is the constant term being the identity operator on M.

Now, let $A(z)$ be a series and $B(z)$ a field on M. Consider the following expression

$$\operatorname*{Res}_{y=0} A(y)B(z)(y-z)^m|_{|y|>|z|} = \sum_{n\in\mathbb{Z}} \left(\sum_{i=0}^{\infty} (-1)^i \binom{m}{i} A_{m-i} B_{n+i} \right) z^{-n-1} \tag{1.3.1}$$

for a fixed integer m. Then, for any $v \in M$, the second sum in the right applied to v is a finite sum for each n since finitely many $B_{n+i}v$ are nonzero. Moreover $B_{n+i}v$ vanish for sufficiently large n. Therefore (1.3.1) defines a field on M.

Next, assume further that $A(z)$ is a field, and consider

$$\operatorname*{Res}_{y=0} B(z)A(y)(y-z)^m|_{|y|<|z|} = \sum_{n\in\mathbb{Z}} \left(\sum_{i=0}^{\infty} (-1)^{m+i} \binom{m}{i} B_{m+n-i} A_i \right) z^{-n-1} \tag{1.3.2}$$

for a fixed integer m. Then, for any $v \in M$, the second sum in the right applied to v is a finite sum for each n since finitely many $A_i v$ are nonzero. Let $A_0 v, \ldots, A_\ell v$ be the nonzero vectors. Then, for each $0 \le i \le \ell$, we have $B_{m+n_i-i}A_i v = 0$ for sufficiently large n_i. Therefore, taking the maximum of n_i, $(0 \le i \le \ell)$, we see that the above summation is zero for sufficiently large n. Hence (1.3.2) gives rise to a field on M.

Notation 1.3.3. To simplify the presentation of the paper, we often omit the region of the expansion of $(y-z)^m$ if there is no danger of confusion; The region is determined by the order of $A(y)$ and $B(z)$. Namely, we always regard

$$A(y)B(z)(y-z)^m \quad \text{and} \quad B(z)A(y)(y-z)^m$$

as the series obtained by expanding $(y-z)^m$ convergent in the regions

$$|y| > |z| \quad \text{and} \quad |y| < |z|$$

respectively. Hence $A(y)B(z)(y-z)^m \ne B(z)A(y)(y-z)^m$ for $m < 0$ even if $A(y)B(z) = B(z)A(y)$. We also obey this rule in case more than two series are involved.

If $B(z)$ is a field, then

$$\begin{aligned}\operatorname*{Res}_{y=0} \partial A(y)B(z)(y-z)^m &= -m\operatorname*{Res}_{y=0} A(y)B(z)(y-z)^{m-1} \\ &= \partial\left(\operatorname*{Res}_{y=0} A(y)B(z)(y-z)^m\right) - \operatorname*{Res}_{y=0} A(y)\partial B(z)(y-z)^m\end{aligned} \tag{1.3.3}$$

and if $A(z)$ and $B(z)$ are fields, then

$$\begin{aligned}\operatorname*{Res}_{y=0} B(z)\partial A(y)(y-z)^m &= -m\operatorname*{Res}_{y=0} B(z)A(y)(y-z)^{m-1} \\ &= \partial\left(\operatorname*{Res}_{y=0} B(z)A(y)(y-z)^m\right) - \operatorname*{Res}_{y=0} \partial B(z)A(y)(y-z)^m.\end{aligned} \tag{1.3.4}$$

We note that *normally ordered product*[5]

$$\overset{\circ}{\circ} A(z)B(z) \overset{\circ}{\circ} = A(z)_-B(z) + B(z)A(z)_+$$

makes sense if $A(z)$ and $B(z)$ are fields.

1.4 Residue products of fields

Now let us explain the residue products[6] ([BG], [Li, Lemma 3.1.4], [LZ1, Definition 2.1]), indexed by integers, which assign a field to an ordered pair of fields for each integer m.

Definition 1.4.1. For two fields $A(z)$ and $B(z)$ on M, the m-th residue product, ($m \in \mathbb{Z}$), is defined by

$$A(z)_{(m)}B(z) = \operatorname*{Res}_{y=0} A(y)B(z)(y-z)^m - \operatorname*{Res}_{y=0} B(z)A(y)(y-z)^m. \tag{1.4.1}$$

Explicitly, (1.4.1) is written as

$$A(z)_{(m)}B(z) = \sum_{n\in\mathbb{Z}} (A_{(m)}B)_n z^{-n-1},$$

where

$$(A_{(m)}B)_n = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} (A_{m-i}B_{n+i} - (-1)^m B_{m+n-i}A_i).$$

We remark that the residue product $A(z)_{(m)}B(z)$ for nonnegative m makes sense even if $A(z)$ or $B(z)$ is not a field.

[5] cf. [BBS, (A.2)], [SY, (5.5)]

[6] In the literature, it is simply called the n-th product. However, to distinguish it from the abstract products of a vertex algebra, we have added the adjective *residue*.

Note 1.4.2. In the physics notation, the residue products are expressed as (cf. [BG, (2.11)]):

$$A(z)_{(m)}B(z) = \oint_{C_z} \frac{dy}{2\pi\sqrt{-1}} R(A(y)B(z))(y-z)^m$$
$$= \oint_{|y|>|z|} \frac{dy}{2\pi\sqrt{-1}} A(y)B(z)(y-z)^m - \oint_{|y|<|z|} \frac{dy}{2\pi\sqrt{-1}} B(z)A(y)(y-z)^m$$

where R denotes the *radial ordering* defined by

$$R(A(y)B(z)) = \begin{cases} A(y)B(z), & |y| > |z|, \\ B(z)A(y), & |y| < |z|, \end{cases}$$

and C_z is a small contour around z.

Note that if $A(z)$ and $B(z)$ are fields, then $A(z)_{(m)}B(z)$ is again a field on M by the consideration in the preceding subsection.

We always understand that the derivative precedes the residue product:

$$\partial A(z)_{(m)}B(z) = (\partial A(z))_{(m)}B(z), \quad A(z)_{(m)}\partial B(z) = A(z)_{(m)}(\partial B(z)).$$

Then, by (1.3.3) and (1.3.4), we have

$$(1.4.2) \quad \partial A(z)_{(m)}B(z) = -mA(z)_{(m-1)}B(z) = \partial(A(z)_{(m)}B(z)) - A(z)_{(m)}\partial B(z).$$

In particular, let us consider the (-1) st product:

$$A(z)_{(-1)}B(z) = \sum_{n\in\mathbb{Z}} \left(\sum_{i=0}^{\infty} (A_{-i-1}B_{n+i} + B_{n-i-1}A_i) \right) z^{-n-1}.$$

It coincides with the normally ordered product

$$:A(z)B(z): = A(z)_- B(z) + B(z)A(z)_+.$$

Therefore, using (1.4.2), we have

$$A(z)_{(-k-1)}B(z) = :\partial^{(k)}A(z)B(z):$$

for a nonnegative integer k. Note that the normally ordered product is neither commutative nor associative in general. However, we have the following property (cf. [BBS, p.365], [SY, p.292]):

Proposition 1.4.3. *Let $A(z), B(z)$ and $C(z)$ be fields. Then*

$$(A(z)_{(-1)}B(z))_{(-1)}C(z) - (B(z)_{(-1)}A(z))_{(-1)}C(z)$$
$$= A(z)_{(-1)}(B(z)_{(-1)}C(z)) - B(z)_{(-1)}(A(z))_{(-1)}C(z))$$

From this, we easily deduce that the bracket

$$[\![A(z), B(z)]\!] = A(z)_{(-1)}B(z) - B(z)_{(-1)}A(z)$$

satisfies the Jacobi identity

$$[\![A(z), [\![B(z), C(z)]\!]]\!] = [\![[\![A(z), B(z)]\!], C(z)]\!] + [\![B(z), [\![A(z), C(z)]\!]]\!]$$

whereas $[\![B(z), A(z)]\!] = -[\![A(z), B(z)]\!]$ is obvious.

Let us close this subsection with the following:

Proposition 1.4.4. *Let $A(z)$ be a field and $I(z)$ the identity field on a vector space M. Then*

$$A(z)_{(m)}I(z) = \begin{cases} 0, & (m \geq 0), \\ \partial^{(-m-1)}A(z), & (m \leq -1), \end{cases} \tag{1.4.3}$$

$$I(z)_{(m)}A(z) = \begin{cases} 0, & (m \neq -1), \\ A(z), & (m = -1). \end{cases} \tag{1.4.4}$$

We note, in particular,

$$A(z)_{(m)}I(z) = \begin{cases} 0, & (m \geq 0), \\ A(z), & (m = -1), \end{cases} \tag{1.4.5}$$

which is a part of the axioms for a vertex algebra (see Subsection 4.1).

