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Explicit resolution of three dimensional terminal singularities

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Dedicated to Prof. Shigefumi Mori on his 60th birthday

Abstract.

We prove that any three dimensional terminal singularity $P \in X$ can be resolved by a sequence of divisorial extractions with minimal discrepancies which are weighted blowups over points.

§1. Introduction

Terminal singularities are the smallest category that minimal model program could work in higher dimension. In fact, the development of minimal model program in dimension three was built on the understanding of three dimensional terminal singularities: Reid set up some fundamental results on canonical and terminal singularities (cf. [20, 21, 22]), Mori classified three dimensional terminal singularities explicitly (cf. [18]) and then Kollár and Mori proved the existence of flips by classifying "extremal neighborhood" (cf. [19, 13]), which is essentially the classification of singularities on a rational curve representing extremal ray. Together with the termination of flips of Shokurov (cf. [23]), one has the minimal model program in dimension three.

It is interesting, and perhaps of fundamental importance, to know those birational maps explicitly in minimal model program. For example, if X is a non-singular threefold and $X \to W$ is a divisorial contraction to a point then W could have simple singularities like $(x^2 + y^2 + z^2 + u^2 = 0)$, $(x^2 + y^2 + z^2 + u^3 = 0)$, or a quotient singularity $\frac{1}{2}(1,1,1)$ (cf. [17]). It is expected that the singularities get worse by further contractions.

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On the other way around, given a germ of three-dimensional terminal singularity $P \in X$, it is expected that one can have a resolution by successive divisorial extractions. For example, given a terminal quotient singularity $P \in X$, one has the "economical resolution" by Kawamata blowups successively. In [5], Hayakawa shows the following

Theorem 1. *For a terminal singularity $P \in X$ of index $r > 1$, there exists a partial resolution*

$$X_n \to \ldots \to X_1 \to X_0 = X \ni P$$

such that X_n is Gorenstein and each $f_i : X_{i+1} \to X_i$ is a divisorial contraction to a point $P_i \in X_i$ of index $r_i > 1$ with minimal discrepancy $1/r_i$. All these maps f_i are weighted blowups.

It is natural to ask whether one can resolve terminal singularities of index 1 in a similar manner, after Markushevich's result that there exists a divisorial contraction with discrepancy 1 over any cDV point which is a weighted blowup (cf. [16]).

Definition 1.1. *Given a three-dimensional terminal singularity $P \in X$. We say that there exists a feasible resolution for $P \in X$ if there is a sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X \ni P,$$

such that X_n is non-singular and each $X_{i+1} \to X_i$ is a divisorial contraction to a point with minimal discrepancy, i.e. a contraction to a point $P_i \in X_i$ of index $r_i \geq 1$ with discrepancy $1/r_i$.

The purpose of this note is prove that a feasible resolution exists for three dimensional terminal singularities.

Theorem 2 (Main Theorem)**.** *Given a three-dimensional terminal singularity $P \in X$. There exists a feasible resolution for $P \in X$.*

One might expect to construct such resolution by finding a divisorial contraction with discrepancy 1 over a terminal singularity of index 1 and combining with Theorem 1. However, there are some technical difficulties.

First of all, given a divisorial contraction $Y \to X \ni P$ with discrepancy 1 over a terminal singularity $P \in X$ of index 1, then Y usually have singularities of higher indexes. Resolving these higher index points by Hayakawa's result, one might pickup some extra singularities of index 1 in the process. However, the studies of singularities of index 1 was not there in Hayakaya's work.

Another difficulty is that singular Riemann-Roch is not sensitive to Gorenstein points. Therefore, the powerful technique introduced by

Kawakita (cf. [8, 9, 10]) to study singularities and divisorial contraction by using singular Riemann-Roch formula is not valid.

What we did in this note is basically pick convenient weighted blowups, keep good track of terminal singularities, and put them into a right hierarchy. The hierarchy is as following: 1. terminal quotient singularities; 2. cA points; 3. cA/r points; 4. cD and $cAx/2$ poitns; 5. $cAx/4$, $cD/2$, and $cD/3$ points ; 6. cE_6 points; 7. $cE/2$ points; 8. cE_7 points; 9. cE_8 points.

These involves careful elaborative case-by-case studies. The reader might find that the structure is very similar to part of Kollár work in [14]. Indeed, a lots of materials can be found in the preprints of Hayakawa [6, 7], in which he tried to classified all divisorial contractions with discrepancy 1 over a cD or cE point. Many of our choices of resolutions are inspired by his works. This work can not be done without his work in [6, 7]. For reader's convenience and for the sake of self-contained, we choose to reproduce the proofs that we needed here. The existence of divisorial contractions and explicit descriptions are already given by Hayakawa in his series of works. What is really new in this article is that we choose those convenient weighted blowups and work out the inductive process.

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§2. Preliminaries

2.1. weighted blowups

We will need weighted blowup which are divisorial extraction with minimal discrepancy. For this purpose, we first fix some notations.

Let $N = \mathbb{Z}^n$ and $v_0 = \frac{1}{r}(a_1, ..., a_n) \in \frac{1}{r}\mathbb{Z}^n$. We write $\overline{N} := N + \mathbb{Z}v_0$. Let σ be the cone of first quadrant, i.e. the cone generated by the standard basis $e_1, ..., e_n$ and Σ be the fan consists of σ and all the subcones of σ. We have that $\mathcal{X}_0 := \mathrm{Spec}\mathbb{C}[\sigma^\vee \cap \overline{M}]$ is a quotient variety of $\mathbb{C}^n$ by the cyclic group $\mathbb{Z}/r\mathbb{Z}$, which we denote it as $\mathbb{C}^n/v_0$ or $\mathbb{C}^n/\frac{1}{r}(a_1, ..., a_n)$.

For any primitive vector $v = \frac{1}{r}(b_1, b_2, ..., b_n) \in \overline{N}$ with $b_i > 0$, we can consider the weighted blowup $\mathcal{X}_1 \to \mathcal{X}_0 := \mathbb{C}^n/v_0$ with weight v, which is the toric variety obtained by subdivision along v. More concretely, let

σ_i be the cone generated by $\{e_1, ..., e_{i-1}, v_1, e_{i+1}, ..., e_n\}$, then

$$\mathcal{X}_1 := \cup_{i=1}^n \mathcal{U}_i,$$

where $\mathcal{U}_i = \mathrm{Spec}\mathbb{C}[\sigma_i^\vee \cap \overline{M}]$. We always denote the origin of $\mathcal{U}_i$ by Q_i and the exceptional divisor $\mathcal{E} \cong \mathbb{P}((b_1, b_2, ..., b_n))$ by $\mathbb{P}(v)$.

For any semi-invariant $\varphi = \sum \alpha_{i_1,...,i_n} x_1^{i_1}...x_n^{i_n}$, and for any vector $v = \frac{1}{r}(b_1, b_2, ..., b_n) \in \overline{N}$ we define

$$wt_v(\varphi) := \min\{\sum_{j=1}^n \frac{b_j i_j}{r} | \alpha_{i_1,...,i_n} \neq 0\}.$$

Let $X \in \mathcal{X}_0$ be a complete intersection defined by semi-invariants $(\varphi_1 = ... = \varphi_c = 0)$. Let Y be its proper transform in $\mathcal{X}_1$. By abuse the notation, we also call the induced map $f : Y \to X$ the weighted blowups of X with weight v, or denote it as $\mathrm{wBl}_v : Y \to X$. Notice that $Y \cap U_i$ is defined by $\tilde{\varphi}_1 = ... = \tilde{\varphi}_c = 0$ with

$$\tilde{\varphi}_j := \varphi(x_1 x_i^{\frac{b_1}{r}}, \ldots, x_{i-1} x_i^{\frac{b_{i-1}}{r}}, x_i^{\frac{b_i}{r}}, x_{i+1} x_i^{\frac{b_{i+1}}{r}}, \ldots, x_n x_i^{\frac{b_n}{r}}) x_i^{-wt_v(\varphi)},$$

for each j. Let $E := \mathcal{E} \cap Y \subset \mathbb{P}(v)$ denote the exceptional divisor and $U_i := \mathcal{U}_i \cap Y$.

Let $X = (\varphi_1 = \varphi_2 = ... = \varphi_c = 0) \subset \mathbb{C}^n/v_0$ be an irreducible variety such that $o = P \in X$ is the only singularities. Let $Y \to X \ni P$ be a weighted blowup with weight v and exceptional divisor $E = \mathcal{E} \cap Y \subset \mathbb{P}(v)$. We are interesting in $\mathrm{Sing}(Y)$. We may decompose it into

$$\mathrm{Sing}(Y) = \mathrm{Sing}(Y)_{\mathrm{ind}=1} \cup \mathrm{Sing}(Y)_{\mathrm{ind}>1},$$

where $\mathrm{Sing}(Y)_{\mathrm{ind}=1}$ (resp. $\mathrm{Sing}(Y)_{\mathrm{ind}>1}$) denotes the locus of singularities of index $= 1$ (resp. > 1). Clearly, the locus of points of index > 1 in $\mathcal{X}_1$ coincide with $\mathrm{Sing}(\mathbb{P}(v))$. Hence we have

$$\mathrm{Sing}(Y)_{\mathrm{ind}>1} = Y \cap \mathrm{Sing}(\mathbb{P}(v)) = E \cap \mathrm{Sing}(\mathbb{P}(v)).$$

We will need the following Lemma to determine singularities on Y of index 1.

Lemma 3. *Keep the notation as above. Consider* $\mathrm{wBl}_v : Y \to X$.

(1) *If* $\mathcal{U}_i \cong \mathbb{C}^n$, *then* $\mathrm{Sing}(Y) \cap U_i \subset \mathrm{Sing}(E) \cap U_i$.
(2) *If* Y *is a terminal threefold, then* $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \subset \mathrm{Sing}(E)$

Proof. For each i, we may write $\varphi_i = \varphi_{i,h} + \varphi_{i,r}$, where $\varphi_{i,h}$ (resp. $\varphi_{i,r}$) denotes the homogeneous part (remaining part) of φ_i with weight equals to $wt_v(\varphi_i)$.

In order to prove (2), it suffices to check that $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_j \subset \mathrm{Sing}(E) \cap U_j$ for each j. Without loss of generality, we work on U_n. For simplicity on notations, we also assume that X is a hypersurface.

Let $\rho_n : \mathbb{C}^n \to U_n \cong \mathbb{C}^n/\mu_r$ be the canonical projection. On $\mathbb{C}^n$, $\rho_n^{-1}(Y)$ is defined by semi-invariant

$$\tilde{\varphi} = \widetilde{\varphi}_h + \widetilde{\varphi}_r = \varphi_h(x_1, ..., x_{n-1}, 1) + \widetilde{\varphi}_r,$$

with $x_n | \widetilde{\varphi}_r$ and $\rho_i^{-1}(E) \subset \rho_n^{-1}(U_n)$ is defined by $\widetilde{\varphi}_h = x_n = 0$.

Note that $\mathrm{Sing}(Y) \subset E$ for $f : Y \to X$ is isomorphic away from $p \in X$. Also note that a quotient of a three dimensional smooth point can not be terminal singularity of index 1. Therefore,

$$\begin{aligned}
\rho_n^{-1}(\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_n) \quad & = \rho_n^{-1}(\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_n \cap E) \\
& = \rho_n^{-1}(\mathrm{Sing}(Y \cap U_n)_{\mathrm{ind}=1}) \cap \rho_n^{-1}(E) \\
& \subset \mathrm{Sing}(\rho_n^{-1}(Y \cap U_n)) \cap \rho_n^{-1}(E) \\
& = \tilde{\varphi} = \tilde{\varphi}_{x_1} ... = \tilde{\varphi}_{x_n} = x_n = 0 \\
& \subset \tilde{\varphi} = \tilde{\varphi}_{x_1} = ... = \tilde{\varphi}_{x_{n-1}} = x_n = 0 \\
& = \tilde{\varphi}_h = \tilde{\varphi}_{h,x_1} = ... = \tilde{\varphi}_{h,x_{n-1}} = 0 \\
& = \mathrm{Sing}(\rho_n^{-1}(E \cap U_n)),
\end{aligned}$$

where $\tilde{\varphi}_{x_i}$ denotes $\frac{\partial \tilde{\varphi}}{\partial x_i}$.

Since ρ_n is étale, therefore, $\rho_n(\mathrm{Sing}(\rho_n^{-1}(E \cap U_n))) \subset \mathrm{Sing}(E \cap U_n)$. The statement follows for hypersurface. The same argument works for higher codimension as well.

The proof for (1) also follows from the similar argument. Q.E.D.

Corollary 4. *Let Y be a terminal threefold obtained by $\mathrm{wBl}_v : Y \to X$ with weight v. Suppose that E is a quasi-smooth weighted complete intersection in $\mathbb{P}(v)$, then $\mathrm{Sing}(Y)_{\mathrm{ind}=1} = \emptyset$.*

Proof. If E is a quasi-smooth weighted complete intersection in $\mathbb{P}(v)$, then $\mathrm{Sing}(E) = E \cap \mathrm{Sing}(\mathbb{P}(v))$. Therefore,

$$\mathrm{Sing}(Y)_{\mathrm{ind}=1} \subset \mathrm{Sing}(E) \subset \mathrm{Sing}(\mathbb{P}(v)).$$

However, $\mathrm{Sing}(\mathbb{P}(v))$ consists of quotient singularities of index > 1. This implies in particular that $\mathrm{Sing}(Y)_{\mathrm{ind}=1} = \emptyset$. Q.E.D.

2.2. weighted blowup of threefolds

Given a threefold terminal singularity $P \in X = (\varphi = 0) \subset \mathbb{C}^4/v_0$ of index r, we usually consider weighted blowup $\mathrm{wBl}_v : Y \to X$ with weight $v = \frac{1}{r}(a_1, a_2, a_3, a_4)$ and $a_i \in \mathbb{Z}_{>0}$ [1]. It worths to determine when a weighted blowup is a divisorial contraction.

Theorem 5. *Let $P \in X = (\varphi = 0) \subset \mathbb{C}^4$ be a germ of three dimensional terminal singularity and $f = \mathrm{wBl}_v : Y \to X$ with weight $v = \frac{1}{r}(a_1, a_2, a_3, 1)$ with exceptional divisor $E \subset \mathbb{P}(v)$. Suppose that*

- *E is irreducible;*
- *$\frac{1}{r}\sum a_i - wt_v(\varphi) - 1 = \frac{1}{r}$;*
- *either $Y \cap U_4$ is terminal or E has Du Val singularities on U_4.*

Then $Y \to X$ is a divisorial contraction.

