Advanced Studies in Pure Mathematics 73, 2017
Hyperbolic Geometry and Geometric Group Theory
pp. 135–158

On suspensions, and conjugacy of a few more automorphisms of free groups

François Dahmani

Abstract.

In a previous work, we remarked that the conjugacy problem for pairs of atoroidal automorphisms of a free group was solvable by mean of the isomorphism problem for hyperbolic groups and an orbit problem for the automorphism group of their suspensions (*i.e.* their semidirect product with $\mathbb{Z}$ for the relevant structural automorphism).

We consider the same problem a few more automorphisms of free groups, those that produce relatively hyperbolic suspensions that do not split over a parabolic subgroup.

§ Introduction

Let F be a finitely presented group (we will soon assume that it is free), $\mathrm{Aut}(F)$ be its automorphism group, and $\mathrm{Out}(F) = \mathrm{Aut}(F)/\mathrm{Inn}(F)$ be its outer automorphism group.

Given two semi-direct products, $F \rtimes_\alpha \langle t \rangle$ and $F \rtimes_\beta \langle t' \rangle$, their structural automorphisms α and β are conjugated in $\mathrm{Out}(F)$ if and only if there is an isomorphism $F \rtimes_\beta \langle t \rangle \to F \rtimes_\alpha \langle t' \rangle$ that preserves the fiber (which is F) and the orientation (*i.e.* sends tF on $t'F$).

This suggests a way of analysing the conjugacy problem in a class of elements of $\mathrm{Out}(F)$ through an isomorphism problem in a class of semidirect products of F.

A motivating case is that of a free group. Though a solution to the conjugacy problem of automorphisms of free groups was announced by Lustig [Lu-00, Lu-01], it might still be desirable to find short complete solutions for specific classes of elements in $\mathrm{Out}(F_n)$.

In [D] we considered the case of atoroidal automorphisms. In that case, the semi-direct product $F \rtimes_\alpha \langle t \rangle$ is a hyperbolic group [B], and

Received February 19, 2015.
Revised October 26, 2015.
Partially supported by the ANR (2011-BS01-013-02) and the Institut Universitaire de France.

since the isomorphism problem for hyperbolic groups is solvable [Se, DGr, DG-11], the conjugacy problem for atoroidal automorphisms of free groups was reduced to an orbit problem for $\mathrm{Out}(F \rtimes_\alpha \langle t\rangle)$ in $H_1(F \rtimes_\alpha \langle t\rangle, \mathbb{Z})$ (this orbit problem was an interpretation of the condition that there is an isomorphism $F \rtimes_\beta \langle t\rangle \to F \rtimes_\alpha \langle t'\rangle$ that preserves the fiber and the orientation, once is given an abstract isomorphism $F \rtimes_\beta \langle t\rangle \to F \rtimes_\alpha \langle t'\rangle$). We solved this orbit problem by showing that $\mathrm{Out}(F \rtimes_\alpha \langle t\rangle)$ is a virtually abelian group, and by interpreting the orbit problem as a system of linear Diophantine equations.

In view of [DGr], [DG-15], it is natural to ask whether one can approach the conjugacy problem of larger classes of automorphisms, namely those producing proper relatively hyperbolic suspensions.

Definition 0.1. *Let $\phi \in \mathrm{Aut}(F)$, and $F_1, \dots, F_k$ finitely generated proper subgroups of F. We say that the automorphism ϕ is hyperbolic relative to $\{F_1, \dots, F_k\}$ if there exists integers $m_1, \dots, m_k > 0$ and elements $f_1, \dots, f_k \in F$ such that, for all i, $t^{m_i} f_i$ normalises F_i, and such that the group $(F \rtimes_\phi \langle t\rangle)$ is hyperbolic relative to $\{(F_i \rtimes \langle t^{m_i} f_i\rangle), i = 1, \dots, k\}$.*

The case of automorphism of free groups is once again particularly interesting, since according to [GLu], all non-polynomial automorphisms of free groups should produce interesting relatively hyperbolic suspensions (of course it could be interesting to consider also a free product of nice groups).

Thus from now on F is a free group.

One says that a subgroup F_0 of F is *polynomial* for a given automorphism ϕ if every conjugacy class of elements in F_0 has polynomial growth under iterates of ϕ (more explicitly, that means that for all $\gamma \in F_0$, the length of a cyclically reduced representative of $\phi^n(\gamma)$ is bounded above by a polynomial in n). We say that an automorphism is polynomial if F itself is polynomial. For an *outer* automorphism Φ of F, we say that a subgroup F_0 of F is polynomial for Φ if there is an automorphism ϕ in the class of Φ for which F_0 is polynomial.

In [L, Prop 1.4] Levitt proves that for any outer automorphism Φ of a free group F, there is a finite family of finitely generated subgroups of F, polynomial for Φ, such that all polynomial subgroups of Φ are conjugated into one of them (see also [GLu, Prop. 3.2]).

The aim of this note is thus to explore to what extend the method used in [D] can be extended to larger classes of automorphisms of free groups, and in particular to (some) non-polynomial automorphisms.

However, I ultimately had to restrict the study to those automorphisms whose suspension does not split over a parabolic subgroup. I also have to concede that this attempt uses three results unpublished at the time of writing (this issue will be made clear in a few lines).

The main result of this attempt is the following.

Theorem 0.2. *There is an (explicit) algorithm that, given two automorphisms ϕ_1, ϕ_2 of a finitely generated free group F, terminates if both produce proper relatively hyperbolic suspensions, relative to suspensions of polynomial subgroups, without parabolic splitting, and it indicates whether ϕ_1 and ϕ_2 are conjugated in* $\mathrm{Out}(F)$.

The arguments presented below involve other tools than in [D], in particular Dehn fillings, and growth of conjugacy classes under iterations of automorphisms. They rely on a certain number of currently unpublished results, so I would like to make this reliance clear. First, there is the main result of Gautero and Lustig paper [GLu]. This is used twice; to produce examples to which the results might apply (so, in some sense, as a motivation), and to compute explicitly the polynomial subgroups (actually, this is to certify that an exponentially growing automorphism is indeed exponentially growing). Then there is the splitting computation of Touikan [T]. And finally, there is the solution ot the isomorphism problem of some rigid relatively hyperbolic groups, by Guirardel and myself, [DG-15].

The authors wants to say merci to the referee, for useful comments and corrections.

§1. Preliminary

1.1. General

Since this note is a sequel to [D], we assume that the reader has access to that previous paper, and we will freely use its content. For readability, though, we briefly introduce now a few items that we need from that paper. First is a variation on some classical fact.

As in [D] we will call a semidirect product with $\mathbb{Z}$, $F \rtimes_\phi \mathbb{Z}$ a suspension of F by ϕ, whose fiber is F and whose orientation is defined by Ft (this is to distinguish it from the suspension by ϕ^{-1} which is the same group, but with reverse orientation).

Lemma 1.1. *(see for instance* [D, Lemma 2.3]*) Let ϕ_1 and ϕ_2 be two automorphisms of F. The following assertions are equivalent.*

(1) *ϕ_1 and ϕ_2 are conjugate in* $\mathrm{Out}(F)$*;*
(2) *there is an isomorphism between their suspensions $(F \rtimes_{\phi_1} \langle t \rangle)$ and $(F \rtimes_{\phi_2} \langle t' \rangle)$ that preserves the fiber F (in both directions) and the orientation (i.e sends t in Ft');*

(3) *there is an isomorphism between their suspensions that preserves the orientation and sends the fiber inside the fiber;*

(4) *there is an isomorphism between their suspensions whose factorization through the abelianisations preserves the orientation, and sends the image of the fiber inside the image of the fiber.*

When talking about a splitting of a group, we mean a graph-of-group decomposition (often noted $\mathbb{X}$) as defined in Bass-Serre theory. This is presented in many places, beginning with Serre's famous book. We recall very briefly our conventions for defining a splitting, and its automorphisms. One is given an underlying graph (unoriented, with possible double edges, and loops) X whose set of vertices and set and oriented edges we denote respectively by V and E, and whose involution on the oriented edges (reversion of orientation) we denote by $\bar{\ } : e \mapsto \overline{e}$, and terminaison map $t : E \to V$. One is given groups for each vertices, denoted $\Gamma_v, v \in V$, and for each edge $e \in E$, another group Γ_e, with $\Gamma_e = \Gamma_{\bar{e}}$, and an injective morphism $i_e : \Gamma_e \hookrightarrow \Gamma_{t(e)}$. The Bass group is the group generated by all vertex groups and all edges with the relations that $\bar{e} = e^{-1}$ and that $\bar{e} i_{\bar{e}}(g) e = i_e(g)$ everywhere it is defined. The fundamental group of the graph of group at a vertex v_0 is the subgroup of the Bass group of all elements of the form $e_1\gamma_1 e_2\gamma_2 \dots e_r\gamma_r$ where $e_i \in E$, $\gamma_i \in \Gamma_{t(e_i)}$ for all i, and such that consecutive e_i define a loop at v_0 in the graph X (that is, for all i, $t(\overline{e_{i+1}}) = t(e_i)$, and $t(e_r) = t(\overline{e_1}) = v_0$).