2 Mutually local fields

2.1 Locality of fields

The notion of locality for two-dimensional chiral quantum fields is related to Wightman's axioms for quantum field theory (cf. [K, Chapter 1]). We adopt the following formulation in the language of formal Laurent series[7]:

[7]The condition (2.1.1) was first considered by Dong-Lepowsky [DL, (7.24)] under the term *commutativity*, while the same was considered in earlier papers in the language of operator valued rational functions (cf. [G],[FLM],[FHL]).

Definition 2.1.1. Two series $A(z)$ and $B(z)$ are called *mutually local* if

$$A(y)B(z)(y-z)^n = B(z)A(y)(y-z)^n \tag{2.1.1}$$

holds for some nonnegative integer n. In this case, we also say $A(z)$ is local to $B(z)$.

In terms of Fourier modes, (2.1.1) is written as

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}A_{p+n-i}B_{q+i} = \sum_{i=0}^{n}(-1)^i\binom{n}{i}B_{q+i}A_{p+n-i},$$

or equivalently

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\left(A_{p+n-i}B_{q+i} - (-1)^n B_{q+n-i}A_{p+i}\right) = 0$$

where p, q run over all integers. Note that $A(z)$ or $B(z)$ need not be local to itself.

Let us introduce the following notion:

Definition 2.1.2. The *order* of locality of series $A(z)$ and $B(z)$ is the minimum of the nonnegative integers n satisfying (2.1.1).

Thus, $A(z)$ and $B(z)$ are mutually local at order n_0 if and only if (2.1.1) holds precisely for $n \geq n_0$. Then, by the definition of the residue products, we have

$$A(z)_{(n)}B(z) = 0, \quad (n \geq n_0). \tag{2.1.2}$$

Note that it may happen that $A(z)_{(n)}B(z) = 0$ holds for some $0 \leq n < n_0$. It is easy to see that if $A(z)$ and $B(z)$ are mutually local at order n_0 (≥ 1), then $\partial A(z)$ and $B(z)$ are mutually local at order $n_0 + 1$.

Now we turn to the study of the locality of many series. We say that series $A^1(z), \ldots, A^\ell(z)$ are mutually local if all the distinct pairs $A^i(z)$ and $A^j(z)$, $(i \neq j)$, are mutually local.

The following proposition is a generalization of [Li, proof of Proposition 3.2.7]. Recall our convention in Notation 1.3.3

Proposition 2.1.3. *Let $A^1(z), \ldots, A^\ell(z)$ be mutually local series and let m_{ij} be the order of locality of $A^i(z)$ and $A^j(z)$ for each $i < j$. Then*

$$[\cdots[[A^1(z_1), A^2(z_2)], A^3(z_3)], \cdots, A^\ell(z_\ell)]\prod_{i<j}(z_i - z_j)^{n_{ij}} = 0$$

holds if $\sum_{i<j} n_{ij} \geq \sum_{i<j} m_{ij} - (\ell - 2)$.

Proof. We show by induction on ℓ. The case $\ell = 2$ is nothing but the definition of the locality. Suppose $\ell > 2$. Then, by expanding as

$$\begin{aligned}(z_i - z_\ell)^{n_{i\ell}} &= (z_i - z_j + z_j - z_\ell)^{n_{i\ell}-m_{i\ell}}(z_i - z_\ell)^{m_{i\ell}} \\ &= \sum_{s=0}^{n_{i\ell}-m_{i\ell}} \binom{n_{i\ell}-m_{i\ell}}{s}(z_i - z_j)^{n_{i\ell}-m_{i\ell}-s}(z_j - z_\ell)^s(z_j - z_\ell)^{m_{i\ell}}\end{aligned}$$

for an appropriate j if $n_{i\ell} \geq m_{i\ell}$, and by repeating this procedure, the left-hand side of the desired equality is written as a linear combination of terms of the form

$$[\cdots[[A^1(z_1), A^2(z_2)], A^3(z_3)], \cdots, A^\ell(z_\ell)]\prod_{i<j}(z_i - z_j)^{p_{ij}} \tag{2.1.3}$$

where the exponents p_{ij} satisfy

$$\sum_{1\leq i<j\leq \ell-1} p_{ij} \geq \sum_{1\leq i<j\leq \ell-1} m_{ij} - (\ell - 3), \quad \text{or} \quad p_{i\ell} \geq m_{i\ell} \text{ for all } i.$$

In the former case, (2.1.3) vanishes by the inductive hypothesis, while in the latter case, by the locality of $A^i(z)$ and $A^\ell(z)$. □

In particular, we have

Lemma 2.1.4. *Let $A(z), B(z)$ and $C(z)$ be mutually local series and let k_0, ℓ_0 and m_0 be the order of locality of $A(z)$ and $C(z)$, $B(z)$ and $C(z)$, and $A(z)$ and $B(z)$ respectively. Then, for any integers k, ℓ, m,*

$$\begin{aligned}&(y-z)^n\Big(A(x)B(y)C(z) - B(y)A(x)C(z)\Big)(x-y)^m(y-z)^\ell(x-z)^k \\ &\quad = (y-z)^n\Big(C(z)A(x)B(y) - C(z)B(y)A(x)\Big)(x-y)^m(y-z)^\ell(x-z)^k\end{aligned}$$

holds for all $n \in \mathbb{N}$ satisfying $n \geq k_0 + \ell_0 + m_0 - k - \ell - m - 1$.

An immediate consequence of this is ([Li, Proposition 3.2.7])

Proposition 2.1.5. *If $A(z), B(z)$ and $C(z)$ are mutually local fields, then $A(z)_{(m)}B(z)$ and $C(z)$ are local.*

Here the order of locality of $A(z)_{(m)}B(z)$ and $C(z)$ is at most $k_0+\ell_0+m_0-m-1$ for $m < m_0$ whereas $A(z)_{(m)}B(z) = 0$ for $m \geq m_0$.

Another consequence of the locality is the following:

Proposition 2.1.6. *Let $A^1(z), \dots, A^\ell(z)$ be mutually local fields. Then, for any $u \in M$,*

$$A^1_{p_1} \cdots A^\ell_{p_\ell} u = 0, \quad (p_1 + \cdots + p_\ell \geq n, p_1, \dots, p_\ell \in \mathbb{Z}), \tag{2.1.4}$$

for sufficiently large n.

Proof. Since $A^i(z)$ are fields, we have $A^i_p u = 0, (p \geq n_i)$, for sufficiently large n_i. Let m_{ij} be the order of locality of $A^i(z)$ and $A^j(z)$ and set $n = \sum_{i=1}^{\ell} n_i + \sum_{1 \leq i < j \leq \ell} m_{ij} - \ell + 1$. We shall prove (2.1.4) for this n by induction on ℓ. It trivially holds in the case $\ell = 1$. We suppose $\ell > 1$. Then, by the locality, we have

$$\begin{aligned} A^1(z_1) \cdots A^{\ell-1}(z_{\ell-1}) A^\ell(z_\ell) \prod_{i=1}^{\ell-1} (z_i - z_\ell)^{m_{i\ell}} \\ = A^\ell(z_\ell) A^1(z_1) \cdots A^{\ell-1}(z_{\ell-1}) \prod_{i=1}^{\ell-1} (z_i - z_\ell)^{m_{i\ell}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\sum_{k_1, \dots, k_{\ell-1} \geq 0} \prod_{i=1}^{\ell-1} (-1)^{k_i} \binom{m_{i\ell}}{k_i} A^1_{p_1 - k_1} \cdots A^{\ell-1}_{p_{\ell-1} - k_{\ell-1}} A^\ell_{p_\ell + k_1 + \cdots + k_{\ell-1}} u \\ &= \sum_{k_1, \dots, k_{\ell-1} \geq 0} \prod_{i=1}^{\ell-1} (-1)^{k_i} \binom{m_{i\ell}}{k_i} A^\ell_{p_\ell + \sum_{i=1}^{\ell-1} (m_{i\ell} - k_i)} A^1_{p_1 - m_{i\ell} + k_1} \cdots A^{\ell-1}_{p_{\ell-1} - m_{\ell-1} + k_{\ell-1}} u \end{aligned} \tag{2.1.5}$$

for any $p_1, \dots, p_\ell \in \mathbb{Z}$.

