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Singular spaces with trivial canonical class

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Dedicated to Professor Shigefumi Mori on the occasion of his 60th birthday

Abstract.

The classical Beauville-Bogomolov Decomposition Theorem asserts that any compact Kähler manifold with numerically trivial canonical bundle admits an étale cover that decomposes into a product of a torus, and irreducible, simply-connected Calabi-Yau– and holomorphic-symplectic manifolds. The decomposition of the simply-connected part corresponds to a decomposition of the tangent bundle into a direct sum whose summands are integrable and stable with respect to any polarisation.

Building on recent extension theorems for differential forms on singular spaces, we prove an analogous decomposition theorem for the tangent sheaf of projective varieties with canonical singularities and numerically trivial canonical class.

In view of recent progress in minimal model theory, this result can be seen as a first step towards a structure theory of manifolds with Kodaira dimension zero. Based on our main result, we argue that the natural building blocks for any structure theory are two classes of canonical varieties, which generalise the notions of irreducible Calabi-Yau– and irreducible holomorphic-symplectic manifolds, respectively.

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§1. Introduction

1.A. Introduction and main result

The minimal model program aims to reduce the birational study of projective manifolds with Kodaira dimension zero to the study of associated *minimal models*, that is, normal varieties X with terminal singularities whose canonical divisor is numerically trivial. The ideal case, where the minimal variety X is smooth, is described in the fundamental Decomposition Theorem of Beauville and Bogomolov.

Theorem 1.1 (Beauville-Bogomolov Decomposition, [Bea83] and references there)**.** *Let X be a compact Kähler manifold whose canonical divisor is numerically trivial. Then there exists a finite étale cover, $X' \to X$ such that X' decomposes as a product*

$$X' = T \times \prod_{\nu} X_{\nu} \tag{1.1.1}$$

where T is a compact complex torus, and where the X_ν are irreducible and simply-connected Calabi-Yau– or holomorphic-symplectic manifolds.

Remark 1.2. The decomposition (1.1.1) induces a decomposition of the tangent bundle $T_{X'}$ into a direct sum whose summands have vanishing Chern class, and are integrable in the sense of Frobenius' theorem. Those summands that correspond to the X_ν are slope-stable with respect to any ample polarisation.

In view of recent progress in minimal model theory, an analogue of Theorem 1.1 for minimal models would clearly be a substantial step towards a complete structure theory for varieties of Kodaira dimension zero. However, since the only known proof of Theorem 1.1 heavily uses Kähler-Einstein metrics and the solution of the Calabi conjecture, a full generalisation of Beauville-Bogomolov Decomposition Theorem 1.1 to the singular setting is difficult.

Main result The main result of our paper is the following Decomposition Theorem for the tangent sheaf of minimal varieties with vanishing first Chern class. Presenting the tangent sheaf as a direct sum of integrable subsheaves which are stable with respect to any polarisation, Theorem 1.3 can be seen as an infinitesimal analogue of the Beauville-Bogomolov Decomposition 1.1 in the singular setting.

Theorem 1.3 (Decomposition of the tangent sheaf)**.** *Let X be a normal projective variety with at worst canonical singularities, defined over the complex numbers. Assume that the canonical divisor of X is numerically trivial: $K_X \equiv 0$. Then there exists an Abelian variety A as well as a projective variety $\widetilde{X}$ with at worst canonical singularities, a finite cover $f : A \times \widetilde{X} \to X$, étale in codimension one, and a decomposition*

$$\mathscr{T}_{\widetilde{X}} \cong \bigoplus \mathscr{E}_i$$

such that the following holds.

(1.3.1) *The $\mathscr{E}_i$ are integrable saturated subsheaves of $\mathscr{T}_{\widetilde{X}}$, with trivial determinants.*

Further, if $g : \widehat{X} \to \widetilde{X}$ is any finite cover, étale in codimension one, then the following properties hold in addition.

(1.3.2) *The sheaves $(g^* \mathscr{E}_i)^{**}$ are slope-stable with respect to any ample polarisation on $\widehat{X}$.*

(1.3.3) *The irregularity of $\widehat{X}$ is zero, $h^1\big(\widehat{X}, \mathscr{O}_{\widehat{X}}\big) = 0$.*

The decomposition found in Theorem 1.3 satisfies an additional uniqueness property. For a precise statement, see Remark 7.5 on page 93. Taking g to be the identity, we see that the irregularity of $\widetilde{X}$ is zero, and that the summands $\mathscr{E}_i$ are stable with respect to any polarisation.

Other results In the course of the proof, we show the following two additional results, pertaining to stability of the tangent bundle, and to wedge products of differentials forms that are defined on the smooth locus of a minimal model. We feel that these results might be of independent interest.

Proposition 1.4 (Stability of $\mathscr{T}_X$ does not depend on polarisation, Proposition 5.7)**.** *Let X be a normal projective variety having at worst canonical singularities. Assume that K_X is numerically equivalent to zero. If the tangent sheaf $\mathscr{T}_X$ is slope-stable with respect to one polarisation, then it is also stable with respect to any other polarisation.* *Q.E.D.*

Proposition 1.5 (Non-degeneracy of the wedge product, Proposition 6.1)**.** *Let X be a normal n-dimensional projective variety X having at worst canonical singularities. Denote the smooth locus of X by X_{reg}. Suppose that the canonical divisor is trivial. If $0 \le p \le n$ is any number, then the natural pairing given by the wedge product on X_{reg},*

$$\bigwedge : H^0\big(X_{\mathrm{reg}},\, \Omega^p_{X_{\mathrm{reg}}}\big) \times H^0\big(X_{\mathrm{reg}},\, \Omega^{n-p}_{X_{\mathrm{reg}}}\big) \longrightarrow \underbrace{H^0\big(X_{\mathrm{reg}},\, \omega_{X_{\mathrm{reg}}}\big)}_{\cong H^0\left(X,\,\omega_X\right) \cong \mathbb{C}},$$

is non-degenerate. *Q.E.D.*

Singular analogues of Calabi-Yau and irreducible symplectic manifolds Based on the Decomposition Theorem 1.3, we will argue in Section 8 that the natural building blocks for any structure theory of projective manifolds with Kodaira dimension zero are canonical varieties with *strongly stable* tangent sheaf. Strong stability, introduced in Definition 7.2 on page 92, is a formalisation of condition (1.3.2) that appears in the Decomposition Theorem 1.3.

In dimension no more than five, we show that canonical varieties with strongly stable tangent sheaf fall into two classes, which naturally generalise the notions of irreducible Calabi-Yau– and irreducible holomorphic-symplectic manifolds, respectively. There is ample evidence to suggest that this dichotomy should hold in arbitrary dimension.

Outline of the paper The proof of Theorem 1.3 relies on recent extension results for differential forms on singular spaces, which we recall in Section 2 below. There are three additional preparatory sections, Sections 3–5, where we recall structure results for varieties with trivial canonical bundle, and discuss stability properties of the tangent sheaf on varieties with numerically trivial canonical divisor. Some of the material in these sections is new.

Using the extension result together with Hodge-theoretic arguments, we will show in Section 6 that the wedge-product induces perfect pairings of reflexive differential forms. This will later on be used to split the inclusion of certain subsheaves of the tangent sheaf. The results obtained there generalise ideas of Bogomolov [Bog74], but are new in the singular setting, to the best of our knowledge. With these preparations in place, Theorem 1.3 is then shown in Section 7.

Based on our main results, the concluding Section 8 discusses possible approaches towards a more complete structure theory of singular varieties with trivial canonical bundle, and proves first results in this direction.

Global Convention Throughout the paper we work over the complex number field. In the discussion of sheaves, the word "stable" will always mean "slope-stable with respect to a given polarisation". Ditto for semistability.

1.B. Acknowledgements

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§2. Reflexive differentials on normal spaces

2.A. Differentials, reflexive tensor operations

Given a normal variety X, we denote the sheaf of Kähler differentials by Ω^1_X. The tangent sheaf will be denoted by $\mathscr{T}_X = (\Omega^1_X)^*$. Reflexive differentials, as defined below, will play an important role in the discussion.

Definition 2.1 (Reflexive differential forms, cf. [GKKP11, Sect. 2.E]). Let X be a normal variety, let X_{reg} be the smooth locus of X and $\imath : X_{\mathrm{reg}} \hookrightarrow X$ its open embedding into X. If $0 \le p \le \dim X$ is any number, we denote the reflexive hull of the p^{th} exterior power of Ω^1_X by

$$\Omega^{[p]}_X := \left(\wedge^p \Omega^1_X\right)^{**} = \imath_* \Omega^p_{X_{\mathrm{reg}}}.$$

We refer to sections in $\Omega^{[p]}_X$ as *reflexive p-forms on X*.

Remark 2.2 (Reflexive differentials and dualising sheaf). In the setting of Definition 2.1, recall that the Grothendieck dualising sheaf ω_X is always reflexive. We obtain that

$$\Omega^{[\dim X]}_X = \omega_X = \mathscr{O}_X\big(K_X\big).$$

Notation 2.3 (Reflexive tensor operations). Let X be a normal variety and $\mathscr{A}$ a coherent sheaf of $\mathscr{O}_X$-modules, of rank r. For $n \in \mathbb{N}$, set $\mathscr{A}^{[n]} := (\mathscr{A}^{\otimes n})^{**}$. Further, set $\det \mathscr{A} = \left(\wedge^r \mathscr{A}\right)^{**}$. If $\pi : X' \to X$ is a morphism of normal varieties, set $\pi^{[*]}(\mathscr{A}) := \left(\pi^* \mathscr{A}\right)^{**}$.

2.B. Extension results for differential forms on singular spaces

One of the main ingredients for the proof of the Decomposition Theorem 1.3 is the following extension result for differential forms on singular spaces, recently shown in joint work of the authors and Sándor Kovács. In its simplest form, the extension theorem asserts the following.

Theorem 2.4 (Extension Theorem, [GKKP11, Thm. 1.5])**.** *Let X be a quasi-projective complex algebraic variety and D an effective $\mathbb{Q}$-divisor on X such that the pair (X, D) is Kawamata log terminal (klt). If $\pi : \widetilde{X} \to X$ is any resolution of singularities and $0 \le p \le \dim X$ any number, then the push-forward sheaf $\pi_* \Omega^p_{\widetilde{X}}$ is reflexive.* *Q.E.D.*

Using Definition 2.1, the Extension Theorem 2.4 can be reformulated, saying that $\pi_* \Omega^p_{\widetilde{X}} = \Omega^{[p]}_X$ for all $p \le \dim X$. Equivalently, if $E \subset \widetilde{X}$ denotes the π-exceptional set, then the Extension Theorem asserts that for any open set $U \subset X$, any p-form on $\pi^{-1}(U) \setminus E$ extends across E, to give a p-form on $\pi^{-1}(U)$. In other words, it asserts that the natural restriction map

$$\Omega^p_{\widetilde{X}}\big(\pi^{-1}(U)\big) \to \Omega^p_{\widetilde{X}}\big(\pi^{-1}(U) \setminus E\big)$$

is surjective. We refer to the original papers [GKKP11, Sect. 1] and [GKK10] or to the survey [Keb13] for an in-depth discussion.

§3. Irregularity and Albanese map of canonical varieties

The Albanese map is one important tool in the study of varieties with trivial canonical divisor. The following invariant is relevant in its investigation.

Definition 3.1 (Augmented irregularity). Let X be a normal projective variety. We denote the irregularity of X by $q(X) := h^1\big(X, \mathscr{O}_X\big)$ and define the *augmented irregularity* $\widetilde{q}(X) \in \mathbb{N} \cup \{\infty\}$ as

$$\widetilde{q}(X) := \max\big\{q(\widetilde{X}) \mid \widetilde{X} \to X \text{ a finite cover, étale in codimension one}\big\}.$$

Remark 3.2 (Irregularity and the Albanese map). If X is a projective variety with canonical singularities, recall from [Kaw85, Sect. 8] that the Albanese map $\alpha_X : X \to \mathrm{Alb}(X)$ is well defined, and that $\dim \mathrm{Alb}(X) = q(X)$. Better still, if $\pi : \widetilde{X} \to X$ is any resolution of singularities, then the Albanese map $\alpha_{\widetilde{X}}$ of $\widetilde{X}$ agrees with α_X. In other words, there exists a commutative diagram as follows,

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\alpha_{\widetilde{X}}", "\text{Albanese map}"'] \arrow[d, "\pi,\ \text{desing.}"'] & \mathrm{Alb}(\widetilde{X}) \arrow[d, equal] \\
X \arrow[r, "\alpha_X", "\text{Albanese map}"'] & \mathrm{Alb}(X).
\end{tikzcd}$$

The following result of Kawamata describes the Albanese map of varieties whose canonical divisor is numerically trivial. As we will see in Corollary 3.6, this often reduces the study of varieties with trivial canonical class to those with $\widetilde{q}(X) = 0$.