If this fundamental group (of the graph of group $\mathbb{X}$) is isomorphic to a certain group G we say that $\mathbb{X}$ is a splitting of G. We call a splitting non-trivial if the action of its fundamental group on the Bass-Serre tree has no global fixed point.

An automorphism of the graph of groups $\mathbb{X}$ is a tuple $(\Phi_X, (\phi_v), (\phi_e), (\gamma_e))$ where Φ_X is a automorphism of the underlying graph X, for all vertices v, $\phi_v : \Gamma_v \to \Gamma_{\Phi_X(v)}$ is an isomorphism, for all edges e, $\phi_e : \Gamma_e \to \Gamma_{\Phi_X(e)}$ is also an isomorphism, and $\gamma_e \in \Gamma_{\Phi_X(t(e))}$ satisfies

$$\text{(1)} \qquad \text{Bass Diagram: } \phi_{t(e)} \circ i_e = \mathrm{ad}_{\gamma_e} \circ i_{\Phi_X(e)} \circ \phi_e,$$

for $\mathrm{ad}_{\gamma_e} : x \mapsto \gamma_e^{-1} x \gamma_e$. One might like to read the condition as: "each attaching map $i_e, e \in E$ commute with the isomorphisms $\phi_v, \phi_e, v \in V, e \in E$ up to conjugation in the target vertex group".

The small modular group of a splitting was used in [D]. We suggest reading [D, §1] and more precisely §1.2 *loc. cit.* for a slightly broader discussion about it. The small modular group of a splitting $\mathbb{X}$ of a group G, denoted by $Mod_{\mathbb{X}}$ is a subgroup of the automorphism group of $\mathbb{X}$ consisting of those for which $\Phi_X = \mathrm{Id}_X$, $\phi_v \in \mathrm{Inn}(\Gamma_v)$ and $\phi_e = \mathrm{Id}_{\Gamma_e}$ for all v, e. It is generated by the union of two families of automorphisms, the oriented Dehn twists (for which the ϕ_v are all the identity, and only one element $\gamma_e \in Z_{\Gamma_{t(e)}}(i_e(\Gamma_e))$ is non trivial), and the inert twists (for which $\phi_v = \mathrm{ad}_{\gamma_v}$ and γ_e is the same γ_v if $t(e) = v$). The image of the small modular group in the outer automorphism group of G consists of the group generated by Dehn twists over edges of the splitting $\mathbb{X}$. Its image in the automorphism group of the abelianisation of G is generated by Dehn twists over non separating edges of $\mathbb{X}$.

Given a suspension $F \rtimes \langle t \rangle$, we define the map $\delta : F \rtimes \langle t \rangle \to \mathbb{Z}$ to be the quotient by F. Of course, it factorises through the abelianisation of $F \rtimes \langle t \rangle$. We insist in seeing the targer of δ as $\mathbb{Z}$ to be able to interpret $\delta(\gamma)$ as an integer.

Given a splitting $\mathbb{X}$ of $F \rtimes \langle t \rangle$, and a choice of base point in $\mathbb{X}$, we can realise each element γ of $F \rtimes \langle t \rangle$ as its expression in the Bass group, and for each (oriented) edge e of X we may define $n(\gamma, e)$ as the number of occurences of e in the reduced form of this expression, minus the number of occurences of $\overline{e}$.

We obtained the following result.

Proposition 1.2. *(See* [D, 2.3]*) Let G be a finitely generated group that can be expressed as a semi-direct product $F \rtimes \langle t \rangle$.*

Given a splitting $\mathbb{X}$ of G, and for each $e \in E$, a generating set $S_e \subset \Gamma_{t(e)}$ for $Z_{\Gamma_{t(e)}}(i_e(\Gamma_e))$ and a family $(\gamma_j)_{0 \leq j \leq j_0}$ of elements of G, one can decide whether there is an element $\eta \in Mod_{\mathbb{X}}$ whose image $\bar{\eta}$ in $\mathrm{Aut}(H_1(G))$ sends $\bar{\gamma}_j$ in $\bar{F}$ for all $j < j_0$ and $\bar{\gamma}_{j_0}$ inside $\bar{t}\bar{F}$.

More precisely, there is such an element η if and only if the explicit Diophantine linear system of equations

$$\forall j, \sum_{e \in E^+ s_e \in S_e} r_{s_e} n(\gamma_j, e) \delta(s_e) = -\delta(\gamma_j) + \mathrm{dirac}_{j=j_0} \tag{2}$$

(with unknowns r_{s_e}) has a solution.

1.2. On polynomial growth

The following preliminary result is useful. It follows from the recent algorithmic construction of relative train tracks, by Feighn and Handel [FH, Theorem 2.1]. We give a different proof below (certainly not of the same scope as the mentioned reference) for the curiosity of the reader.

Proposition 1.3. *There is an algorithm that, provided with a free group F and an automorphism ϕ, terminates and indicates whether F is polynomial for ϕ.*

Proof. First, we will give a procedure certifying that an automorphism is polynomial, and then a procedure certifying that an automorphism is of exponential growth on some conjugacy class.

By [BFH-00, Coro 5.7.6], if ϕ is a polynomially growing automorphism, then there is n such that ϕ^n is unipotent in $GL(H_1(F))$. It is then sufficient to devise a procedure certifying whether a unipotent automorphism is polynomially growing. Then we use (a weak aspect of) Theorem [BFH-05, 3.11]: if the automorphism is polynomially growing, there exists a topological representative $\tau : G \to G$ of ϕ on a graph G, and a filtration of G, $\{v\} = G_0 \subset \cdots \subset G_n = G$ such that any edge e in $G_i \setminus G_{i-1}$ is sent on a path ec where c is a path in G_{i-1}. Also, if such a representative exists, then, for every edge $e \in G_i$, the length of $\phi^n(e)$ can be bounded by a polynomial in n depending only on i (this can be seen by induction; it is obvious for $i = 0$ or 1 since G_0 contains no edge, and if it is true for $i-1$, let $P_{(i-1)}$ the corresponding polynomial, and for $e \in G_i$, with $\phi(e) = ec$, we can write $\phi^n(e) = e\phi(c)\phi(c)^2 \dots \phi(c)^{n-1}$, and the total length is bounded by $\sum_{k\leq n-1} P_{(i-1)}(k)^{|c|}$, hence by $\sum_{k\leq n-1} P_{(i-1)}(k)^M$ for $M = \max\{|\phi(e)|, e \in G_i\}$, which is polynomial in n). Thus, ϕ is polynomial if and only if there is such a topological representative. This can be certified by enumeration of topological representatives, since the condition used is easily algorithmically checked.

We now need a procedure that produces a certificate that ϕ is not polynomialy growing when it is the case.

For that, we'll use that ϕ is not polynomially growing if and only if the suspension has a proper relative hyperbolic structure. One direction of this equivalence ($\Longrightarrow$) is the content of [GLu] (actually [GLu] describes the relative hyperbolic structure). We present now an argument for the other direction. If the suspension is a proper relatively hyperbolic group (with at least one hyperbolic element), then there are hyperbolic elements in each coset of the fiber: this follows from [O-06b, Lemma 4.4] (I also find rather pleasant the following proof: a simple random walk on the relatively hyperbolic group $F \rtimes \mathbb{Z}$ will a.s. walk on only finitely many non-hyperbolic elements (apply Borel-Cantelli Lemma, with the exponentially decreasing probability to walk on a parabolic element, e.g. [Si]), and, by recurrence on $\mathbb{Z}$, it will walk infinitely many times on the preimage of any chosen coset). Thus, there is a hyperbolic element f_h in the fiber F, and another $t' = tf_0$ in the coset tF.

The relative distance of F and t'^kF grows therefore linarly in k, and by exponential divergence (in the hyperbolic coned-off graph), the shortest path, in t'^kF, from t'^k to $f_h t'^k$ has exponential relative length in k, hence exponential absolute length. This makes f_h an exponentially growing element for the automorphism $\phi \circ \mathrm{ad}_{f_0}$. This means that this automorphism cannot be polynomialy growing (as it is visible on the topological representative τ of a polynomially growing automorphism that no element can be exponentially growing). Therefore ϕ has an exponentially growing *conjugacy class* in F.

By [DG-13], and enumeration of the proper subgroups of F, if there exists a proper relative hyperbolic structure, one can eventually find it and thus a certificate that the automorphism is not polynomially growing (here a certificate is the data of an exponentially growing conjugacy class, with a proof that it is exponentially growing).

Q.E.D.