Moreover, $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_4 \subset \mathrm{Sing}(E) \cap U_4$. *For any $R \in \mathrm{Sing}(Y)_{ind=1} \cap U_4$, R is at worst of type cA (resp. cD, cE_6, cE_7, cE_8) if $R \in \mathrm{Sing}(E)$ is of type A (resp. D, E_6, E_7, E_8).*

Proof. Suppose that E is irreducible, then $K_Y = f^*K_X + a(E,X)E$ with $a(E,X) = \frac{1}{r}\sum a_i - wt_v(\varphi) - 1$. Let $D = (x_4 = 0) \subset Div(X)$ and D_Y be its proper transform in Y. One has $f^*D = D_Y + \frac{1}{r}E$. Hence $\frac{1}{r} = \frac{1}{r}\sum a_i - 1 - wt_v(\varphi)$ implies that

$$f^*(K_X + D) = K_Y + D_Y$$

and hence $D_Y \sim_X -K_Y$.

Let $g : Z \to Y$ be a resolution of Y. For any exceptional divisor F in Z such that $g(Z) \subset D_Y$. One has $g^*D_Y = D_Z + mF + \dots$ for some $m > 0$. It follows that $a(F, Y) = m > 0$. Hence Y is terminal if $Y - D_Y = Y \cap U_4$ is terminal.

In fact, by Lemma 3.(1), one sees that $\mathrm{Sing}(Y) \cap U_4 \subset \mathrm{Sing}(E) \cap U_4$. If $E \cap U_4$ is DuVal, then $\mathrm{Sing}(E) \cap U_4$ is isolated hence so is $\mathrm{Sing}(Y) \cap U_4$. More precisely, for $R \in \mathrm{Sing}(E) \cap U_4$ with local equation

$$\psi := \varphi_h(x_1, ..., x_4)|_{x_4=1}$$

of type A (resp. D, E), the local equation for $R \in Y \cap U_4$ is of the form

$$\psi + x_4 g(x_1, ..., x_4),$$

which is a compound DuVal equation. Therefore, if R is singular in Y, then R is a at worst isolated cDV of type cA (resp. cD, cE) [2]. By results of Reid [20], Kollár and Shephard-Barron [15], an isolated cDV

[1] Divisorial contractions to a point of index $r > 1$ have been studied extensively by Hayakawa and are known to be weighted blowups

[2] If there are lower degree terms appearing in g, then the singularity R could be simpler or even non-singular

singularity is terminal. Hence Y is terminal and therefore $f : Y \to X$ is a divisorial contraction. Q.E.D.

In the sequel, all weighted blowups with discrepancy $1/r$ over a terminal singularity of index r can easily checked to be divisorial contractions with minimal discrepancy $1/r$ by the above Theorem 5 or by direct computation.

We will need some further easy but handy Lemmas.

Lemma 6. *Let* $\mathrm{wBl}_v : Y \to X$ *be a divisorial contraction. If* $wt_v(x^2) = wt_v(\varphi)$*, then* $\mathrm{Sing}(Y) \cap U_1 = \emptyset$*.*

Proof. Now E is defined by $(\Phi : \mathbf{x}^2 + f(\mathbf{y}, \mathbf{z}, \mathbf{u}) = 0) \subset \mathbb{P}(v)$. It is clear that $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_1 \subset \mathrm{Sing}(E) \cap U_1 = \emptyset$. Notice also that Q_1 is the only point in U_1 with index > 1 and $Q_1 \notin Y$. Hence $\mathrm{Sing}(Y) \cap U_1 = \emptyset$. Q.E.D.

Lemma 7. *Let* $\mathrm{wBl}_v : Y \to X$ *be a divisorial contraction. If* $x_i^m x_j \in \varphi$ *with* $wt_v(x_i^m x_j) = wt_v(\varphi)$ *or* $x_i^m \in \varphi$ *with* $wt_v(x_i^m) = wt_v(\varphi) + 1$*, then* $Y \cap U_i$ *is non-singular away from* Q_i *and* Q_i *is either non-singular or a terminal quotient singularity of index* $wt_v(x_i)$*. In particular,* $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_i = \emptyset$*.*

Proof. On U_i, $Y \cap U_i$ is given by $(\tilde{\varphi} = 0) \subset \mathbb{C}^4/\mathbb{Z}_{a_i}$ with

$$\tilde{\varphi} = \begin{cases} x_j + \text{others}, & \text{if } wt_v(x_i^m x_j) = wt_v(\varphi); \\ x_i + \text{others}, & \text{if } wt_v(x_i^m) = wt_v(\varphi) + 1. \end{cases}$$

Hence $Y \cap U_i \cong \mathbb{C}^3/\mathbb{Z}_{a_i}$ and the statement follws. Q.E.D.

Lemma 8. *Consider* $X = (\varphi = 0) \subset \mathbb{C}^4$*. Suppose that* $R \in X$ *is an isolated singularity and* $xy \in \varphi$ *or* $x^2 + y^2 \in \varphi$*. Then* R *is of* cA*-type.*

Proof. Up to a unit, we may assume that $\varphi = xy + xg(x, z, u) + yh(y, z, u) + \bar{\varphi}(z.u)$. Since $\frac{\partial^2 \varphi}{\partial x \partial y}(R) = 1 \neq 0$, the local expansion near R is of the form $\bar{x}\bar{y} + \bar{f}(\bar{z}, \bar{u})$ where $\bar{x} = x - x(R)$ respectively. Also $mult_o \bar{f} \geq 2$. Hence it is a cA point. Q.E.D.

Corollary 9. *Consider* $X = (\varphi = 0) \subset \mathbb{C}^4/\mathbb{Z}_r$*. Suppose that* $R \in \mathrm{Sing}(X)_{\mathrm{ind}=1}$ *is an isolated singularity and either* $xy \in \varphi$ *or* $x^2 + y^2 \in \varphi$*. Then* R *is of* cA*-type.*

Proof. Let $\pi : \mathbb{C}^4 \to \mathbb{C}^4/\mathbb{Z}_r$ is the quotient map. Since R is a index 1 point, $\pi^{-1}(R)$ does not intersects the fixed locus of the $\mathbb{Z}_r$ action. This implies in particular that $\mathbb{Z}_r$ acts on $\pi^{-1}(R)$ freely and each point of $Q \in \pi^{-1}(R)$ is singular in $\pi^{-1}(X)$. By Lemma 8, Q is of type cA and hence so is R. Q.E.D.

By the similar argument, one can also see the following

Lemma 10. *Consider $X = (\varphi = 0) \subset \mathbb{C}^4/\mathbb{Z}_r$ with $r = 1, 2, 3$. Suppose that $\varphi = x^2 + f(y, z, u)$ with f_3, the 3-jet of f, is nonzero and not a cube. Let $R \in \mathrm{Sing}(X)$ be an isolated singularity. Then R could only be of type $cA, cA/r, cD, cD/r$ or a terminal quotient singularity.*

§3. resolution of cA and cA/r points

Lemma 11. *Let $f : Y \to X$ be the economic resolution of a terminal quotient singularity $P \in X$. Then this is a feasible resolution for $P \in X$.*

Proof. Given a terminal quotient singularity $P \in X$ of type $\frac{1}{r}(s, r-s, 1)$ with $s < r$ and $(s, r) = 1$, we start by considering the Kawamata blowup $Y \to X$ (cf. [11]), i.e. weighted blowup with weight $v = \frac{1}{r}(s, r-s, 1)$ over P. It is clearly a divisorial contraction with minimal discrepancy $\frac{1}{r}$.

Note that $\mathrm{Sing}(Y)$ consists of at most two points Q_1, Q_2 of index $s, r-s$ respectively. By induction on r, we get a resolution $Y = Y_{r-1} \to \ldots \to Y \to X \ni P$. It is easy to see that this is the economic resolution. Q.E.D.

Theorem 12. *There is a feasible resolution for any cA points.*

Proof. For any cA point $p \in X$, there is an embedding $j : X \subset \mathbb{C}^4$ such that $P \in X$ is defined by $(\varphi : xy + f(z, u) = 0) \subset \mathbb{C}^4$. We fix this embedding once and for all and define

$$\tau(P) := \min\{i + j | z^i u^j \in f(z, u)\}.$$

We may and do assume that $z^\tau \in f$. We write $f = f_\tau + f_{>\tau}$, where f_τ denote the homogeneous part of weight τ.

We need to introduce

$$\tau^\sharp(P) := \min\{i + j | z^i u^j \in f(z, u), i \leq 1\}.$$

Since $P \in X$ is isolated, one has that f contains a term of the type zu^{p-1} or u^p for some p. Hence $\tau^\sharp(P)$ is well-defined. Notice also that $\tau(P) \leq \tau^\sharp(P)$.

We shall prove by induction on τ and $\tau^\sharp$.

Case 1. $\tau = 2$.

By easy change of coordinates, we may and do assume that $f(z, u) = z^2 + u^b$. We take $Y \to X$ to be the weighted blowup with weights $(1, 1, 1, 1)$ (or the usual blowup over P). It is clear that $\mathrm{Sing}(Y) = \{Q_4\}$, which is defined by

$$(\tilde{\varphi} : xy + z^2 + u^{b-2} = 0) \subset \mathbb{C}^4.$$

By induction on b, we are done.

Case 2. $\tau > 2$.

We may write $f_\tau = \prod (z-\alpha_t u)^{l_t}$ since $z^\tau \in f$. We take $\mathrm{wBl}_v : Y_1 \to X$ with weights $v = (1, \tau-1, 1, 1)$. It is clear that $\mathrm{Sing}(Y)_{\mathrm{ind}>1} = \{Q_2\}$, which is a terminal quotient singularity of index $\tau - 1$. Hence it remains to consider index 1 points.

We have that

$$E = (\mathbf{xy} + \prod (\mathbf{z} - \alpha_t \mathbf{u})^{l_t} = 0) \subset \mathbb{P}(1, \tau-1, 1, 1).$$

Now $\mathrm{Sing}(E) = \{R_t = (0,0,\alpha_t,1)\}_{l_t \geq 2}$. In fact, for any R_t with $l_t \geq 2$, one sees that $R_t \subset E$ is a singularity of A-type, it follows that if R_t is singular in Y, then it is of type cA with $\tau(R_t) \leq l_t$.

Subcase 2-1. f_τ factored into more than one factors.

Then $\tau(R_t) \leq l_t < \tau$, then we are done by induction on τ.

Subcase 2-2. If f_τ factored into only one factor.

We may assume $f_\tau = z^\tau$ by changing coordinates. It is easy to see that $\tau(Q_4) \leq \tau^\sharp(Q_4) < \tau^\sharp(P)$ hence

$$\tau(Q_4) + \tau^\sharp(Q_4) < \tau(P) + \tau^\sharp(P).$$

Then we are done by induction on $\tau + \tau^\sharp$. Q.E.D.

Corollary 13. *There is a feasible resolution for any cA/r point.*

Proof. Given $P \in X$ defined by

$$(\varphi : xy + f(z^r, u) = xy + \sum a_{ij} z^{ir} u^j = 0) \subset \mathbb{C}^4 / \frac{1}{r}(s, r-s, 1, r).$$

Let

$$\begin{cases} \kappa^\sharp(\varphi) := \min\{k | u^k \in f\}, \\ \kappa(\varphi) := \min\{i+j | z^{ir} u^j \in f\}. \end{cases}$$

We shall prove by induction on $\kappa^\sharp + \kappa$. Note that there is some $u^k \in f$ otherwise P is not isolated. Thus $\kappa^\sharp + \kappa$ is finite and $\kappa \leq \kappa^\sharp$.

1. $\kappa^\sharp = 1, \kappa = 1$.

Then $P \in X$ is a terminal quotient singularity. We are done.

2. $\kappa^\sharp + \kappa > 2$.

We always consider $Y \to X$ the weighted blowup with weights $\frac{1}{r}(s, \kappa r - s, 1, r)$, which is a divisorial contraction by [4]. Computation on each charts similarly, one sees the following:

(1) $Y \cap U_1$ is singular only at Q_1, which is a terminal quotient singularity of index s (non-singular on U_1 if $s = 1$).

(2) $Y \cap U_2$ is singular only at Q_2, which is a terminal quotient singularity of index $\kappa r - s$ (non-singular on U_2 if $\kappa r - s = 1$).

(3) $Y \cap U_3$ is defined by $xy + f(z, uz)z^{-\kappa} = 0 \subset \mathbb{C}^4$. Hence $\mathrm{Sing}(Y) \cap U_3$ must be of type cA by Lemma 8. There exists a feasible resolution over these points.

(4) it remains to consider Q_4, which is locally defined by

$$(\tilde{\varphi} : xy + \sum a_{ij} z^{ir} u^{i+j-\kappa} = 0) \subset \mathbb{C}^4 / \frac{1}{r}(s, r-s, 1, r).$$

In fact, one sees that $\kappa^\sharp(Q_4) = \kappa^\sharp(P) - \kappa(P)$ and $\kappa(Q_4) \leq \kappa(P)$.

By induction on $\kappa^\sharp + \kappa$, we have a feasible resolution over Q_4. Together with feasible resolution over other singularities on Y, we have a feasible resolution over Y and hence over X. Q.E.D.

§4. resolution of cD and $cAx/2$ points

Given a cD_n point $P \in X$ which is defined by $(\varphi : x^2 + y^2 z + z^{n-1} + ug(x, y, z, u) = 0) \subset \mathbb{C}^4$ for some $n \geq 4$. We start by considering the *normal form* of cD singularities.

Definition 4.1. *We say that a cD point $P \in X$ admits a normal form if there is an embedding*

$$(\varphi : x^2 + y^2 z + \lambda y u^l + f(z, u) = 0) \subset \mathbb{C}^4$$

with the following properties:

(1) *$l \geq 2$. (We adapt the convention that $l = \infty$ if $\lambda = 0$.)*

(2) *$zu^{p-1} \in f$ or $u^p \in f$ for some $p > 0$ if $\lambda = 0$.*

(3) *$z^{q-1}u \in f$ or $z^q \in f$ for some $q > 0$.*

An isolated singularity $P \in X$ given by this form (with $l \geq 0$ and possibly not of cD type) is called a cD-like singularity, which is terminal.

For a cD-like singularity $P \in X$, we define

$$\begin{cases} \mu^\sharp(P \in X) := \min\{2i + j | z^i u^j \in \varphi, i = 0 \text{ or } 1\}; \\ \mu(P \in X) := \min\{2i + j | z^i u^j \in \varphi\}; \\ \mu^\flat(P \in X) := \min\{\mu(P \in X), 2l - 2\}; \\ \tau^\sharp(P \in X) := \min\{i + j | z^i u^j \in \varphi, i = 0 \text{ or } 1\}. \end{cases}$$

Clearly, one has $\mu^\flat \leq \mu \leq \mu^\sharp \leq \infty$. Also $\mu^\sharp, \tau^\sharp < \infty$ if $\lambda = 0$.

Lemma 14. *Given a cD-like point $P \in X$ defined by*

$$(\varphi\colon x^2 + y^2 z + \lambda y u^l + f(z, u) = 0) \subset \mathbb{C}^4,$$

with $\mu^\flat \leq 3$. Then there exists a feasible resolution for $P \in X$.