Proposition 3.3 (Fibre space structure of the Albanese map, [Kaw85, Prop. 8.3])**.** *Let X be a normal n-dimensional projective variety X with at worst canonical singularities. Assume that K_X is numerically trivial. Then K_X is torsion, the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is surjective and has the structure of an étale-trivial fiber bundle.*

In other words, there exists a positive number $m \in \mathbb{N}^+$ such that $\mathscr{O}_X(m \cdot K_X) \cong \mathscr{O}_X$. Furthermore, there exists a finite étale cover $B \to \mathrm{Alb}(X)$ from an Abelian variety B to $\mathrm{Alb}(X)$ such that the fiber product over $\mathrm{Alb}(X)$ decomposes as a product

$$X \times_{\mathrm{Alb}(X)} B \cong F \times B,$$

where F is a normal projective variety. *Q.E.D.*

Remark 3.4. If X is a projective variety with canonical singularities and numerically trivial canonical class, Proposition 3.3 implies that $q(X) = \dim \mathrm{Alb}(X) \leq \dim X$. If $\widetilde{X} \to X$ is any finite cover, étale in codimension one, then $\widetilde{X}$ will likewise have canonical singularities [KM98, Prop. 5.20] and numerically trivial canonical class. In summary, we see that $\widetilde{q}(X) \leq \dim X$. The augmented irregularity of canonical varieties with numerically trivial canonical class is therefore finite.

Remark 3.5. In the setting of Proposition 3.3, the canonical map

$$F \times B \cong X \times_{\mathrm{Alb}(X)} B \to X$$

is étale. The variety $F \times B$ is thus canonical by [KM98, Prop. 5.20]. Since B is smooth, this automatically implies that F is canonical. If the canonical divisor of X is trivial, then F will likewise have a trivial canonical divisor.

A variant of the following corollary has appeared as [Pet94, Thm. 4.2].

Corollary 3.6 (Structure of varieties with numerically trivial canonical class, cf. [Kaw85, Cor. 8.4])**.** *Let X be a normal n-dimensional projective variety with at worst canonical singularities. Assume that K_X is numerically trivial. Then there exist projective varieties A, Z and a morphism $\nu : A \times Z \to X$ such that the following holds.*

(3.6.1) *The variety A is Abelian.*
(3.6.2) *The variety Z is normal and has at worst canonical singularities.*
(3.6.3) *The canonical class of Z is trivial, $\omega_Z \cong \mathscr{O}_Z$.*
(3.6.4) *The augmented irregularity of Z is zero, $\widetilde{q}(Z) = 0$.*
(3.6.5) *The morphism ν is finite, surjective and étale in codimension one.*

Proof. We construct a sequence of finite surjective morphisms,

$$F \times B \xrightarrow[\text{étale}]{\gamma} X^{(2)} \xrightarrow[\text{étale in codim. one}]{\beta} X^{(1)} \xrightarrow[\text{index-one cover}]{\alpha} X,$$

as follows. Recall from Proposition 3.3 that the canonical divisor of X is torsion, and let $\alpha : X^{(1)} \to X$ be the associated index-one cover, cf. [KM98, Sect. 5.2] or [Rei87, Sect. 3.5]. We have seen in Remark 3.4 that the variety $X^{(1)}$ has a trivial canonical divisor and at worst canonical singularities. Remark 3.4 also shows that $\widetilde{q}(X^{(1)})$ is finite. This implies that there exists a finite morphism $\beta : X^{(2)} \to X^{(1)}$, étale in codimension one, such that

$$\widetilde{q}\left(X^{(1)}\right) = q\left(X^{(2)}\right) = \widetilde{q}\left(X^{(2)}\right).$$

Again, $X^{(2)}$ has trivial canonical divisor and canonical singularities. Next, let $\gamma : F \times B \to X^{(2)}$ be the étale morphism obtained by applying Proposition 3.3 and Remark 3.5 to the variety $X^{(2)}$. The variety B then satisfies the following,

$$\dim B = \dim \operatorname{Alb} X^{(2)} = q\left(X^{(2)}\right) = \widetilde{q}\left(X^{(2)}\right). \tag{3.6.6}$$

Remark 3.5 also asserts that F has a trivial canonical divisor and canonical singularities.

To finish the proof, it suffices to show that $\widetilde{q}(F) = 0$. If not, we could apply Proposition 3.3 and Remark 3.5 to F, obtaining an étale map $\delta : (F' \times B') \times B \to F \times B$, where B' is Abelian, and of positive dimension. The composed morphism $\gamma \circ \delta : F' \times (B' \times B) \to X^{(2)}$ is again étale, showing that

$$\widetilde{q}\left(X^{(2)}\right) \geq q\big(F' \times (B' \times B)\big) \geq \dim B + \dim B' > \dim B,$$

thus contradicting Equation (3.6.6) above. This shows that $\widetilde{q}(F) = 0$ and finishes the proof of Corollary 3.6. Q.E.D.

§4. A criterion for numerical triviality

The proof of the Decomposition Theorem 1.3 of rests on an analysis of the tangent sheaf of Kawamata log terminal spaces and of its destabilising subsheaves. We will show that the determinant of any destabilising subsheaf is trivial, at least after passing to a suitable cover. This part of the proof is based on a criterion for numerical triviality, formulated in Proposition 4.2. The criterion generalises the following result which goes back to Kleiman.

Lemma 4.1 (Kleiman's criterion for numerical triviality)**.** *Let Z be an irreducible, normal projective variety of dimension $n \geq 2$ and D a $\mathbb{Q}$-Cartier divisor on Z. If $D \cdot H_1 \cdots H_{n-1} = 0$ for all $(n-1)$-tuples of ample divisors $H_1, \ldots, H_{n-1}$ on Z, then D is numerically trivial.*

Proof. Passing to a sufficiently high multiple of D, we can assume without loss of generality that D is Cartier, and thus linearly equivalent to the difference of two ample Cartier divisors, $D \sim H_{1,1} - H_{1,2}$. Let $H_2, \ldots, H_{n-1}$ be arbitrary ample divisors. Recall from [Kle66, Prop. 3 on page 305] that to prove numerical triviality of D, it suffices to show that the following two equalities

$$D \cdot H_1 \cdot H_2 \cdots H_{n-1} = 0 \quad \text{and} \tag{4.1.1}$$

$$D \cdot D \cdot H_2 \cdots H_{n-1} = 0 \tag{4.1.2}$$

hold. Equation (4.1.1) holds by assumption. For Equation (4.1.2), observe that

$$D \cdot D \cdot H_2 \cdots H_{n-1} = D \cdot H_{1,1} \cdot H_2 \cdots H_{n-1} - D \cdot H_{1,2} \cdot H_2 \cdots H_{n-1},$$

where both summands are zero, again by assumption. Q.E.D.

Proposition 4.2, the main result of this section, is a variant of this criterion, adapted to the discussion of $\mathbb{Q}$-factorialisations, where the ample divisors H_i are replaced by big and nef divisors which are obtained as the pull-back of ample divisors via the $\mathbb{Q}$-factorialisation map.

Proposition 4.2 (Criterion for numerical triviality on $\mathbb{Q}$-factorialisations)**.** *Let $\phi : Z' \to Z$ be a small birational morphism of irreducible, normal projective varieties of dimension $n \geq 2$.*

Let D' be a pseudoeffective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Z' and assume that there are ample Cartier divisors $H_1, \dots, H_{n-1}$ on Z such that

$$D' \cdot \phi^*(H_1) \cdots \phi^*(H_{n-1}) = 0. \tag{4.2.1}$$

If Z' is $\mathbb{Q}$-factorial, then D' is numerically trivial.

Remark 4.3 (Small morphisms). In Proposition 4.2 and elsewhere in this paper, we call a birational morphism ψ of normal, irreducible projective varieties *small* if its exceptional set has codimension at least two.

Proof of Proposition 4.2. Let $B' \subsetneq Z'$ be the ϕ-exceptional set, and $B := \phi(B') \subsetneq Z$ its image. It suffices to prove Proposition 4.2 for a multiple of D'. We will therefore assume without loss of generality that D' is an integral Cartier divisor. To show that D' is numerically trivial, we aim to apply Kleiman's criterion for numerical triviality, Lemma 4.1. To this end, let $A_1, \dots, A_{n-1}$ be arbitrary ample Cartier divisors on Z'. Choose numbers $a_1, \dots, a_{n-1} \in \mathbb{N}^+$ such that the $a_i A_i$ are very ample, choose general elements $\Theta_i \in |a_i A_i|$ and consider the complete intersection curve $\Gamma' := \Theta_1 \cap \cdots \cap \Theta_{n-1} \subsetneq Z'$. By general choice, the curve Γ' is smooth and will not intersect with the small set B'. Lemma 4.1 asserts that in order to establish numerical triviality of D' it suffices to show that

$$\Gamma' \cdot D' = a_1 \cdots a_{n-1} \cdot A_1 \cdots A_{n-1} \cdot D' = 0. \tag{4.3.1}$$

To establish (4.3.1), consider the image $\Gamma := \phi(\Gamma')$, which is a smooth curve contained in $Z \setminus B$. The curve Γ is not necessarily a complete intersection curve for $H_1, \dots, H_{n-1}$, but can be completed to become a complete intersection curve, as follows. Choosing sufficiently large numbers $m_1, \dots, m_{n-1} \in \mathbb{N}^+$, we can assume that the linear sub-systems

$$V_i := \{\Delta \in |m_i H_i| \,:\, \Gamma \subset \operatorname{supp} \Delta\}$$

are positive-dimensional, basepoint-free outside of Γ, and separate points outside of Γ. We can also assume that the sheaves $\mathscr{I}_\Gamma \otimes \mathscr{O}_Z(m_i H_i)$ are spanned. Choose general elements $\Delta_i \in V_i$ and consider the complete intersection curve

$$\Gamma_{\text{complete}} := \Delta_1 \cap \cdots \cap \Delta_{n-1} \subset Z.$$

The curve Γ_{complete} is reduced, avoids B and clearly contains Γ. We can thus write

$$\Gamma_{\text{complete}} = \Gamma \cup \Gamma_{\text{rest}} \quad \text{and} \quad \phi^{-1}(\Gamma_{\text{complete}}) = \Gamma' \cup \phi^{-1}(\Gamma_{\text{rest}}),$$

where Γ_{rest} is an irreducible movable curve on Z. To end the argument, observe that

$$\begin{aligned} 0 &= m_1 \cdots m_{n-1} \cdot D' \cdot \phi^*(H_1) \cdots \phi^*(H_{n-1}) && \text{by Assumption (4.2.1)} \\ &= D' \cdot \phi^*(\Gamma_{\text{complete}}) = \underbrace{D' \cdot \Gamma'}_{\geq 0} + \underbrace{D' \cdot \phi^*(\Gamma_{\text{rest}})}_{\geq 0}. \end{aligned}$$

Since D' is pseudoeffective, and since both Γ' and $\phi^{-1}(\Gamma_{\text{rest}})$ are movable, it follows that both summands are non-negative, hence zero. This shows Equation (4.3.1) and finishes the proof of Proposition 4.2. Q.E.D.

§5. The tangent sheaf of varieties with trivial canonical class

5.A. Semistability of the tangent sheaf

To prepare for the proof of the Decomposition Theorem 1.3, we study stability notions of the tangent sheaf. For canonical varieties with numerically trivial canonical divisor, we show that the tangent sheaf $\mathscr{T}_X$ is semistable with respect to any polarisation, and that $\mathscr{T}_X$ is stable with respect to one polarisation if and only if it is stable with respect to any other.

The following result of Miyaoka is crucial. We will refer to Theorem 5.1 at several places throughout the present paper.

Theorem 5.1 (Generic semipositivity, [Miy87a, Miy87b])**.** *Let X be a normal projective variety of dimension $n > 1$. Assume that X is* not *uniruled. Let $H_1, \ldots, H_{n-1}$ be ample line bundles on X. Then there exists a number $M \in \mathbb{N}^+$ such that for all $m_1, \ldots, m_{n-1} > M$ the following holds.*

(5.1.1) *The linear systems $|m_j H_j|$ are basepoint-free.*

(5.1.2) *If $C = D_1 \cap \cdots \cap D_{n-1} \subsetneq X$ is a curve cut out by general elements $D_j \in |m_j H_j|$, then C is smooth, X is smooth along C, and $\Omega_X^{[1]}|_C$ is a nef vector bundle on C.* Q.E.D.

Notation 5.2. The conclusion of Theorem 5.1 is often rephrased by saying that $\Omega_X^{[1]}$ is *generically nef with respect to $H_1, \ldots, H_{n-1}$*.

Theorem 5.3 (Mehta-Ramanathan theorem, cf. [MR82, Rem. 6.2] and [Fle84])**.** *In the setup of Theorem 5.1, if one chooses the number M large enough, then $\mathscr{T}_X$ is semistable with respect to the polarisation $(H_1, \ldots, H_{n-1})$ if and only if its restriction $\mathscr{T}_X|_C$ is semistable as a vector bundle on the curve C.* Q.E.D.