Proposition 1.4. *Let F be a free group. There is an explicit algorithm that, given $\phi \in \mathrm{Aut}(F)$ expressed on a basis of F, terminates and produces a basis for each group of a collection $F_1, \dots, F_k$ of maximal polynomial subgroups of F for Φ the class of ϕ in $\mathrm{Out}(F)$, and computes minimal exponents $m_i > 0$ and elements f_i so that $t^{m_i} f_i$ normalises F_i (see definition 0.1).*

Proof. Note that there is a unique relative hyperbolic structure for $F \rtimes_\phi \langle t \rangle$ whose parabolic groups are suspensions of polynomial subgroups. Indeed, the polynomial subgroups must be parabolic, by the observation made in the proof of 1.3. Moreover, by [GLu], $F \rtimes_\phi \langle t \rangle$ is indeed relatively hyperbolic to the subgroups that we need to compute.

We enumerate the tuples (S, m, f), where S is a finite subset of F, m is an integer, and $f \in F$. For each of them, by the usual Stallings' folding process, we may find a basis of $\langle S \rangle$, and we may check whether $\langle S \rangle$ is stable by conjugation by $t^m f$ and $f^{-1} t^{-m}$. If so, $\langle S \rangle$ is normalized by $t^m f$; in that case, using Proposition 1.3, we may certify whether, the product $\langle S \rangle \rtimes \langle t^m f \rangle$ is a suspension of a polynomial automorphism on $\langle S \rangle$. For any collection of such subgroups, we may use [DG-13] in order to certify that $F \rtimes_\phi \langle t \rangle$ is relatively hyperbolic. When this happens, the algorithm is done. Q.E.D.

Recall that a splitting of a relatively hyperbolic group is peripheral if, in the Bass-Serre tree, all parabolic subgroups are elliptic. Let us say that $\phi \in \mathrm{Aut}(F)$ is relatively hyperbolic with no parabolic splitting (RH-noPS for short) if it is properly hyperbolic relative to a collection of polynomial subgroups of F, and the suspension $F \rtimes_\phi \langle t \rangle$ has no non-trivial peripheral splitting over a subgroup of a parabolic subgroup.

Let us say that $\phi \in \mathrm{Aut}(F)$ is relatively hyperbolic with no elementary splitting (RH-noES for short) if it is properly hyperbolic relative to a collection of polynomial subgroups of F, and the suspension $F \rtimes_\phi \langle t \rangle$

has no peripheral splitting over a cyclic or parabolic subgroup, except the trivial one.

An unsatisfying aspect of this work is that I am unable to provide an algorithm certifying whether an element of $\mathrm{Aut}(F)$ is RH-noPS. But if it is, then we can do something.

§2. Conjugacy problems

2.1. Conjugacy of two relatively hyperbolic automorphisms without elementary splitting

Proposition 2.1. *Let F be a free group. There is an (explicit) algorithm that, given two automorphisms, ϕ_1, ϕ_2, terminates if both ϕ_i are RH-noPS, and provides*

- *either an isomorphism $F \rtimes_{\phi_1} \langle t \rangle \to F \rtimes_{\phi_2} \langle t \rangle$ preserving fiber, and orientation;*
- *or a certificate that $F \rtimes_{\phi_1} \langle t \rangle$ and $F \rtimes_{\phi_2} \langle t \rangle$ are not isomorphic by an isomorphism preserving fiber, orientation;*
- *or a non-trivial peripheral splitting of either $F \rtimes_{\phi_i} \langle t \rangle$ over a cyclic subgroup, which is either maximal cyclic or parabolic.*

The algorithm in question may terminate even if one of the ϕ_i is not RH-noPS. It never lies though.

The following application is immediate, given Lemma 1.1.

Corollary 2.2. *The conjugacy problem for RH-noES elements of* $\mathrm{Out}(F)$ *is solvable: there is an algorithm that given two automorphisms that are RH-noES, decides whether or not they are conjugated.*

Let us now prove Proposition 2.1

Proof. First, by Proposition 1.4, we may assume that we know explicitly both relative hyperbolic structures of $G_i = F \rtimes_{\phi_i} \langle t \rangle$ with presentations, as suspensions of subgroups of F, of the parabolic subgroups (that are non-virtually cyclic). Let us write $P_{1,j} = F_{1,j} \rtimes \langle r_j \rangle, j = 1, \dots, k$ conjugacy representatives of maximal parabolic subgroups of G_1, with $P_{1,j} \cap F = F_{1,j}$ (recall that we have explicit presentations of the groups $P_{1,j}$ as such suspensions). Similarily, we have $P_{2,j} = F_{2,j} \rtimes \langle r'_j \rangle, j = 1, \dots, k'$ conjugacy representatives of maximal parabolic subgroups of G_2. If $k \neq k'$, G_1 and G_2 cannot be isomorphic, hence we can assume that $k = k'$.

In parallel, we then perform the three following searchs (so-called procedures, below).

The first procedure is the enumeration of morphisms $G_1 \to G_2 \to G_1$. It stops when mutually inverse isomorphisms preserving orientation and sending the fiber into the fiber are found.

The second procedure is as follows. For incrementing integers m, we compute $N_{1,j}^{(m)}$ the intersection of all subgroups of $F_{1,j}$ of index $\leq m$. Note that for each j, $N_{1,j}^{(m)}$ is a sequence of normal subgroups of $P_{1,j}$, with trivial intersection (as m goes to infinity). Denote by $\langle\langle \cup_j N_{1,j}^{(m)} \rangle\rangle$ the normal closure in G_1 of their union (over j for a fixed m). Then we try to certify, using [P], that $\bar{G}_1^{(m)} = G_1 / \langle\langle \cup_j N_{1,j}^{(m)} \rangle\rangle$ is hyperbolic.

We will denote by $K_1^{(m)}$ the kernel $\langle\langle \cup_j N_{1,j}^{(m)} \rangle\rangle$ in G_1 of the previous quotient.

Similarily, we compute $\bar{G}_2^{(m)}$ and check that it is hyperbolic. Since $P_{1,j}/N_{1,j}^{(m)}$ is virtually cyclic, by virtue of the Dehn Filling theorem [O-07, Thm. 1.1], for m large enough these groups are indeed hyperbolic. So this step of the second procedure will eventually provide groups $\bar{G}_i^{(m)}$, ($i = 1, 2$, m sufficiently large), that are certified hyperbolic. For all m, $\cup_j N_{1,j}^{(m)}$ is contained in F which is normal in G_1. Hence the whole group $K_1^{(m)}$ is contained in F, and $\bar{G}_1^{(m)}$ is naturally a suspension $\bar{G}_1^{(m)} = (F/K_1^{(m)}) \rtimes \langle \bar{t} \rangle$. The second procedure then calls the algorithm of Theorem [D, 3.2] in order to decide whether $\bar{G}_1^{(m)}$ and $\bar{G}_2^{(m)}$ are isomorphic by a fiber and orientation preserving isomorphism. This is done in parallel for all incrementing m for which the groups are certified hyperbolic. The second procedure stops if an integer m is found so that $\bar{G}_1^{(m)}$ and $\bar{G}_2^{(m)}$ are not isomorphic by a fiber and orientation preserving isomorphism.

The third procedure is as follows. For both $i = 1, 2$, one enumerates presentations of $G_i = F \rtimes_{\phi_i} \langle t \rangle$ by Tietze transformations, and, for each one exhibiting a splitting of G_i over a cyclic subgroup as an amalgamation, we check whether the splitting is non-trivial (it suffices to check that both factors have a generator that does not commute with the cyclic subgroup) and, for each one of the form of an HNN extension, $\langle H, t \,|\, tct^{-1} = c', \mathcal{R}_H \rangle$ we check whether the stable letter t is non trivial (so the presentation is genuinely that of an HNN-extension over a cyclic group). If we discover a non-trivial cyclic splitting, we may check whether its cyclic edge subgroup is maximal cyclic or parabolic, using [O-06a, Thms 5.6 and 5.17]. We then enumerate the conjugates of the parabolic subgroups, and if we find that each parabolic subgroup has a conjugate contained in a vertex group, this third procedure stops, and outputs the splitting, with the relevant conjugations of parabolic subgroups.

Now that we described the three procedures, we discuss the implication of their termination.

If the first procedure terminates, then by Lemma 1.1, there exists a fiber-and-orientation preserving isomorphism, and the two given automorphisms of F are conjugated in $\mathrm{Out}(F)$. If the second procedure terminates, there cannot exist any isomorphism $G_1 \to G_2$ preserving fiber and orientation (it would preserve the class of parabolic subgroups, characterised by being polynomial, and hence pass to the characteristic quotients). If the third procedure terminates, we have found a non-trivial splitting of either $F \rtimes_{\phi_i} \langle t \rangle$ over a cyclic subgroup.