Now, suppose $p_1 + \cdots + p_\ell \geq n$. If $p_\ell \geq n_\ell$, then $A^1_{p_1} \cdots A^{\ell-1}_{p_{\ell-1}} A^\ell_{p_\ell} u = 0$. If $p_\ell < n_\ell$, then we have

$$\sum_{i=1}^{\ell-1} (p_i - m_{i\ell}) \geq \sum_{i=1}^{\ell-1} n_i + \sum_{1 \leq i < j \leq \ell-1} m_{ij} - \ell + 2$$

so that the right-hand side of (2.1.5) vanishes by the induction assumption. Hence we have

$$\begin{aligned} \sum_{k_1, \dots, k_{\ell-1} \in \mathbb{N}} (-1)^{k_1 + \cdots + k_{\ell-1}} \binom{m_{1\ell}}{k_1} \cdots \binom{m_{\ell-1\,\ell}}{k_{\ell-1}} \\ \times A^1_{p_1 - k_1} \cdots A^{\ell-1}_{p_{\ell-1} - k_{\ell-1}} A^\ell_{p_\ell + k_1 + \cdots + k_{\ell-1}} u = 0. \end{aligned}$$

Therefore, by induction on $n_\ell - p_\ell$ for fixed $p_1 + \cdots + p_\ell (\geq n)$, we have (2.1.4). □

2.2 Operator product expansion

This subsection is devoted to the explanation of the notion of operator product expansion from the point of view of Kac [K, Subsection 2.3], however, we reformulate it so that we do not use the delta function.

Let $A(z)$ and $B(z)$ be mutually local series on a vector space M. Then

$$[A(y), B(z)](y-z)^m = 0$$

for some $m \in \mathbb{N}$ and we have

$$[A(y)_+, B(z)](y-z)^m = -[A(y)_-, B(z)](y-z)^m.$$

Since the left-hand side does not have terms of degree greater than $m-1$ in y whereas the right does not have terms of negative degree in y, they are equal to a polynomial of degree $m-1$ in y. Hence we may write

$$[A(y)_+, B(z)](y-z)^m = \sum_{i=0}^{m-1} C^i(z)(y-z)^{m-i-1},$$

$$-[A(y)_-, B(z)](y-z)^m = \sum_{i=0}^{m-1} C^i(z)(y-z)^{m-i-1}$$

where $C^i(z)$ are some series. Now, since the difference

$$[A(y)_+, B(z)] - \sum_{i=0}^{m-1} C^i(z)/(y-z)^{i+1}|_{|y|>|z|}$$

has only finitely many terms of positive degree in y, and it vanishes if we multiply it by $(y-z)^m$, it must be identically zero by Lemma 1.1.1. Therefore, we have

$$[A(x)_+, B(z)] = \sum_{i=0}^{m-1} \left. \frac{C^i(z)}{(y-z)^{i+1}} \right|_{|y|>|z|}$$

and similarly

$$-[A(y)_-, B(z)] = \sum_{i=0}^{m-1} \left. \frac{C^i(z)}{(y-z)^{i+1}} \right|_{|y|<|z|}.$$

Therefore

$$A(y)B(z) = \sum_{i=0}^{m-1} \left. \frac{C^i(z)}{(y-z)^{i+1}} \right|_{|y|>|z|} + {:}A(y)B(z){:}, \tag{2.2.1}$$

$$B(z)A(y) = \sum_{i=0}^{m-1} \left. \frac{C^i(z)}{(y-z)^{i+1}} \right|_{|y|<|z|} + {:}A(y)B(z){:} \tag{2.2.2}$$

where $\mathop{\circ}\limits_{\circ} A(y)B(z) \mathop{\circ}\limits_{\circ} = A(y)_{-}B(z) + B(z)A(y)_{+}$.

Conversely, if (2.2.1) and (2.2.2) hold, then it is obvious that the series $A(z)$ and $B(z)$ are mutually local.

Now, it follows from (2.2.1) and (2.2.2) that $A(z)_{(j)}B(z) = C^j(z)$ by the definition of the residue products.

Hence we have obtained ([K,Theorem 2.3])

Theorem 2.2.1 (Operator product expansion). *Let $A(z)$ and $B(z)$ be series on a vector space. They are mutually local if and only if both*

$$A(y)B(z) = \sum_{i=0}^{m-1} \frac{A(z)_{(i)}B(z)}{(y-z)^{i+1}}\bigg|_{|y|>|z|} + \mathop{\circ}\limits_{\circ} A(y)B(z) \mathop{\circ}\limits_{\circ} \quad \text{and}$$

$$B(z)A(y) = \sum_{i=0}^{m-1} \frac{A(z)_{(i)}B(z)}{(y-z)^{i+1}}\bigg|_{|y|<|z|} + \mathop{\circ}\limits_{\circ} A(y)B(z) \mathop{\circ}\limits_{\circ}$$

hold for some $m \in \mathbb{N}$.

The two equalities in the theorem are often abbreviated into the single expression

$$A(y)B(z) \sim \sum_{i=0}^{m-1} \frac{A(z)_{(i)}B(z)}{(y-z)^{i+1}},$$

which is called the *operator product expansion* (OPE), and the right-hand side is called the *contraction*.

Remark 2.2.2. The first equality of Theorem 2.2.1 holds without the assumption of the locality ([LZ1, Proposition 2.3]):

$$A(y)B(z) = \sum_{i=0}^{\infty} \frac{A(z)_{(i)}B(z)}{(y-z)^{i+1}}\bigg|_{|y|>|z|} + \mathop{\circ}\limits_{\circ} A(y)B(z) \mathop{\circ}\limits_{\circ},$$

where the sum in the right is indeed a finite sum for each degree in y.

Next let us further expand the remainder $\mathop{\circ}\limits_{\circ} A(y)B(z) \mathop{\circ}\limits_{\circ}$ in case $A(z)$ and $B(z)$ are fields. To this end, we prepare the notion of a field in two variables: A series $A(y,z) = \sum_{p,q\in\mathbb{Z}} A_{p,q}y^{-p-1}z^{-q-1}$ is a field if, for any $u \in M$, there exists integers p_0 and q_0 such that

$$A_{p,q}u = 0, \quad \text{if} \quad p \geq p_0 \quad \text{or} \quad q \geq q_0.$$

In other words, $A(y,z)$ is a field if and only if $A(y,z)u \in M((y,z))$.

If $A(z)$ and $B(z)$ are fields, then the normally ordered product ${}^{\circ}_{\circ} A(y)B(z) {}^{\circ}_{\circ}$ is a field. If $A(y, z)$ is a field, then

$$A(z, z) = \sum_{p,q \in \mathbb{Z}} A_{p,q} z^{-p-q-2} = \sum_{n \in \mathbb{Z}} \Big(\sum_{p+q=n-1} A_{p,q} \Big) z^{-n-1}$$

makes sense and is a field.

Lemma 2.2.3. *If $A(y, z)$ is a field, then there exists a unique field $R(y, z)$ such that*

$$A(y, z) - A(z, z) = (y - z)R(y, z).$$

In fact, the series

$$R(y, z) = \sum_{p \le -1, q \in \mathbb{Z}} \Big(\sum_{i=0}^{\infty} A_{p-i,q+i} \Big) y^{-p-1} z^{-q-1} - \sum_{p \ge 0, q \in \mathbb{Z}} \Big(\sum_{i=0}^{\infty} A_{p+i,q-i-1} \Big) y^{-p-1} z^{-q-1}$$

is a field satisfying $A(y, z) - A(z, z) = (y - z)R(y, z)$. The uniqueness is obvious by Lemma 1.1.1.

By successive use of this lemma, we obtain ([K, Proposition 3.1])

Proposition 2.2.4 (Taylor's formula). *If $A(y, z)$ is a field, then, for any positive integer N, there exists a unique field $R_N(y, z)$ such that*

$$A(y, z) = \sum_{i=0}^{N-1} \partial_y^{(i)} A(y, z)|_{y=z} (y - z)^i + R_N(y, z)(y - z)^N.$$

In particular, for a field $A(z)$, we have

$$A(y) = \sum_{i=0}^{N-1} \partial_z^{(i)} A(z)(y - z)^i + R_N(y, z)(y - z)^N \tag{2.2.3}$$

for some field $R_N(y, z)$.