Proof. If there is a linear or quadratic term in f, then P is non-singular or of cA-type by Lemma 8. In particular, feasible resolution exists. We thus assume that $l \geq 2$ and we may write

$$f(z,u) = f_3(z,u) + f_{\geq 4}(z,u),$$

where $f_3(z,u)$ (resp. $f_{\geq 4}(z,u)$) is the 3-jet (resp. 4 and higher jets) of $f(z,u)$.

Case 1. $l \geq 3$.

If $\mu \leq 2$, then P is at worst of type cA. Thus we may and do assume that $\mu = 3$ and hence $u^3 \in f_3 \neq 0$. Clearly, $\varphi_3 = y^2 z + f_3$ is irreducible.

Subcase 1-1. f_3 is factored into more than one factors.

We consider $\mathrm{wBl}_v : Y \to X$ with weight $v = (2,1,1,1)$. One can verify that $E = (\mathbf{y}^2\mathbf{z} + f_3(\mathbf{z},\mathbf{u}) = 0) \subset \mathbb{P}(2,1,1,1)$ is irreducible. By Lemma 7, one has that $\mathrm{Sing}(Y) \cap U_i$ is non-singular away from Q_i for $i = 1,2$. In fact, Y is non-singular at Q_2. Therefore, $Y \to X$ is a divisorial contraction by Theorem 5.

Since $Q_4 \notin Y$, it remains to consider $Y \cap U_3$, which is defined by

$$\begin{aligned} \tilde{\varphi}: \quad & x^2 z + y^2 + \lambda y z^{l-2} u^l + \tilde{f}_3(z,u) + \tilde{f}_{\geq 4} \\ & = x^2 z + y^2 + \lambda y z^{l-2} u^l + \prod (u - \alpha_t)^{l_t} + \tilde{f}_{\geq 4}, \end{aligned}$$

where $\tilde{f}_3(z,u)$ (resp. $\tilde{f}_{\geq 4}(z,u)$) denotes the proper transform of $f_3(z,u)$ (resp. $f_{\geq 4}(z,u)$). More explicitly, $\tilde{f}_3(z,u) = f_3(z,zu) z^{-wt_v(\varphi)}$.

Let R be a singular point in $\mathrm{Sing}(Y) \cap U_3$. If f_3 is factored into more than one factors, then $l_t \leq 2$ for all t. It follows that R is at worst of cA type by Lemma 8. Notice also that $\mathrm{Sing}(Y)_{\mathrm{ind}>1} = \{Q_1\}$ which is a quotient singularity of index 2. Thus feasible resolution exists.

Subcase 1-2. f_3 is factored into one factor.

We thus assume that $f_3 = (u - \alpha z)^3$. Change coordinate by

$$\begin{cases} \bar{u} = u - \alpha z; \\ \bar{y} = y + \frac{\lambda}{2} \sum_{j \geq 1} C^l_j \alpha^j z^{j-1} \bar{u}^{l-j}; \end{cases}$$

then we have

$$\varphi = x^2 + \bar{y}^2 z + \lambda \bar{y} \bar{u}^l + \bar{u}^3 + \bar{f}_{\geq 4}(z, \bar{u}).$$

Therefore, we may and do assume that $f_3 = u^3$ in the normal form.

We consider again $\mathrm{wBl}_v : Y \to X$ with weight $v = (2,1,1,1)$. Since

$$E = (\mathbf{y}^2\mathbf{z} + \mathbf{u}^3 = 0) \subset \mathbb{P}(2,1,1,1),$$

therefore $Y \to X$ is a divisorial contraction by Theorem 5 (where U_4 is replaced by U_2). Moreover, $\mathrm{Sing}(E) \subset (\mathbf{y} = \mathbf{u} = 0)$, hence $\mathrm{Sing}(E) \subset$

$U_1 \cup U_3$. Together with Lemma 7, it remains to consider Q_3, which is a cD point given by

$$\tilde{\varphi} : x^2 z + y^2 + \lambda y u^l z^{l-2} + u^3 + \tilde{f}_{\geq 4}.$$

Change coordinate by $\bar{y} := x, \bar{x} := y + \frac{1}{2}\lambda u^l = z^{l-2}$, one sees that Q_3 is cD-like given by

$$\tilde{\varphi} : \bar{x}^2 + \bar{y}^2 z + u^3 + \tilde{f}_{\geq 4} - \frac{1}{4}\lambda^2 u^{2l} z^{2l-4}.$$

Clearly, Q_3 is still in Subcase 1-2 and $\tau(Q_3) \leq \tau(P) - 2$. By induction on τ, we conclude that feasible resolution exists for this case.

Case 2. $l = 2$.
Subcase 2-1. $f_3 = 0$.
We consider $\mathrm{wBl}_v : Y \to X$ with weight $v = (2, 2, 1, 1)$. Now

$$E = (\mathbf{x}^2 + \lambda \mathbf{y}\mathbf{u}^2 + f_4(\mathbf{z}, \mathbf{u}) = 0) \subset \mathbb{P}(2, 2, 1, 1)$$

is clearly irreducible. By considering $Y \cap U_4$, which is nonsingular, one has that $Y \to X$ is a divisorial contraction by Lemma 7 and Theorem 5.

One sees that $\mathrm{Sing}(E) \subset (\mathbf{x} = \mathbf{u} = 0)$, hence $\mathrm{Sing}(E) \subset U_2 \cup U_3$. Notice also that $Q_1 \notin Y$ and $Q_2 \in Y$ is a $cA/2$ point. Together with Lemma 8, it remains to consider Q_3, which is a cD-like point and still in Subcase 2-1. Clearly, $\tau(Q_3) \leq \tau(P) - 2$. By induction on τ, we conclude that feasible resolution exists for this case.
Subcase 2-2. $f_3 \neq 0$ and $\varphi_3 = y^2 z + \lambda y u^2 + f_3$ is irreducible.
We consider $\mathrm{wBl}_v : Y \to X$ with weights $v = (2, 1, 1, 1)$. Now

$$E = (\mathbf{y}^2\mathbf{z} + \lambda \mathbf{y}\mathbf{u}^2 + f_3(\mathbf{z}, \mathbf{u}) = 0) \subset \mathbb{P}(2, 1, 1, 1).$$

One has that $Y \to X$ is a divisorial contraction by the same reason. By Lemma 7, it's clear that $\mathrm{Sing}(Y) \cap (U_1 \cup U_2 \cup U_4) = \{Q_1\}$, which is a quotient singularity of index 2.

It remains to consider Q_3. If f_3 is factored into more than one factors then the same argument in Subcase 1-1 works. We thus assume that $f_3 = (\beta u + \alpha z)^3$. In fact, one sees that Q_3 is singular only when $f_3 = u^3$. Argue as in Subcase 1-2. We have a feasible resolution for $P \in X$.
Subcase 2-3. $f_3 \neq 0$ and $\varphi_3 = y^2 z + \lambda y u^2 + f_3$ is reducible.
In this situation, $y^2 z + \lambda y u^2 + f_3 = (y + l(z, u))(zy + \lambda u^2 - l(z, u)z)$ for some linear form $l(z, u) \neq 0$. Let $\bar{y} = y + l(z, u)$, then we have

$$\varphi_3 = \bar{y}^2 z + \lambda \bar{y} u^2 - 2\bar{y} z l(z, u).$$

We consider weighted blowup $Y \to X$ with weights $(2,2,1,1)$. Now

$$E = (\mathbf{x}^2 + \lambda \mathbf{y}\mathbf{u}^2 - 2\mathbf{y}\mathbf{z}l(\mathbf{z},\mathbf{u}) + f_4(\mathbf{z},\mathbf{u}) = 0) \subset \mathbb{P}(2,2,1,1),$$

is clearly irreducible. By considering $Y \cap U_4$, which is nonsingular by Lemma 7, one has that $Y \to X$ is a divisorial contraction by Theorem 5. Since $l(z,u) \neq 0$, one sees that $Y \cap U_2$ has at worst $cA/2$ singularities and $Y \cap U_3$ has at worst cA singularities. Therefore feasible resolution exists. Q.E.D.

By [16, Proposition 1.3], we have that

(1) if $P \in X$ is cD_4, then $\varphi = x^2 + \varphi_3(y,z,u) + \varphi_{\geq 4}(y,z,u)$ with $\varphi_3(y,z,u)$ is not divisible by a square of a linear form;
(2) if $P \in X$ is cD_n with $n \geq 5$, then $\varphi = x^2 + y^2z + \varphi_{\geq 4}(y,z,u)$.

Therefore, the plan is as following: for cD_4 points, the parallel argument as in Lemma 14 works. For $cD_n \geq 5$ points, which always admits normal forms, we prove by induction on $\mu^\flat$. We will need to consider $cAx/2$ points simultaneously in the induction.

Proposition 15. *There is a feasible resolution for any cD_4 singularity.*

Proof. We have $y^2z, z^3 \in \varphi_3$. Replacing z by $z+u$ and completing square, we may and do assume that $\varphi_3 = y^2z + \lambda yu^2 + f_3(z,u)$, with $z^3 \in f_3$ and

$$\varphi = x^2 + y^2z + \lambda yu^2 + f_3(z,u) + \varphi_{\geq 4}(y,z,u).$$

Case 1. $\lambda = 0$ and φ_3 is irreducible.
We can work as in Subcase 1-1 and 1-2 of Lemma 14.
Case 2. $\lambda = 0$ and φ_3 is reducible.
In this situation, $y^2z + f_3 = z(y^2 + q(z,u))$ for some quadratic form $q(z,u) \neq 0$. We consider weighted blowup $Y \to X$ with weights $v = (2,1,2,1)$. Now

$$E = (\mathbf{x}^2 + \mathbf{y}^2\mathbf{z} + \text{possibly others} = 0) \subset \mathbb{P}(2,2,1,1),$$

is clearly irreducible. By considering $Y \cap U_2$, which is nonsingular by Lemma 7, one has that $Y \to X$ is a divisorial contraction by Theorem 5.

Since $z^3 \in \varphi_3$, one sees that $Y \cap U_3$ define by

$$(\tilde{\varphi} : x^2 + y^2 + z^2 + \text{others} = 0) \subset \mathbb{C}^4/\frac{1}{2}(2,1,1,1),$$

which has at worst $cA/2$ singularities.

It remains to consider Q_4. Since $P \in X$ is isolated, one sees that there exists yu^p, zu^p or $u^p \in \varphi$ for some p. It follows that there exists yu^{p-1}, zu^{p-2} or $u^{p-3} \in \tilde{\varphi}$ in $Y \cap U_4$. Hence feasible resolution exists by induction on p.

Case 3. $\lambda \neq 0$. We can work as in Subcase 2-2 and 2-3 of Lemma 14. Note that $z^3 \in f_3$ hence Subcase 2-1 can not happen. Q.E.D.

Lemma 16. *A singularity $P \in X$ of type $cD_{n\geq 5}$ admits a normal form with $l \geq 3$.*

Proof. It is straightforward to solve for formal power series $\bar{y} = y + y_2 + y_3 + \dots$ and $\bar{z} = z + z_2 + z_3 + \dots$ satisfying

$$x^2 + \bar{y}^2\bar{z} + \lambda\bar{y}u^l + f(\bar{z}, u) = x^2 + y^2 z + g_{\geq 4}(y, z, u),$$

where $y_k = y_k(z, u)$ and $z_k = z_k(y, z, u)$ are the k-th jets and $l = \min\{k | yu^k \in g_{\geq 4}(y, z, u)\}$. By Artin's Approximation Theorem [1], this gives an embedding as desired.

Observe that $\varphi = 0$ is singular along the line $(x = y = z = 0)$ if $z^2 | f(z, u)$. Similarly, $\varphi = 0$ is singular along the line $(x = y = u = 0)$ if $u^2 | f(z, u)$. Since $P \in X$ is isolated, it follows that $z^{q-1}u \in \varphi$ or $z^q \in \varphi$ for some $q > 0$ and $zu^{p-1} \in \varphi$ or $u^p \in \varphi$ for some $p > 0$ if $\lambda = 0$. Q.E.D.

In order to obtain a feasible resolution for $cD_{n\geq 5}$ points in general, we will need to consider $cAx/2$ point as well. Given a $cAx/2$ point $P \in X$, with an embedding

$$(\varphi\colon x^2 + y^2 + f(z, u) = 0) \subset \mathbb{C}^4/\frac{1}{2}(1, 0, 1, 1),$$

we define

$$\tau(P \in X) := \min\{i + j | z^i u^j \in f\}.$$

Note that $f(z, u)$ is $\mathbb{Z}_2$-invariant and hence consists of even degree terms only. We set $\tau' := \tau/2 \in \mathbb{Z}$.

For inductive purpose, we start by considering points with τ small.

Lemma 17. *Given a $cAx/2$-like point P defined by*

$$(\varphi\colon x^2 + y^2 + f(z, u) = 0) \subset \mathbb{C}^4/\frac{1}{2}(1, 0, 1, 1),$$

with $\tau \leq 2$. Suppose that P is terminal. Then P is non-singular or $cA/2$. In any case, feasible resolutions exist for such points.

Proof. If $\tau = 0$, then P is clearly non-singular. If $\tau = 2$, then we may assume that $z^2 \in f(z,u)$. Hence it is a $cA/2$ point. By Corollary 13, feasible resolution exists. Q.E.D.

We are now ready to handle $cAx/2$ and cD points.

Proposition 18. *Given a $cAx/2$ point $P \in X$ with $\tau(P \in X) = \tau_0 \geq 4$. Suppose that feasible resolutions exist for cD-like point with $\mu^\flat < \tau_0$ and feasible resolutions exist for $cAx/2$-like point with $\tau < \tau_0$. Then there is a feasible resolution for $P \in X$.*

Proof. Let $f_{\tau_0}(z,u) = \sum_{i+j=\tau_0} a_{ij} z^i u^j$. It can be factored into $\prod_{t\in T}(\alpha_t z + \beta_t u)^{m_t}$, where m_t denotes the multiplicities with $\sum_t m_t = \tau_0$.

Case 1. $f_{\tau_0}(z,u)$ is not a perfect square.
Depending on the parity of $\tau_0/2$, we first consider $\mathrm{wBl}_v : Y \to X$ with weights $v = \frac{1}{2}(\frac{\tau_0}{2}, \frac{\tau_0}{2}+1, 1, 1)$ or $\frac{1}{2}(\frac{\tau_0}{2}+1, \frac{\tau_0}{2}, 1, 1)$. It is a divisorial contraction with minimal discrepancy $\frac{1}{2}$ (cf. [4, Theorem 8.4]). Without loss of generality, we study the first case.

Now $E = (\mathbf{x}^2 + f_{\tau_0}(\mathbf{z},\mathbf{u}) = 0) \subset \mathbb{P}(\frac{\tau_0}{2}, \frac{\tau_0}{2}+1, 1, 1)$. Easy computation yields the following:

(1) $Y \cap U_1$ is non-singular and $Y \cap U_2$ is singular only at Q_2, which is a terminal quotient singularity of index $\frac{\tau_0}{2}+1$.