Now we need to show that there is always at least one procedure that terminates, *i.e.* the following lemma.

Lemma 2.3. *Assume that G_1 and G_2 are RH-noPS. If the third procedure and the second procedure never terminate, then the first procedure terminates.*

This Lemma is actually a consequence of a result obtained in a collaboration of Vincent Guirardel and the author.

Proof. We assume that for all m large enough, $\phi_m : \bar{G}_1^{(m)} \to \bar{G}_2^{(m)}$ is a fiber and orientation preserving isomorphism.

Observe that eventually, $P_{1,j}/N_{1,j}^{(m)}$ contains a large finite subgroup and an infinite order element normalizing it. In $G_2^{(m)}$ (for large m), all finite subgroups lie in conjugates of $P_{2,\ell}/N_{2,\ell}^{(m)}$ (by [DG-15, Lemma 4.3]), and since $G_2^{(m)}$ is eventually hyperbolic relative to this collection of subgroups, any infinite order element normalizing a finite subgroup is in the same conjugate of $P_{2,\ell}/N_{2,\ell}^{(m)}$. Thus, eventually each ϕ_m must send, for each j, the group $P_{1,j}/N_{1,j}^{(m)}$ on some conjugate of some $P_{2,\ell}/N_{2,\ell}^{(m)}$, $(\ell \leq k)$. Then, Theorem [DG-15, Thm. 5.2] states that either G_1 or G_2 has a peripheral splitting over a maximal cyclic, or a parabolic subgroup (which must be parabolic if we assume that the third procedure does not terminate, hence in contradiction with the assumption of the lemma), or there is an isomorphism $\phi : G_1 \to G_2$ that commutes with infinitely many ϕ_m, up to composition with a conjugation in $\bar{G}_2^{(m)}$ (in other words, it makes a diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{\phi_\infty} & G_2 \\ \downarrow & & \downarrow \\ \bar{G}_1^{(m)} & \xrightarrow{\mathrm{ad}_{\bar{g}} \circ \phi_m} & \bar{G}_2^{(m)} \end{array}$$

commute, where $\mathrm{ad}_{\bar{g}}$ is a conjugation). In this second circumstance, ϕ has to preserve the fiber and orientation, since the ϕ_m do, and the kernels are co-final. Q.E.D.

Q.E.D.

There is a slightly stronger version of Proposition 2.1 that we will need for the next part.

Given a suspension $F \rtimes \langle t \rangle$, a transverse cyclic peripheral structure is a tuple of elements of the form $(t^{k_j} f_j)_{j=1,\dots r}$, for $k_j \neq 0$ and $f_j \in F$.

A fiber-and-orientation preserving isomorphism between suspensions equipped with such structures is said to preserve the structure if it sends the conjugacy classes of the first exactly on the conjugacy classes of the second.

Let us amend our definition of RH-noPS, and say that a suspension with a transverse cyclic peripheral structure is RH-noPS relative to the transverse peripheral structure if it is properly hyperbolic relative to a collection of polynomial subgroups of F, and the suspension $F \rtimes_\phi \langle t \rangle$ has no non-trivial peripheral splitting over a subgroup of a parabolic subgroup, in which each element of the transverse peripheral structure (which is a group) is conjugated to a factor.

The following Proposition is, as we said, similar to Proposition 2.1. The difference is in the presence of the transverse cyclic peripheral structure (a minor difference) but also in the fact that we had ambitionned to get the full list of fiber-and-orientation preserving isomorphisms. This ambition is not realized unfortunately, but enough is granted for the application in the next part.

We keep the notation $K_1^{(m)}, K_2^{(m)}$ for the normal subgroups introduced in the proof of Proposition 2.1.

Proposition 2.4. *Let F be a free group. There is an (explicit) algorithm that, given two automorphisms, ϕ_1, ϕ_2, and two transverse cyclic peripheral structures $\mathcal{P}_1, \mathcal{P}_2$ of $F \rtimes_{\phi_1} \langle t \rangle$ and $F \rtimes_{\phi_2} \langle t \rangle$ respectively, terminates if both ϕ_i are RH-noPS relative to their transverse peripheral structures, and provides*

(1) *either a list of isomorphisms $F \rtimes_{\phi_1} \langle t \rangle \to F \rtimes_{\phi_2} \langle t \rangle$ preserving fiber, orientation, and transverse cyclic peripheral structure, and an integer m such that for each $p \in \mathcal{P}_2$, the centralizer of $\bar{p}$ in $\overline{G_2}^{(m)} = G_2/K_2^{(m)}$ is the image of the centralizer of p in G_2 and such that, for any other such isomorphism ψ, there is one, ϕ, in the list, an element $g \in G_2$, such that for all $h \in G_1$, there is $z_h \in K_m$ for which $\psi(h)^g = \phi(h) z_h$.*

(2) *or a certificate that* $F \rtimes_{\phi_1} \langle t \rangle$ *and* $F \rtimes_{\phi_2} \langle t \rangle$ *are not isomorphic by an isomorphism preserving fiber, orientation, and transverse cyclic peripheral structure;*

(3) *or a non-trivial peripheral splitting of either* $F \rtimes_{\phi_i} \langle t \rangle$ *over a cyclic subgroup, in which each element of the transverse peripheral structure is elliptic.*

The first point means that the list contains all isomorphisms up to conjugacy in G_2 and multiplication by a large element of F.

Proof. As in the proof of Proposition 2.1, we use three procedures.

The first procedure is the enumeration of morphisms $G_1 \to G_2 \to G_1$. This procedure has an incrementing list $\mathcal{L}$, which is empty at the beginning. Every time mutually inverse isomorphisms preserving the transverse peripheral structure, the orientation and sending the fiber into the fiber are found, such that the isomorphism $G_1 \to G_2$ is not conjugated to any item of the list $\mathcal{L}$, the procedure stores $G_1 \to G_2$ into $\mathcal{L}$. We precise below when this first procedure is set to stop.

The second one slightly differs from Proposition 2.1. We still compute $\bar{G}_1^{(m)}$ and $\bar{G}_2^{(m)}$ (and the images of the transversal peripheral structure in them), and try to certify that they are hyperbolic and that the assumption of point 1, on the centralizers, is satisfied (which happens if m is large enough, by Lemma 2.12). Let us call this "certification α for m". When this is the case, using [DG-11, Coro. 3.4], we check whether these groups are rigid (in the sense that they have no peripheral splitting over a virually cyclic group with infinite center, in which each element of the transverse peripheral structure is conjugated in a vertex group) and if they are we proceed and compute by [DG-11, Coro. 3.5] the complete list of isomorphisms $\bar{G}_1^{(m)} \to \bar{G}_2^{(m)}$ up to conjugacy in $\bar{G}_2^{(m)}$. Once this is done, we check which one of them are fiber-and-orientation preserving, and preserve the transverse peripheral structure, and we record them in a list $\mathcal{L}_m$.

The second procedure is set to stop if an m is found so that there is no fiber-and-orientation preserving isomorphisms $\bar{G}_1^{(m)} \to \bar{G}_2^{(m)}$ that preserves the transversal peripheral structure.

The first procedure (which was run in parallel with the second) is set to stop if a list $\mathcal{L}$ is found and an integer m is found so that the following three conditions are satisfied. First, "certification α" for m is done. Second, $\mathcal{L}_m$ is completely computed, and third, the currently computed list $\mathcal{L}$ surjects on $\mathcal{L}_m$, by the natural quotient map.

The third procedure looks for a non-trivial peripheral splitting over a cyclic subgroup which is maximal cyclic or parabolic, in which each element of the transverse peripheral structure is conjugated to a vertex group (it is similar to that of Proposition 2.1).

Observe that, if the first procedure stops, we have in $\mathcal{L}$ a list of isomorphisms $F \rtimes_{\phi_1} \langle t \rangle \to F \rtimes_{\phi_2} \langle t \rangle$ preserving fiber (by Lemma 1.1 ($3 \implies 2$)), orientation, and transverse cyclic peripheral structure, such that any other such isomorphism differs from one in the list by a conjugation, and the multiplication by elements in the fiber (in the sense of 2.4-(1)). Observe also that if the second procedure stops, we have (as in 2.1) a certificate that $F \rtimes_{\phi_1} \langle t \rangle$ and $F \rtimes_{\phi_2} \langle t \rangle$ are not isomorphic by an isomorphism preserving fiber, orientation, and transverse cyclic peripheral structure.

Again, we can conclude by the following lemma.