Therefore ([K,Theorem 3.1])

Theorem 2.2.5. *Let $A(z)$ and $B(z)$ be fields on a vector space. If they are mutually local at order m, then, for any positive integer N, there exists a unique field*

$R_N(y,z)$ *such that*

$$A(y)B(z) = \sum_{i=-N}^{m-1} \frac{A(z)_{(i)}B(z)}{(y-z)^{i+1}}\bigg|_{|y|>|z|} + R_N(y,z)(y-z)^N,$$

$$B(z)A(y) = \sum_{i=-N}^{m-1} \frac{A(z)_{(i)}B(z)}{(y-z)^{i+1}}\bigg|_{|y|<|z|} + R_N(y,z)(y-z)^N.$$

This result is seen to be an interpretation in the language of formal Laurent series of the expressions

$$A(y)B(z) = \sum_{i=-\infty}^{m-1} \frac{A(z)_{(i)}B(z)}{(y-z)^{-i-1}}, \quad B(z)A(y) = \sum_{i=-\infty}^{m-1} \frac{A(z)_{(i)}B(z)}{(y-z)^{-i-1}}.$$

2.3 Tensor product of fields

Now, to illustrate the role of the OPE, we consider the tensor products of fields.

Let $A(z)$ and $A'(z)$ be fields on vector spaces M and M' respectively. Then the tensor product

$$A(z) \otimes A'(z) = \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} A_k \otimes A_{n-k-1} \right) z^{-n-1}$$

is a field on $M \otimes M'$ as easily verified.

Suppose that $A(z)$ and $B(z)$ are fields on M mutually local at order m_0 and that $A'(z)$ and $B'(z)$ are fields on M' mutually local at order m'_0. Then by Theorem 2.2.5, we have

$$\begin{aligned}
&(A(y) \otimes A'(y))(B(z) \otimes B'(z)) \\
&= \left(\sum_{i=-N}^{m_0-1} \frac{A(z)_{(i)}B(z)}{(y-z)^{i+1}}\bigg|_{|y|>|z|} + R_N(y,z)(y-z)^N \right) \\
&\qquad \otimes \left(\sum_{i=-N'}^{m'_0-1} \frac{A'(z)_{(i)}B'(z)}{(y-z)^{i+1}}\bigg|_{|y|>|z|} + R'_{N'}(y,z)(y-z)^{N'} \right),
\end{aligned}$$

and the similar expression for $(B(z) \otimes B'(z))(A(y) \otimes A'(y))$ where $|y| > |z|$ is replaced by $|y| < |z|$. In particular, they coincide after multiplied by $(y-z)^{m_0+m'_0}$. Moreover, by taking sufficiently large N and N', we may compute the residue products of $A(z) \otimes A'(z)$ and $B(z) \otimes B'(z)$. Thus,

Theorem 2.3.1. *Let $A(z)$ and $B(z)$ be fields on M mutually local at order m_0 and let $A'(z)$ and $B'(z)$ be fields on M' local at order m_0'. Then the tensor products $A(z) \otimes A'(z)$ and $B(z) \otimes B'(z)$ are fields on $M \otimes M'$ local at order $m_0 + m_0'$ with*

$$(A(z) \otimes A'(z))_{(m)}(B(z) \otimes B'(z)) = \sum_{i \in \mathbb{Z}} (A(z)_{(i)}B(z)) \otimes (A'(z)_{(m-i-1)}B'(z))$$

for any $m \in \mathbb{Z}$.

Note that we have

$$\begin{aligned}
&(A(y) \otimes A'(y))(B(z) \otimes B'(z)) \\
&\qquad = \sum_{i=-\infty}^{m_0-1} \frac{A(z)_{(i)}B(z)}{(y-z)^{i+1}} \otimes \sum_{i=-\infty}^{m_0'-1} \frac{A'(z)_{(i)}B'(z)}{(y-z)^{i+1}} \\
&\qquad = \sum_{m=-\infty}^{m_0+m_0'-1} \frac{\sum_{i \in \mathbb{Z}} (A(z)_{(i)}B(z)) \otimes (A'(z)_{(m-i-1)}B'(z))}{(y-z)^{m+1}}
\end{aligned}$$

formally.

2.4 Affine Lie algebras and the OPE of currents

Let us describe affine Lie algebras as an example of OPE.

Let $\mathfrak{g}$ be a Lie algebra with a symmetric bilinear form $(\,|\,) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbf{k}$ which is invariant (or associative) in the sense that

$$([X, Y]|Z) = (X|[Y, Z])$$

holds for any $X, Y, Z \in \mathfrak{g}$. Set

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbf{k}} \mathbf{k}[t, t^{-1}] \oplus \mathbf{k}K$$

and define a bilinear map $[\,,\,] : \hat{\mathfrak{g}} \times \hat{\mathfrak{g}} \longrightarrow \hat{\mathfrak{g}}$ by setting

$$\begin{aligned}
[X \otimes t^m, Y \otimes t^n] &= [X, Y] \otimes t^{m+n} + m\delta_{m+n,0}(X|Y)K, \\
[K, X \otimes t^n] &= [X \otimes t^n, K] = [K, K] = 0.
\end{aligned}$$

Then $[\,,\,]$ gives a Lie algebra structure on $\hat{\mathfrak{g}}$ called the *affine Lie algebra* associated to $\mathfrak{g}$ and $(\,|\,)$. A $\hat{\mathfrak{g}}$-module on which the center K acts by a scalar k is called a $\hat{\mathfrak{g}}$-module at level k.

Let M be a $\hat{\mathfrak{g}}$-module at level k. We denote the action of $X \otimes t^n$ on M by X_n, and set

$$X(z) = \sum_{n \in \mathbb{Z}} X_n z^{-n-1}, \quad X \in \mathfrak{g}.$$

The series $X(z), X \in \mathfrak{g}$, are called the (nonabelian) *currents* associated to $\hat{\mathfrak{g}}$. Then

$$\begin{aligned}
&[X(y)_+, Y(z)] \\
&\quad = \sum_{\ell \geq 0} \sum_{m \in \mathbb{Z}} [X_\ell, Y_m] y^{-\ell-1} z^{-m-1} \\
&\quad = \sum_{\ell \geq 0, m \in \mathbb{Z}} [X,Y]_{\ell+m} y^{-\ell-1} z^{-m-1} + \sum_{\ell \geq 0, m \in \mathbb{Z}} \ell \delta_{\ell+m,0} (X|Y) k y^{-\ell-1} z^{-m-1} \\
&\quad = \sum_{\ell \geq 0, n \in \mathbb{Z}} [X,Y]_n z^{-n-1} y^{-\ell-1} z^{\ell} + \sum_{\ell \geq 0} (X|Y) k y^{-\ell-1} \ell z^{\ell-1} \\
&\quad = \left.\frac{[X,Y](z)}{y-z}\right|_{|y|>|z|} + \left.\frac{(X|Y)k}{(y-z)^2}\right|_{|y|>|z|}
\end{aligned}$$

and

$$-[X(y)_-, Y(z)] = \left.\frac{[X,Y](z)}{y-z}\right|_{|y|<|z|} + \left.\frac{(X|Y)k}{(y-z)^2}\right|_{|y|<|z|}.$$

Hence $X(z)$ and $Y(z)$ are mutually local, and

$$\begin{aligned}
X(y)Y(z) &= \left.\frac{[X,Y](z)}{y-z}\right|_{|y|>|z|} + \left.\frac{(X|Y)k}{(y-z)^2}\right|_{|y|>|z|} + {:}X(y)Y(z){:} \\
Y(z)X(y) &= \left.\frac{[X,Y](z)}{y-z}\right|_{|y|<|z|} + \left.\frac{(X|Y)k}{(y-z)^2}\right|_{|y|<|z|} + {:}X(y)Y(z){:}.
\end{aligned} \tag{2.4.1}$$

Namely, the OPE is given by

$$X(y)Y(z) \sim \frac{[X,Y](z)}{y-z} + \frac{(X|Y)k}{(y-z)^2}.$$

Therefore, the m-th residue products, $(m \geq 0)$, are given by

$$X(z)_{(m)}Y(z) = \begin{cases} 0, & (m \geq 2), \\ (X|Y)kI(z), & (m = 1), \\ [X,Y](z), & (m = 0). \end{cases}$$

Conversely, suppose given a linear map

$$\mathfrak{g} \longrightarrow (\operatorname{End} M)[[z, z^{-1}]], \quad X \longmapsto X(z)$$

satisfying the OPE above. Then we recover the commutation relation

$$[X_m, Y_n] = [X, Y]_{m+n} + m\delta_{m+n,0}(X|Y)k$$

easily from the OPE (2.4.1).