The well-known semistability of the tangent bundle is an immediate consequence of Miyaoka's generic semipositivity result.

Proposition 5.4 (Semistability of the tangent sheaf)**.** *Let X be a normal projective variety having at worst canonical singularities. If K_X is numerically trivial, then $\mathscr{T}_X$ is semistable with respect to any polarisation.*

Proof. Let $(H_1, \dots, H_{n-1})$ be arbitrary ample Cartier divisors on X. Choose numbers $M, m_1, \dots, m_{n-1}$ and construct a general complete intersection curve $C \subsetneq X$ as in Theorems 5.1 and 5.3.

It follows from numerical triviality of K_X and from the assumption on the singularities that X is not uniruled. Theorem 5.1 therefore asserts that the restriction $\Omega_X^{[1]}|_C$ is nef. Since additionally $K_X \cdot C = 0$ by assumption, it follows that the bundle $\mathscr{T}_X|_C$ is semistable. The Mehta-Ramanathan Theorem 5.3 then shows that $\mathscr{T}_X$ is semistable with respect to $(H_1, \dots, H_{n-1})$. Q.E.D.

Remark 5.5. Although semistable, the tangent sheaf of a variety with trivial canonical bundle might not be stable. To give an easy example, the tangent sheaf of the product of two such varieties is not stable.

5.B. Pseudoeffectivity of quotients of $\Omega_X^{[p]}$

The proof of our main result uses the following criterion for the pseudoeffectivity of Weil divisors on $\mathbb{Q}$-factorial spaces. In case where X is smooth, this has been shown by Campana-Peternell.

Proposition 5.6 (Pseudoeffectivity of quotients of $\Omega_X^{[p]}$, cf. [CP11, Thm. 0.1])**.** *Let X be a normal, $\mathbb{Q}$-factorial projective variety and let D be a Weil divisor on X. Assume that there exists a number $0 \le p \le \dim X$ and a non-trivial sheaf morphism $\psi : \Omega_X^{[p]} \to \mathscr{O}_X(D)$. If X is not uniruled, then D is pseudoeffective.*

Proof. Assume that X is not uniruled, and that a non-trivial sheaf morphism $\psi : \Omega_X^{[p]} \to \mathscr{O}_X(D)$ is given. The existence of a resolution of singularities combined with a classical result of Rossi [Ros68, Thm. 3.5] shows that there exists a strong log resolution of singularities $\pi : \widetilde{X} \to X$ with the additional property that $\pi^{[*]}\mathscr{O}_X(D)$ is locally free. The π-exceptional set $E \subsetneq \widetilde{X}$ is then of pure codimension one, and has simple normal crossing support. Finally, write $\pi^{[*]}\mathscr{O}_X(D) = \mathscr{O}_{\widetilde{X}}(\widetilde{D})$, where $\widetilde{D}$ is a divisor on $\widetilde{X}$ that agrees with the strict transform $\pi_*^{-1}(D)$ outside of the π-exceptional set E.

Since the two sheaves $\pi^{[*]}\Omega_X^{[p]}$ and $\Omega^p_{\widetilde{X}}$ are isomorphic outside of E, there exists a number $m \in \mathbb{N}^+$ and a sheaf morphism

$$\widetilde{\psi} : \Omega^p_{\widetilde{X}} \to \mathscr{O}_{\widetilde{X}}(\widetilde{D} + mE)$$

which agrees on $\widetilde{X} \setminus E$ with the pull-back of ψ. If $\mathscr{L} := (\operatorname{Image} \widetilde{\psi})^{**}$ denotes the reflexive hull of the image sheaf, then $\mathscr{L}$ is invertible by [OSS80, Lem. 1.1.15]. Recalling that $\Omega^p_{\widetilde{X}}$ is a quotient of $(\Omega^1_{\widetilde{X}})^{\otimes p}$, it follows from [CP11, Thm. 0.1] that the line bundle $\mathscr{L}$ is in fact pseudoeffective. It follows that $\widetilde{D} + mE$ is pseudoeffective as well, since it contains the pseudoeffective subsheaf $\mathscr{L}$.

Since X is assumed to be $\mathbb{Q}$-factorial, it is clear that the cycle-theoretic push-forward of any pseudoeffective divisor on $\widetilde{X}$ is $\mathbb{Q}$-Cartier and again pseudoeffective. Using that $\pi_* \widetilde{D} = D$, this shows our claim.
Q.E.D.

5.C. Stability of the tangent sheaf

While Theorem 5.1 and Proposition 5.4 are fairly standard today, the following result, which shows that stability of the tangent bundle is independent of the chosen polarisation, is new to the best of our knowledge. We feel that it might be of independent interest.

Proposition 5.7 (Stability of $\mathscr{T}_X$ does not depend on polarisation)**.** *Let X be a normal projective variety having at worst canonical singularities. Assume that K_X is numerically equivalent to zero. Let $h_1 := (H_1^{(1)}, \ldots, H_{n-1}^{(1)})$ and $h_2 := (H_1^{(2)}, \ldots, H_{n-1}^{(2)})$ be two sets of ample polarisations. If $\mathscr{T}_X$ is h_1-stable, then it is also h_2-stable.*

Proof. Suppose that $\mathscr{T}_X$ is *not* h_2-stable. Even though not stable, recall from Proposition 5.4 that $\mathscr{T}_X$ is semistable with respect to h_2. Choose a Jordan-Hölder filtration of $\mathscr{T}_X$ with respect to h_2 and let $0 \subsetneq \mathscr{S} \subsetneq \mathscr{T}_X$ be its first term, cf. [HL97, Sect. 1.5]. The sheaf $\mathscr{S}$ is then h_2-stable, and saturated as a subsheaf of $\mathscr{T}_X$. Semistability of $\mathscr{T}_X$ implies that $\mathscr{S}$ has h_2-slope zero. In other words, $\mu_{h_2}(\mathscr{S}) = 0$.

Next, let $\phi : X' \to X$ be a $\mathbb{Q}$-factorialisation, that is, a small birational morphism where X' is $\mathbb{Q}$-factorial and has at worst canonical singularities. The existence of ϕ is established in [BCHM10, Lem. 10.2]. Consider the reflexive pull-back sheaf $\mathscr{S}' := \phi^{[*]}\mathscr{S}$. Since $\mathscr{S}'$ injects into $\mathscr{T}_{X'}$ outside of the small π-exceptional set, and since both sheaves are reflexive, we obtain an injection $\mathscr{S}' \hookrightarrow \mathscr{T}_{X'} = \phi^{[*]}\mathscr{T}_X$.

We claim that $\det \mathscr{S}'$ is numerically trivial on X'. To this end, recall from Proposition 5.6 that $\big(\det \mathscr{S}'\big)^*$ is pseudoeffective. On the other hand, since $\mathscr{S}'$ and $\phi^*(\mathscr{S})$ agree outside of the π-exceptional set, we obtain

$$\big(\det \mathscr{S}'\big)^* \cdot \phi^*\big(H_1^{(2)}\big) \cdots \phi^*\big(H_{n-1}^{(2)}\big) = -\mu_{h_2}(\mathscr{S}) = 0.$$

Together with Proposition 4.2, these two observations show that $(\det \mathscr{S}')^*$ and $\det \mathscr{S}'$ are numerically trivial, as claimed.

Using numerical triviality of $\det \mathscr{S}'$, the same line of reasoning now gives

$$\mu_{h_1}(\mathscr{S}) = \big(\det \mathscr{S}'\big) \cdot \phi^*\big(H_1^{(1)}\big) \cdots \phi^*\big(H_{n-1}^{(1)}\big) = 0,$$

showing that $\mathscr{T}_X$ is *not* h_1-stable. This contradiction concludes the proof. Q.E.D.

§6. Differential forms on varieties with trivial canonical class

6.A. Non-degeneracy of the wedge product

Differential forms on smooth varieties with trivial canonical class were studied by Bogomolov [Bog74]. In this section we apply the Extension Theorem 2.4 to study reflexive differential forms on singular varieties with trivial canonical classes, following an approach discussed in [Pet94]. The results obtained in this section will play an important role in the proof of the Decomposition Theorem 1.3, which is given in the subsequent Section 7.

Proposition 6.1 (Non-degeneracy of the wedge product)**.** *Let X be a normal n-dimensional projective variety X having at worst canonical singularities. Suppose that $\omega_X \cong \mathscr{O}_X$. If $0 \le p \le n$ is any number, then the natural pairing given by the wedge product,*

$$\bigwedge : H^0\big(X,\, \Omega_X^{[p]}\big) \times H^0\big(X,\, \Omega_X^{[n-p]}\big) \longrightarrow H^0\big(X,\, \omega_X\big) \cong \mathbb{C},$$

is non-degenerate.

Remark 6.2. If $\imath : X_{\mathrm{reg}} \hookrightarrow X$ is the inclusion of the smooth locus into X, recall from Definition 2.1 that $\Omega_X^{[p]} = \imath_* \Omega^p_{X_{\mathrm{reg}}}$. Given a non-zero form $\eta \in H^0\big(X_{\mathrm{reg}},\, \Omega^p_{X_{\mathrm{reg}}}\big)$, Proposition 6.1 simply says that there exists a "complementary" form $\phi \in H^0\big(X_{\mathrm{reg}},\, \Omega^{n-p}_{X_{\mathrm{reg}}}\big)$, such that $\eta \wedge \phi$ extends to a non-zero, hence nowhere vanishing section of $\mathscr{O}_X \cong \omega_X \cong \Omega_X^{[n]}$.

Remark 6.3. Proposition 6.1 has been shown in relevant cases in [Pet94, Prop. 5.8]. Our proof of Proposition 6.1 follows [Pet94] closely.

6.A.1. *Proof of Proposition 6.1* We end the present Section 6.A with a proof of Proposition 6.1. To improve readability, the proof is subdivided into six, mostly independent steps.

Step 1 in the proof of Proposition 6.1: Setup of notation We choose a desingularisation $\pi : \widetilde{X} \to X$ of X. Denote the π-exceptional set by $E \subset \widetilde{X}$ and fix a non-zero section $\sigma \in H^0\big(X, \omega_X\big)$. Since ω_X is invertible, and since X has canonical singularities, it follows immediately from the definition that the pull-back of σ is a holomorphic n-form on $\widetilde{X}$, possibly with zeroes along the exceptional set, say

$$\tau := \pi^*(\sigma) \in H^0\big(\widetilde{X},\, \omega_{\widetilde{X}}\big).$$

Because $H^0\big(\widetilde{X},\, \omega_{\widetilde{X}}\big) \cong H^0\big(X, \omega_X\big) = \mathbb{C} \cdot \sigma$, the form τ clearly spans the vector space $H^0\big(\widetilde{X},\, \omega_{\widetilde{X}}\big)$. By the Extension Theorem 2.4 we have $\pi_* \Omega^p_{\widetilde{X}} = \Omega^{[p]}_X$. To prove Proposition 6.1, it is therefore sufficient to prove the following claim.

Claim 6.4. Given any holomorphic p-form $\alpha \in H^0\big(\widetilde{X},\, \Omega^p_{\widetilde{X}}\big)$ there exists a “complementary” form $\beta \in H^0\big(\widetilde{X},\, \Omega^{n-p}_{\widetilde{X}}\big)$ such that $\alpha \wedge \beta = \tau$.

Step 2 in the proof of Proposition 6.1: Dolbeault cohomology on $\widetilde{X}$ Following standard notation, let $\mathcal{A}^{a,b}$ denote the sheaf of $\mathbb{C}$-valued differentiable forms of type (a,b) on $\widetilde{X}$. Taking products and wedge products with σ and τ, respectively, we obtain sheaf morphisms,

$$\begin{array}{rccc} \psi_X : & \mathscr{O}_X & \to & \omega_X \\ & f & \mapsto & f \cdot \sigma \end{array} \qquad \begin{array}{rccc} \psi_{\widetilde{X}} : & \mathscr{O}_{\widetilde{X}} & \to & \omega_{\widetilde{X}} \\ & f & \mapsto & f \cdot \tau \end{array}$$

$$\begin{array}{rccc} \psi_0 : & \mathcal{A}^{0,0} & \to & \mathcal{A}^{n,0} \\ & f & \mapsto & f \cdot \tau \end{array} \qquad \begin{array}{rccc} \psi_q : & \mathcal{A}^{0,q} & \to & \mathcal{A}^{n,q} \\ & \alpha & \mapsto & \alpha \wedge \tau. \end{array}$$

where $0 < q \leq n$. Observe that ψ_X is isomorphic by assumption.

Since τ is holomorphic, its exterior derivative vanishes $\overline{\partial}\tau = 0$. This immediately implies relations

$$\overline{\partial} \circ \psi_q = \psi_{q+1} \circ \overline{\partial} \quad \text{for all } 0 \leq q \leq n. \tag{6.5.1}$$

In particular, the sheaf morphisms ψ_q induce well-defined morphisms between Dolbeault cohomology groups,

$$\phi_q : H^{0,q}\big(\widetilde{X}\big) \to H^{n,q}\big(\widetilde{X}\big) \quad \text{for all } 0 \leq q \leq n.$$

We will later see in Step 4 of this proof that the morphisms ϕ_q are in fact isomorphic.