Lemma 2.5. *Assume that G_1 and G_2 are RH-noPS relative to their transverse cyclic peripheral structures. If the third procedure and the second procedure never terminate, then the first procedure terminates (i.e there is a finite list of isomorphisms (preserving the transverse peripheral structure, the orientation and sending the fiber into the fiber) from G_1 to G_2, and an integer m as in (2.4-(1)) such that any isomorphism preserving fiber, orientation, and peripheral structure $\bar{G}_1^{(m)} \to \bar{G}_2^{(m)}$ is the image of an isomorphism $G_1 \to G_2$ in the list).*

Proof. We endow G_1 and G_2 with the relatively hyperbolic structure consisting of their parabolic subgroups, and for each hyperbolic element in the transverse peripheral structure, the conjugates of the maximal cyclic subgroup containing it. This still makes a relatively hyperbolic group (see [O-06b, 1.7] for instance). In this context, the assumption that the third procedure does not stops says that the groups G_1 and G_2 have no splitting which is peripheral (for this extended peripheral structure), over an elementary subgroup.

Observe that, eventually, the groups $\bar{G}_1^{(m)}$ and $\bar{G}_2^{(m)}$ are rigid (in the sense of the second procedure). Indeed, if it is not the case for the sequence $(\bar{G}_1^{(m)})$ for instance, then after passing to a subsequence, for each m in the subsequence, there are infinitely many automorphisms of $\bar{G}_1^{(m)}$ preserving the transverse peripheral structure, the orientation and the fiber (namely the iterates of a Dehn twist over an edge group of a splitting falsifying rigidity). From there, by [DG-15, Corolary 5.10], one gets that G_1 must have a splitting of a type contradicting the previous paragraph.

Therefore, for sufficiently large m, the second procedure can compute the list $\mathcal{L}_m$ of isomorphisms preserving fiber orientation, and peripheral structure between $\bar{G}_1^{(m)}$ and $\bar{G}_2^{(m)}$. Since the second procedure does not stop, for all sufficiently large m, this list is not empty.

Assume that the first procedure does not stop. Then, for all finite list $\mathcal{L}$ of isomorphisms $G_1 \to G_2$ (preserving transverse peripheral structure, fiber and orientation), and all n there exists $m > n$ and $\psi_m : \bar{G}_1^{(m)} \to \bar{G}_2^{(m)}$, an isomorphism preserving fiber orientation, and peripheral structure, that does not commute with any element of the list $\mathcal{L}$, even after conjugation by an element of $\bar{G}_2^{(m)}$. Choosing a sequence of lists $\mathcal{L}$ exhausting the set of isomorphisms $G_1 \to G_2$ (with the prescribed preservation property), and a increasing sequence of integers n, one gets a sequence of ψ_m as above, for a sequence of integers m going to infinity.

We may use Theorem [DG-15, 5.2] to extract a subsequence of the isomorphisms ψ_m, and to find an isomorphism $\psi_\infty : G_1 \to G_2$ commuting with the composition of ψ_m with a conjugation (for all m in the extracted sequence), as our previous use of it in Lemma 2.3. By cofinality, such a ψ_∞ has to preserve the fiber and the orientation. Since the transverse peripheral structure consists exactly of the cyclic groups among the parabolic groups of the relatively hyperbolic structures under consideration, we deduce that ψ_∞ globally preserves the transverse cyclic structure. By [DG-15, Lemma 3.13], we know that, for sufficiently large m, non conjugate parabolic subgroups in G_2 map on non conjugate subgroups in $\bar{G}_2^{(m)}$. Since each ψ_m preserves the transverse peripheral structure, and (after extraction of subsequence) commute with ψ_∞, it follows that ψ_∞ sends the transverse peripheral structures of G_1 to that of G_2. Therefore ψ_∞ is eligible for being recorded in the list of the first procedure, and thus eventually appears in the sequence of our lists that served to define the ψ_m. But that is a contradiction, by definition of the ψ_m.

Q.E.D.

Q.E.D.

2.2. Conjugacy of two relatively hyperbolic automorphisms without parabolic splitting

The class of RH-noPS automorphisms, is larger than that of RH-noES. We can treat it as well, but it requires a little care.

Recall that if $F \rtimes_{\phi_1} \langle t \rangle$ splits as a graph of groups, then vertex groups and edge groups are suspension of subgroups of F that are vertex groups and edge groups (respectively) of a graph of group decomposition of F (see [D, Lemma 2.6]).

Proposition 2.6. *Let $\mathbb{X}_1, \mathbb{X}_2$ be graph-of-groups decompositions of $F \rtimes_{\phi_1} \langle t\rangle$ and $F \rtimes_{\phi_2} \langle t\rangle$ respectively, over cyclic subgroups.*

Assume that $\Phi_X : X_1 \to X_2$ is an isomorphism of the underlying graphs of the decompositions, and that m is an integer, and, for all vertex $v \in X_1^{(0)}$, $\mathcal{L}_v$ is a list of isomorphisms from $\Gamma_{1,v}$ the vertex group of v in $\mathbb{X}_1$ to $\Gamma_{2,\Phi_X(v)}$ the group of $\Phi_X(v)$ in $\mathbb{X}_2$, that are all as in (2.4-(1)) (for $G_i = \Gamma_{i,v}$) for the transverse peripheral structure consisting of the adjacent edge subgroups, and the fiber and orientation defined by [D, Lemma 2.6].

The following are equivalent.

(1) *There is an isomorphism $\Phi : \pi_1(\mathbb{X}_1, \tau) \to \pi_1(\mathbb{X}_2, \Phi_X(\tau))$ that is fiber and orientation preserving, that induces a graph-of-groups isomorphism, and that induces Φ_X at the level of graphs.*

(2) *There is $\Phi_0 = (\Phi_X, (\phi_v), (\phi_e), (\gamma_e))$ an isomorphism of graphs of groups, such that ϕ_v is fiber-and-orientation preserving, and such that the linear diophantine equation (2) for the γ_i being the images of a fixed basis of F, and of t, by Φ_0, has a solution.*

(3) *There is $\Phi'_0 = (\Phi_X, (\phi'_v), (\phi_e), (\gamma'_e))$ an isomorphism of graphs of groups, such that $\phi'_v \in \mathcal{L}_v$, and such that the linear diophantine equation (2) for the γ_i being the images of a fixed basis of F, and of t, by Φ'_0, has a solution.*

Proof. The first point implies the second: by Lemma [D, 2.6], all vertex groups are suspensions of their intersections with F, therefore if Φ preserves the fiber and orientation, so do all ϕ_v, and the equation (2) admits an obvious solution (the null solution).

The second point implies the first, because, by Proposition 1.2, the system (2) has a solution if and only if there is a modular graph-of-group automorphism of $\pi_1(\mathbb{X}_2, \Phi_X(\tau))$ that sends $\Phi_0(F)$ exactly on F, and preserves orientation.

The third point obviously implies the second one.

We need to show that the second point implies the third one. This is a more subtle part. It basically says that if there is an isomorphism of graph of groups whose orbit under the small modular group intersects the set of fiber-preserving isomorphisms, then there is one that is accessible to us, which might not quite be in the same orbit for the action of the small modular group, but that share the property that its orbit intersects the set of fiber-preserving isomorphisms. The key is in the assertions, in Lemma 2.7 that the different products are in F, and this is ensured by the (*a priori* non immediate) assumption (2.4-(1)).

So, we have $\Phi_0 = (\Phi_X, (\phi_v), (\phi_e), (\gamma_e))$ an isomorphism of graphs of groups, as in the second point.

By assumption on $\mathcal{L}_v$, the given isomorphisms ϕ_v differ from isomorphisms in $\mathcal{L}_v$ by conjugation (in the target), and multiplication by elements of F that are also in the vertex group G_v. By composing with inert twists (vanishing in $\mathrm{Out}(F \rtimes_{\phi_2} \langle t \rangle)$), we can assume that each ϕ_v has same image as an element of $\mathcal{L}_v$ in the Dehn Filling reduction $\overline{\Gamma_v}^{(m)} \to \overline{\Gamma_{\phi_X(v)}}^{(m)}$.

Lemma 2.7. *There is an isomorphism of graphs of groups*

$$\Phi_0' = (\Phi_X, (\phi_v'), (\phi_e), (\gamma_e'))$$

such that $\phi_v' \in \mathcal{L}_v$ *has same image as* ϕ_v *in the Dehn Filling reduction* $\overline{\Gamma_v}^{(m)} \to \overline{\Gamma_{\phi_X(v)}}^{(m)}$, *and such that for all edge* e, $(\gamma_e')^{-1}\gamma_e \in F$, *and also* $\gamma_e(\gamma_e')^{-1} \in F$, *and* $(\gamma_e')\gamma_e^{-1} \in F$.

Proof. We thus construct Φ_0' using these elements $\phi_v' \in \mathcal{L}_v$. The morphisms ϕ_e are given by the marking of the cyclic edge groups. We need that there exists elements γ_e' completing the collection into an isomorphism of graph-of-groups, but this is actually the condition that, for $v = o(e)$, ϕ_v' preserve the peripheral structure of the adjacent cyclic edge groups. Note that one can choose the γ_e' up to a multiplication on the left by an element of $\Gamma_{o(e)}$ centralising $i_e(\Gamma_e)$.