Therefore,

Proposition 2.4.1. *A linear map*

$$\mathfrak{g} \longrightarrow (\operatorname{End} M)[[z, z^{-1}]], \quad X \longmapsto X(z) = \sum_{n \in \mathbb{Z}} X_n z^{-n-1}$$

defines a $\hat{\mathfrak{g}}$-module structure at level k on M by

$$\hat{\mathfrak{g}} \longrightarrow \operatorname{End} M, \quad X \otimes t^n \longmapsto X_n$$

if and only if the series $X(z)$ and $Y(z)$ are mutually local with the OPE

$$X(y)Y(z) \sim \frac{[X, Y](z)}{y - z} + \frac{(X|Y)k}{(y - z)^2}$$

for any $X, Y \in \mathfrak{g}$.

2.5 Virasoro algebra and the OPE of the energy-momentum tensor

Let $\mathcal{V}ir$ denote the vector space spanned by $\{L_n \mid n \in \mathbb{Z}\} \cup \{C\}$:

$$\mathcal{V}ir = (\oplus_{n \in \mathbb{Z}} \mathbf{k}\, L_n) \oplus \mathbf{k}\, C.$$

Define a bilinear map $[\, ,\,] : \mathcal{V}ir \times \mathcal{V}ir \longrightarrow \mathcal{V}ir$ by setting

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}\, C, \\ [C, L_m] &= [L_n, C] = [C, C] = 0. \end{aligned}$$

Then $[\, ,\,]$ gives a Lie algebra structure on $\mathcal{V}ir$ called the *Virasoro algebra*. A $\mathcal{V}ir$-module on which the center C acts by a scalar c is called a $\mathcal{V}ir$-module of *central charge* c.

Let M be a $\mathcal{V}ir$-module of central charge c. We denote the action of L_n on M by the same symbol, and let

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}.$$

Then

$$
\begin{aligned}
&[T(y)_+, T(z)] \\
&= \sum_{\ell\geq -1, k\in\mathbb{Z}} [L_\ell, L_m] y^{-\ell-2} z^{-m-2} \\
&= \sum_{\ell\geq -1, m\in Z} (\ell - m) L_{\ell+m} y^{-\ell-2} z^{-m-2} + \sum_{\ell\geq -1, m\in Z} \frac{\ell^3-\ell}{12} \delta_{\ell+m,0} c y^{-\ell-2} z^{-m-2} \\
&= \sum_{\ell\geq -1, n\in\mathbb{Z}} (2\ell - n) L_n y^{-\ell-2} z^{\ell-n-2} + \sum_{\ell\geq -1} \frac{\ell^3-\ell}{12} c y^{-\ell-2} z^{\ell-2} \\
&= \sum_{n\in\mathbb{Z}} (-n-2) L_n z^{-n-3} \sum_{\ell\geq -1} y^{-\ell-2} z^{\ell+1} \\
&\qquad + 2 \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \sum_{\ell\geq -1} (\ell+1) y^{-\ell-2} z^\ell + \frac{c}{2} \sum_{\ell\in\mathbb{Z}} \binom{\ell+1}{3} y^{-\ell-2} z^{\ell-2} \\
&= \left.\frac{\partial T(z)}{y-z}\right|_{|y|>|z|} + \left.\frac{2T(z)}{(y-z)^2}\right|_{|y|>|z|} + \left.\frac{c/2}{(y-z)^4}\right|_{|y|>|z|}.
\end{aligned}
$$

and

$$
-[T(y)_-, T(z)] = \left.\frac{\partial T(z)}{y-z}\right|_{|y|<|z|} + \left.\frac{2T(z)}{(y-z)^2}\right|_{|y|<|z|} + \left.\frac{c/2}{(y-z)^4}\right|_{|y|<|z|}.
$$

Hence $T(z)$ is local to itself, and

$$
\begin{aligned}
T(y)T(z) &= \left.\frac{\partial T(z)}{y-z}\right|_{|y|>|z|} + \left.\frac{2T(z)}{(y-z)^2}\right|_{|y|>|z|} + \left.\frac{c/2}{(y-z)^4}\right|_{|y|>|z|} + {:}T(y)T(z){:}, \\
T(z)T(y) &= \left.\frac{\partial T(z)}{y-z}\right|_{|y|<|z|} + \left.\frac{2T(z)}{(y-z)^2}\right|_{|y|<|z|} + \left.\frac{c/2}{(y-z)^4}\right|_{|y|<|z|} + {:}T(y)T(z){:}.
\end{aligned}
$$

Namely the OPE is given by

$$
T(y)T(z) \sim \frac{\partial T(z)}{y-z} + \frac{2T(z)}{(y-z)^2} + \frac{c/2}{(y-z)^4}.
$$

Therefore, the m-th residue products, $(m \geq 0)$, are given by

$$
T(z)_{(m)}T(z) = \begin{cases} 0, & (m \geq 4), \\ c/2 I(z), & (m = 3), \\ 0, & (m = 2), \\ 2T(z), & (m = 1), \\ \partial T(z), & (m = 0). \end{cases}
$$

Conversely, if a series $T(z)$ on M satisfies the OPE above, then we recover the commutation relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}c.$$

Therefore,

Proposition 2.5.1. *A series $T(z)$ on a vector space M defines a $\mathcal{V}ir$-module structure of central charge c on M by the Fourier modes of*

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

if and only if $T(z)$ is local to itself with the OPE

$$T(y)T(z) \sim \frac{\partial T(z)}{y-z} + \frac{2T(z)}{(y-z)^2} + \frac{c/2}{(y-z)^4}.$$

Such a series $T(z)$ appears as the (chiral) energy-momentum tensor in conformal field theory.

3 Borcherds identity for local fields

In this section, we show the identity satisfied by three mutually local fields with respect to the residue products, which is a consequence of the usual Jacobi identity

$$[[A(x), B(y)], C(z)] = [A(x), [B(y), C(z)]] - [B(y), [A(x), C(z)]].$$

The strategy is, roughly speaking, to multiply this Jacobi identity by the rational function

$$(x-y)^r(y-z)^q(x-z)^p$$

and take the residue after expanding it in various regions. Special cases of such derivation are considered by Li [Li, p.166] and Kac [K, Proposition 2.3 (c) and Proposition 3.3 (c)]. However, to execute it in full generality, one has to be careful about divergence, which can be avoided by clever use of the locality.

3.1 Binomial identities

Let us consider the rational function

$$F(x,y,z) = (x-y)^r(y-z)^q(x-z)^p, \quad p,q,r \in \mathbb{Z},$$

which has the expansions

$$F_0(x,y,z) = \sum_{i=0}^{\infty} \binom{p}{i} (x-y)^{r+i}(y-z)^{p+q-i}, \qquad (|y-z| > |x-y|),$$

$$F_1(x,y,z) = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} (x-z)^{p+r-i}(y-z)^{q+i}, \qquad (|x-z| > |y-z|),$$

$$F_2(x,y,z) = \sum_{i=0}^{\infty} (-1)^{r+i} \binom{r}{i} (y-z)^{q+r-i}(x-z)^{p+i}, \qquad (|y-z| > |x-z|)$$

convergent in the respective regions.

Expanding these series again into series in x, y, z, and comparing them with the corresponding expansions of $F(x,y,z)$, we have, for example,

$$F(x,y,z)|_{|x|>|y|>|z|} = F_1(x,y,z)|_{|x|>|y|>|z|},$$

for all integers p, q and r. To be precise, we have

$$\begin{aligned} &\sum_{i,j,k\geq 0} (-1)^{i+j+k} \binom{r}{i}\binom{q}{j}\binom{p}{k} x^{p+r-i-k} y^{q+i-j} z^{j+k} \\ &\qquad = \sum_{i,j,k\geq 0} (-1)^{i+j+k} \binom{r}{i}\binom{p+r-i}{j}\binom{q+i}{k} x^{p+r-i-j} y^{q+i-k} z^{j+k} \end{aligned}$$

which is equivalent to a set of identities for binomial coefficients (See Appendix C for detail).