(2) $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_3 \subset \{R_t\}_{m_t \geq 2, (\alpha_t,\beta_t)\neq(1,0)}$. Each R_t is defined by $x^2 + y^2 z + unit \cdot \bar{u}^{m_t} + \tilde{f}_{\geq \tau_0}(z,\bar{u}) = 0 \subset \mathbb{C}^4$, where $\bar{u} := u + \alpha_t/\beta_t$. This is a cD point with $\mu^\flat(R_t) \leq m_t$.

(3) Similarly, $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_4 \subset \{R_t\}_{m_t \geq 2, (\alpha_t,\beta_t)\neq(0,1)}$. Each R_t is defined by $x^2 + y^2 u + unit \cdot \bar{z}^{m_t} + \tilde{f}_{\geq \tau_0}(\bar{z},u) = 0 \subset \mathbb{C}^4$, where $\bar{z} := z + \beta_t/\alpha_t$. This is a cD point with $\mu^\flat(R_t) \leq m_t$.

As a summary, one sees that $\mathrm{Sing}(Y) \subset \{Q_2, R_t\}_{m_t \geq 2}$. Notice $|T| = 1$ would implies that f_{τ_0} is a perfect square, which is a contradiction. Hence we may assume that $|T| > 1$ and therefore $m_t < \tau_0$ for all t. We can take a feasible resolution for each R_t and Q_2 to obtain the required feasible resolution for $P \in X$.

Case 2. $f_{\tau_0}(z,u) = (h_{\tau_0/2}(z,u))^2$ is a perfect square.
We need to make a coordinate change so that $P \in X$ is rewritten as

$$(x^2 + 2xh_{\tau_0/2}(z,u) + y^2 + f_{\tau_0+1}(z,u) + f_{>\tau_0+1}(z,u) = 0) \subset \mathbb{C}^4.$$

Depending on the parity of τ_0, we consider $\mathrm{wBl}_v : Y_1 \to X$ with weights $v = \frac{1}{2}(\frac{\tau_0}{2}+2, \frac{\tau_0}{2}+1, 1, 1)$ or $\frac{1}{2}(\frac{\tau_0}{2}+1, \frac{\tau_0}{2}+2, 1, 1)$. Without loss of generality, we study the first case. Now $E = (\mathbf{y}^2 + 2\mathbf{x}h_{\tau_0/2} + f_{\tau_0+1}(\mathbf{z},\mathbf{u}) = 0) \subset \mathbb{P}(\frac{\tau_0}{2}+2, \frac{\tau_0}{2}+1, 1, 1)$.

Easy computation yields the following:

(1) $Y \cap U_2$ is non-singular and $Y_1 \cap U_1$ is singular only at Q_1, which is a terminal quotient singularity of index $\frac{\tau_0}{2}+2$.

(2) $Y \cap U_3$ is defined by

$$(\tilde{\varphi} : x^2 z + 2x h_{\tau_0/2}(1,u) + y^2 + \tilde{f}_{\geq \tau_0+1} = 0) \subset \mathbb{C}^4.$$

For any singularity $R \in \mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_3$, we write $R = (0,0,\alpha,\beta)$ and consider that coordinate change that $\bar{x} := y, \bar{y} := x, \bar{z} := z-\alpha, \bar{u} := u-\beta$. Then R is at worst a cD-like point and $\mu^\flat(R) \leq \tau_0 - 2$.

(3) The same holds for singularity in $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_4$.

As a summary, one sees that $\mathrm{Sing}(Y)$ consist of a terminal quotient singularity Q_1 and possibly some cD-like points R_t with $\mu^\flat(R_t) < \tau_0$. We can take a feasible resolution for each R_t and Q_1 to obtain the required feasible resolution for $P \in X$. Q.E.D.

Proposition 19. *Given a cD-like point with $\mu^\flat(P \in X) = \mu_0$. Suppose that feasible resolutions exist for $cAx/2$-like point with $\tau \leq \mu_0$ and feasible resolutions exist for cD-like point with $\mu^\flat < \mu_0$. Then there is a feasible resolution for $P \in X$.*

Proof. We always fix a normal form once and for all. By Lemma 14, we may assume that $\mu_0 \geq 4$. We set $\mu' := \lfloor \frac{\mu_0}{2} \rfloor$. We consider divisorial contraction $Y \to X$ with weights $(\mu', \mu'-1, 2, 1)$. Recall that $P \in X$ is given by

$$(\varphi : x^2 + y^2 z + \lambda y u^l + \sum a_{ij} z^i u^j = 0).$$

We may write $f_{2\mu'} := \sum_{2i+j=2\mu'} a_{ij} z^i u^j = \prod_t (\alpha_t z + \beta_t u^2)^{m_t}$.

Case 1. $\lambda = 0$.

It is straightforward to see that only singularity on $U_1 \cup U_2$ is Q_2, which is a terminal quotient singularity. On $U_3 \cup U_4$, for any singularity $R \in \mathrm{Sing}(Y)_{\mathrm{ind}=1} \subset \mathrm{Sing}(E)$, then R correspond to a factor $(\alpha_t z + \beta_t u^2)^{m_t}$ with $m_t \geq 2$. We distinguishes the following three subcases.

1-1. $R \neq Q_3, Q_4$.

By changing coordinates $\bar{z} := z + \frac{\beta_t}{\alpha_t}$, one sees that R is a cA point. Hence feasible resolution over R exists.

1-2. $R = Q_3$.

Since $Y \cap U_3$ is defined by

$$(\tilde{\varphi} : x^2 + y^2 + \sum a_{ij} z^{2i+j-2\mu'} u^j = 0) \subset \mathbb{C}^4 / \frac{1}{2}(1,0,1,1).$$

One sees that Q_3 is a $cAx/2$-like point with

$$\tau(Q_4) = \min\{2i + j - 2\mu' + j | a_{ij} \neq 0\} \leq j_0 \leq \mu(P) = \mu^\flat(P) = \mu_0.$$

Feasible resolution over Q_3 exists by our hypothesis.
1-3. $R = Q_4$.
Since $Y \cap U_4$ is defined by

$$(\tilde{\varphi} : x^2 + y^2 z + \sum a_{ij} z^i u^{2i+j-2\mu'} = 0) \subset \mathbb{C}^4.$$

Then Q_4 is a cD-like point with $\lambda = 0$ as well. Since $\mu(P) = 2i_0 + j_0$ for some $z^{i_0} u^{j_0} \in \varphi$, one sees that

$$\mu(Q_4) = \min\{2i + j - 2\mu' + 2i | a_{ij} \neq 0\} \leq \mu(P) - 2\mu' + 2i_0 \leq \mu(P), \quad (\dagger)$$

where the last inequality follows from $\mu' = \lfloor \frac{\mu_0}{2} \rfloor \geq \lfloor \frac{2i_0}{2} \rfloor$.

One can easily check that

$$\begin{array}{l} \mu^\flat(Q_4) = \mu(Q_4) \leq \mu(P) = \mu^\flat(P); \\ \mu^\sharp(Q_4) \leq \mu^\sharp(P) + 2 - 2\mu' < \mu^\sharp(P). \end{array}$$

By inductively on $\mu^\sharp$, there exist feasible resolution for $P \in X$.
Case 2. $\lambda \neq 0$.
Subcase 2-1. $2l - 2 = \mu(P)$.
We proceed as in Case 1 and see that $\mathrm{Sing}(Y) = \{Q_2, Q_3\}$, where Q_2 is a terminal quotient singularity and $Q_3 \in Y \cap U_3$ is given by

$$(\tilde{\varphi} : x^2 + y^2 - \frac{1}{4}\lambda^2 u^{2l} + \sum a_{ij} z^{2i+j-2\mu'} u^j = 0) \subset \mathbb{C}^4/\frac{1}{2}(1, 0, 1, 1),$$

after completing the square. One sees that Q_3 is a $cAx/2$-like point with

$$\tau(Q_4) \leq \min\{2i + j - 2\mu' + j | a_{ij} \neq 0\} \leq \mu(P) = \mu^\flat(P) = \mu_0.$$

Feasible resolution over Q_3 exists by our hypothesis.
Subcase 2-2. $2l - 2 > \mu(P)$.
We can proceed as in Case 1 all the way to equation $\dagger$. Therefore, $l(Q_4) = l(P) - \mu' - 1$ and

$$\mu^\flat(Q_4) = \min\{2l - 2\mu' - 4, \mu(Q_4)\} \leq \min\{2l - 2, \mu(P)\} = \mu^\flat(P).$$

Inductively, we are reduced to either $\mu^\flat < \mu_0$ or $2l - 2 = \mu$. Hence feasible resolution exists.
Subcase 2-3. $2l - 2 < \mu(P)$.
For any $z^i u^j \in f$, one has $2i + j > 2l - 2$ and hence $i + j \geq l$. We consider $\mathrm{wBl}_v : Y \to X$ with $v = (l, l, 1, 1)$ instead.

By Lemma 7, $Y \cap U_4$ is nonsingular and hence $Y \to X$ is a divisorial contraction by Theorem 5. One sees that $\mathrm{Sing}(Y) = \{Q_2, Q_3\}$, where Q_2 is a terminal quotient singularity and $Q_3 \in Y \cap U_3$ is given by

$$(\tilde{\varphi} : x^2 + y^2 + \lambda y u^l + \sum a_{ij} z^{i+j-2l} u^j = 0) \subset \mathbb{C}^4,$$

which is a cD-like point. For a cD-like point, we introduce

$$\rho^\sharp(P \in X) := \min\{i + j | z^i u^j \in \varphi, j = 0 \text{ or } 1\},$$

which is finite and $\frac{1}{2}\mu \leq \rho^\sharp \leq \mu$. Compare Q_4 with P, we have $\rho^\sharp(Q_4) \leq \rho^\sharp(P) + 1 - 2l$. Repeat the process t-times, we have $Q_{t,4} \in Y_t \to \ldots \to Y_1 = Y \to X \ni P$ such that for t sufficiently large

$$\mu(Q_{t,4}) \leq 2(\mu(P) + t(1 - 2l)) < \mu_0(P).$$

Hence we are reduced to the situation $\mu^\flat(Q_{t,4}) \leq \mu(Q_{t,4}) < \mu_0$, or Subcase 2-1, or Subcase 2-2 in finite steps and feasible resolution over $P \in X$ exists by our hypothesis.

Q.E.D.

Combining all the above results in this section, we have the following:

Theorem 20. *There is a feasible resolution for any singularity of type cD or $cAx/2$.*

§5. resolution of $cAx/4, cD/2, cD/3$ points

In [5], Hayakawa shows that there is a partial resolution

$$X_n \to \ldots \to X_1 \to X \ni P,$$

for a point $P \in X$ of index $r > 1$ such that X_n has only terminal singularities of index 1 and each map is a divisorial contraction with minimal discrepancies. If $\mathrm{Sing}(X_n)_{\mathrm{ind}=1}$ is either of type cA or cD, then feasible resolution exists by the result of previous sections.

In fact, the partial resolution was constructed by picking any divisorial contraction with minimal discrepancy at each step. Therefore, for our purpose, it suffices to pick one divisorial contraction $Y \to X$ over a given higher index point $P \in X$ of type $cAx/4$, $cD/2$, $cD/3$, or $cE/2$ and verify that $\mathrm{Sing}(Y)_{\mathrm{ind}=1}$ is either of type cA or cD.

Lemma 21. *Given $P \in X$ of type $cAx/4$, there is a divisorial contraction $Y \to X$ with discrepancy $\frac{1}{4}$ such that $\mathrm{Sing}(Y)_{\mathrm{ind}=1}$ is of type cA or cD.*

Proof. We may write $P \in X$ as

$$(\varphi : x^2+y^2+f(z,u) = x^2+y^2+\sum_{i+j=2l+1\geq 3} a_{ij}z^{2i}u^j = 0) \subset \mathbb{C}^4/\frac{1}{4}(1,3,1,2).$$

Let $\sigma(P \in X) := \min\{i+j|a_{ij} \neq 0\}$, then we may write $f(z,u) = f_\sigma(z,u) + f_{>\sigma}(z,u)$.

Case 1. $f_\sigma(z,u)$ is not a perfect square.
Depending on parity of $\frac{\sigma-1}{2}$, we consider $\mathrm{wBl}_v : Y \to X$ with weights $v = \frac{1}{4}(\sigma+2,\sigma,1,2)$ or $\frac{1}{4}(\sigma,\sigma+2,1,2)$. By [4, Theorem 7.4], this is the only divisorial contraction.

Without loss of generality, we study the first weight. By Lemma 7, 8, we have $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_i$ is empty for $i = 1,2$. Moreover, Q_4 is a $cD/2$-like point of index 2. It suffices to consider U_3.

In U_3, $Y \cap U_3$ is defined by

$$(\tilde{\varphi} : x^2z + y^2 + \tilde{f}(z,u) = 0) \subset \mathbb{C}^4.$$

Therefore, it is immediate to see that $\mathrm{Sing}(Y) \cap U_3$ is either of type cA or cD.

Case 2. $f_\sigma(z,u) = -h(z,u)^2$ is a perfect square.
Depending on parity of $\frac{\sigma-1}{2}$, we need to make a coordinate change so that $P \in X$ is written as

$$(\varphi : x^2 + 2xh(z,u) + y^2 + f_{>\sigma}(z,u) = 0) \subset \mathbb{C}^4/\frac{1}{4}(1,3,1,2),$$

or

$$(\varphi : y^2 + 2yh(z,u) + x^2 + f_{>\sigma}(z,u) = 0) \subset \mathbb{C}^4/\frac{1}{4}(1,3,1,2).$$

We consider weighted blowup $Y \to X$ with weights $\frac{1}{4}(\sigma+4,\sigma+2,1,2)$ or $\frac{1}{4}(\sigma+2,\sigma+4,1,2)$ respectively. By [4, Theorem 7.4], this is a divisorial contraction.

Without loss of generality, we study the first weight. By Lemma 7, we have $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_i$ is empty for $i = 1,2$. Then $R \neq Q_4$ for Q_4 is a $cD/2$-like point of index 2. It suffices to consider U_3. Indeed, $Y \cap U_3$ is defined by

$$(\tilde{\varphi} : x^2z + 2xh_\sigma(1,u) + y^2 + \tilde{f}(z,u) = 0) \subset \mathbb{C}^4.$$

Therefore, it is immediate to see that $\mathrm{Sing}(Y) \cap U_3$ is either of type cA or cD. Q.E.D.