Once such elements γ_e' are chosen, Φ_0' is defined. Recall that on a Bass generator $e \in X^{(1)} \setminus \tau$, $\Phi_0'(e) = (\gamma_{\bar e}')^{-1}\Phi_X(e)(\gamma_e')$. We need to compute how Φ_0' differs from Φ_0 on Bass generators.

Let us call c_e the marked generator of the edge group Γ_e. To make notations readable, we will still write c_e for $i_e(c_e)$.

One has $c_{\Phi_X(e)} = \phi_v(c_e)^{\gamma_e}$ by Bass diagram 1. By virtue of ϕ_v' preserving the peripheral structure (for $v = o(e)$) this is also $= (\phi_v'(c_e))^{h_e})^{\gamma_e}$, for some $h_e \in \Gamma_{\phi_X(v)}$ which can be chosen up to left multiplication by an element centralising $\phi_v'(c_e)$. By Bass diagram 1 (for ϕ_v') this is $= ((c_{\Phi_X(e)})^{(\gamma_e')^{-1}})^{h_e})^{\gamma_e}$. It follows that $(\gamma_e')^{-1}h_e\gamma_e$ centralises $c_{\Phi_X(e)}$, and lies in $\Gamma_{\phi_X(v)}$, for $v = o(e)$.

Recall that γ_e' can be chosen up to a multiplication on the left by an element of $\Gamma_{o(e)}$ centralising $i_e(\Gamma_e)$. By a right choice of the collection of γ_e' (or, in different words, by the right application of Dehn twists), we may assume that $(\gamma_e')^{-1}h_e\gamma_e = 1$.

By virtue of ϕ_v' coinciding with ϕ_v in the Dehn filling $\overline{\Gamma_v}^{(m)}$, the image $\bar h_e$ of h_e actually centralises $\overline{\phi_v'(c_e)}$. By assumption on m (see 2.4 -(1)), the centralizers of the transverse peripheral structure in $\bar{G_2}^m$ are the images of the centralizers in G_2, so this makes $h_e = z_e f_e$ for

z_e centralising $\phi'_v(c_e)$ and f_e in F. Thus, we may choose it so that $z_e = 1$, hence $h_e \in F$. Finally, since F is normal, and $(\gamma'_e)^{-1} h_e \gamma_e = 1$, it follows that $(\gamma'_e)^{-1}\gamma_e \in F$. The two other relations are obtained respectively by conjugating by γ_e^{-1} (F is normal) and taking the inverse of the later. Q.E.D.

We can resume the proof of the Proposition (second point implies the third one). Finally, we need to check that the automorphism Φ'_0 provided by the Lemma is suitable. We can compare the images of the Bass generator e, namely $\Phi_0(e)$ to $\Phi'_0(e)$.

Lemma 2.8. *For all edge e, $\Phi'_0(e)^{-1}\Phi_0(e) \in F$, and for all edge e', the number $n(\Phi'_0(e)^{-1}\Phi_0(e), e')$ that counts the number of occurences of e' in the normal form of $\Phi'_0(e)^{-1}\Phi_0(e)$, minus the number of occurences of $\overline{e'}$ (see §1.1), is* 0.

Proof. Recall that

$$\Phi_0(e) = \gamma_{\bar{e}}^{-1}\phi_X(e)\gamma_e \quad \text{and} \quad \Phi'_0(e) = (\gamma'_{\bar{e}})^{-1}\phi_X(e)\gamma'_e.$$

The difference is therefore

$$\Phi'_0(e)^{-1}\Phi_0(e) = (\gamma'_e)^{-1}\phi_X(e)^{-1}(\gamma'_{\bar{e}}\gamma_{\bar{e}}^{-1})\phi_X(e)\gamma_e$$

which can also be written $\Phi'_0(e)^{-1}\Phi_0(e) = (\gamma'_e)^{-1}(\gamma'_{\bar{e}}\gamma_{\bar{e}}^{-1})^{\phi_X(e)}\gamma_e$ and slightly less naturally,

$$\Phi'_0(e)^{-1}\Phi_0(e) = (\gamma_e(\gamma'_e)^{-1}(\gamma'_{\bar{e}}\gamma_{\bar{e}}^{-1})^{\phi_X(e)})^{\gamma_e}.$$

Since we established that $\gamma_e(\gamma'_e)^{-1} \in F$, $\gamma'_{\bar{e}}\gamma_{\bar{e}}^{-1} \in F$ and F is normal, $\Phi'_0(e)^{-1}\Phi_0(e) \in F$.

Moreover, since all factors in the product

$$(\gamma_e(\gamma'_e)^{-1}(\gamma'_{\bar{e}}\gamma_{\bar{e}}^{-1})^{\phi_X(e)})^{\gamma_e}$$

are in vertex groups, except $\phi_X(e)$ and $\phi_X(e)^{-1}$ (which both appear once). Therefore, for all edge e', the quantity $n(\gamma'_{\bar{e}}\gamma_{\bar{e}}^{-1}, e')$ is 0. Q.E.D.

We may now finish the argument and show that the system of equations (2) for the γ_i being the images by Φ'_0 of a basis of F, and of t, has a solution.

Consider an element f of the given basis of F. Write the normal form in $\pi_1(\mathbb{X}_1, \tau)$ as $f = g_0 e_1 g_1 e_2 \dots e_n g_{n+1}$. The normal form of its image by Φ_0 in $\pi_1(\mathbb{X}_2, \Phi_X(\tau))$ is thus

$$\Phi_0(f) = \phi_{v_0}(g_0)\,\Phi_0(e_0)\,\dots\,\Phi_0(e_n)\,\phi_{v_{n+1}}(g_{n+1}).$$

By assumption (2.4-(1)), each $\phi_{v_i}(g_i)$ differs from ϕ'_{v_i} by a conjugation (say by ξ_i), and the multiplication on the right by an element (say $f_{i,\ell}$) of F, that lies in a vertex group (hence $n(f_{i,\ell}, e') = 0$ for all e'). We also established in the lemma that $\Phi'_0(e_i)^{-1}\Phi_0(e_i) \in F$ (call it $f_{i,r}$), and that $n(f_{i,r}, e') = 0$ for all e').

We thus get

$$\Phi_0(f) = (\phi'_{v_0}(g_0))^{\xi_0}\, f_{0,\ell}\, \Phi'_0(e_0) f_{0,r}\, \dots$$

$$\dots (\phi'_{v_n}(g_n))^{\xi_n}\, f_{n,\ell}\, \Phi'_0(e_n)\, f_{n,r}\, (\phi'_{v_{n+1}}(g_{n+1}))^{\xi_{n+1}}\, f_{n+1,\ell}.$$

We first observe that for all e, $n(\Phi_0(f), e) = n(\Phi'_0(f), e)$ since all $n(f_{i,\ell}, e) = 0$ and all $n(f_{i,r}, e) = 0$.

In $H_1(F \rtimes_{\phi_2} \langle t \rangle)$ this normal form turns into

$$\overline{\Phi_0(f)} = \left(\prod_{i=0}^{n+1} \overline{\phi'_{v_i}(g_i)}\right) \times \left(\prod_{i=0}^{n} \overline{\Phi'_0(e_i)}\right) \times (f_{tot}),$$

where f_{tot} is the image of $\prod_i f_{i,\ell} f_{i,r}$ in H_1, hence is in the image of F.

Of course,

$$\overline{\Phi'_0(f)} = \left(\prod_{i=0}^{n+1} \overline{\phi'_{v_i}(g_i)}\right) \times \left(\prod_{i=0}^{n} \overline{\Phi'_0(e_i)}\right).$$

In the notations of equation (2), $\delta(\overline{\Phi'_0(f)}) = \delta(\overline{\Phi_0(f)})$ because $\delta(f_{tot}) = 0$ as f_{tot} is in F. Moreover, we noticed that for all edge e, $n(\Phi'_0(f), e) = n(\Phi_0(f), e)$. Therefore the two systems of Diophantine equations (2) for the γ_i being the images of a fixed basis of F by Φ_0 and for the γ_i being the images of a fixed basis of F by Φ'_0, are syntaxically the same system of equations. Tautologically, if one has a solution, the other aslo.

Q.E.D.

For the last (and main) result, we'll need the theory of JSJ decompositions for relatively hyperbolic groups. The theory initiated by Rips and Sela is developed by Guirardel and Levitt in a very stable and useful formulation. We refer to [GL-10, GL-11], from which we recall the following existence and characteristic result.