Step 3 in the proof of Proposition 6.1: Dolbeault and sheaf cohomology on $\widetilde{X}$ The sheaf morphisms ψ_X and $\psi_{\widetilde{X}}$ induce additional morphisms between sheaf cohomology groups,

$$H^q\big(\psi_X\big) : H^q\big(X, \mathscr{O}_X\big) \to H^q\big(X, \omega_X\big) \qquad \text{and}$$
$$H^q\big(\psi_{\widetilde{X}}\big) : H^q\big(\widetilde{X}, \mathscr{O}_{\widetilde{X}}\big) \to H^q\big(\widetilde{X}, \omega_{\widetilde{X}}\big),$$

for all $0 \leq q \leq n$. Again, observe that the morphisms $H^q\big(\psi_X\big)$ are isomorphic by assumption. We will see in Step 4 of this proof that the morphisms $H^q\big(\psi_{\widetilde{X}}\big)$ are isomorphic as well.

The morphisms ϕ_q and $H^q\big(\psi_{\widetilde{X}}\big)$ are closely related. In fact, it follows from (6.5.1) that the sheaf morphisms $\psi_\bullet$ align to give a morphism between the Dolbeault resolutions of $\mathscr{O}_{\widetilde{X}}$ and $\omega_{\widetilde{X}}$, respectively. In other words, there exists a commutative diagram as follows,

$$\begin{tikzcd}
\mathscr{O}_{\widetilde{X}} \arrow[r, hook] \arrow[d, "\psi_{\widetilde{X}}"'] & \mathcal{A}^{0,0} \arrow[r, "\overline{\partial}"] \arrow[d, "\psi_0"'] & \mathcal{A}^{0,1} \arrow[r, "\overline{\partial}"] \arrow[d, "\psi_1"'] & \mathcal{A}^{0,2} \arrow[r, "\overline{\partial}"] \arrow[d, "\psi_2"'] & \mathcal{A}^{0,3} \arrow[r, "\overline{\partial}"] \arrow[d, "\psi_3"'] & \cdots \\
\omega_{\widetilde{X}} \arrow[r, hook] & \mathcal{A}^{n,0} \arrow[r, "\overline{\partial}"'] & \mathcal{A}^{n,1} \arrow[r, "\overline{\partial}"'] & \mathcal{A}^{n,2} \arrow[r, "\overline{\partial}"'] & \mathcal{A}^{n,3} \arrow[r, "\overline{\partial}"'] & \cdots
\end{tikzcd} \tag{6.5.2}$$

The following is then a standard consequence of homological algebra, see for instance [Dem09, Ch. IV, §6, eq. (6.3)].

Conclusion 6.6. There exist commutative diagrams

$$\begin{tikzcd}
H^{0,q}\big(\widetilde{X}\big) \arrow[r, "\phi_q"] \arrow[d, "\text{Dolbeault isom.}"', "\cong"] & H^{n,q}\big(\widetilde{X}\big) \arrow[d, "\cong"', "\text{Dolbeault isom.}"] \\
H^q\big(\widetilde{X}, \mathscr{O}_{\widetilde{X}}\big) \arrow[r, "H^q(\psi_{\widetilde{X}})"] & H^q\big(\widetilde{X}, \omega_{\widetilde{X}}\big)
\end{tikzcd}$$

for all indices $0 \leq q \leq n$. Q.E.D.

Step 4 in the proof of Proposition 6.1: cohomology on $\widetilde{X}$ and on X Next, we aim to compare cohomology groups on $\widetilde{X}$ with those on X. More precisely, we claim the following.

Claim 6.7. Given any index $0 \le q \le n$, there exist morphisms $\rho_{\mathscr{O}}$, ρ_ω forming a commutative diagram as follows,

$$
\begin{tikzcd}
H^q\big(\widetilde{X},\, \mathscr{O}_{\widetilde{X}}\big) \arrow[r, "H^q(\psi_{\widetilde{X}})"] & H^q\big(\widetilde{X},\, \omega_{\widetilde{X}}\big) \\
H^q\big(X,\, \mathscr{O}_X\big) \arrow[u, "\rho_{\mathscr{O}}", "\cong"'] \arrow[r, "H^q(\psi_X)", "\cong"'] & H^q\big(X,\, \omega_X\big). \arrow[u, "\cong", "\rho_\omega"']
\end{tikzcd}
\tag{6.7.1}
$$

In particular, the morphisms $H^q(\psi_{\widetilde{X}})$ are isomorphic for all $0 \le q \le n$.

Proof. Since π has connected fibers, and since X has only canonical singularities, we have canonical identifications

$$\pi_* \mathscr{O}_{\widetilde{X}} \cong \mathscr{O}_X \qquad \text{and} \qquad \pi_* \omega_{\widetilde{X}} \cong \omega_X.$$

Observe that the section σ, seen as a section in $\pi_* \omega_{\widetilde{X}}$, will be identified with the differential form τ. Using these identifications, we need to show that there exist two morphisms

$$\rho_{\mathscr{O}} : H^q\big(X,\, \mathscr{O}_X\big) \to H^q\big(\widetilde{X},\, \mathscr{O}_{\widetilde{X}}\big) \quad \text{and} \quad \rho_\omega : H^q\big(X,\, \omega_X\big) \to H^q\big(\widetilde{X},\, \omega_{\widetilde{X}}\big),$$

which make Diagram (6.7.1) commutative and are isomorphic. While this can be concluded from universal properties and spectral sequences, we found it more instructive to give an elementary construction using Čech cohomology.

To this end, choose an open affine cover $(U_i)_{i \in I}$ of X, which will be acyclic for any coherent sheaf, and let $\rho_{\mathscr{O}}$, ρ_ω be the compositions of the vertical arrows in the following natural diagram.

$$
\begin{tikzcd}
\check H^q\big(\widetilde X,\, \mathscr O_{\widetilde X}\big) \arrow[r, "\check H^q(\psi_{\widetilde X})"] & \check H^q\big(\widetilde X,\, \omega_{\widetilde X}\big) \\
\check H^q\big((\pi^{-1}U_i)_{i\in I},\, \mathscr O_{\widetilde X}\big) \arrow[r, "\check H^q((\pi^{-1}U_i)_{i\in I},\, \psi_{\widetilde X})"] \arrow[u, "r_{\widetilde X,\mathscr O}"] \arrow[d, leftrightarrow, "\cong"'] & \check H^q\big((\pi^{-1}U_i)_{i\in I},\, \omega_{\widetilde X}\big) \arrow[u, "r_{\widetilde X,\omega}"'] \arrow[d, leftrightarrow, "\cong"] \\
\check H^q\big((U_i)_{i\in I},\, \pi_*\mathscr O_{\widetilde X}\big) \arrow[r, "\check H^q((U_i)_{i\in I},\, \pi_*\psi_{\widetilde X})"] \arrow[d, leftrightarrow, "\cong"'] & \check H^q\big((U_i)_{i\in I},\, \pi_*\omega_{\widetilde X}\big) \arrow[d, leftrightarrow, "\cong"] \\
\check H^q\big((U_i)_{i\in I},\, \mathscr O_X\big) \arrow[r, "\check H^q((U_i)_{i\in I},\, \psi_X)"] \arrow[d, "r_{X,\mathscr O}"', "\cong"] & \check H^q\big((U_i)_{i\in I},\, \omega_X\big) \arrow[d, "\cong"', "r_{X,\omega}"] \\
\check H^q\big(X,\, \mathscr O_X\big) \arrow[r, "\check H^q(\psi_X)", "\cong"'] \arrow[uuuu, bend left=40, "\check\rho_{\mathscr O}"] & \check H^q\big(X,\, \omega_X\big) \arrow[uuuu, bend right=40, "\check\rho_\omega"']
\end{tikzcd}
$$

Here, the morphisms $r_{\bullet,\bullet}$ are the standard refinement morphisms that map Čech cohomology groups defined with respect to a specific open covering into Čech cohomology. Identifying Čech and sheaf cohomology, commutativity of Diagram (6.7.1) is then immediate.

To prove that $\rho_{\mathscr O}$ and ρ_ω are isomorphic, it suffices to show that the refinement morphisms, $r_{\widetilde X,\mathscr O}$ and $r_{\widetilde X,\omega}$ are isomorphic. We do that by showing that the covering $(\pi^{-1}U_i)_{i\in I}$ is acyclic for both $\mathscr O_{\widetilde X}$ and $\omega_{\widetilde X}$. That, however, follows immediately from the following two well-known vanishing results which hold for all indices $q > 0$,

$$
\begin{aligned}
R^q\pi_*\mathscr O_{\widetilde X} &= 0 \quad \text{because } X \text{ has rational singularities, [KM98, Thm. 5.22]}\\
R^q\pi_*\omega_{\widetilde X} &= 0 \quad \text{Grauert-Riemenschneider vanishing, [KM98, Cor. 2.68].}
\end{aligned}
$$

The finishes the proof of Claim 6.7. Q.E.D.

Combining Conclusion 6.6 and Claim 6.7, we arrive at the following statement, which summarises the results obtained so far.

Conclusion 6.8. The morphisms $\phi_q : H^{0,q}\big(\widetilde X\big) \to H^{n,q}\big(\widetilde X\big)$ are isomorphic for all $0 \le q \le n$. Q.E.D.

Step 5 in the proof of Proposition 6.1: Serre duality Given any index $0 \le p \le n$, consider the complex bilinear form ρ obtained as the composition of the following morphisms,

$$
\begin{aligned}
H^0\big(\widetilde{X},\, \Omega^p_{\widetilde{X}}\big) \times H^0\big(\widetilde{X},\, \Omega^{n-p}_{\widetilde{X}}\big) &\xrightarrow{\text{Dolbeault isom.}} H^{p,0}(\widetilde{X}) \times H^{n-p,0}(\widetilde{X}) \\
&\xrightarrow{\text{Id} \times \text{conjugation}} H^{p,0}(\widetilde{X}) \times H^{0,n-p}(\widetilde{X}) \\
&\xrightarrow{\text{Id} \times \phi_{n-p}} H^{p,0}(\widetilde{X}) \times H^{n,n-p}(\widetilde{X}) \\
&\xrightarrow{\text{Id} \times \text{conjugation}} H^{p,0}(\widetilde{X}) \times H^{n-p,n}(\widetilde{X}) \\
&\xrightarrow{\quad s \quad} \mathbb{C},
\end{aligned}
$$

where s is the perfect pairing given by Serre duality, cf. [Dem09, Ch. VI, Thm. 7.3].

Recall from Conclusion 6.8 that with the exception of s all maps used in the definition of ρ are isomorphisms. It follows that ρ is a perfect pairing. Unwinding the definition, ρ is given in elementary terms as follows,

$$
\rho : H^0\big(\widetilde{X},\, \Omega^p_{\widetilde{X}}\big) \times H^0\big(\widetilde{X},\, \Omega^{n-p}_{\widetilde{X}}\big) \to \mathbb{C}, \qquad \big(\alpha, \beta\big) \mapsto \int_X \alpha \wedge \beta \wedge \overline{\tau}.
$$

Step 6 in the proof of Proposition 6.1: End of proof We are now ready to prove Claim 6.4. Assume we are given a non-zero form $\alpha \in H^0\big(\widetilde{X},\, \Omega^p_{\widetilde{X}}\big)$. Using that ρ is a perfect pairing, we can therefore find a form $\beta \in H^0\big(\widetilde{X},\, \Omega^{n-p}_{\widetilde{X}}\big)$ such that

$$
\rho(\alpha, \beta) = \int_X \alpha \wedge \beta \wedge \overline{\tau} = 1 \tag{6.8.1}
$$

Equation (6.8.1) implies that $\alpha \wedge \beta$ is a non-vanishing element of $H^0\big(\widetilde{X},\, \omega_{\widetilde{X}}\big)$. Since $H^0\big(\widetilde{X},\, \omega_{\widetilde{X}}\big)$ is one-dimensional, there exists a scalar $\lambda \in \mathbb{C}^*$ such that

$$
\tau = \lambda \cdot (\alpha \wedge \beta) = \alpha \wedge (\lambda \cdot \beta).
$$

This finishes the proof of Claim 6.4 and hence of Proposition 6.1. Q.E.D.

6.B. Hodge duality for klt spaces

If X is a projective manifold, Hodge theory gives a complex-linear isomorphism between the spaces $H^0\big(X,\, \Omega^p_X\big)$ and $\overline{H^p\big(X,\, \mathscr{O}_X\big)}$. We show that the same statement holds for reflexive differentials if X has canonical singularities, or more generally if X is the base space of a klt pair.