Lemma 22. *Given $P \in X$ of type $cD/2$, there is a divisorial contraction $Y \to X$ with discrepancy $\frac{1}{2}$ such that* $\mathrm{Sing}(Y)_{\mathrm{ind}=1}$ *is of type cA or cD.*

Proof. By Mori's classification [17, 22], one has that $P \in X$ is given by $(\varphi = 0) \subset \mathbb{C}^4/\frac{1}{2}(1,1,0,1)$ with φ being one the following

$$\begin{cases} x^2 + yzu + y^{2a} + u^{2b} + z^c, & a \geq b \geq 2, c \geq 3 \\ x^2 + y^2z + \lambda yu^{2l+1} + f(z,u^2). \end{cases}$$

Case 1. $\varphi = x^2 + yzu + y^{2a} + u^{2b} + z^c$.
We take weighted blowup $Y \to X$ with weights $v = \frac{1}{2}(3,1,2,1)$ (resp. $\frac{1}{2}(3,1,2,3)$) if $a = b = 2$ (resp. $a \geq 3$), which is a divisorial contraction by [5]. Note that $wt_v(yzu) = wt_v(\varphi)$. Hence uz (resp. yu, yz) appears in the equation of $Y \cap U_2$ (resp. U_3, U_4). By Corollary 9, we conclude that $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_i$ is of type cA for $i = 2,3,4$. Together with Lemma 6, then we are done with this case.
Case 2. $\varphi = x^2 + y^2z + \lambda yu^{2l+1} + f(z,u^2)$.
We may write $f(z,u^2) = \sum a_{ij} z^i u^{2j} \in (z^3, z^2u^2, u^4)\mathbb{C}\{z,u^2\}$. We define

$$\begin{cases} \sigma := \min\{2i + 2j | z^i u^{2j} \in f\}; \\ \sigma^\flat := \min\{2l-1, \sigma\}. \end{cases}$$

Note that we have $l \geq 1$ and $\sigma \geq 2$ for $P \in X$ is a $cD/2$ point.
Subcase 2-1. $l = 1$.
We consider weighted blowup $Y \to X$ with weight $v = \frac{1}{2}(2,1,2,1)$, which is a divisorial contraction by [5]. Now

$$E = (\mathbf{x}^2 + \mathbf{y}^2\mathbf{z} + \lambda \mathbf{y}\mathbf{u}^3 + f_{wt_v=2} = 0) \subset \mathbb{P}(2,1,2,1).$$

By Lemma 6, 7, one sees that $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_i$ is empty for $i = 1,2,4$. Since Q_3 is at worst of type $cAx/2$. We are done.
Subcase 2-2. $l \geq 2$ and $\sigma = 2$.
One has $f_{\sigma=2} = u^4$, in particular, $u^4 \in f$. We consider weighted blowup $Y \to X$ with weight $v = \frac{1}{2}(2,1,2,1)$ again. Now

$$E = (\mathbf{x}^2 + \mathbf{y}^2\mathbf{z} + \mathbf{u}^4 = 0) \subset \mathbb{P}(2,1,2,1).$$

We thus have $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \subset \mathrm{Sing}(E) \subset \{Q_3\}$. However, Q_3 is a point of index 2. We are done.
Subcase 2-3. $\sigma^\flat \geq 3$.
Let

$$\sigma' := 2\lfloor \frac{\sigma^\flat - 1}{2} \rfloor + 1 = \begin{cases} \sigma^\flat & \text{if } \sigma^\flat \text{ is odd;} \\ \sigma^\flat - 1 & \text{if } \sigma^\flat \text{ is even;} \end{cases}$$

We consider weighted blowup $Y \to X$ with weight $v = \frac{1}{2}(\sigma', \sigma' - 2, 4, 1)$, which is a divisorial contraction by [5]. Clearly, by Lemma 6, 7, one sees that $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_i$ is empty for $i = 1, 2$. Since Q_3 is a point of index 4, it remains to consider U_4. Now $Y \cap U_4$ is given by

$$(\tilde{\varphi} : x^2 + y^2 z + \lambda y u^{(2l-1-\sigma')/2} + \tilde{f} = 0) \subset \mathbb{C}^4.$$

It follows that $Y \cap U_4$ is at worst of type cD. We are done. Q.E.D.

Lemma 23. *Given $P \in X$ of type $cD/3$, there is a divisorial contraction $Y \to X$ with discrepancy $\frac{1}{3}$ such that $\mathrm{Sing}(Y)_{\mathrm{ind}=1}$ is of type cA or cD.*

Proof. By Mori's classification [17, 22], one has that $P \in X$ is given as $(\varphi = 0) \subset \mathbb{C}^4/\frac{1}{3}(0, 2, 1, 1)$ with φ being one of the following:

$$\begin{cases} x^2 + y^3 + zu(z+u); & \\ x^2 + y^3 + zu^2 + yg(z,u) + h(z,u); & g \in \mathfrak{m}^4, h \in \mathfrak{m}^6; \\ x^2 + y^3 + z^3 + yg(z,u) + h(z,u); & g \in \mathfrak{m}^4, h \in \mathfrak{m}^6. \end{cases}$$

Case 1. φ is one of the first two cases.
By [4, Theorem 9.9, 9.14, 9.20], the weighted blowup $Y \to X$ with weight $\frac{1}{3}(3, 2, 4, 1)$ is a divisorial contraction. Now

$$E = \left\{ \begin{array}{l} \mathbf{x}^2 + \mathbf{y}^3 + \mathbf{z}\mathbf{u}^2 = 0 \text{ or} \\ \mathbf{x}^2 + \mathbf{y}^3 + \mathbf{z}\mathbf{u}^2 + \lambda \mathbf{y}\mathbf{u}^4 + \lambda' \mathbf{u}^6 = 0 \end{array} \right\} \subset \mathbb{P}(3, 2, 4, 1),$$

for some λ, λ' respectively. It is easy to check that $\mathrm{Sing}(E) = Q_3$ and hence $\mathrm{Sing}(Y)_{\mathrm{ind}=1}$ is empty for the first two cases.
Case 2. $\varphi = x^2 + y^3 + z^3 + yg(z,u) + h(z,u)$.
Subcase 2-1. Either $u^4 \in g$ or $u^6 \in h$.
Then we consider the weighted blowup with weight $\frac{1}{3}(3, 2, 4, 1)$ again, which is a divisorial contraction (cf. [4, Theorem 9.20]. Now

$$E = (\mathbf{x}^2 + \mathbf{y}^3 + \lambda \mathbf{y}\mathbf{u}^4 + \lambda' \mathbf{u}^6 = 0) \subset \mathbb{P}(3, 2, 4, 1),$$

for some $(\lambda, \lambda') \neq (0, 0)$. One sees that $\mathrm{Sing}(E) = Q_3$ and hence $\mathrm{Sing}(Y)_{\mathrm{ind}=1}$ is empty.
Subcase 2-2. $u^4 \notin g$, $u^6 \notin h$ and either $zu^5 \in h$ or $u^9 \in h$.
Then we consider the weighted blowup with weight $\frac{1}{3}(3, 2, 4, 1)$ which is a divisorial contraction. Now

$$E = (\mathbf{x}^2 + \mathbf{y}^3 = 0) \subset \mathbb{P}(3, 2, 4, 1).$$

One sees that $\mathrm{Sing}(E) \subset U_3 \cup U_4$. However, the equation of $Y \cap U_4$ contains the term zu or u and hence contains at worst cA points by

Lemma 8. Together with the fact that Q_3 is a $cAx/4$ point, we are done with this case.

Subcase 2-3. $u^4 \notin g$, all $zu^5, u^6, u^9 \notin h$.
Then we consider the weighted blowup $Y \to X$ with weight $\frac{1}{3}(6,5,4,1)$, which is a divisorial contraction by [4, Theorem 9.25].

By Lemma 7, $Y \cap U_2$ is nonsingular away from Q_2, which is a quotient singularity of index 5. Together with $\mathrm{Sing}(Y) \cap U_1 = \emptyset$ and $Q_3 \notin Y$, it remains to check $Y \cap U_4$, which is defined by

$$(\tilde{\varphi} : x^2 + y^3 u + z^3 + \text{others} = 0) \subset \mathbb{C}^4.$$

which is at worst of type cE_6. In fact, this corresponds to Case 1 and 2 of the proof of Theorem 34. Notice that in the proof, we use weighted blowups wBl_v with $v = (2,2,1,1), (3,2,1,1)$ or $(3,2,2,1)$. After weighted blowup, there could have singularities of type $cA, cD, cA/2$, $cAx/2$, and terminal quotients. We thus concludes that feasible resolution exists for this case. Q.E.D.

We thus conclude the section by the following:

Theorem 24. *There is a feasible resolution for any singularity of type $cAx/4$, $cD/3$, or $cD/2$.*

§6. resolution of cE and $cE/2$ points

Recall that a cE point has the following description.

$$(\varphi : x^2 + y^3 + f(y,z,u) = x^2 + y^3 + yg(z,u) + h(z,u) = 0) \subset \mathbb{C}^4.$$

An isolated singularity with the above desription is called a cE-like singularity.

For a polynomial (resp. formal power series) $G(z,u) \in \mathbb{C}[z,u]$ (resp. $\mathbb{C}[[z,u]]$), we define

$$\tau(G) := \min\{j + k | z^j u^k \in G\}.$$

For cE singularity, one has $\tau(g) \geq 3$ and $\tau(h) \geq 4$. Moreover, either $\tau(g) = 3$ or $\tau(h) \leq 5$. More precisely,

(1) It is cE_6 if $\tau(h) = 4$ and $\tau(g) \geq 3$.
(2) It is cE_7 if $\tau(h) \geq 5$ and $\tau(g) = 3$.
(3) It is cE_8 if $\tau(h) = 5$ and $\tau(g) \geq 4$.

Remark 25. An isolated cE-like singularity is at worst of type cD (resp. cE_6, cE_7, cE_8) if $\tau(g) \leq 2$ or $\tau(h) \leq 3$ (resp. $\tau(h) \leq 4$, $\tau(g) \leq 3$, $\tau(h) \leq 5$).

26. Notations and Conventions

1. We fix the notation that $g_3(z,u) := g_{\tau=3}(z,u), h_4(z,u) := h_{\tau=4}(z,u)$ and $h_5(z,u) := h_{\tau=5}(z,u)$. In the case of cE_6, $\tau(h) = 4$. By replacing z, u and up to a constant, we may and do assume that

$$h_4 \in \{z^4, z^4 + z^3u, z^4 + 2z^3u + z^2u^2, z^4 + z^2u^2, z^4 + zu^3\}.$$

In particular, $z^4 \in h_4$.

In the case of cE_7, $\tau(g) = 3$. We may and do assume that

$$g_3 \in \{z^3, z^3 + z^2u, z^3 + zu^2\}.$$

In particular, $z^3 \in g_3$.

In the case of cE_8, $\tau(h) = 5$. We may and do assume that

$$h_5 \in \{z^5, z^5 + z^4u, z^5 + 2z^4u + z^3u^2, z^5 + z^3u^2, \\ z^5 + 2z^4u - z^3u^2 - 2z^2u^3, z^5 + z^2u^3, z^5 + zu^4\}.$$

In particular, $z^5 \in h_5$.

2. We define

$$\tau^*(\varphi) := \min\{p | y^iz^ju^p \in \varphi \text{ with } i + j \leq 1\}.$$

Since $P \in X$ is isolated, there is a term yu^p, zu^p or u^p in φ otherwise P is singular along a line $(x = y = z = 0)$. Hence $\tau^*(\varphi)$ is a well-defined integer.

3. For a weight $v = (a, b, k, 1)$, we denote it v_l with $l = a+b+k-1$. In our discussion, we always consider weight v_l such that $v_l(\varphi) = l$.

4. Fix a weight $v = \frac{1}{r}(a, b, k, 1)$ with $r = 1, 2$, we write

$$\varphi = x^2 + y^3 + yg_v + yg_{v+1} + yg_> + h_v + v_{v+1} + h_>,$$

where g_v (resp. g_{v+1}) is the homogeneous part of $g(z,u)$ such that $wt_v(yg_v) = wt_v(\varphi)$ (resp. $wt_v(yg_v) = wt_v(\varphi) + 1$) and $g_>$ is the remaining part with greater weight, and h_v (resp. h_{v+1}) is the homogeneous part of $h(z,u)$ with v-weight equal to $wt_v(\varphi)$ (resp. $wt_v(\varphi) + 1$) and $h_>$ is the remaining part with greater weight.

5. For simplicity of notation, sometime we may denote by g_m or h_m for the v-homogeneous part with v-weight equal to m. This notation should not be confused with g_3 nor g_v.

6.1. Some preparation

The general strategy is as following. For a given cE or $cE/2$ singularity $P \in X$. We consider weighted blowup $Y \to X$ with weight

$v = \frac{1}{r}(a, b, k, 1)$ and $r = 1, 2$ such that $\frac{1}{2}(a+b+k+1) - wt_v(\varphi) = 1 + \frac{1}{r}$. This is a weighted blowup with discrepancy $\frac{1}{r}$ if E is irreducible. We check that $\mathrm{Sing}(Y) \cap U_4$ is isolated and each $R \in \mathrm{Sing}(Y) \cap U_4$ is terminal. Then the weighted blowup $Y \to X$ is a divisorial contraction with discrepancy $\frac{1}{r}$ by Theorem 5.

Moreover, we check that each singular point $R \in \mathrm{Sing}(Y)_{\mathrm{ind}=1}$ is "milder" than $P \in X$ in the sense that either it is of milder type, or it can only admit smaller weight. We can prove the existence of feasible resolution by induction on types and weights.

27. We work on $Y \cap U_4$.

Now $Y \cap U_4$ is defined by $\tilde{\varphi}$, which can be written as

$$\begin{aligned}\tilde{\varphi} \quad &= x^2 u^{wt_v(x^2) - wt_v(\varphi)} + y^3 u^{wt_v(y^3) - wt_v(\varphi)} \\ &+ y g_v(z,1) + y u g_{v+1}(z,1) + y\widetilde{g_>} + h_v(z,1) + u h_{v+1}(z,1) + \widetilde{h_>},\end{aligned}$$

such that $u^2 | \widetilde{g_>}$ and $u^2 | \widetilde{h_{>v}}$.

Lemma 28. *Suppose that $wt_v(x^2) = wt_v(\varphi)$ or $wt_v(\varphi) + 1$ and $wt_v(y^3) = wt_v(\varphi)$. Then $\mathrm{Sing}(Y) \cap U_4$ is isolated UNLESS:*

$$\textit{There is } s(z,u) \textit{ such that} \begin{cases} g_v = -3s(z,u)^2, \\ h_v = 2s(z,u)^3, \\ h_{v+1} = -s(z,u) g_{v+1}. \end{cases} \qquad \natural$$

Proof. If $2wt_v(x) = wt_v(\varphi)$, we have $\tilde{\varphi}_x = 2x$. If $2wt_v(x) = wt_v(\varphi) + 1$, then $Y \cap U_1$ is non-singular away from Q_1 by Lemma 7. Hence we have $\mathrm{Sing}(Y) \cap U_4 \subset (x = 0)$ in both cases. Moreover, $\mathrm{Sing}(Y) \subset E$, hence we have $\mathrm{Sing}(Y) \cap U_4 \subset (u = 0)$.