Proposition 2.9. [GL-10, Thm. 13.1, Coro. 13.2] *Let $(G, \mathcal{P})$ be a torsion free relatively freely indecomposable, relatively hyperbolic group. The canonical JSJ splitting of $(G, \mathcal{P})$ is a finite graph-of-groups decomposition of $(G, \mathcal{P})$ with edge groups elementary (cyclic or parabolic), bipartite, such that the groups of a vertex of one color (black) are elementary, and those of the other color (white) are either fundamental groups* *of surfaces with boundary, the adjacent edge groups being associated to the boundary components, or groups that inherit a relatively hyperbolic structure for which there is no non-trivial elementary splitting in which the adjacent edge groups are conjugated into factors (the later are called rigid).*

The canonical JSJ splitting is such that any automorphism of $(G, \mathcal{P})$ *induces an automorphism of the splitting, and is also such that any other elementary splitting of* $(G, \mathcal{P})$ *has a common refinement with it.*

Here, a refinement is an equivariant blow-up of vertices in the Bass-Serre tree.

Recall that in the case of a suspension of a finitely generated group, no vertex group can be a surface group (see [D, 2.11]). Moreover, in the case of a suspension of a free group by an automorphism that is RH-noPS, then all black vertex groups are cyclic, non-parabolic, by definition of RH-noPS.

Proposition 2.10. *Given a finitely generated free group and an automorphism* ϕ *of* F *that is RH-noPS, one can compute the canonical JSJ splitting of* $F \rtimes_\phi \mathbb{Z}$.

Proof. Let $(G, \mathcal{P})$ be the relatively hyperbolic structure for $F \rtimes_\phi \mathbb{Z}$. The computation of the relative hyperbolicity structure was done in 1.4. We can thus enumerate the non-trivial bipartite peripheral cyclic splitting of $(G, \mathcal{P})$. Assume that we are proposed such a bipartite cyclic splitting of $(G, \mathcal{P})$. We need to decide whether or not this proposed splitting is the JSJ splitting. Recall that vertex groups are suspensions of finitely generated subgroups of F, and we may find a presentation of the semidirect product for each of them, by enumeration. We use a result of Touikan here, specifically, [T, Thm C.] that allows to check whether or not, the white vertex groups are rigid (in the sense of Proposition 2.9). Note that since we assume the absence of parabolic splitting, we do not need to satisfy the assumption of [T] on algorithmic tractabiliy of parabolic subgroups. One can also easily check whether the black vertex groups are cyclic, and edge groups are maximal cyclic in their white adjacent vertex group. If all these conditions are verified for a splitting $\mathbb{X}$, the splitting has, by the previous proposition, a common refinement with the JSJ splitting $\mathbb{X}_0$. However, because of lack of surface groups in $\mathbb{X}$ and $\mathbb{X}_0$, for all vertex group G_v in either of them, equipped with the peripheral structure of its adjacent edge groups, all splittings of G_v over cyclic groups in which the peripheral structure is elliptic, are trivial.

Thus considering $\mathbb{T}$ the Bass-Serre tree of a common refinement of $\mathbb{X}$ and $\mathbb{X}_0$, it is apparant that the edges to be collaped to obtain the tree of $\mathbb{X}$ and the tree of $\mathbb{X}_0$ are exactly those with an end of valence 1, and after them, those with an end of valence 2 with same stabilizer (*i.e.* the redundant vertices; note that there is a choice, but either choice lead to the same tree). Since the two collapses toward $\mathbb{X}$ and $\mathbb{X}_0$ can be made by the same choices od edges to collapse, it follows that $\mathbb{X}$ and $\mathbb{X}_0$ coincide. Q.E.D.

Proposition 2.11. *Let F be a finitely generated free group. There is an (explicit) algorithm that given two automorphisms ϕ_1, ϕ_2 of F terminates if both are RH-noPS, and indicates whether they are conjugated in* $\mathrm{Out}(F)$.

Proof. One can compute the JSJ decomposition of both suspensions, by the previous proposition. The situation reduces to the case where an isomorphism of underlying graphs of the JSJ splittings is chosen, and we need to decide whether there is an isomorphism of graph of groups (inducing that isomorphism of underlying graphs) that preserve the fiber, and the orientation. We also choose a maximal subtree τ of the underlying graph X so that all edges outside this subtree correspond to Bass generators in the fundamental group of the graph of group.

Let us write $\mathbb{X}_1, \mathbb{X}_2$ the JSJ splittings of $G_1 = F \rtimes_{\phi_1} \langle t\rangle$ and $G_2 = F \rtimes_{\phi_2} \langle t\rangle$, and we identify the underlying graphs, according to the choice of isomorphism above (the algorithm has to treat all possible such isomorphisms of graphs in parallel). We write $\Gamma_{v,i}$ for the vertex group of v in $\mathbb{X}_i$.

We remark that any elementary vertex group (or edge group) is cyclic, and transveral to the fiber (Lemma [D, 2.7]), hence the orientation of the suspension provides a canonical marking of each edge group. For each non-elementary vertex group, Lemma [D, 2.6] indicates that it is a suspension, and it is equiped with the thus marked cyclic transverse peripheral structure of its adjacent edge groups. Since, by [D, 2.11], it is not a surface nor a free group, by property of the JSJ decomposition, it is RH-noPS relative to the peripheral structure, and it is possible, thanks to Proposition 2.4 to decide whether there is an isomorphism $\Gamma_{v,1} \to \Gamma_{v,2}$, preserving fiber, orientation, and the (cyclic, transverse) peripheral structure of its adjacent edge groups, thus marked.

If for some vertex there is no such isomorphism, then there cannot be any fiber-and-orientation preserving isomorphism between G_1 and G_2, inducing this graph isomorphism (hence ϕ_1 and ϕ_2 are not conjugated, in view of Lemma 1.1). If, on the contrary, for all such vertices, there exists such an isomorphism, then Proposition 2.4 actually provides a list $\mathcal{L}_v$ of such isomorphisms, that satisfies (2.4-(1)), for all vertex $v \in X^{(0)}$. Then, for any choice $(\Phi_v, v \in X^{(0)})$ in $\prod_v \mathcal{L}_v$, one can extend this collection into an isomorphism of graph-of-groups Φ, by chosing appropriately the images of the Bass generators to be $b_e, e \in X \setminus \tau$ (so that they conjugate the edge group in their origin vertex to the edge dgroup in their target vertex). Let us write $(\Phi_s, s \in \prod_v \mathcal{L}_v)$ the collection thus obtained.

Using 1.2 we can decide whether, given Φ_s for some $s \in \prod_v \mathcal{L}_v$, there is an automorphism of graph-of-groups, in the orbit of Φ_s under the small modular group of G_2, that is fiber-and-orientation preserving. If there is one, then we may stop and declare, in view of Lemma 1.1 that ϕ_1 and ϕ_2 are conjugated.

Assume then that there is none such fiber-and-orientation preserving automorphism in the orbits of all Φ_s, $s \in \prod_v \mathcal{L}_v$. By Proposition 2.6 ("$2 \Longrightarrow 3$"), there is no isomorphism of graph-of-groups preserving fiber and orientation. We are done. Q.E.D.

2.3. A lemma on Dehn fillings

In this paragraph we prove a Lemma that we used above. We refer the reader to the setting of [DGO, §7.].

In particular we will use the parabolic cone-off construction, which is a specific way to cone off horosheres of a system of horoballs of a space associated to a relatively hyperbolic group. Using this specific way allows to get quantitative hyperbolicity estimates, and rotating families, while preserving, almost without distortion, most of what occurs (locally) in a thick part of that space.

Lemma 2.12. *Let $(G, \mathcal{P})$ be a relatively hyperbolic group, and $\gamma \in G$, a hyperbolic element.*

There exists m_0 such that for all $m > m_0$, if h is such that $\bar{h}$ centralises $\bar{\gamma}$ in $\bar{G}^{(m)}$, then, there is $z \in K_m$, and h' centralizing γ such that $h = h'z$.

The bound on m will be explicit (but we do not need this particular aspect), though probably not optimal.

Proof. Consider a hyperbolic space X associated to $(G, \mathcal{P})$, upon which G acts as a geometrically finite group, and for convenience, let us choose it to be a cusped-space as defined by Groves and Manning (see [GM, §3]). Let δ be a hyperbolicity constant for X.

After rescaling, we may assume that the hyperbolicity constant is actually less than a specific constant δ_c furnished by [C, 3.5.2], (that will allow, as we did in [DGO, §7] to satisfy [DGO, 5.38], to ensure quantitatively that the parabolic cone-off construction over a separated system of horoballs is hyperbolic).

By assumption γ is hyperbolic in X, so let ρ_0 be a quasi-axis, and $\|\gamma\|$ the translation lenght of γ on this axis. We choose m_0 such that

$N_{j,m_0} \setminus \{1\}$ does not intersect the ball of P_j of radius $10\sinh(r_U) \times 2^{100\delta+\|\gamma\|}$, for r_U fixed as in [DGO, §5.3], namely $r_U = 5 \times 10^{12}$.