Proposition 6.9 (Hodge duality for klt spaces). *Let X be a normal n-dimensional projective variety X. Suppose that there exists an effective $\mathbb{Q}$-divisor D on X such that (X, D) is klt. Given any number $0 \le p \le n$, there are complex-linear isomorphisms*

$$H^0\big(X,\, \Omega_X^{[p]}\big) \cong H^0\left(X_{\mathrm{reg}},\, \Omega^p_{X_{\mathrm{reg}}}\right) \cong \overline{H^p\big(X,\, \mathscr{O}_X\big)}.$$

Proof. Fix a resolution of singularities $\pi : \widetilde{X} \to X$ of X. Then we have the following chain of complex-linear isomorphisms

$$\begin{aligned} H^0\big(X,\, \Omega_X^{[p]}\big) &\cong H^0\big(\widetilde{X},\, \Omega^p_{\widetilde{X}}\big) && \text{see below} \\ &\cong H^{p,0}\big(\widetilde{X}\big) && \text{Dolbeault isomorphism} \\ &\cong \overline{H^{0,p}\big(\widetilde{X}\big)} && \text{Conjugation} \\ &\cong \overline{H^p\big(\widetilde{X},\, \mathscr{O}_{\widetilde{X}}\big)} && \text{Dolbeault isomorphism} \\ &\cong \overline{H^p\big(X,\, \mathscr{O}_X\big)} && \text{see below.} \end{aligned}$$

The first isomorphism exists because $\pi_* \Omega^p_{\widetilde{X}} = \Omega_X^{[p]}$ by the Extension Theorem 2.4. The last isomorphism exists because X has rational singularities, cf. [KM98, Thm. 5.22]. Q.E.D.

We list a few immediate consequences of the results obtained so far.

Corollary 6.10. *Let X be a normal n-dimensional projective variety X having at worst canonical singularities. Suppose that the canonical sheaf of X is trivial, $\omega_X \cong \mathscr{O}_X$. Then the following holds.*

(6.10.1) *Non-zero forms $\eta \in H^0\big(X_{\mathrm{reg}},\, \Omega^q_{X_{\mathrm{reg}}}\big)$ do not have any zeroes.*

(6.10.2) *For all $0 \le p \le n$, we have complex-linear isomorphisms $H^0\big(X,\, \Omega_X^{[p]}\big) \cong H^0\big(X,\, \Omega_X^{[n-p]}\big)^*$, canonically given up to multiplication with a constant.*

(6.10.3) *If the dimension of X is odd, then $\chi\big(X,\, \mathscr{O}_X\big) = 0$.* *Q.E.D.*

Corollary 6.11 (Existence of forms on canonical varieties with $K_X \equiv 0$). *Let X be a normal n-dimensional projective variety X having at worst canonical singularities. Assume that $\widetilde{q}(X) = 0$ and that the canonical divisor K_X is numerically trivial. Then*

$$h^0\big(X,\, \Omega_X^{[1]}\big) = h^0\big(X,\, \Omega_X^{[n-1]}\big) = 0.$$

Proof. We show that $H^0\big(X,\, \Omega_X^{[n-1]}\big) = 0$. Assume to the contrary, and let σ be a non-zero reflexive $(n-1)$-form on X. Recalling from

Kawamata's analysis of the Albanese map, Proposition 3.3, that K_X is torsion, let $f : \widetilde{X} \to X$ be the associated index-one cover. The morphism f is finite and étale in codimension one, the space $\widetilde{X}$ has canonical singularities, and trivial canonical sheaf $\omega_{\widetilde{X}} \cong \mathscr{O}_{\widetilde{X}}$, cf. [KM98, 5.19 and 5.20]. In particular, the reflexive form σ pulls back to a give non-vanishing reflexive $(n-1)$-form $\widetilde{\sigma}$ on $\widetilde{X}$. Furthermore, observe that the covering space $\widetilde{X}$ satisfies all requirements made in Proposition 6.9 and Corollary 6.10. This shows

$$\begin{aligned} 0 = \widetilde{q}(X) \geq q(\widetilde{X}) &= h^1\big(\widetilde{X}, \mathscr{O}_{\widetilde{X}}\big) \\ &= h^0\big(\widetilde{X}, \Omega^{[1]}_{\widetilde{X}}\big) && \text{by Proposition 6.9} \\ &= h^0\big(\widetilde{X}, \Omega^{[n-1]}_{\widetilde{X}}\big) && \text{by Corollary (6.10.2),} \end{aligned}$$

contradicting the existence of $\widetilde{\sigma}$. The same argument also shows $H^0\big(X, \Omega^{[1]}_X\big) = 0$, finishing the proof of Corollary 6.11. Q.E.D.

6.C. Existence of complementary sheaves

We conclude the present Section 6 with a final corollary which generalises [Pet94, Lem. 5.11]; see also [Bog74, p. 581]. It shows that saturated subsheaves of $\mathscr{T}_X$ with trivial determinant often have a complementary subsheaf which presents $\mathscr{T}_X$ as a direct product. Corollary 6.12 is thus an important ingredient in the proof of our main result, the Decomposition Theorem 1.3.

Corollary 6.12 (Existence of complementary subsheaves in $\mathscr{T}_X$)**.** *Let X be a normal projective variety with trivial canonical sheaf $\omega_X \cong \mathscr{O}_X$, having at worst canonical singularities. Let $\mathscr{E} \subsetneq \mathscr{T}_X$ be a saturated subsheaf with trivial determinant,* $\det \mathscr{E} \cong \mathscr{O}_X$. *Then there exists a subsheaf $\mathscr{F} \subsetneq \mathscr{T}_X$ with trivial determinant such that*

$$\mathscr{T}_X \cong \mathscr{E} \oplus \mathscr{F}.$$

We will prove Corollary 6.12 in the remainder of the present Section 6.C. For convenience, the proof is subdivided into four steps.

Step 1 in the proof of Corollary 6.12: Setup We consider the obvious quotient sequence

$$0 \longrightarrow \mathscr{E} \xrightarrow{\ \alpha\ } \mathscr{T}_X \xrightarrow{\ \beta\ } \underbrace{\mathscr{T}_X/\mathscr{E}}_{=:\mathscr{Q}} \longrightarrow 0. \tag{6.12.1}$$

Since $\mathscr{E}$ is saturated in the reflexive sheaf $\mathscr{T}_X$, it is itself reflexive. Further, the associated quotient $\mathscr{Q}$ is a torsion free sheaf, say of rank $r > 0$.

We aim to split sequence (6.12.1) in codimension one. To be more precise, let $Z \subsetneq X$ be the smallest set such that $X^\circ := X \setminus Z$ is smooth and $\mathscr{Q}|_{X^\circ}$ is locally free. Since X is normal, and since torsion-free sheaves on manifolds are locally free in codimension one, [OSS80, p. 148], it follows that Z is small, that is, $\operatorname{codim}_X Z \geq 2$. If we can find a splitting of Sequence (6.12.1) on X° and write $\mathscr{T}_{X^\circ} \cong \mathscr{E}|_{X^\circ} \oplus \mathscr{Q}|_{X^\circ}$, it will follow from reflexivity that $\mathscr{T}_X \cong \mathscr{E} \oplus \mathscr{Q}^{**}$, and the proof of Corollary 6.12 will be finished.

Step 2 in the proof of Corollary 6.12: Construction of the splitting In order to construct the splitting, recall the assumptions that $\det \mathscr{E} \cong \mathscr{O}_X$ and $\omega_X \cong \mathscr{O}_X$. As a consequence, we have triviality of determinants, $\det \mathscr{Q} \cong \det \mathscr{Q}^* \cong \mathscr{O}_X$, see [Kob87, Ch. V, Prop. 6.9] for details. Let $\eta_{\mathscr{Q}} \in H^0\big(X, \det \mathscr{Q}^*\big)$ be any non-vanishing section.

Taking duals on X°, Sequence (6.12.1) gives injections

$$\begin{array}{rccc} \beta^* : & \mathscr{Q}^*|_{X^\circ} & \to & \Omega^1_{X^\circ} \\ \wedge^r \beta^* : & \wedge^r \mathscr{Q}^*|_{X^\circ} & \to & \Omega^r_{X^\circ} \\ \det \beta^* : & \det \mathscr{Q}^* & \to & \Omega^{[r]}_X. \end{array}$$

We obtain a non-trivial reflexive form

$$\eta := \big(\det \beta^*\big)(\eta_{\mathscr{Q}}) \in H^0\big(X, \Omega^{[r]}_X\big) \setminus \{0\}.$$

Denoting the dimension of X by n, Proposition 6.1 asserts the existence of a complementary reflexive form $\mu \in H^0\big(X, \Omega^{[n-r]}_X\big)$ such that $\eta \wedge \mu$ gives a nowhere-vanishing section of ω_X. The triviality of $\det \mathscr{Q}|_{X^\circ}$ and of $\omega_{X^\circ} = \det \mathscr{T}^*_{X^\circ}$ thus gives isomorphisms of sheaves,

$$\begin{array}{rccc} \delta_{\mathscr{Q}} : & \mathscr{Q}|_{X^\circ} & \to & \wedge^{r-1} \mathscr{Q}^*|_{X^\circ} \\ & q & \mapsto & \eta_{\mathscr{Q}}(q, \cdot) \\ & & & \\ \delta_{\mathscr{T}_X} : & \mathscr{T}_{X^\circ} & \to & \Omega^{n-1}_{X^\circ} \\ & \vec{v} & \mapsto & \big(\eta \wedge \mu\big)(\vec{v}, \cdot). \end{array} \tag{6.12.2}$$

Remark 6.13. If $r = 1$, then $\wedge^{r-1} \mathscr{Q}^*|_{X^\circ} = \mathscr{O}_{X^\circ}$ is simply the sheaf of functions.

Using the isomorphisms (6.12.2) and the complementary form μ, we can now define a sheaf morphism $\phi : \mathscr{Q}|_{X^\circ} \to \mathscr{T}_{X^\circ}$ as the composite of the following natural maps

$$\mathscr{Q}|_{X^\circ} \xrightarrow{\delta_{\mathscr{Q}}} \wedge^{r-1} \mathscr{Q}^*|_{X^\circ} \xrightarrow{\wedge^{r-1}\beta^*} \Omega^{r-1}_{X^\circ} \xrightarrow{\wedge \mu} \Omega^{n-1}_{X^\circ} \xrightarrow{\delta^{-1}_{\mathscr{T}_X}} \mathscr{T}_{X^\circ}. \tag{6.13.1}$$

Remark 6.14. In case where $r = 1$, the sheaves $\wedge^{r-1}\mathscr{Q}^*|_{X^\circ}$ and $\Omega^{r-1}_{X^\circ}$ both equal the trivial sheaf $\mathscr{O}_{X^\circ}$. The morphism $\wedge^{r-1}\beta^*$ is then the identity map.

To end the proof of Corollary 6.12, it will now suffice to prove the following claim.

Claim 6.15. The morphism $\phi : \mathscr{Q}|_{X^\circ} \to \mathscr{T}_{X^\circ}$ defines a splitting of Sequence (6.12.1) over the open set X°.

Step 3 in the proof of Corollary 6.12: preparation for proof of Claim 6.15 It suffices to show Claim 6.15 locally, over sufficiently small open sets $U \subseteq X^\circ$. We will prove Claim 6.15 by explicit computation, choosing frames for the bundles $\mathscr{E}$, $\mathscr{T}_X$ and $\mathscr{Q}$ to write down the morphism ϕ and all relevant differential forms. Indeed, choosing U small enough, we can find frames

$$\begin{array}{ll} e_1, \ldots, e_{n-r}, & \ldots \text{frame of } \mathscr{E}|_U, \\ \vec{q}_1, \ldots, \vec{q}_r, \alpha(e_1), \ldots, \alpha(e_{n-r}) & \ldots \text{frame of } \mathscr{T}_X|_U, \text{ and} \\ \beta(\vec{q}_1), \ldots, \beta(\vec{q}_{n-r}) & \ldots \text{frame of } \mathscr{Q}|_U. \end{array}$$

To simplify notation, set

$$\vec{e}_i := \alpha(e_i) \in \mathscr{T}_X(U) \quad \text{and} \quad q_j := \beta(\vec{q}_j) \in \mathscr{Q}(U).$$

We denote the dual frames by

$$\begin{array}{ll} e_1^*, \ldots, e_{n-r}^* & \ldots \text{frame of } \mathscr{E}^*|_U, \\ \vec{q}_1^{\,*} \ldots, \vec{q}_r^{\,*}, \vec{e}_1^{\,*}, \ldots, \vec{e}_{n-r}^{\,*} & \ldots \text{frame of } \Omega^1_X|_U, \text{ and} \\ q_1^*, \ldots, q_r^* & \ldots \text{frame of } \mathscr{Q}^*|_U. \end{array}$$

Observe that $\alpha^*(\vec{e}_i^{\,*}) = e_i^*$ and $\beta^*(q_j^*) = \vec{q}_j^{\,*}$, for all indices i and j. Scaling the frame $\vec{q}_1, \ldots, \vec{q}_r$ appropriately, we may assume that

$$\eta_{\mathscr{Q}}|_U = q_1^* \wedge \cdots \wedge q_r^* \quad \text{and} \quad \eta|_U = \vec{q}_1^{\,*} \wedge \cdots \wedge \vec{q}_r^{\,*}. \tag{6.15.1}$$

Scaling the frame $e_1, \ldots, e_{n-r}$, we can then find forms $\sigma_1, \ldots \sigma_r \in \Omega^{n-r-1}_X(U)$ such that the complementary form μ can be written as

$$\mu|_U = \vec{e}_1^{\,*} \wedge \cdots \wedge \vec{e}_{n-r}^{\,*} + \sum_{i=1}^{r} \vec{q}_i^{\,*} \wedge \sigma_i. \tag{6.15.2}$$

Remark 6.16. We do not claim that Equation (6.15.2) defines the forms σ_i uniquely. In fact, there will almost always be several ways to write $\mu|_C$ in this way. If $n - r - 1 = 0$, then the σ_i are just functions.