Therefore, we have

$$\begin{aligned}\mathrm{Sing}(Y) \cap U_4 \quad &\subset (x = u = 0) \cap (\tilde{\varphi} = \tilde{\varphi}_y = \tilde{\varphi}_u = 0) \\ &\subset (x = u = 0) \cap \Sigma,\end{aligned}$$

where Σ is defined as

$$\begin{cases} y^3 + y g_v + h_v = 0, \\ 3y^2 + g_v = 0, \\ y g_{v+1} + h_{v+1} = 0. \end{cases}$$

If g_v is not a perfect square, then $3y^2 + g_v$ is irreducible and hence Σ is finite. If g_v is a perfect square, then we write it as $g_v = -3s^2$. One sees that Σ is finite unless $y - s$ or $y + s$ divides the above three polynomials. The statement now follows. Q.E.D.

Lemma 29. *Suppose more generally that*

$$\varphi = x^2 + y^3 + s(z,u)y^2 + yg_v + yg_{v+1} + yg_> + h_v + h_{v+1} + h_>.$$

Suppose that $wt_v(x^2) = wt_v(\varphi)$ *and* $wt_v(y^3) = wt_v(\varphi) + 1$. *Then* $\mathrm{Sing}(Y) \cap U_4$ *is isolated UNLESS* $g_v = h_v = h_{v+1} = 0$.

Proof. Since $wt_v(y^3) = wt_v(\varphi) + 1$, then $Y \cap U_2$ is non-singular away from Q_2. Hence we have $\mathrm{Sing}(Y) \cap U_4 \subset (x = y = 0)$. Moreover, $\mathrm{Sing}(Y) \subset E$, hence we have $\mathrm{Sing}(Y) \cap U_4 \subset (u = 0)$.

Therefore, we have

$$\begin{aligned} Siny(Y) \cap U_4 \quad &\subset (x = y = u = 0) \cap (\tilde{\varphi} = \tilde{\varphi}_y = \tilde{\varphi}_u = 0) \\ &\subset (x = y = u = 0) \cap (h_v(z,1) = g_v(z,1) = \tilde{\varphi}_u = 0) \\ &\subset (x = y = u = 0) \cap (h_v(z,1) = g_v(z,1) = h_{v+1}(z,1) = 0). \end{aligned}$$

The statement now follows. Q.E.D.

30. We study the most common case that

$$wt_v(x^2) = wt_v(y^3) = wt_v(\varphi).$$

Suppose furthermore that $\natural$ does not hold, then $\mathrm{Sing}(Y) \cap U_4$ is isolated. We now study the possible type of these singularities.

Notice that we have at least one of g_v, h_v, h_{v+1} is non-zero, otherwise $\natural$ holds.

Case 1. $h_v \neq 0$.

We write

$$h_v(z,u) = \lambda z^{m'} u^{n'} \prod (z - \alpha_t u^k)^{l'_t}.$$

Then

$$k \cdot \tau(h) \geq wt_v(h_\tau) \geq wt_v(\varphi) = n' + k(m' + \sum l'_t). \qquad \dagger_h$$

In particular,

$$\tau(h) \geq m' + \sum l'_t.$$

Then $E \cap U_4$ is defined by

$$\begin{aligned} &(x^2 + y^3 + yg_v(z,1) + h_v(z,1) = 0) \\ &= (x^2 + y^3 + yg_v(z,1) + \lambda' z^{m'} \prod (z - \alpha'_t)^{l'_t} = 0) \subset U_4 \cong \mathbb{C}^4, \end{aligned}$$

which is irreducible.

It is easy to see that

- if $m' + \sum l'_t \leq 2$ then $E \cap U_4$ is at worst Du Val of A-type and hence $\mathrm{Sing}(Y) \cap U_4$ is at worst of cA type;

- if $m' + \sum l'_t = 3$ then $E \cap U_4$ is at worst Du Val of D-type and hence $\mathrm{Sing}(Y) \cap U_4$ is at worst of cD type;
- if $m' + \sum l'_t = 4$ then $E \cap U_4$ is at worst Du Val of E_6-type and hence $\mathrm{Sing}(Y) \cap U_4$ is at worst of cE_6 type;
- if $m' + \sum l'_t = 5$ then $E \cap U_4$ is at worst Du Val of E_8-type and hence $\mathrm{Sing}(Y) \cap U_4$ is at worst of cE_8 type;

Notice also that if $P \in X$ is of type cE_6 (resp. cE_8), then $\mathrm{Sing}(Y) \cap U_4$ is at worst of cE_6 (resp. cE_8).

In any event, $Y \cap U_4$ is terminal. By Theorem 5, $Y \to X$ is a divisorial contraction with discrepancy 1.

Case 2. $g_v \neq 0$.
We write
$$g_v(z, u) = \lambda z^m u^n \prod (z - \alpha_t u^k)^{l_t}.$$

Then
$$k \cdot \tau(g) \geq wt_v(g_\tau) \geq wt_v(\varphi) - b = n + k(m + \sum l_t). \qquad \dagger_g$$

In particular,
$$\tau(g) \geq m + \sum l_t.$$

Then $E \cap U_4$ is defined by
$$\begin{aligned}&(x^2 + y^3 + yg_v(z,1) + h_v(z,1) = 0)\\ &= (x^2 + y^3 + \lambda y z^m \prod (z - \alpha_t)^{l_t} + h_v(z,1) = 0) \subset U_4 \cong \mathbb{C}^4,\end{aligned}$$

which is irreducible.

It is easy to see that

- if $m + \sum l_t \leq 1$ then $E \cap U_4$ is at worst Du Val of A-type and hence $\mathrm{Sing}(Y) \cap U_4$ is at worst of cA type;
- if $m + \sum l_t = 2$ then $E \cap U_4$ is at worst Du Val of D-type and hence $\mathrm{Sing}(Y) \cap U_4$ is at worst of cD type;
- if $m + \sum l_t = 3$ then $E \cap U_4$ is at worst Du Val of E_7-type and hence $\mathrm{Sing}(Y) \cap U_4$ is at worst of cE_7 type.

Notice also that if $P \in X$ is of type cE_7, then $\mathrm{Sing}(Y) \cap U_4$ is at worst of cE_7.

Case 3. $h_v = 0, h_{v+1} \neq 0$.
We write
$$h_{v+1}(z, u) = \lambda'' z^{m''} u^{n''} \prod (z - \alpha_t u^k)^{l''_t}.$$

Then
$$k \cdot \tau(h) \geq wt_v(h_\tau) \geq wt_v(\varphi) = n'' + k(m'' + \sum l''_t) - 1. \qquad \dagger'_h$$

In particular, if $k > 1$, then we still have

$$\tau(h) \geq m'' + \sum l''_t.$$

The same conclusion as in Case 1 still holds.

In any event, $Y \cap U_4$ is terminal. By Theorem 5, $Y \to X$ is a divisorial contraction.

As a summary, we conclude that

Theorem 31. *Given $P \in X$ a cE point defined by $(\varphi : x^2 + y^3 + yg(z,u) + h(z,u) = 0)$. Let $Y \to X$ be a weight blowup with weight $v = (a,b,k,1)$ that $k > 1$. Suppose that $wt_v(x^2) = wt_v(y^3) = wt_v(\varphi)$ and $\natural$ does not hold. Then $Y \to X$ is a divisorial contraction. Also, any singularity on $Y \cap U_4$ is at worst of type cE_6 (resp. cE_7, cE_8) if $P \in X$ is of type cE_6 (resp. cE_7, cE_8).*

Remark 32. Consider the case that $P \in X$ is of type cE_6. Suppose the worst case that Y has a singularity R of type cE_6. This happens only when $h_v = z^4$ or $h_v = (z - \alpha_t u)^4$ for $m' + \sum l'_t \leq 4$. If $h_v = (z - \alpha'_t u^k)^4$. By considering the weight-invariant coordinate change that $\bar{z} = z - \alpha'_t u^k$, we may and do assume that $R = Q_4$ and Q_4 is the unique singularity in U_4.

We can make the same assumption if $P \in X$ is of type cE_7, cE_8.

Proposition 33. *Let $Y \to X$ be a weighted blowup of a cE point with weight $v = (a,b,k,1)$. Suppose that $wt_v(x^2) = wt_v(y^3) = wt_v(\varphi)$. If any one of $g_v = 0$, $h_v = 0$, $n > 0$, or $n' > 0$ holds, then* $\mathrm{Sing}(E) \cap U_2 - U_4 = \emptyset$.

In particular, if $\natural$ does not hold and $v = v_{30}, v_{24}, v_{18}, v_{12}$, then $Y \to X$ is a divisorial contraction and $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \subset U_4$.

Proof. In affine coordinate U_2, E is defined by

$$\tilde{\phi} : x^2 + 1 + g_v(z,u) + h_v(z,u) = 0,$$

with $g_v(z,u)$ (resp. $h_v(z,u)$) being homogeneous with respect to the weight v of weight $wt_v(\varphi) - b$ (resp. $wt_v(\varphi)$). It follows that

$$\begin{cases} kz\frac{\partial g_v(z,u)}{\partial z} + u\frac{\partial g_v(z,u)}{\partial u} = (wt_v(\varphi) - b) \cdot g_v(u,v), \\ kz\frac{\partial h_v(z,u)}{\partial z} + u\frac{\partial h_v(z,u)}{\partial u} = wt_v(\varphi) \cdot h_v(u,v) \end{cases}$$

It follows that

$$\psi_1 := wt_v(\varphi)\tilde{\phi} - ax\tilde{\phi}_x - kz\tilde{\phi}_z - u\tilde{\phi}_u = wt_v(\varphi) + bg_v(u,v),$$

where $g_v(u,v) = \lambda z^m u^n \prod(z-\alpha_t u^k)^{l_t}$. Note that ψ_1 must be satisfied at any singular point $\mathrm{Sing}(E)\cap U_2$. If $\lambda = 0$, then one sees that $\mathrm{Sing}(E)\cap U_2 = \emptyset$.

If $n > 0$, then $\mathrm{Sing}(E)\cap U_2 \not\subset U_4$ otherwise $u = 0$ will leads to a contradiction.

If we consider

$$\psi_2 := (wt_v(\varphi) - b)\tilde{\phi} - (a - b/2)x\tilde{\phi}_x - kz\tilde{\phi}_z - u\tilde{\phi}_u$$

$$= (wt_v(\varphi) - b) - bh_v(z,u),$$

where $h_v(u,v) = \lambda' z'^m u'^n \prod(z - \alpha'_t u^k)^{l'_t}$. Then one sees similarly that $\mathrm{Sing}(E)\cap U_2 = \emptyset$ if $\lambda' = 0$ and $\mathrm{Sing}(E)\cap U_2 \not\subset U_4$ if $n' > 0$.

We now prove the second statement. Since $\natural$ does not hold, hence $Y \to X$ is a divisorial contraction. Therefore, $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \subset \mathrm{Sing}(E)$. We have that $\mathrm{Sing}(E)\cap U_1 = \emptyset$. Notice that either $g_v = 0$ or $n > 0$ for v_{12}, v_{24}, v_{30}. Also one has either $h_v = 0$ or $n' > 0$ for v_{18}. Therefore, $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap (U_1 \cup U_2 \cup U_4) \subset U_4$. Finally Q_3 is of index > 1. This completes the proof. Q.E.D.

6.2. Resolution of cE_6 points

In this subsection, we shall prove that

Theorem 34. *There is a feasible resolution for any cE_6 singularity.*

Proof. We will need to consider weighted blowup $Y \to X$ with the following weights $v_{12} = (6,4,3,1)$, $v_8 = (4,3,2,1)$, $v_6 = (3,2,2,1)$, $v_4 = (2,2,1,1)$ and $v_5 = (3,2,1,1)$. It is sufficient to show that $\mathrm{Sing}(Y)_{\mathrm{ind}>1}$ is not of type $cE/2$ and there exists feasible resolution on $\mathrm{Sing}(Y)_{\mathrm{ind}=1}$.
Case 1. $wt_{v_{12}}(f) < 12$, $wt_{v_8}(f) < 8$ and $wt_{v_6}(f) < 6$.
Subcase 1-1. h_4 is not a perfect square.

We consider the weighted blowup $Y \to X$ with weight $v_4 = (2,2,1,1)$. It is clear that E is irreducible if h_4 is not a perfect square.

Since $wt_{v_6}(f) < 6$, we must have a term $\theta \in f$ such that $wt_{v_6}(\theta) < 6$. One has that $\theta = yu^3$ or $\theta = u^5$.
Claim. $Y \to X$ is a divisorial contraction.
To see this, if $\theta = u^5$, then it follows that $Y \cap U_4$ is nonsingular and thus $Y \to X$ is a divisorial contraction by Theorem 5. If $\theta = yu^3$, then $\natural$ does not hold and hence $Y \cap U_4$ has at worst singularities of type cA. Therefore, $Y \to X$ is a divisorial contraction by Theorem 5.

Clearly, $Y \cap U_1$ is nonsingular. Moreover, $Y \cap U_2$ has singularity of type $cAx/2$ at Q_2 and at worst of type cA for points other than Q_2. Since $z^4 \in h_4$, we have $Q_3 \notin Y$. Therefore, feasible resolution exists for this case.

Subcase 1-2. h_4 is a perfect square, i.e. $h_4 = z^4$ or $z^4 + 2z^3u + z^2u^2$.

Since $wt_{v_6}(f) < 6$, we have either yu^3 or u^5 in f. Write $h_4 = -q(z,u)^2$. Consider the coordinate change $\bar{x} := x - q(z,u)$, we have

$$\bar{\varphi} := \bar{x}^2 + 2\bar{x}q(z,u) + y^3 + yg(z,u) + h_{\tau\geq 5}(z,u).$$

We consider weighted blowup with weight $v_5 = (3,2,1,1)$ instead. Note that we still have either yu^3 or $u^5 \in \bar{\varphi}$.

By Lemma 7, $Y \cap U_i$ is nonsingular away from Q_i for $i = 1,2,4$ and Q_1, Q_2 are terminal quotient singularity of index $3,2$ respectively. By Theorem 5, $Y \to X$ is a divisorial contraction. It remains to consider Q_3. Since $z^4 \in h_4$, we have $\bar{x}z^2 \in \bar{\varphi}$. Hence $Y \cap U_3$ is also nonsingular by Lemma 7. We thus conclude that $\mathrm{Sing}(Y) = \mathrm{Sing}(Y)_{\mathrm{ind}>1}$ consists of Q_1, Q_2, which are terminal quotient singularities of index $3,2$ respectively.

Case 2 $wt_{v_{12}}(f) < 12$, $wt_{v_8}(f) < 8$ and $wt_{v_6}(f) \geq 6$.

We consider the weighted blowup $Y \to X$ with weight $v_6 = (3,2,2,1)$. It is clear that E is irreducible. There is a term $\theta \in f$ with $wt_{v_8}(\theta) < 8$ and $wt_{v_6}(\theta) \geq 6$. One sees that

$$\theta \in \{yzu^2, yu^4, z^3u, z^2u^2, z^2u^3, zu^4, zu^5, u^6, u^7\}.$$

It follows in particular that at least one of g_v, h_v, h_{v+1} is non-zero.

Claim. $Y \to X$ is a divisorial contraction.