We choose L_0 such that in a δ-hyperbolic space, all L_0-local geodesic is a quasigeodesic (for some constants). And we choose L_1 so that any quasigeodesic with these constants is at distance $\leq L_1$ from a geodesic with same end points. The constant L_3 is set to be $\geq 50 \times 16 \times 900$.

Consider $\mathcal{H}_0$ the 2-separated invariant system of horoballs of X (associated to $\mathcal{P}$), and in this system, consider the system of horoballs at depth $(50\delta + 20\|\gamma\| + L_0 + L_1 + L_3)$, which we call $\mathcal{H}$. This way, ρ_0, which has a fundamental domain for γ of length $\|\gamma\|$, and which intersects $X \setminus \mathcal{H}_0$, does not get $(L_0 + L_1 + L_3)$-close to an horoball of $\mathcal{H}$.

We then consider the parabolic cone-off $\mathcal{C}(X, \mathcal{H})$, as defined in [DGO, §7, Def. 7.2], for this pair, and for a radius of cone $r_U = 5 \times 10^{12}$.

By [DGO, Lem. 7.4], this parabolic cone-off is δ_p-hyperbolic, (for a value of δ_p estimated to be 16×900 in [DGO, §5.3]) and moreover the image of ρ_0 is a L_0-local geodesic. By our choices of constants it follows that a quasi-axis ρ of γ in the parabolic cone-off does not approach any of the cones by a distance $50\delta_p$.

We now work only in the parabolic cone-off.

Observe also that, for the chosen m, any (non-trivial) element of $N_{j,m}$ translate on the corresponding horosphere of $\mathcal{H}$ by a distance of at least $10\sinh(r_U)$ (measured in the graph distance of the horosphere). Therefore, for the chosen m, K_m is the group of a very rotating family at the apices of the parabolic cone-off, in the sense of [DGO, §5.1].

Let $x_0 \in \rho$. The segment $[x_0, \gamma x_0]$ is contained in ρ, and thus does not get $50\delta_p$ close to an apex.

Consider the segment $[x_0, hx_0]$ and for all apex a on it, define the two points a_- and a_+ on $[x_0, a]$ and $[a, hx_0]$ (subsegments of $[x_0, hx_0]$) at distance $27\delta_p$ from a (they exist since the cones have much larger radius than $27\delta_p$, and x_0 is not in a cone).

By multiplying h by elements of K_m, we may assume that $d(a_-, a_+) = d(a_-, (\mathrm{Fix}(a) \cap K_m)a_+)$, for every apex a in the segment $[x_0, hx_0]$.

By assumption, $h\gamma h^{-1}\gamma^{-1} \in K_m$, and we can assume that it is non trivial (otherwise there is nothing to prove). By [DGO, Lemma 5.10 (pointed Greendlinger lemma)], the segment $[h\gamma x_0, \gamma h x_0]$ contains an apex a_0 and a $5\delta_p$-shortening pair: a pair of points at distance $27\delta_p$ from the same apex, such that the image of one by an element fixing the apex and in the rotating group is at distance $\leq 5\delta_p$ from the other.

Hyperbolicity in the pentagon $(x_0, hx_0, h\gamma x_0, \gamma h x_0, \gamma x_0)$, together with the absence of apices in $[x_0, \gamma x_0]$, and in its image after translation by h, $[hx_0, h\gamma x_0]$, shows that at least one segment among $[x_0, hx_0]$ and $[\gamma x_0, \gamma hx_0]$ must get at distance δ_p from a_0 and $5\delta_p$-follows two arcs of $[a_0, \gamma hx_0]$ and $[a_0, h\gamma x_0]$ (both subsegments of $[h\gamma x_0, \gamma hx_0]$) for at least $50\delta_p$. By properties of rotating families, we see that one of the two segments $[x_0, hx_0], [\gamma x_0, \gamma hx_0]$ must contain the apex a_0. Assume that only one of them contains a_0, and let us say that it is $[x_0, hx_0]$ (if it is the other, the argument is identical). Hyperbolicity forces the $5\delta_p$-shortening pair of $[h\gamma x_0, \gamma hx_0]$ at a_0 to be $5\delta_p$-close to a_- and a_+. Therefore some element of $K_m \cap \mathrm{Fix}(a_0)$ takes a_+ to a point at distance at most $20\delta_p$ from a_-, thus contradicting the above minimality condition.

It follows that both $[x_0, hx_0]$ and $[\gamma x_0, \gamma hx_0]$ (which is its image by γ) must contain a_0.

But then, the image of a_0 by γ is at distance at most $\|\gamma\|$ from a_0. By separation of apices, it must then be a_0, and γ fixes an apex, contrarily to our assumption.

Q.E.D.

References

[BFH-00] M. Bestvina, M. Feighn, M. Handel, The Tits alternative for Out(Fn). I. Dynamics of exponentially-growing automorphisms. Ann. of Math. (2) 151 (2000), no. 2, 517–623.

[BFH-05] The Tits alternative for Out(Fn). II. A Kolchin type theorem. Ann. of Math. (2) 161 (2005), no. 1, 1–59.

[B] P. Brinkmann, Hyperbolic automorphisms of free groups. Geom. Funct. Anal. 10 (2000), no. 5, 1071–1089.

[C] R. Coulon, Asphericity and small cancellation theory for rotation families of groups, Groups Geom. Dyn. **5**, (2011), no.4 729–765.

[DG-11] F. Dahmani, V. Guirardel, The isomorphism problem for all hyperbolic groups. Geom. Funct. Anal. 21 (2011), no. 2, 223–300.

[DG-13] F. Dahmani, V. Guirardel, Presenting parabolic subgroups, Alg. Geom. Top. 13 (2013) 3203-3222.

[DG-15] F. Dahmani, V. Guirardel, Recognizing a relatively hyperbolic group by its Dehn Fillings, arXiv:1506.03233

[DGO] F. Dahmani, V. Guirardel, D. Osin, Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces, Memoirs Amer. Math. Soc. **245**, no.1156 (2017).

[DGr] F. Dahmani, D. Groves, The isomorphism problem for toral relatively hyperbolic groups. Publ. Math. Inst. Hautes Études Sci. No. 107 (2008), 211–290.

[D] F. Dahmani, On suspensions, and conjugacy of hyperbolic automorphisms, Trans. Amer. Math. Soc. **368** (2016), 5565–5577.

[FH] M. Feighn, M. Handel, Algorithmic constructions of relative train track maps and CTs, arXiv:1411.6302

[GLu] F. Gautero, M. Lustig, The mapping torus of a free group automorphism is hyperbolic relative to the canonical subgroups of polynomial growth. Preprint 2007

[GM] D. Groves, J. Manning, Dehn filling in relatively hyperbolic groups, Israel J. Math., 168 (2008), 317-429.

[GL-10] V. Guirardel, G. Levitt, JSJ decompositions: definitions, existence, uniqueness II. Compatibility and acylindricity. arXiv:1002.4564.

[GL-11] V. Guirardel, G. Levitt, Trees of cylinders and canonical splittings. Geom. Topol., 15(2) (2011) 977–1012.

[L] G. Levitt, Counting growth types of automorphisms of free groups. Geom. Funct. Anal. 19, no.4, 1119-1146 (2009).

[Lu-00] M. Lustig. Structure and conjugacy for automorphisms of free groups I. MPI-Bonn preprint series 2000, No. 241; http://www.mpim-bonn.mpg.de/preprints

[Lu-01] M. Lustig. Structure and conjugacy for automorphisms of free groups II. MPI-Bonn preprint series 2001, No. 4; http://www.mpim-bonn.mpg.de/preprints

[O-06a] D. Osin Relatively hyperbolic groups, Intrinsic geometry, algebraic properties, and algorithmic problems, Mem. Amer. Math. Soc. 179 (2006) no. 843. vi+100pp.

[O-06b] D. Osin, Elementary subgroups of relatively hyperbolic groups, and bounded generation, Internat. J. Algebra Comput. 16 (2006) no.1 99-118.

[O-07] D. Osin, Peripheral fillings of relatively hyperbolic groups. Invent. Math. 167 (2007), no. 2, 295–326.

[P] P. Papasoglu, An algorithm detecting hyperbolicity. Geometric and computational perspectives on infinite groups (Minneapolis, MN and New Brunswick, NJ, 1994), 193–200, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 25, Amer. Math. Soc., Providence, RI, 1996.

[Se] Z. Sela, The isomorphism problem for hyperbolic groups, Ann. Math., 141 (1995), 217–283.

[Si] A. Sisto, Contracting elements and random walks, to appear in Crelle's Journal.

[T] N. Touikan, Finding tracks in 2-complexes, to appear in Trans. Amer. Math. Soc.

François Dahmani,

Université Grenoble Alpes, Institut Fourier, F-38000 Grenoble, France.

e-mail: francois.dahmani@univ-grenoble-alpes.fr