Remark 6.17. On the open set U, Equations (6.15.1) and (6.15.2) together imply that the globally defined form $\eta \wedge \mu$ is given as

$$\big(\eta \wedge \mu\big)|_U = \vec{q}_1^{\,*} \wedge \cdots \wedge \vec{q}_r^{\,*} \wedge \vec{e}_1^{\,*} \wedge \cdots \wedge \vec{e}_{n-r}^{\,*}.$$

If $n - r - 1 > 0$, then the forms σ_i of Equation (6.15.2) can be decomposed further, writing them as sums of pure tensors that only involve $\vec{e}_1^{\,*}, \dots, \vec{e}_{n-r}^{\,*}$, and tensors that involve $\vec{q}_1^{\,*}, \dots, \vec{q}_r^{\,*}$,

$$\sigma_i = \sum_{j=1}^{n-r} a_{ij} \cdot \vec{e}_1^{\,*} \wedge \cdots \xcancel{\vec{e}_j^{\,*}} \cdots \wedge \vec{e}_{n-r}^{\,*} + \sum_{k=1}^{r} \vec{q}_k^{\,*} \wedge \tau_{ik}, \tag{6.17.1}$$

for suitable functions $a_{ij} \in \mathscr{O}_{X^\circ}(U)$ and forms $\tau_{ik} \in \Omega_X^{n-r-2}(U)$.

Remark 6.18. Again, we do not claim that Equation (6.17.1) defines the forms τ_{ik} uniquely. In contrast, note that the functions a_{ij} are uniquely determined by (6.15.2) and (6.17.1).

Step 4 in the proof of Corollary 6.12: proof of Claim 6.15 and end of proof We will prove Claim 6.15 only in case where $n-r-1 > 0$. The case where $n - r = 1$ follows exactly the same pattern, but is easier. To be precise, we will prove that

$$\begin{array}{rccc} \phi|_U : & \mathscr{Q}|_U & \to & \mathscr{T}_X|_U \\ & q_\ell & \mapsto & \vec{q}_\ell + \sum_{j=1}^{n-r} \pm a_{\ell j} \cdot \vec{e}_j \end{array} \tag{6.18.1}$$

where the $a_{\ell j} \in \mathscr{O}_{X^\circ}(U)$ are the functions introduced in Equation (6.17.1) above. Therefore, $\beta|_U \circ \phi|_U = \mathrm{id}_{\mathscr{Q}|_U}$, establishing Claim 6.15. By definition of ϕ, (6.18.1) is equivalent to showing that (6.18.2)

$$\delta_{\mathscr{T}_X}^{-1}\Big(\big((\wedge^{r-1}\beta^*)\,(\delta_{\mathscr{Q}}(q_\ell))\big) \wedge \mu\Big) = \vec{q}_\ell + \sum_{j=1}^{n-r} \pm a_{\ell j} \cdot \vec{e}_j \quad \text{for all indices } \ell.$$

The computation proving (6.18.2) uses the following elementary observation.

Observation 6.19. It follows immediately from (6.15.1) and from Remark 6.17 that the sheaf morphisms $\delta_{\mathscr{Q}}$ and $\delta_{\mathscr{T}_X}$ introduced in (6.12.2) have the following explicit description on U,

$$\begin{array}{llll} \delta_{\mathscr{Q}}|_U : & \mathscr{Q}|_U & \to & \wedge^{r-1}\mathscr{Q}^*|_U \\ & q_\ell & \mapsto & (-1)^{\ell+1}\cdot q_1^* \wedge \cdots \cancel{q_\ell^*} \cdots \wedge q_r^* \end{array}$$

$$\begin{array}{llll} \delta_{\mathscr{T}_X}|_U : & \mathscr{T}_X|_U & \to & \Omega_X^{n-1}|_U \\ & \vec{q}_\ell & \mapsto & (-1)^{\ell+1} \cdot \vec{q}_1^{\,*} \wedge \cdots \cancel{\vec{q}_\ell^{\,*}} \cdots \wedge \vec{q}_r^{\,*} \wedge \vec{e}_1^{\,*} \wedge \cdots \wedge \vec{e}_{n-r}^{\,*} \\ & \vec{e}_\ell & \mapsto & (-1)^{r+\ell+1} \cdot \vec{q}_1^{\,*} \wedge \cdots \wedge \vec{q}_r^{\,*} \wedge \vec{e}_1^{\,*} \wedge \cdots \cancel{\vec{e}_\ell^{\,*}} \cdots \wedge \vec{e}_{n-r}^{\,*}. \end{array}$$

With Observation 6.19 in place, Equation (6.18.2) is now shown easily by direct computation as follows.

$$\begin{aligned} \mathsf{A} &:= \delta_{\mathscr{Q}}(q_\ell) \\ &\overset{\text{Obs. 6.19}}{=} (-1)^{\ell+1}\cdot q_1^* \wedge \cdots \cancel{q_\ell^*} \cdots \wedge q_r^* \end{aligned}$$

$$\begin{aligned} \mathsf{B} &:= \big(\wedge^{r-1}\beta^*\big)(\mathsf{A}) \\ &\overset{\text{Defn. of } \beta^*}{=} (-1)^{\ell+1}\cdot \vec{q}_1^{\,*} \wedge \cdots \cancel{\vec{q}_\ell^{\,*}} \cdots \wedge \vec{q}_r^{\,*} \end{aligned}$$

$$\begin{aligned} \mathsf{C} &:= \mathsf{B} \wedge \mu \\ &\overset{\text{by (6.15.2)}}{=} (-1)^{\ell+1}\cdot \vec{q}_1^{\,*} \wedge \cdots \cancel{\vec{q}_\ell^{\,*}} \cdots \wedge \vec{q}_r^{\,*} \wedge \left(\vec{e}_1^{\,*} \wedge \cdots \wedge \vec{e}_{n-r}^{\,*} + \sum_{i=1}^{r} \vec{q}_i^{\,*} \wedge \sigma_i \right) \\ &= (-1)^{\ell+1}\cdot \vec{q}_1^{\,*} \wedge \cdots \cancel{\vec{q}_\ell^{\,*}} \cdots \wedge \vec{q}_r^{\,*} \wedge \big(\vec{e}_1^{\,*} \wedge \cdots \wedge \vec{e}_{n-r}^{\,*} + \vec{q}_\ell^{\,*} \wedge \sigma_i \big) \\ &\overset{\text{by (6.17.1)}}{=} (-1)^{\ell+1}\cdot \vec{q}_1^{\,*} \wedge \cdots \cancel{\vec{q}_\ell^{\,*}} \cdots \wedge \vec{q}_r^{\,*} \wedge \Bigg(\vec{e}_1^{\,*} \wedge \cdots \wedge \vec{e}_{n-r}^{\,*} \\ &\qquad + \vec{q}_\ell^{\,*} \wedge \left(\sum_{j=1}^{n-r} a_{\ell j} \cdot \vec{e}_1^{\,*} \wedge \cdots \cancel{\vec{e}_j^{\,*}} \cdots \wedge \vec{e}_{n-r}^{\,*} + \sum_{k=1}^{r} \vec{q}_k^{\,*} \wedge \tau_{ik} \right) \Bigg) \\ &= (-1)^{\ell+1}\cdot \vec{q}_1^{\,*} \wedge \cdots \cancel{\vec{q}_\ell^{\,*}} \cdots \wedge \vec{q}_r^{\,*} \wedge \vec{e}_1^{\,*} \wedge \cdots \wedge \vec{e}_{n-r}^{\,*} \\ &\qquad + \sum_{j=1}^{n-r} \pm a_{\ell j} \cdot \vec{q}_1^{\,*} \wedge \cdots \wedge \vec{q}_r^{\,*} \wedge \vec{e}_1^{\,*} \wedge \cdots \cancel{\vec{e}_j^{\,*}} \cdots \wedge \vec{e}_{n-r}^{\,*} \end{aligned}$$

and finally

$$\mathsf{D} := \delta_{\mathscr{T}_X}^{-1}(\mathsf{C}) \overset{\text{Obs. 6.19}}{=} \vec{q}_\ell + \sum_{j=1}^{n-r} \pm a_{\ell j} \cdot \vec{e}_j.$$

This finishes the proof of Equations (6.18.2), (6.18.1), Claim 6.15, and hence of Corollary 6.12. Q.E.D.

§7. Proof of Theorem 1.3

We have divided the proof of Theorem 1.3 into a sequence of steps, each formulated as a separate result. Some of these statements might be of independent interest. The proof of Theorem 1.3 follows quickly from these preliminary steps and is given in Section 7.C.

Theorem 7.1 (Splitting the tangent sheaf of varieties with trivial canonical bundle)**.** *Let X be a normal n-dimensional projective variety with at worst canonical singularities. Assume that $\omega_X \cong \mathscr{O}_X$ and that $\widetilde{q}(X) = 0$. Let $h = (H_1, \ldots, H_{n-1})$ be ample divisors on X and assume that $\mathscr{T}_X$ is not h-stable. Let $0 \subsetneq \mathscr{E} \subsetneq \mathscr{T}_X$ be a saturated destabilising subsheaf, that is, a proper subsheaf with non-negative slope $\mu_h(\mathscr{E}) \geq 0$ and torsion free quotient $\mathscr{T}_X/\mathscr{E}$.*

Then there exists a number $M \in \mathbb{N}^+$ such that $(\det \mathscr{E})^{[M]} \cong \mathscr{O}_X$. Further, there exists a finite cover $f : \widetilde{X} \to X$, étale in codimension one, and a proper subsheaf $\mathscr{F} \subsetneq \mathscr{T}_{\widetilde{X}}$ such that the following holds.

(7.1.1) *The tangent sheaf of $\widetilde{X}$ decomposes as a direct sum, $\mathscr{T}_{\widetilde{X}} \cong \big(f^{[*]}\mathscr{E}\big) \oplus \mathscr{F}$.*

(7.1.2) *Both summands in (7.1.1) have trivial determinant. In other words,* $\det f^{[*]}\mathscr{E} \cong \mathscr{O}_{\widetilde{X}}$ *and* $\det \mathscr{F} \cong \mathscr{O}_{\widetilde{X}}$.

Before proving Theorem 7.1 in Section 7.A below, we note an important corollary, obtained from Theorem 7.1 by iterated application. The following notation, which summarises the conditions spelled out in Condition (1.3.2) of the Decomposition Theorem 1.3, is used in its formulation.

Definition 7.2 (Strong stability). Let X be a normal projective variety of dimension n, and $\mathscr{F}$ a reflexive coherent sheaf of $\mathscr{O}_X$-modules. We call $\mathscr{F}$ *strongly stable*, if for any finite morphism $f : \widetilde{X} \to X$ that is étale in codimension one, and for any choice of ample divisors $\widetilde{H}_1, \ldots, \widetilde{H}_{n-1}$ on $\widetilde{X}$, the reflexive pull-back $f^{[*]}\mathscr{F}$ is stable with respect to $(\widetilde{H}_1, \ldots, \widetilde{H}_{n-1})$.

Corollary 7.3 (Existence of a decomposition)**.** *Let X be a normal projective variety having at worst canonical singularities. Assume that $\omega_X \cong \mathscr{O}_X$. Then there exists a finite cover $f : \widetilde{X} \to X$, étale in codimension one, and a decomposition*

$$\mathscr{T}_{\widetilde{X}} \cong \bigoplus \mathscr{E}_i,$$

where the $\mathscr{E}_i$ *are strongly stable subsheaves of* $\mathscr{T}_{\widetilde{X}}$ *with trivial determinants,* $\det \mathscr{E}_i \cong \mathscr{O}_{\widetilde{X}}$.

Remark 7.4. We note that the summands $\mathscr{E}_i$ in the decomposition established in Corollary 7.3 are automatically saturated. Indeed, as a subsheaf of the torsion-free sheaf $\mathscr{T}_{\widetilde{X}}$ each $\mathscr{E}_i$ is torsion-free. The quotient of $\mathscr{T}_{\widetilde{X}}$ by any of the summands is a direct sum of the remaining summands, hence torsion-free.