Now, let $A(z), B(z)$ and $C(z)$ be fields on a vector space M which are not necessarily mutually local. The above argument shows that we have the following set of identities involving these fields:

Lemma 3.1.1. *For all $p, q, r \in \mathbb{Z}$, we have*

$$\begin{aligned} A(x)B(y)C(z)F(x,y,z) &= A(x)B(y)C(z)F_1(x,y,z), \\ A(x)C(z)B(y)F(x,y,z) &= A(x)C(z)B(y)F_1(x,y,z), \\ B(y)A(x)C(z)F(x,y,z) &= B(y)A(x)C(z)F_2(x,y,z), \\ B(y)C(z)A(x)F(x,y,z) &= B(y)C(z)A(x)F_2(x,y,z), \\ C(z)A(x)B(y)F(x,y,z) &= C(z)A(x)B(y)F_0(x,y,z), \\ C(z)B(y)A(x)F(x,y,z) &= C(z)B(y)A(x)F_0(x,y,z), \end{aligned}$$

where $A(z), B(z), C(z)$ are fields on a vector space M.

Here we have omitted the regions of the expansions according to our convention in Notation 1.3.3.

Note that, for example, the identity

$$A(x)B(y)C(z)F(x,y,z) = A(x)B(y)C(z)F_0(x,y,z)$$

is not valid in general. In fact,

$$F_0(x,y,z)|_{|x|>|y|>|z|} = \sum_{i,j,k\geq 0} (-1)^{j+k} \binom{p}{i} \binom{r+i}{j} \binom{p+q-i}{k} x^{r+i-j} y^{p+q-i+j-k} z^k$$

is divergent for negative p since each of the coefficients becomes an infinite sum. On the other hand, the coefficients are finite sums for nonnegative p and the equality is valid in this case.

Such cases are summarized as follows:

Lemma 3.1.2. *For all $p \in \mathbb{N}$ and $q, r \in \mathbb{Z}$, we have*

$$A(x)B(y)C(z)F(x,y,z) = A(x)B(y)C(z)F_0(x,y,z),$$
$$B(y)A(x)C(z)F(x,y,z) = B(y)A(x)C(z)F_0(x,y,z).$$

For all, $p, q \in \mathbb{Z}$ and all $r \in \mathbb{N}$, we have

$$B(y)C(z)A(x)F(x,y,z) = B(y)C(z)A(x)F_1(x,y,z),$$
$$C(z)B(y)A(x)F(x,y,z) = C(z)B(y)A(x)F_1(x,y,z),$$
$$A(x)C(z)B(y)F(x,y,z) = A(x)C(z)B(y)F_2(x,y,z),$$
$$C(z)A(x)B(y)F(x,y,z) = C(z)A(x)B(y)F_2(x,y,z).$$

3.2 Borcherds identity for non-local fields

If p and r are both nonnegative, then all the identities of Lemma 3.1.1 and Lemma 3.1.2 are valid. Therefore

$$\begin{aligned}
[[A(x),B(y)],C(z)]F_0(x,y,z) &= [[A(x),B(y)],C(z)]F(x,y,z) \\
&= ([A(x),[B(y),C(z)]] - [B(y),[A(x),C(z)]])\,F(x,y,z) \\
&= [A(x),[B(y),C(z)]]F_1(x,y,z) - [B(y),[A(x),C(z)]]F_2(x,y,z).
\end{aligned}$$

Thus we have obtained

Theorem 3.2.1. *Let $A(z), B(z)$ and $C(z)$ be fields on a vector space. Then, for any $p, r \in \mathbb{N}$ and any $q \in \mathbb{Z}$,*

$$\sum_{i=0}^{\infty} \binom{p}{i} [[A(x), B(y)], C(z)](x-y)^{r+i}(y-z)^{p+q-i}$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} [A(x), [B(y), C(z)]](x-y)^{p+r-i}(y-z)^{q+i}$$
$$- \sum_{i=0}^{\infty} (-1)^{r+i} \binom{r}{i} [B(y), [A(x), C(z)]](y-z)^{q+r-i}(x-z)^{p+i}.$$

Taking $\operatorname*{Res}_{y=0} \operatorname*{Res}_{x=0}$ of the both sides, we have[8]

Corollary 3.2.2. *Let $A(z), B(z)$ and $C(z)$ be fields on a vector space. Then, for any $p, r \in \mathbb{N}$ and any $q \in \mathbb{Z}$,*

$$\sum_{i=0}^{\infty} \binom{p}{i} \left(A(z)_{(r+i)}B(z)\right)_{(p+q-i)} C(z)$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \left(A(z)_{(p+r-i)}(B(z)_{(q+i)}C(z)) - (-1)^r B(z)_{(q+r-i)}(A(z)_{(p+i)}C(z))\right).$$

This result is not true in general if p or r is negative. It is because of the failure of Lemma 3.1.2 for such indices. However, if we assume the locality of the fields $A(z), B(z)$ and $C(z)$, then the theorem is generalized to arbitrary integers p, q, r, as we will see in the following two sections.

3.3 Consequences of locality

Now assume that the fields $A(z), B(z)$ and $C(z)$ are mutually local. Then, by Lemma 2.1.4, we have

$$(y-z)^n [A(x), B(y)] C(z)(x-y)^k (y-z)^\ell (x-z)^m$$
$$= (y-z)^n C(z)[A(x), B(y)](x-y)^k (y-z)^\ell (x-z)^m$$

for sufficiently large n. This reduces a calculation involving the left-hand side to that involving the right-hand side.

In this way, we obtain the following lemma.

[8] A special case ($r = 0$) of this result is described by Kac [K, Proposition 3.3 (c)]. General case is deduced from this case by the inductive structure of the Borcherds identity (cf. Proposition 4.3.1).

Lemma 3.3.1. *If $A(z), B(z)$ and $C(z)$ are mutually local, then for any $p, q, r \in \mathbb{Z}$,*

$$[A(x), B(y)]C(z)F(x,y,z) = \sum_{i=0}^{\infty} \binom{p}{i} [A(x), B(y)]C(z)(x-y)^{r+i}(y-z)^{p+q-i},$$

$$[B(y), C(z)]A(x)F(x,y,z) = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} [B(y), C(z)]A(x)(x-z)^{p+r-i}(y-z)^{q+i},$$

$$[A(x), C(z)]B(y)F(x,y,z) = \sum_{i=0}^{\infty} (-1)^{r+i} \binom{r}{i} [A(x), C(z)]B(y)(y-z)^{q+r-i}(x-z)^{p+i}.$$

Here the right-hand side of each equality is a finite sum because of the locality.

Proof. By Lemma 2.1.4 and Lemma 3.1.2,

$$\begin{aligned}
&(y-z)^n \sum_{i=0}^{\infty} \binom{p}{i} [A(x), B(y)]C(z)(x-y)^{r+i}(y-z)^{p+q-i} \\
&= (y-z)^n \sum_{i=0}^{\infty} \binom{p}{i} C(z)[A(x), B(y)](x-y)^{r+i}(y-z)^{p+q-i} \\
&= (y-z)^n C(z)[A(x), B(y)]F_0(x,y,z) \\
&= (y-z)^n C(z)[A(x), B(y)]F(x,y,z) \\
&= (y-z)^n [A(x), B(y)]C(z)F(x,y,z)
\end{aligned}$$

for sufficiently large n. Therefore the series

$$\begin{aligned}
D(y,z) = \sum_{i=0}^{\infty} \binom{p}{i} \Big(&[A(x), B(y)]C(z)(x-y)^{r+i}(y-z)^{p+q-i} \\
&- [A(x), B(y)]C(z)F(x,y,z) \Big)
\end{aligned}$$

satisfies $(y-z)^n D(y,z)v = 0$ for any vector $v \in M$. Moreover, since $D(y,z)v$ has only finitely many terms of negative degree in z, we must have $D(y,z)v = 0$ by Lemma 1.1.1. Thus we have shown the first equality of the lemma. The other equalities are proved similarly. □

Remark 3.3.2. Note that the right-hand side of the first equality, for example, makes sense if $A(z)$ and $B(z)$ are mutually local. However, in order the equality to hold, we need not only this but also the locality of $A(z)$ and $C(z)$, and of $B(z)$ and $C(z)$ in general.

3.4 Borcherds identity for local fields

Now we arrive at the main result of this part.