To see this, suppose first that $\natural$ holds, then $g_v = -3s(z,u)^2$ for some $s(z,u) \neq 0$. We may assume that $s(z,u) = u^2$ and hence $yu^4 \in \varphi$. Then $Y \cap U_4$ is nonsingular by Lemma 7 and hence $Y \to X$ is a divisorial contraction by Theorem 5.

Suppose that $\natural$ does not hold. Then $\mathrm{Sing}(Y) \cap U_4$ is isolated. In U_4, the corresponding term $\tilde{\varphi}$ of θ in $\tilde{\varphi}$ is

$$\tilde{\theta} \in \{yz, y, z^3, z^2, z^2u, z, zu, 1, u\}.$$

Hence $\mathrm{Sing}(Y) \cap U_4$ is at worst of type cD. By Theorem 5, $Y \to X$ is a divisorial contraction. This proved the Claim.

We consider $Y \cap U_3$. We have that $z^4 \in h_4$ and hence Q_3 is at worst of type $cA/2$. By Corollary 9, $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_3$ is at worst of cA type. By Lemma 6, $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_1 = \emptyset$. Together with $Q_2 \notin Y$, we concludes that $\mathrm{Sing}(Y)_{\mathrm{ind}=1}$ is at worst of type cD and $\mathrm{Sing}(Y)_{\mathrm{ind}>1} = \{Q_3\}$, of type $cA/2$. Feasible resolution exists for this case.

Case 3 $wt_{v_{12}}(f) < 12$ and $wt_{v_8}(f) \geq 8$.

We consider the weighted blowup $Y \to X$ with weight $v_8 = (4,3,2,1)$.

1. Note that $\tau(h) = 4$ and $wt_{v_8}(h) \geq 8$, we thus have $h_4 = z^4 \in h_v \neq 0$. By Lemma 29, $\mathrm{Sing}(Y) \cap U_4$ is isolated. Also, one has $Q_3 \notin Y$.

2. Since $wt_{v_{12}}(f) < 12$ and $wt_{v_8}(f) \geq 8$, there is a term $\theta = y^i z^j u^k \in f$ with $wt_{v_{12}}(\theta) < 12$ and $wt_{v_8}(\theta) \geq 8$. Hence the corresponding term $\tilde{\theta} = y^i z^j u^{k'} \in \tilde{\varphi}$ satisfying

$$i + j + k' = i + j + (3i + 2j + k - 8) \leq 3.$$

One can verify that $\mathrm{Sing}(Y) \cap U_4$ is at worst of type cE_6 with h_4 has at least two factors. Hence if there is a cE_6 points then it is in Case 1 or 2. By Theorem 5, $Y \to X$ is a divisorial contraction.

3. By Lemma 7, $\mathrm{Sing}(Y) \cap U_2 = \{Q_2\}$ and Q_2 is a terminal quotient singularity of index 3. Also $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_1 = \emptyset$ by Lemma 6.

4. We summarize that $\mathrm{Sing}(Y)_{\mathrm{ind}>1}$ consists of Q_2, which is a quotient singularity of index 3 and two terminal quotient singularities of index 2 in the line $(\mathbf{y} = \mathbf{u} = 0) \subset E$ and $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \subset U_4$ are at worst of type cE_6 in Case 1 or 2.

Case 4. $wt_{v_{12}}(f) \geq 12$.

We consider $Y \to X$ the weighted blowup with weight $v_{12} = (6, 4, 3, 1)$.

1. Since $wt_{v_{12}}(h) \geq 12$ and $\tau(h) = 4$, we have $z^4 \in h_v$. It follows that $Q_3 \notin Y$, $h_v \neq 2s^3$, and thus $\natural$ does not holds. By Theorem 31 and Proposition 33, the weighted blowup $Y \to X$ is a divisorial contraction with discrepancy 1. Moreover, $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \subset U_4$ are at worst of type cD or there is only a unique point $R \in Y$ of type of type cE_6.

2. It follows that feasible resolution exists for $P \in X$ unless that $\mathrm{Sing}(Y)_{\mathrm{ind}=1} = R \in Y$ is of type cE_6. In fact, if it is of type cE_6, we may assume that $R = Q_4$ (cf. Remark 32).

Clearly,

$$\begin{cases} \tau^*(\tilde{\varphi}) < \tau^*(\varphi), \\ wt_{v_{12}}(\tilde{\varphi}) \leq wt_{v_{12}}(\varphi). \end{cases}$$

The existence of feasible resolution is thus reduced to cE_6 singularities with $wt_{v_{12}} < 12$ by induction on τ^*.

3. We remark that $\mathrm{Sing}(Y)_{\mathrm{ind}>1}$ consists of terminal quotient singularities on the line $(\mathbf{z} = \mathbf{u} = 0)$ and $(\mathbf{y} = \mathbf{u} = 0)$ of index $2, 3$ respectively.

This exhausts all cases of type cE_6. We thus conclude that for a given $P \in X$ of type cE_6, there is a feasible partial resolution $Y_s \to \ldots \to Y_1 = Y \to X$ such that $\mathrm{Sing}(Y_s)_{\mathrm{ind}=1}$ are at worst cD and $\mathrm{Sing}(Y_s)_{\mathrm{ind}>1}$ can only be of type $cA/2, cA/2$ or terminal quotient. Hence feasible resolution exists for Y_s and hence for $P \in X$. Q.E.D.

6.3. Resolution of $cE/2$ points

It is convenient to consider $cE/2$ points before we move into the cE_7 and cE_8 singularities. Given a $cE/2$ point $P \in X$, which is given by

$$(\varphi = x^2 + y^3 + \sum a_{ij} y z^j u^k + \sum b_{jk} z^j u^k = 0) \subset \mathbb{C}^4/\frac{1}{2}(1, 0, 1, 1),$$

with $h_4 := \sum_{j+k=4} b_{jk} z^j u^k \neq 0$.

We will consider weighted blowup with weights $v_1 = \frac{1}{2}(3,2,3,1)$ or $v_2 = \frac{1}{2}(5,4,3,1)$. Note that $wt_{v_1}(z) = wt_{v_2}(z), wt_{v_1}(u) = wt_{v_2}(u)$. Hence may simply denote it as $wt_{3,1}(G)$ for $G \in \mathbb{C}[[z,u]]$.

Theorem 35. *There is a feasible resolution for any $cE/2$ singularity.*

Proof. We first consider weighted blowup $Y \to X$ with weight $v = \frac{1}{2}(3,2,3,1)$. As before, we can rewrite φ as

$$\varphi = x^2 + y^3 + yg_v + yg_{v+1} + yg_> + h_v + h_{v+1} + h_>.$$

Notice that Lemma 28 still holds in the current situation.

Case 1. $wt_{3,1}(h_4) = 3$, i.e. $u^4 \notin h_4, zu^3 \in h_4$.

It is straightforward to see that E is irreducible and $Y \cap U_4$ is nonsingular, hence $Y \to X$ is a divisorial contraction. Also $Y \cap U_3$ has singularity Q_3 of type $cD/3$, might have terminal quotient singularity of index 3 along the line $\mathbf{y} = \mathbf{u} = 0$ and might have singularity at worst of type cD. There is no other singularity.

Case 2. $wt_{3,1}(h_4) = 4$, i.e. $u^4, zu^3 \notin h_4, z^2u^2 \in h_4$.

Since $z^2u^2 \in h_{v+1}$, one sees that $\natural$ does not hold and hence $Y \cap U_4$ has only isolated singularities.

It is straightforward to see that $Y \cap U_4$ might have singularities at worst of type cD, hence $Y \to X$ is a divisorial contraction. On $Y \cap U_3$, there are a singularity Q_3 of type $cD/3$, possibly terminal quotient singularities of index 3 along the line $(\mathbf{y} = \mathbf{u} = 0)$ and possibly singularities at worst of type cD. There is no other singularity outside $U_3 \cup U_4$.

Case 3. $wt_{3,1}(h_4) \geq 5$ and $\natural$ does not hold.

The similar argument works. Indeed, there is a term $\theta \in \varphi$ among $\{yu^4, yzu^3, yu^6, zu^5, u^6, u^8\}$. The corresponding term in $\tilde{\varphi}$ the equation of $Y \cap U_4$ is among $\{y, yzu, yu, zu, 1, u\}$. It is easy to see that singularities are at worst of type cD or $cD/3$ as in Case 2.

Case 4. $wt_{3,1}(h_4) \geq 5$ and $\natural$ holds.

We then consider a coordinate change that $\bar{y} := y - \lambda u^2$ for some λ so that we may rewrite $P \in X$ as

$$\bar{\varphi} = x^2 + \bar{y}^3 + 3s\bar{y}^2 + \bar{y}g_{v+1} + \bar{y}g_{v+2} + \bar{y}g_> + \bar{h}_{v+2} + \bar{h}_{v+3} + \bar{h}_>$$

similarly.

Since $wt_{3,1}(h_4) \geq 5$, one has either z^3u or $z^4 \in \varphi$. It follows that either z^3u or $z^4 \in \bar{\varphi}$.

Subcase 4-1. Suppose that there is a term $\theta = y^i z^j u^k \in \bar{\varphi}$ such that $6i + 5j + k \leq 16$. We consider weighted blowup $Y \to X$ with weight

$\frac{1}{2}(5,4,3,1)$ instead. By Lemma 29, $Y \cap U_4$ is isolated. The corresponding term $\tilde{\theta} = y^i z^j u^{k'}$ in $Y \cap U_4$ satisfying

$$i + j + k' = j + (3j + k - 10)/2 \leq 3.$$

One sees that $Y \cap U_4$ has at worst cE_6 singularities. Hence $Y \to X$ is a divisorial contraction.

Moreover, $\mathrm{Sing}(Y) \cap U_i = \{Q_i\}$ for $i = 2, 3$, which is a terminal quotient singularity of index 4 and 3. Also $Y \cap U_1$ is non-singular. Therefore feasible resolutions exist.

Subcase 4-2. Suppose that there is no term $\theta = y^i z^j u^k \in \bar{\varphi}$ such that $6i + 5j + k \leq 16$. We consider weighted blowup $Y \to X$ with weight $v_3 = \frac{1}{2}(9,6,5,1)$ instead. Note that in this situation, $wt_{v_3}\bar{\varphi} = 9$ and $z^4 \in \bar{\varphi}$. It is easy to see that $\mathrm{Sing}(Y) \cap U_4$ is isolated by Lemma 28 or by direct computation. Indeed, $Y \cap U_4$ has at worst singularities of type cE_6. Hence $Y \to X$ is a divisorial contraction.

Moreover, $\mathrm{Sing}(Y)_{\mathrm{ind}>1} = \{Q_3\}$ which is of index 5. Another higher index point is a point $R \in (\mathbf{z} = \mathbf{u} = 0)$, which is terminal quotient of index 3. We thus conclude that a feasible resolution exists for any $cE/2$ point by Theorem 34 and results in previous sections. Q.E.D.

6.4. Resolution of cE_7 points

In this subsection, we consider cE_7 points.

Theorem 36. *There is a feasible resolution for any cE_7 singularity.*

Proof. We shall consider weights $v_{18} = (9,6,4,1)$, $v_{14} = (7,5,3,1)$, $v_{12} = (6,4,3,1)$, $v_9 = (5,3,2,1)$, $v_8 = (4,3,2,1)$, $v_6 = (3,2,2,1)$, $v_5 = (3,2,1,1)$ and discuss as in cE_6 case.

Case 1. $wt_{v_{18}}(f) < 18, \ldots, wt_{v_6}(f) < 6$.
We consider weighted blowup with weight $v_5 = (3,2,1,1)$.

Since $z^3 \in g_v$, we have that $yz^3 \in f$, E is irreducible, and $Y \cap U_3$ is non-singular. Hence, $Y \to X$ is a divisorial contraction by Theorem 5.

By Lemma 7, $Y \cap U_i$ is non-singular away from Q_i for $i = 1, 2, 3$. Notice that there is a term θ with $wt_{v_6}(\theta) < 6$ and $wt_{v_5}(\theta) \geq 5$. It follows that θ is yu^3 or u^5. Hence Q_4 is either non-singular or $Q_4 \notin Y$. Therefore, $\mathrm{Sing}(Y) = \mathrm{Sing}(Y)_{\mathrm{ind}>1} = \{Q_1, Q_2\}$, which are terminal quotient points of index 3 and 2 respectively.

Case 2. $wt_{v_{18}}(f) < 18, \ldots, wt_{v_8}(f) < 8$, $wt_{v_6}(f) \geq 6$.
We consider weighted blowup with weight $v_6 = (3,2,2,1)$ and proceed as in Case 2 of cE_6, then $\mathrm{wBl}_{v_6} : Y \to X$ is a divisorial contraction and $\mathrm{Sing}(Y) \cap U_4$ is at worst of type cD.

We consider $Y \cap U_3$. Since $yz^3 \in \varphi$, one sees that Q_3 is at worst of type $cD/2$. By Corollary 10, $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_3$ is at worst of type cD.

By Lemma 6, $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_1 = \emptyset$. Together with $Q_2 \notin Y$, we conclude that $\mathrm{Sing}(Y)_{\mathrm{ind}=1}$ is at worst of type cD and $\mathrm{Sing}(Y)_{\mathrm{ind}>1} = \{Q_3\}$, of type $cD/2$. Feasible resolution exists for this case.

Case 3. $wt_{v_{18}}(f) < 18, \ldots, wt_{v_9}(f) < 9$, $wt_{v_8}(f) \geq 8$.
We consider weighted blowup with weight $v_8 = (4, 3, 2, 1)$.

There is a term $\theta = y^i z^j u^k$ satisfying $wt_{v_9}(\theta) < 9$ and $wt_{v_8}(\theta) \geq 8$. Hence either g_v or h_v contains θ and is non-zero. By Lemma 29, $\mathrm{Sing}(Y) \cap U_4$ is isolated.

By the same argument as in Case 3 of cE_6, one sees that $\mathrm{Sing}(Y) \cap U_4$ is at worst of type cE_6. This implies in particular that $Y \to X$ is a divisorial contraction.

Since both y^3, yz^3 are in φ, by Lemma 7, one has $Y \cap U_i$ is nonsingular away from Q_i for $i = 2, 3$. Together with $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_1 = \emptyset$, we are done.

Case 4. $wt_{v_{18}}(f) < 18, \ldots, wt_{v_{12}}(f) < 12$, $wt_{v_9}(f) \geq 9$.
We consider weighted blowup with weight $v_9 = (5, 3, 2, 1)$. One has $z^3 \in g_v \neq -3s^2$. Hence $\natural$ does not hold and $\mathrm{Sing}(Y) \cap U_4$ is isolated by Lemma 28.

We consider $Y \cap U_4$. Since $wt_{v_{12}}(\theta) < 12$ and $wt_{v_9}(\theta) \geq 9$ for some $\theta = y^i z^j u^k \in \varphi$, we have $\tilde{\theta} = y^i z^j u^{k'} \in \tilde{\varphi}$ with $i + j + k' = 4i + 3j + k - 9 \leq 2$. It follows easily that $Y \cap U_4$ has at worst singularity of type cA by Lemma 8. Therefore, $Y \to X$ is a divisorial contraction with discrepancy 1.