Proof of Corollary 7.3. We need to find a cover $f : \widetilde{X} \to X$ and a decomposition $\mathscr{T}_{\widetilde{X}} \cong \bigoplus \mathscr{E}_i$, such that all factors $\mathscr{E}_i$ are strongly stable. Since the rank of $\mathscr{T}_X$ is finite, there exists a finite cover $f : \widetilde{X} \to X$, étale in codimension one, with a proper decomposition

$$\mathscr{T}_{\widetilde{X}} \cong \bigoplus_{i \geq 1} \mathscr{E}_i, \tag{7.4.1}$$

in which the number of direct summands is maximal. We claim that each summand is then automatically strongly stable.

We argue by contradiction and assume that there exists a further finite cover, $g : \widehat{X} \to \widetilde{X}$, étale in codimension one, and a list of ample divisors $\widehat{h} = (\widehat{H}_1, \dots, \widehat{H}_{n-1})$ on $\widehat{X}$ such that the reflexive pull-back of one of the summands, say $\widehat{\mathscr{E}}_1 := g^{[*]}\mathscr{E}_1$, is not stable with respect to $\widehat{h}$. Let $0 \subsetneq \widehat{\mathscr{S}} \subsetneq \widehat{\mathscr{E}}_1$ be a $\widehat{h}$-destabilising subsheaf. By [Kob87, Prop. 7.6(b)], whose proof carries over without change from the smooth to the singular setup, we may assume that $\widehat{\mathscr{S}}$ is saturated in $\widehat{\mathscr{E}}_1$ and therefore also in $\mathscr{T}_{\widehat{X}}$. Since $\widehat{\mathscr{E}}_1$ and $\mathscr{T}_{\widehat{X}}$ both have vanishing $\widehat{h}$-slope, the sheaf $\widehat{\mathscr{S}}$ is also a destabilising subsheaf for $\mathscr{T}_{\widehat{X}}$. Replacing $\widehat{X}$ by a further cover, if necessary, Theorem 7.1 therefore allows to assume without loss of generality that the tangent sheaf splits, say $\mathscr{T}_{\widehat{X}} = \widehat{\mathscr{S}} \oplus \widehat{\mathscr{Q}}$. Since the sheaves $\mathscr{T}_{\widehat{X}}$, $\widehat{\mathscr{E}}_1$, $\widehat{\mathscr{S}}$ and $\widehat{\mathscr{Q}}$ are all locally free on the smooth locus of $\widehat{X}$, elementary linear algebra gives a decomposition

$$\widehat{\mathscr{E}}_1|_{\widehat{X}_{\mathrm{reg}}} = \widehat{\mathscr{S}}|_{\widehat{X}_{\mathrm{reg}}} \ \oplus\ (\widehat{\mathscr{E}}_1 \cap \widehat{\mathscr{Q}})|_{\widehat{X}_{\mathrm{reg}}}.$$

Taking double duals, we obtain a decomposition $\widehat{\mathscr{E}}_1 = \widehat{\mathscr{S}}^{**} \oplus (\widehat{\mathscr{E}}_1 \cap \widehat{\mathscr{Q}})^{**}$, which contradicts maximality of the decomposition (7.4.1) and therefore finishes the proof of Corollary 7.3. Q.E.D.

Remark 7.5 (Uniqueness of the decomposition). Given a variety X as in Corollary 7.3, let $f_1 : \widetilde{X}_1 \to X$ and $f_2 : \widetilde{X}_1 \to X$ be two finite morphisms, étale in codimension one, such that the tangent bundles split into strongly stable summands,

$$\mathscr{T}_{\widetilde{X}_1} \cong \bigoplus_{i=1}^{N} \mathscr{E}_i^1 \quad \text{and} \quad \mathscr{T}_{\widetilde{X}_2} \cong \bigoplus_{j=1}^{M} \mathscr{E}_j^2.$$

Let $\widehat{X}$ be an irreducible component of the normalisation of the fibered product $\widetilde{X}_1 \times_X \widetilde{X}_2$. We obtain a diagram

$$\begin{tikzcd} \widehat{X} \arrow[r, "g_1"] \arrow[d, "g_2"'] & \widetilde{X}_1 \arrow[d, "f_1"] \\ \widetilde{X}_2 \arrow[r, "f_2"'] & X, \end{tikzcd}$$

where g_1, g_2 are again finite and étale in codimension one. Since $\mathscr{T}_{\widehat{X}} \cong g_1^{[*]} \mathscr{T}_{\widetilde{X}_1} \cong g_2^{[*]} \mathscr{T}_{\widetilde{X}_2}$, we obtain decompositions

$$\mathscr{T}_{\widehat{X}} \;\cong\; \bigoplus_{i=1}^{N} g_1^{[*]} \mathscr{E}_i^1 \;\cong\; \bigoplus_{j=1}^{M} g_2^{[*]} \mathscr{E}_j^2. \tag{7.5.1}$$

Choosing any ample polarisation on $\widehat{X}$, stability of the summands implies that any morphism $\widehat{\mathscr{E}}_i^1 \to \widehat{\mathscr{E}}_j^2$ must either be trivial, or an isomorphism. It follows that the decompositions (7.5.1) satisfy the following extra conditions

(7.5.2) the number of summands in the decompositions agrees, $N = M$, and

(7.5.3) up to permutation of the summands we have isomorphisms $g_1^{[*]} \mathscr{E}_i^1 \cong g_2^{[*]} \mathscr{E}_i^2$ for all $i \in \{1, \ldots, N\}$.

In that sense, the decomposition found in Corollary 7.3 is unique.

7.A. Proof of Theorem 7.1

Since the proof of Theorem 7.1 is somewhat long, we have subdivided it into four relatively independent steps, given in Sections 7.A.1–7.A.4 below. Figure 7.1 on the facing page gives an overview of the spaces and morphisms constructed in the course of the proof.

7.A.1. *Step 1: The subsheaf $\mathscr{E} \subsetneq \mathscr{T}_X$* To start the proof of Theorem 7.1, we discuss the structure of the saturated destabilising sheaf $\mathscr{E} \subsetneq \mathscr{T}_X$. First note that due to torsion-freeness of $\mathscr{E}$ and of $\mathscr{T}_X/\mathscr{E}$, the sheaf $\mathscr{E}$ is a sub-vectorbundle of $\mathscr{T}_X$ outside of a set of codimension two. Next, we compute its slope.

$$\begin{tikzcd}[column sep=huge]
\widetilde{X}' \arrow[r, "g\text{, index-one cover for } \mathscr{E}'", "\text{finite, étale in codim. one}"'] \arrow[d, equal] & X' \arrow[r, "\phi\text{, } \mathbb{Q}\text{-factorialisation}", "\text{small, birational}"'] & X \arrow[d, equal] \\
\widetilde{X}' \arrow[r, "\psi\text{, Stein factorisation I}", "\text{conn. fibers, small, birational}"'] & \widetilde{X} \arrow[r, "f\text{, Stein factorisation II}", "\text{finite, étale in codim. one}"'] & X
\end{tikzcd}$$

Fig. 7.1. Spaces and morphisms constructed in the proof of Theorem 7.1

Lemma 7.6 (Slopes of destabilising subsheaves)**.** *In the setup of Theorem 7.1, any destabilising subsheaf of $\mathscr{T}_X$ has slope zero. In particular, $\mu_h(\mathscr{E}) = 0$.*

Proof. Let $\mathscr{G}$ be any destabilising subsheaf of $\mathscr{T}_X$. Since K_X is assumed to be trivial, it follows that $\mathscr{G}$ has non-negative slope, $\mu_h(\mathscr{G}) \geq 0$. On the other hand, we know from Proposition 5.4 that $\mathscr{T}_X$ is h-semistable. Consequently, we have $\mu_h(\mathscr{G}) = 0$, as claimed. Q.E.D.

7.A.2. *Step 2: The $\mathbb{Q}$-factorialisation of X* Let $\phi : X' \to X$ be a $\mathbb{Q}$-factorialisation of X, that is, a small birational morphism where X' is $\mathbb{Q}$-factorial and has only canonical singularities. The existence of ϕ is established in [BCHM10, Lem. 10.2]. Since ϕ is small, it is clear that $\omega_{X'} \cong \phi^*\omega_X \cong \mathscr{O}_{X'}$, and that $\widetilde{q}(X') = 0$. Set $\mathscr{E}' := \phi^{[*]}(\mathscr{E})$. Since $\mathscr{E}'$ injects into the tangent sheaf $\mathscr{T}_{X'}$ away from a set of codimension two, it follows from reflexivity that $\mathscr{E}'$ injects into $\mathscr{T}_{X'}$ everywhere. We can therefore view it as a proper subsheaf $\mathscr{E}' \subsetneq \mathscr{T}_{X'}$. Notice that $\mathscr{E}'$ is saturated in $\mathscr{T}_{X'}$, since ϕ is small.

Claim 7.7. To prove statements (7.1.1) and (7.1.2) of Theorem 7.1, it suffices to find a finite cover $g : \widetilde{X}' \to X'$, étale in codimension one, and a decomposition

$$\mathscr{T}_{\widetilde{X}'} \cong \mathscr{F}' \oplus \left(g^{[*]}\mathscr{E}'\right) \tag{7.7.1}$$

where $\det \mathscr{F}' \cong \det\left(g^{[*]}\mathscr{E}'\right) \cong \mathscr{O}_{\widetilde{X}'}$.

Proof. As indicated in Figure 7.1, consider the Stein factorisation of the composed morphism $\phi \circ g$,

$$\begin{tikzcd}[column sep=huge]
\widetilde{X}' \arrow[r, "\psi\text{, small birational}"'] \arrow[rr, bend left=15, "\phi\circ g"] & \widetilde{X} \arrow[r, "f\text{, finite, étale in codim. 1}"'] & X.
\end{tikzcd}$$

The reflexive push-forward of (7.7.1),

$$\mathscr{T}_{\widetilde{X}} \cong \big(\psi_* \mathscr{T}_{\widetilde{X}'}\big)^{**} \cong \Big(\psi_*\big(\mathscr{F}' \oplus g^{[*]}\mathscr{E}'\big)\Big)^{**} \cong \underbrace{\big(\psi_*\mathscr{F}'\big)^{**}}_{=:\mathscr{F}} \oplus \underbrace{\big(\psi_*(g^{[*]}\mathscr{E}')\big)^{**}}_{\cong f^{[*]}\mathscr{E}},$$

then yields a decomposition on $\widetilde{X}$ that satisfies both (7.1.1) and (7.1.2), thus finishing the proof of Claim 7.7. Q.E.D.

7.A.3. *Step 3: High reflexive powers of* $\det \mathscr{E}$ *and* $\det \mathscr{E}'$ *are trivial* We need to show that high reflexive powers of $\det \mathscr{E}$ and $\det \mathscr{E}'$ are trivial. To this end, recall that the reflexive sheaf $\det \mathscr{E}'$ is a Weil divisorial sheaf on X'. In other words, there exists a Weil divisor D' on X' so that $\det \mathscr{E}' \cong \mathscr{O}_{X'}(D')$. Since X' is $\mathbb{Q}$-factorial, there exists be a number $m \in \mathbb{N}^+$ such that mD' is actually Cartier.

As a first step towards showing triviality of a sufficiently high multiple, we prove numerical triviality of D'.

Lemma 7.8. *Setting as above, then the* $\mathbb{Q}$*-Cartier divisor* D' *is numerically trivial.*

Proof. We aim to apply Proposition 4.2 in order to conclude that D' is numerically trivial. As a first step in this direction, recall from Lemma 7.6 that $\mu_h(\mathscr{E}) \cdot h = 0$. Consequently, we have the following equality of intersection numbers of $\mathbb{Q}$-Cartier divisors,

$$(7.8.1) \qquad 0 = D' \cdot \phi^*(H_1) \cdots \phi^*(H_{n-1}) = -D' \cdot \phi^*(H_1) \cdots \phi^*(H_{n-1}).$$

Secondly, observe that the inclusion $\mathscr{E}' \hookrightarrow \mathscr{T}_{X'}$ yields an inclusion $\det \mathscr{E}' \hookrightarrow \wedge^{[p]} \mathscr{T}_{X'}$. Dualising, we obtain a non-zero morphism

$$\Omega^{[p]}_{X'} \to (\det \mathscr{E}')^* \cong \mathscr{O}_{X'}(-D').$$

Since K_X is trivial and since X has only canonical singularities, X is not uniruled and Proposition 5.6 therefore implies that the $\mathbb{Q}$-Cartier divisor $-D'$ is pseudoeffective on X'. Due to Equation (7.8.1), Proposition 4.2 applies to show that $-D'$ and hence D' is numerically trivial indeed. Q.E.D.

Next up, we conclude from numerical triviality that high reflexive powers of $\det \mathscr{E}$ and $\det \mathscr{E}'$ are trivial.

Corollary 7.9. *Setting as above. Then there exists a number* $M \in \mathbb{N}^+$ *such that* $(\det \mathscr{E}')^{[M]} \cong \mathscr{O}_{X'}$ *and* $(\det \mathscr{E})^{[M]} \cong \mathscr{O}_X$.