Theorem 3.4.1. *Let $A(z), B(z)$ and $C(z)$ be fields on a vector space. If they are mutually local, then for any $p, q, r \in \mathbb{Z}$,*

$$\begin{aligned}\sum_{i=0}^{\infty}\binom{p}{i}&[[A(x),B(y)],C(z)](x-y)^{r+i}(y-z)^{p+q-i}\\&=\sum_{i=0}^{\infty}(-1)^i\binom{r}{i}\Big([A(x),[B(y),C(z)]](x-z)^{p+r-i}(y-z)^{q+i}\\&\qquad\qquad-(-1)^r[B(y),[A(x),C(z)]](y-z)^{q+r-i}(x-z)^{p+i}\Big).\end{aligned}$$

Proof. By Lemma 3.1.1 and Lemma 3.3.1, we have

$$\begin{aligned}&\sum_{i=0}^{\infty}\binom{p}{i}\left([A(x),B(y)]C(z)-C(z)[A(x),B(y)]\right)(x-y)^{r+i}(y-z)^{p+q-i}\\&=\left([A(x),B(y)]C(z)-C(z)[A(x),B(y)]\right)F(x,y,z)\\&=(A(x)[B(y),C(z)]-[B(y),C(z)]A(x))F(x,y,z)\\&\qquad-(B(y)[A(x),C(z)]-[A(x),C(z)]B(y))\,F(x,y,z)\\&=\sum_{i=0}^{\infty}(-1)^i\binom{r}{i}\left(A(x)[B(y),C(z)]-[B(y),C(z)]A(x)\right)(x-z)^{p+r-i}(y-z)^{q+i}\\&-\sum_{i=0}^{\infty}(-1)^{r+i}\binom{r}{i}\left(B(y)[A(x),C(z)]-[A(x),C(z)]B(y)\right)(y-z)^{q+r-i}(x-z)^{p+i},\end{aligned}$$

as desired. □

Taking $\operatorname*{Res}_{y=0}\operatorname*{Res}_{x=0}$, we have[9]

Corollary 3.4.2 (Borcherds identity for local fields). *Let $A(z), B(z)$ and $C(z)$ be fields on a vector space. If they are mutually local, then for any $p, q, r \in \mathbb{Z}$,*

$$\begin{aligned}&\sum_{i=0}^{\infty}\binom{p}{i}\left(A(z)_{(r+i)}B(z)\right)_{(p+q-i)}C(z)\\&=\sum_{i=0}^{\infty}(-1)^i\binom{r}{i}\left(A(z)_{(p+r-i)}(B(z)_{(q+i)}C(z))-(-1)^rB(z)_{(q+r-i)}(A(z)_{(p+i)}C(z))\right).\end{aligned}$$

[9]This result is implicitly shown in the work of Li. In fact, by [Li, proof of Proposition 3.2.9], we can apply [Li, proof of Proposition 2.2.4] to $a = A(z), b = B(z), c = C(z)$, and $T = \partial_z$; the result follows (cf. Proposition 3.4.3 and Note 6.2.2).

The following special case of the Borcherds identity holds under a weaker assumption ([Li, proof of Proposition 3.2.9]).

Proposition 3.4.3 (Li). *Let $A(z), B(z)$ and $C(z)$ be fields. If $A(z)$ and $B(z)$ are mutually local at order n_0, then*

$$\sum_{i=0}^{\infty}(-1)^i\binom{n}{i}A(z)_{(p+n-i)}(B(z)_{(q+i)}C(z))$$
$$=\sum_{i=0}^{\infty}(-1)^{n+i}\binom{n}{i}B(z)_{(q+n-i)}(A(z)_{(p+i)}C(z))$$

for all $n \geq n_0$.

Proof. By Lemma 3.1.1 and 3.1.2 for $r = n \geq 0$, we have

$$\begin{aligned}\sum_{i=0}^{\infty}&(-1)^i\binom{n}{i}[A(x),[B(y),C(z)]](x-z)^{p+n-i}(y-z)^{q+i}\\ &= [A(x),[B(y),C(z)]]F_1(x,y,z) = [A(x),[B(y),C(z)]]F(x,y,z),\\ \sum_{i=0}^{\infty}&(-1)^{n+i}\binom{n}{i}[B(y),[A(x),C(z)]](y-z)^{q+n-i}(x-z)^{p+i}\\ &= [B(y),[A(x),C(z)]]F_2(x,y,z) = [B(y),[A(x),C(z)]]F(x,y,z).\end{aligned}$$

They coincide by the locality of $A(z)$ and $B(z)$. Take $\operatorname*{Res}_{y=0}\operatorname*{Res}_{z=0}$ to get the result. □

3.5 Skew symmetry

Let us discuss the identity satisfied by two fields with respect to the residue products. We first consider the non-local case:

Proposition 3.5.1. *Let $A(z)$ and $B(z)$ be fields. Then for any integer m, we have*

$$\left(B(z)_{(m)}A(z)\right)_+ = \sum_{i=0}^{\infty}(-1)^{m+i+1}\partial^{(i)}\left(A(z)_{(m+i)}B(z)\right)_+.$$

Proof. Let n be a nonnegative integer. Then

$$
\begin{aligned}
&z^n[B(y),A(z)](y-z)^m = (-1)^{m+1}z^n[A(z),B(y)](z-y)^m\\
&= (-1)^{m+1}\sum_{i=0}^{n}\binom{n}{i}y^{n-i}(z-y)^i[A(z),B(y)](z-y)^m\\
&= (-1)^{m+1}\sum_{i=0}^{n}(\partial_y^{(i)}y^n)[A(z),B(y)](z-y)^{m+i}\\
&= (-1)^{m+1}\sum_{i=0}^{n}(-1)^i y^n\partial_y^{(i)}\left([A(z),B(y)](z-y)^{m+i}\right) + \quad \text{(total derivative)}.
\end{aligned}
$$

Take $\operatorname*{Res}_{z=0}\operatorname*{Res}_{y=0}$ of the both sides, to get

$$
\left(B_{(m)}A\right)_n = \sum_{i=0}^{n}(-1)^{m+i+1}\partial^{(i)}\left(A_{(m+i)}B\right)_n.
$$

Since this holds for all nonnegative integers n, we have the result. □

If the two fields are mutually local, then the equality holds also for the negative parts:

Proposition 3.5.2. *Let $A(z)$ and $B(z)$ be fields. If $A(z)$ and $B(z)$ are mutually local, then for any integer m, we have,*

$$
B(z)_{(m)}A(z) = \sum_{i=0}^{\infty}(-1)^{m+i+1}\partial^{(i)}\left(A(z)_{(m+i)}B(z)\right).
$$

This is a special case of Corollary 3.4.2. In fact, substituting the identity fields $I(z)$ for $C(z)$ and setting $p=-1, q=0$ and $r=m$ in the corollary and using Proposition 1.4.4, we have the result (see (4.2.6) in Subsection 4.2).

An alternative proof is as follows (cf. [K, Proof of Proposition 3.3 (b)]). Since $A(z)$ and $B(z)$ are local,

$$
A(z)B(y) = \sum_{j=-N}^{m_0-1}\left.\frac{A(y)_{(j)}B(y)}{(z-y)^{j+1}}\right|_{|z|<|y|} + R_N(z,y)(z-y)^N
$$

by the operator product expansion, where $N>-m$. Here we have used Theorem 2.2.1 of which the roles of y and z are swapped. Applying Taylor's formula (2.2.3) to $A(z)_{(j)}B(y)$, we rewrite the right-hand side as

$$\sum_{j=-N}^{m_0-1} \frac{(-1)^{j+1}}{(y-z)^{j+1}}\bigg|_{|y|>|z|} \left(\sum_{i=0}^{M-1} \partial^{(i)}(A(z)_{(j)}B(z))(y-z)^i + S_M^j(y,z)(y-z)^M \right)$$
$$+ R_N(y,z)(y-z)^N$$

where $S_M^j(y,z)$ are some fields. This is written as

$$\sum_{j=-N}^{m_0-1} \sum_{i=0}^{M-1} (-1)^{j+1} \frac{\partial_y^{(i)}(A(y)_{(j)}B(y))}{(y-z)^{j-i+1}}\bigg|_{|y|>|z|} + T_N(y,z)(y-z)^N$$

for some fields $T_N(y,z)$ if M is sufficiently large. Comparing the coefficient to $(y-z)^{-m-1}$ with that in

$$A(z)B(y) = \sum_{k=-N}^{m_0-1} \frac{B(z)_{(k)}A(z)}{(y-z)^{k+1}}\bigg|_{|y|>|z|} + U_N(y,z)(y-z)^N,$$

we have the result. Note that we formally have

$$\sum_{k=-\infty}^{m_0-1} \frac{B(z)_{(k)}A(z)}{(y-z)^{k+1}} = \sum_{j=-\infty}^{m_0-1} \frac{A(y)_{(j)}B(y)}{(z-y)^{j+1}} = \sum_{j=-\infty}^{m_0-1} \sum_{i=0}^{\infty} \frac{(-1)^{j+1}\partial^{(i)}(A(z)_{(i)}B(z))}{(y-z)^{j-i+1}}.$$

3.6 OPE of normally ordered products

To illustrate the role of the Borcherds identity, we will describe some of its consequences when applied to the normally ordered products of fields.