By Lemma 7, one sees that $Y \cap U_i$ is non-singular away from Q_i for $i = 1, 3$. Moreover, $Q_2 \notin Y$. Feasible resolution exists for this case.

Case 5.$wt_{v_{18}}(f) < 18$, $wt_{v_{14}}(f) < 14$, $wt_{v_{12}}(f) \geq 12$.
We consider the weighted blowup with weight $v_{12} = (6, 4, 3, 1)$. One has $z^3 \in g_{v+1} \neq 0$.

Subcase 5-1. Suppose that $\natural$ does not hold.
Then $Y \to X$ is a divisorial contraction and $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \subset U_4$ by Proposition 33.

Indeed, by the discussion in 30, we may assume that $\mathrm{Sing}(Y)_{\mathrm{ind}=1}$ singularities at worst of type cD or a singularity of type cE_7 at Q_4. Clearly, we have

$$\begin{cases} \tau^*(\tilde{\varphi}) < \tau^*(\varphi), \\ wt_{v_{18}}(\tilde{\varphi}) \leq wt_{v_{18}}(\varphi), \\ wt_{v_{14}}(\tilde{\varphi}) \leq wt_{v_{14}}(\varphi). \end{cases}$$

By induction on τ^*, we are reduced to the case that $wt_{v_{12}} < 12$.

Subcase 5-2. Suppose that $\natural$ hold.
Notice that there is $\theta \in \varphi$ with $\varphi_{v_{14}}(\theta) < 14, \varphi_{v_{12}}(\theta) \geq 12$, it is easy to see that $\theta \in yg_v$, h_v or in h_{v+1}. This implies in particular that

$s \neq 0$. We consider a coordinate change that $\bar{y} := y - s(z,u)$ for some $s = \alpha zu + \beta u^4$ so that we may write $P \in X$ as

$$\bar{\varphi} = x^2 + \bar{y}^3 + 3s\bar{y}^2 + \bar{y}g_{v+1} + \bar{y}\bar{g}_> + \bar{h}_>$$

similarly.

We consider weighted blowup with weight $v_{14} = (7,5,3,1)$ instead in this situation. Since $\bar{y}z^3 \in \bar{\varphi}$, by Lemma 29, one sees that $\mathrm{Sing}(Y) \cap U_4$ is isolated. One can check that $\mathrm{Sing}(Y) \cap U_4$ has at worst singularity of type cD for $s \neq 0$. Therefore, $Y \to X$ is a divisorial contraction.

One can easily check that for $i = 2,3$, $\mathrm{Sing}(Y) \cap U_i = \{Q_i\}$, which is terminal quotient of index 5 and 3 respectively. Moreover, $Q_1 \notin Y$ and hence there exists a feasible resolution.

Case 6. $wt_{v_{18}}(f) < 18$, $wt_{v_{14}}(f) \geq 14$.
We consider the weighted blowup with weight $v_{14} = (7,5,3,1)$. Similarly, one has $g_3 = z^3$ and $z^3 \in g_v \neq 0$. By Lemma 29, $\mathrm{Sing}(Y) \cap U_4$ is isolated. Since $wt_{v_{18}}(\theta) < 18$ and $wt_{v_{14}}(\theta) \geq 14$ for some $\theta = y^i z^j u^k \in \varphi$, we have $\tilde{\theta} = y^i z^j u^{k'} \in \tilde{\varphi}$ with $i + j + k' = 6i + 4j + k - 14 \leq 3$. One can verify that any $R \in \mathrm{Sing}(Y) \cap U_4$ is at worst of type cE_6. Therefore $Y \to X$ is a divisorial contraction.

By Lemma 7, we have that $Y \cap U_i$ is nonsingular away from Q_i for $i = 2,3$. Moreover $Q_1 \notin Y$, hence feasible resolution exists for this case.

Case 7. $wt_{v_{18}}(f) \geq 18$
We consider the weighted blowup with weight $v_{18} = (9,6,4,1)$. Since $wt_{v_{18}}(g) \geq 18$ and $\tau(g) = 3$, we have $g_3 = z^3$ and $z^3 \in g_v \neq -3s^2$. It is clear that $\natural$ does not holds. By Proposition 33, $Y \to X$ is a divisorial contraction and $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \subset U_4$.

$\mathrm{Sing}(Y) \cap U_4$ is at worst of type cE_7. If there is a singularity of type cE_7, then we proceed by induction in τ^*. Then it can be reduced to the cases with $wt_{v_{18}} < 18$.

This completes the proof that a feasible resolution exist for cE_7 singularity. Q.E.D.

6.5. Resolution of cE_8 points

In this subsection, we shall prove that

Theorem 37. *There is a feasible resolution for any cE_8 singularity.*

Proof. We will need to consider weights $v_{30} = (15,10,6,1)$, $v_{24} = (12,8,5,1)$,...etc.

Case 1.$wt_{v_{30}}(f) < 30, \ldots, wt_{v_8}(f) < 8$.
We consider weighted blowup with weight $v_6 = (3,2,2,1)$. By 6.2, we have that $wt_{v_6}(f) \geq 6$ always holds and $z^5 \in h$. We consider weighted blowup with weight $v_6 = (3,2,2,1)$ and proceed as in Case 2 of cE_6,

then $\mathrm{wBl}_{v_6} : Y \to X$ is a divisorial contraction and $\mathrm{Sing}(Y) \cap U_4$ is at worst of type cD.

We consider $Y \cap U_3$. We have that $z^5 \in \varphi$. Hence Q_3 is at worst of type $cE/2$ and $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_3$ is at worst of type cE_6. By Lemma 6, $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_1 = \emptyset$. Together with $Q_2 \notin Y$, we conclude that feasible resolution exists for this case.

Case 2. $wt_{v_{30}}(f) < 30, \ldots, wt_{v_9}(f) < 9$, $wt_{v_8}(f) \geq 8$

We consider weighted blowup with weight $v_8 = (4, 3, 2, 1)$. By the same argument as in Case 3 of cE_7, one has that $Y \to X$ is a divisorial contraction with $\mathrm{Sing}(Y) \cap U_4$ at worst of type cE_6.

Since y^3 is in φ, by Lemma 7, one has that $Y \cap U_2$ is nonsingular away from Q_2. Together with $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_1 = \emptyset$ and Q_3 is of type $cAx/2$, we are done.

Case 3. $wt_{v_{30}}(f) < 30, \ldots, wt_{v_{12}}(f) < 12$, $wt_{v_9}(f) \geq 9$

We consider weighted blowup with weight $v_9 = (5, 3, 2, 1)$. Since $h_5 \neq 0$, we have either z^4u or $z^5 \in h$. Now the same argument as in Case 4 of cE_7 goes through.

Case 4. $wt_{v_{30}}(f) < 30, \ldots, wt_{v_{14}}(f) < 14$, $wt_{v_{12}}(f) \geq 12$.

We consider weighted blowup with weight $v_{12} = (6, 4, 3, 1)$. The proof is essentially parallel to Case 5 of cE_7. Note that we have $h_5 = z^5$ or $z^4u \in f$.

Subcase 4-1. Suppose $\natural$ does not hold.

Then $Y \to X$ is a divisorial contraction and $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \subset U_4$ by Theorem 31 and Proposition 33.

Indeed by the discussion in 30, we know that either $\mathrm{Sing}(Y)_{\mathrm{ind}=1}$ consists of singularities at worst of type cE_6 or we may assume that Q_4, the only singularity in U_4, is of type cE_8.

Clearly, we have

$$\begin{cases} \tau^*(\tilde{\varphi}) < \tau^*(\varphi), \\ wt_{v_l}(\tilde{\varphi}) \leq wt_{v_l}(\varphi), \end{cases}$$

for all $l \geq 12$. By induction on τ^*, we are done.

Subcase 4-2. Suppose that $\natural$ hold.

As in Subcase 5-2 of cE_7, we consider a coordinate change and then the weighted blowup with weight $v_{14} = (7, 5, 3, 1)$ instead in this situation. Since $z^5 \in \bar{\varphi}$, by Lemma 29, one sees that $\mathrm{Sing}(Y) \cap U_4$ is isolated. One can check that $\mathrm{Sing}(Y) \cap U_4$ has at worst singularity of type cD for $s \neq 0$. Therefore, $Y \to X$ is a divisorial contraction.

One can easily check that for $i = 2, 3$, $\mathrm{Sing}(Y) \cap U_i = \{Q_i\}$, which is terminal quotient of index 5 and 3 respectively. Moreover, $Q_1 \notin Y$ and hence there exists a feasible resolution.

Case 5.$wt_{v_{30}}(f) < 30, \ldots, wt_{v_{18}}(f) < 18, wt_{v_{14}}(f) \geq 14$
We can proceed as in Subcase 6 of cE_7. Since $z^5 \in h_{v+1} \neq 0$, we still have that $\mathrm{Sing}(Y) \cap U_4$ has isolated singularities by Lemma 29. Thus the same conclusion holds.
Case 6.$wt_{v_{30}}(f) < 30$, $wt_{v_{24}}(f) < 24, wt_{v_{20}}(f) < 20$, $wt_{v_{18}}(f) \geq 18$
We consider weighted blowup with weight $v_{18} = (9, 6, 4, 1)$.

Since there is a term $\theta \in f$ with $wt_{v_{20}}(\theta) < 20$ and $wt_{v_{18}}(\theta) \geq 18$. This implies $\tilde{\theta}$ is in g_v, h_v or h_{v+1}.
Subcase 6-1. Suppose $\natural$ does not hold.
Then $Y \to X$ is a divisorial contraction by Theorem 31.

One sees that $Y \cap U_3$ is given by $(\tilde{\varphi} : x^2 + y^3 + z^2 + \text{other terms} = 0) \subset \mathbb{C}^4/\frac{1}{4}(1, 2, 3, 1)$. Therefore, $\mathrm{Sing}(Y) \cap U_3$ is type $cAx/4$ or cA. We also have $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_1 = \emptyset$, and $Q_2 \notin Y$.

It remains to consider $Y \cap U_4$. Notice that the corresponding term $\tilde{\theta} = y^i z^j u^{k'}$ has $i + j + k' \leq 4$ unless $\theta = z^4u^3, \tilde{\theta} = z^4u$. If $i + j + k' \leq 4$, then $Y \cap U_4$ has singularities at worst of type cE_7 and hence feasible resolution exists. Suppose that $\tilde{\theta} = z^4u \in \tilde{\varphi}$. Hence

$$wt_{v_l}(\tilde{\varphi}) < l,$$

for $l = 30, 24, 20, 18$. Therefore, $Y \cap U_4$ has singularities at worst of type cE_8 in Subcase 1-4.
Subcase 6-2. Suppose that $\natural$ hold.
We first consider a coordinate change that $\bar{y} := y - s(z, u)$ with $s(z, u) \neq 0$ since there is θ in yg_v, h_v or h_{v+1}. Now $P \in X$ is defined as

$$\bar{\varphi} = x^2 + \bar{y}^3 + 3s\bar{y}^2 + \bar{y}g_{v+1} + \bar{y}\bar{g}_> + \bar{h}_>$$

and we consider weighted blowup with weight $v_{20} = (10, 7, 4, 1)$ instead in this situation.

Since $z^5 \in \varphi$ and hence $z^5 \in \bar{\varphi}$, one sees that $\mathrm{Sing}(Y) \cap U_4$ is isolated, by Lemma 29. Since $s = (\alpha zu^2 + \beta u^6) \neq 0$, we have either $\bar{y}^2zu^2$ or $\bar{y}^2u^6 \in \bar{\varphi}$. One can check that $Y \cap U_4$ has at worst singularities of type cD and thus $Y \to X$ is a divisorial contraction.

Together with the fact that $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_1$ is empty, $Y \cap U_2$ is non-singular away from Q_2 and $Q_3 \in \mathrm{Sing}(Y)_{\mathrm{ind}>1}$, one sees that a feasible resolution exists.
Case 7. $v_{30}(f) < 30$, $wt_{v_{24}}(f) < 24$, $wt_{v_{20}}(f) \geq 20$
We consider the weighted blowup with weight $v_{20} = (10, 7, 4, 1)$. Since $z^5 \in h_v$, we have $Q_3 \notin Y$ and $\mathrm{Sing}(Y) \cap U_4$ is isolated by Lemma 29.

We work on U_4. There is a term $\theta = y^i z^j u^k \in f$ with $wt_{v_{24}}(\theta) < 24$ and $wt_{v_{20}}(\theta) \geq 20$. Hence $\tilde{\theta} = y^i z^j u^{k'}$ with $i + j + k' \leq 3$. It follows that $\mathrm{Sing}(Y) \cap U_4$ is at worst of type cE_6.

Therefore, $Y \to X$ is a divisorial contraction. Together with the fact that $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \cap U_1$ is empty, $Y \cap U_2$ is non-singular away from Q_2 and $Q_3 \in \mathrm{Sing}(Y)_{\mathrm{ind}>1}$, one sees that a feasible resolution exists.

Case 8.$wt_{v_{30}}(f) < 30$, $wt_{v_{24}}(f) \geq 24$.

We consider the weighted blowup with weight $v_{24} = (12, 8, 5, 1)$. One notices that $\tau(h) = 5$ implies that $z^5 \in h_{v+1}$. Since $wt_{v_{24}}(g_{v+1}) = 17$ and hence $u^2|g_{v+1}$. It follows that $z^5 \notin s(z,u)g_{v+1}$ and $\natural$ does not hold. Therefore $Y \to X$ is a divisorial contraction and $\mathrm{Sing}(Y)_{\mathrm{ind}=1} \subset U_4$ by Theorem 31 and Proposition 33.

Unless $\mathrm{Sing}(Y) \cap U_4 = \{Q_4\}$ is of type cE_8, we have feasible resolution of Y. If $\mathrm{Sing}(Y) \cap U_4 = \{Q_4\}$ is of type cE_8, then we have

$$\begin{cases} \tau^*(\tilde{\varphi}) < \tau^*(\varphi); \\ wt_{v_{30}}(\tilde{\varphi}) \leq wt_{v_{30}}(\varphi); \\ wt_{v_{24}}(\tilde{\varphi}) \leq wt_{v_{24}}(\varphi). \end{cases}$$

By induction on τ^*, the existence of feasible resolution is thus reduced to the existence of feasible resolution of milder singularity or to the existence of feasible resolution of cE_8 singularities with $wt_{v_{24}} < 24$.

Case 9. $wt_{v_{30}}(f) \geq 30$.

We consider the weighted blowup with weight $v_{30} = (15, 10, 6, 1)$. The similar argument as in Case 8 works.

This completes the proof. Q.E.D.

Proof of Main Theorem. This follows from Theorem 12, 13, 20, 24, 34, 35, 36, 37. Q.E.D.

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