Let $A(z), B(z)$ and $C(z)$ be fields on a vector space. Recall that the normally ordered product of $A(z)$ and $B(z)$ is given by

$$\mathbin{\circ\atop\circ} A(z)B(z) \mathbin{\circ\atop\circ} = A(z)_-B(z) + B(z)A(z)_+ = A(z)_{(-1)}B(z).$$

Now suppose that the fields $A(z), B(z)$ and $C(z)$ are mutually local. Then so are $A(z)$ and $\mathbin{\circ\atop\circ} B(z)C(z) \mathbin{\circ\atop\circ}$, and the Borcherds identity Corollary 3.4.2 for $p=m, q=-1, r=0$ yields

(3.6.1)

$$A(z)_{(m)} \mathbin{\circ\atop\circ} B(z)C(z) \mathbin{\circ\atop\circ} = \mathbin{\circ\atop\circ} B(z)(A(z)_{(m)}C(z)) \mathbin{\circ\atop\circ} + \sum_{i=0}^{\infty} \binom{m}{i} (A(z)_{(i)}B(z))_{(m-i-1)}C(z).$$

Similarly, Corollary 3.4.2 for $p = 0, q = m, r = -1$ reads

(3.6.2)

$$\begin{aligned}&{\circ\atop\circ} A(z)B(z) {\circ\atop\circ}_{(m)} C(z)\\ &\qquad = \sum_{i=0}^{\infty} \Big(A(z)_{(-i-1)}(B(z)_{(m+i)}C(z)) + B(z)_{(m-i-1)}(A(z)_{(i)}C(z)) \Big).\end{aligned}$$

We may likewise compute the residue products of the normally ordered products of many number of fields.

Now, let $\mathfrak{g}$ be a Lie algebra with a symmetric invariant bilinear form $(\,|\,)$ and let $\hat{\mathfrak{g}}$ be the associated affine Lie algebra. Let M be a $\hat{\mathfrak{g}}$-module at level k and suppose that the currents $X(z), (X \in \mathfrak{g})$, are fields on M. Recall that the residue products are given by (see Subsection 2.4)

$$X(z)_{(m)}Y(z) = \begin{cases} 0, & (m \geq 2), \\ (X|Y)kI(z), & (m = 1), \\ [X, Y](z), & (m = 0). \end{cases}$$

Then, by (3.6.1), we have

$$X(z)_{(m)} {\circ\atop\circ} Y(z)Z(z) {\circ\atop\circ} = \begin{cases} 0, & (m \geq 3), \\ ([X, Y]|Z)kI(z), & (m = 2), \\ (X|Y)kZ(z) + (X|Z)kY(z) + [[X, Y], Z](z), & (m = 1), \\ {\circ\atop\circ} [X, Y](z)Z(z) {\circ\atop\circ} + {\circ\atop\circ} Y(z)[X, Z](z) {\circ\atop\circ}, & (m = 0). \end{cases}$$

Now, suppose that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbf{k} = \mathbb{C}$ and let $(\,|\,)$ be a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$, which is unique up to a normalization. Then, for a basis $\{J^i\}$ of $\mathfrak{g}$, we may consider the dual basis $\{J_i\}$ with respect to $(\,|\,)$. Set

$$S(z) = \sum_{i=1}^{\dim \mathfrak{g}} {\circ\atop\circ} J^i(z)J_i(z) {\circ\atop\circ}.$$

It does not depend on the choice of the basis $\{J^i\}$.

To compute the residue products of $S(z)$, let us recall the property of the

Casimir element $Q = \sum_{i=1}^{\dim \mathfrak{g}} J^i J_i \in U(\mathfrak{g})$. For any $X \in \mathfrak{g}$,

$$
\begin{aligned}
[X, Q] &= \sum_j [X, J^j] J_j + \sum_i J^i [X, J_i] \\
&= \sum_{i,j} (J_i | [X, J^j]) J^i J_j + \sum_{i,j} J^i ([X, J_i] | J^j) J_j \\
&= \sum_{i,j} ((J_i | [X, J^j]) + ([X, J_i] | J^j)) J^i J_j \\
&= 0.
\end{aligned}
\tag{3.6.3}
$$

Hence Q acts as a scalar on an irreducible $\mathfrak{g}$-module. In particular, we have

$$
\sum_{i=1}^{\dim \mathfrak{g}} [J^i, [J^i, X]] = 2h^\vee X
\tag{3.6.4}
$$

for some scalar $h^\vee$ independent of $X \in \mathfrak{g}$. If $(\,|\,)$ is normalized so that $(\theta|\theta) = 2$ for the highest root θ, the scalar $h^\vee$ coincides with the dual Coxeter number of $\mathfrak{g}$: $h^\vee = (\theta|\theta + 2\rho)/(\theta|\theta)$, where ρ is half the sum of positive roots, called the Weyl vector. We also note

$$
\sum_{i=1}^{\dim \mathfrak{g}} (J^i | J_i) = \dim \mathfrak{g}, \qquad \sum_{i=1}^{\dim \mathfrak{g}} [J^i, J_i] = 0.
$$

By using these properties, we obtain

$$
\begin{aligned}
X(z)_{(2)} S(z) &= \sum_i ([X, J^i] | J_i) k I(z) \\
&= (X | \sum_i [J^i, J_i]) k I(z) = 0, \\
X(z)_{(1)} S(z) &= \sum_i ((X | J^i) k J_i(z) + (X | J_i) k J^i(z) + [[X, J^i], J^i](z)) \\
&= k(\sum_i (X | J^i) J_i(z) + \sum_i (X | J_i) J^i(z)) + \sum_i [J_i, [J^i, X]](z) \\
&= 2(k + h^\vee) X(z), \\
X(z)_{(0)} S(z) &= \sum_j {:}[X, J^j](z) J_j(z){:} + \sum_i {:}J^i(z) [X, J_i](z){:} \\
&= \sum_{i,j} (J_i | [X, J^j]) {:}J^i(z) J_j(z){:} + \sum_{i,j} ([X, J_i] | J^i) {:}J^i(z) J_j(z){:} \\
&= 0.
\end{aligned}
$$

Therefore,

$$S(z)_{(m)}S(z) = \begin{cases} 0, & (m \geq 4), \\ 2(k+h^\vee)k(\dim \mathfrak{g})\, I(z), & (m=3), \\ 0, & (m=2), \\ 4(k+h^\vee)S(z), & (m=1), \\ 2(k+h^\vee)\partial S(z), & (m=0). \end{cases}$$

Since $S(z)$ is local to itself, we have the following OPE,

$$S(y)S(z) \sim 2(k+h^\vee)\left(\frac{k\dim\mathfrak{g}}{(y-z)^4} + \frac{2S(z)}{(y-z)^2} + \frac{\partial S(z)}{y-z}\right).$$

Hence, if $k \neq -h^\vee$, the field

$$T(z) = \frac{1}{2(k+h^\vee)}S(z) = \frac{1}{2(k+h^\vee)}\sum_{i=1}^{\dim\mathfrak{g}} {}^{\circ}_{\circ} J^i(z)J_i(z) {}^{\circ}_{\circ}$$

satisfies the OPE of Virasoro field

$$T(y)T(z) \sim \frac{c/2}{(y-z)^4} + \frac{2T(z)}{(y-z)^2} + \frac{\partial T(z)}{y-z}$$

where the central charge is given by

$$c = \frac{k\dim\mathfrak{g}}{k+h^\vee}.$$

The field $T(z)$ is called the *Segal-Sugawara form.*

We finally note that if $k = -h^\vee$, we have

$$[S(y), X(z)] = 0 \quad \text{for all } X \in \mathfrak{g}.$$

This case is called the *critical level.*