Proof. We are going to use the assumption that $\widetilde{q}(X) = 0$, which implies that

$$0 = q(X') = h^1\big(X', \mathscr{O}_{X'}\big),$$

hence the Picard group of X' is discrete. The subgroup $\operatorname{Pic}^0(X') \subsetneq \operatorname{Pic}(X')$ of invertible sheaves with numerically trivial Chern class is therefore finite. It follows that there exists a positive natural number k such that $(\det \mathscr{E}')^{[km]} \cong \mathscr{O}_{X'}$. Since the reflexive sheaves $\mathscr{O}_X \cong \phi_*\big((\det \mathscr{E}')^{[km]}\big)$ and $(\det \mathscr{E})^{[km]}$ agree in codimension one, they agree everywhere, and $(\det \mathscr{E})^{[km]}$ is likewise trivial. Q.E.D.

7.A.4. *Step 4: Constructing the splitting on a cover of X'* Since $\det \mathscr{E}'$ is torsion, there exists an index-one cover $g : \widetilde{X}' \to X'$ for $\det \mathscr{E}'$, see for example [KM98, 2.52]. This is a finite morphism from a normal variety, étale in codimension one, such that

$$g^{[*]} \det \mathscr{E}' \cong \mathscr{O}_{\widetilde{X}'}. \tag{7.9.1}$$

Recall from [KM98, 5.20] that $\widetilde{X}'$ has trivial canonical bundle and at worst canonical singularities. Set $\widetilde{\mathscr{E}}' := g^{[*]}\mathscr{E}'$. We aim to construct a splitting of $\mathscr{T}_{\widetilde{X}'}$ using the existence of complementary subsheaves shown in Corollary 6.12 on page 87. The following claim guarantees that the assumptions made in Corollary 6.12 are satisfied in our context.

Claim 7.10. The inclusion $\mathscr{E}' \hookrightarrow \mathscr{T}_{X'}$ induces an inclusion $\widetilde{\mathscr{E}}' \hookrightarrow \mathscr{T}_{\widetilde{X}'}$. With the inclusion understood, $\widetilde{\mathscr{E}}'$ is a saturated subsheaf of $\mathscr{T}_{\widetilde{X}'}$

Proof. Since $\phi \circ g : \widetilde{X}' \to X$ is étale in codimension one, the reflexive sheaf $\widetilde{\mathscr{E}}'$ injects into $\mathscr{T}_{\widetilde{X}'}$ outside of a small set, and is a saturated subsheaf there. Since both $\widetilde{\mathscr{E}}'$ and its saturation in $\mathscr{T}_{\widetilde{X}'}$ are reflexive and agree in codimension one, it follows that $\widetilde{\mathscr{E}}'$ actually is isomorphic to its saturation, as claimed. Q.E.D.

With Claim 7.10 in place, Corollary 6.12 asserts the existence of a sheaf $\widetilde{\mathscr{F}}'$ with trivial determinant such that $\mathscr{T}_{\widetilde{X}'} \cong \widetilde{\mathscr{E}}' \oplus \widetilde{\mathscr{F}}'$. As we have seen in Claim 7.7, this concludes the proof of Theorem 7.1. Q.E.D.

7.B. Integrability of direct summands

In this section, we show that the individual summands in the decomposition stated in Corollary 7.3 are integrable; that is, they define foliations.

Theorem 7.11 (Integrability of direct summands)**.** *Let X be a normal projective variety with at worst canonical singularities. Assume* *that $\omega_X \cong \mathscr{O}_X$. Let $\mathscr{T}_X \cong \bigoplus \mathscr{E}_i$ be a decomposition into reflexive sheaves with trivial determinants. Then all $\mathscr{E}_i$ are integrable.*

Proof. We follow the arguments of [Hör07]. Without loss of generality we assume that $\mathscr{T}_X \cong \mathscr{E}_1 \oplus \mathscr{E}_2$, that is, we assume that $\mathscr{T}_X$ can be decomposed into two summands. We will show that $\mathscr{E}_2$ is integrable. The integrability of $\mathscr{E}_1$ then follows for symmetry reasons. Let r_1 be the rank of $\mathscr{E}_1$, and consider the trivialisable sheaf $\mathscr{L}_1 := \det \mathscr{E}_1$. Since $\mathscr{E}_1^*$ is a direct summand of Ω^1_X, the reflexive sheaf $\mathscr{L}_1 \otimes \Omega^{[r_1]}_X$ has a trivial direct summand. Let $\theta \in H^0\big(X,\, \mathscr{L}_1 \otimes \Omega^{[r_1]}_X\big)$ be the corresponding nowhere vanishing $\mathscr{L}_1$-valued differential form and let $\pi : \widetilde{X} \to X$ be a resolution of singularities. By the Extension Theorem 2.4, the reflexive differential form θ pulls back to a non-trivial section $\widetilde{\theta} \in H^0\big(\widetilde{X},\, \pi^*(\mathscr{L}_1) \otimes \Omega^{r_1}_{\widetilde{X}}\big)$.

At general points of $\widetilde{X}$, where π is isomorphic, the sheaf $\pi^*\mathscr{E}_2$ coincides with the degeneracy sheaf $S_{\widetilde{\theta}}$ of $\widetilde{\theta}$, that is, the sheaf of vector fields $\vec{v}$ such that the contraction

$$i_{\vec{v}}(\widetilde{\theta}) = \widetilde{\theta}(\vec{v}, \cdot) \in H^0\big(\widetilde{X},\, \pi^*(\mathscr{L}_1) \otimes \Omega^{r_1-1}_{\widetilde{X}}\big)$$

vanishes[1]. In this setting, it follows from [Dem02, Main Thm.] that $S_{\widetilde{\theta}}$ is integrable. As a consequence, we obtain that $\mathscr{E}_2$ is integrable at general points of X. Since $\mathscr{E}_2$ is a saturated subsheaf of $\mathscr{T}_X$, it follows that it is integrable everywhere. Q.E.D.

7.C. Proof of Theorem 1.3

We maintain notation and assumptions of Theorem 1.3. Corollary 3.6 implies the existence of an Abelian variety A and of a projective variety X' with at worst canonical singularities, with trivial canonical bundle and $\widetilde{q}(X') = 0$, together with a finite cover $A \times X' \to X$, étale in codimension one. Property (1.3.3) stated in Theorem 1.3 is hence fulfilled for any cover of the form $A \times \widetilde{X} \to A \times X'$, where $\widetilde{X} \to X'$ is a finite cover, étale in codimension one. The existence of such a cover $\widetilde{X} \to X$ and of a decomposition of $\mathscr{T}_{\widetilde{X}}$ satisfying Properties (1.3.1) and (1.3.2) follows by combining Corollary 7.3 and Theorem 7.11. In summary, this finishes the proof of Theorem 1.3. Q.E.D.

Remark 7.12. The decomposition theorem of Beauville-Bogomolov holds for compact Kähler manifolds. Therefore, we should expect a singular version in the non-algebraic context as well. In particular, Theorem 1.3 should hold for Kähler varieties. There are however two main

[1] Degneracy subsheaves are introduced and discussed in more detail in Section 8.B.2 below.

ingredients in our argument which are not yet available in the Kähler context: the Extension Theorem 2.4 and the pseudoeffectivity result Proposition 5.6.

§8. Towards a structure theory

If X is any projective manifold with Kodaira dimension zero, $\kappa(X) = 0$, standard conjectures of minimal model theory predict the existence of a birational contraction[2] map $\lambda : X \dashrightarrow X_\lambda$, where X_λ has terminal singularities and numerically trivial canonical divisor. Generalising the Beauville-Bogomolov Decomposition Theorem 1.1, it is widely expected that X_λ admits a finite cover, étale in codimension one, which can be birationally decomposed into a product

$$T \times \prod X_j,$$

where T is a torus and the X_j are singular versions of Calabi-Yau manifolds and irreducible symplectic manifold, which cannot be decomposed further. Such a decomposition result would clearly be a central pillar to any structure theory for varieties with Kodaira dimension zero. The main result of the present paper, Theorem 1.3, is a first step in this direction.

Section 8.A discusses the remaining problems of turning the decomposition found in the tangent sheaf into a decomposition of the variety. Section 8.B gives a conjectural description of the irreducible pieces coming out of the decomposition, discussing singular analogues of Calabi-Yau and irreducible holomorphic-symplectic varieties, and proving the conjectured description in low dimensions. Finally, fundamental groups of varieties with trivial canonical class, which are crucial for our understanding of this class of varieties, are discussed in the concluding Section 8.C.

Remark 8.1. Corollary 3.6 and Theorem 1.3 allow to restrict our attention to varieties with vanishing augmented irregularity. For most of the present Section 8, we will therefore only consider varieties X with $\widetilde{q}(X) = 0$.

8.A. Decomposing varieties with trivial canonical bundle

In technically correct terms, the setup of our discussion is now summarised as follows.

[2]Following standard use, we call a birational map a *contraction map* if its inverse does not contract any divisors.

Setup 8.2. Let X be a normal $\mathbb{Q}$-factorial projective variety with canonical singularities such that K_X is torsion and $\widetilde{q}(X) = 0$. By Theorem 1.3, there exists a finite cover $f : \widetilde{X} \to X$, étale in codimension one, such that $\omega_X = \mathscr{O}_X$ and such that there exists a decomposition

$$\mathscr{T}_{\widetilde{X}} = \bigoplus \mathscr{E}_i$$

of $\mathscr{T}_{\widetilde{X}}$ into strongly stable integrable reflexive subsheaves.

In view of the desired decomposition of the variety $\widetilde{X}$, this naturally leads to the following problems.

Problem 8.3 (Algebraicity of leaves). *In Setup 8.2, show that the leaves of the foliations $\mathscr{E}_i$ are algebraic, perhaps after passing to another cover.*

Problem 8.4 (Decomposition of the variety). *In the setup of Problem 8.3, show that the algebraicity of the leaves leads to a birational decomposition of $\widetilde{X}$, perhaps after passing to another cover. More precisely, show that there is a birational morphism*

$$g : \widetilde{X} \dashrightarrow \prod X_i,$$

isomorphic outside of a small set $V \subset \widetilde{X}$, such that the following holds.

(8.4.1) *The varieties X_j are smooth, projective with $\kappa(X_j) = 0$ for all j.*

(8.4.2) *If p_j denotes the composition of g with j^{th} projection $\Pi X_i \to X_j$, then $p_j^*(\mathscr{T}_{Y_j}) = \mathscr{E}_j$ over $X \setminus V$ for all j.*

Remark 8.5. Once it is known that the leaves of $\mathscr{E}_j$ are algebraic, one easily obtains rational maps $X \dashrightarrow Y_j$ to smooth projective varieties such that $\mathscr{E}_j = \mathscr{T}_{X/Y_j}$ generically. The main problem is now to show that the equality $\mathscr{E}_j = \mathscr{T}_{X/Y_j}$ holds everywhere, and that $\kappa(Y_j) = 0$.

A solution to Problem 8.4 is not the yet desired final outcome of our decomposition strategy for X: since K_X is (numerically) trivial one clearly aims for a decomposition into varieties with trivial canonical class. Assuming that the minimal model program works for varieties of Kodaira dimension zero, each X_j may be replaced by a minimal model X'_j. As a consequence we would obtain a birational map $g' : X \dashrightarrow \Pi X'_i$. If the singularities of X' are not only canonical but terminal, it follows from [Kaw08] that g' is isomorphic in codimension one and decomposes into a finite sequence of flops. One might hope that each terminal variety with numerically trivial canonical class decomposes into terminal varieties with trivial canonical class and strongly stable tangent bundle, after performing a finite cover, étale in codimension one, and after performing a finite number of flops.

8.B. Classifying the strongly stable pieces: Calabi-Yau and irreducible holomorphic-symplectic varieties

We start with after a short discussion of the notion of strong stability in Section 8.B.1, showing by way of example that varieties with strongly stable tangent sheaves are the "right" objects when building a structure theory for spaces of Kodaira dimension zero. The remainder of the present Section 8.B discusses these spaces in detail.

Sections 8.B.2 and 8.B.3 relate stability properties of the tangent bundle to non-degeneracy of differential forms, and discuss implications for the exterior algebra of reflexive forms. We apply these results in the concluding Section 8.B.4 to show that singular varieties with strongly stable tangent sheaf are in a very strong sense natural analogues of Calabi-Yau and irreducible holomorphic-symplectic manifolds, at least in dimension up to five. There is ample evidence to conjecture that this description holds in general, for strongly stable varieties of arbitrary dimension.

8.B.1. *Strong stability versus stability* At first sight, it seems tempting to consider varieties with stable tangent bundle as the building blocks of varieties with semistable tangent sheaf, such as varieties with trivial canonical bundle. However, the following example shows that *strong stability* is indeed the correct notion in our setup.

Example 8.6 (A variety with stable, but not strongly stable tangent sheaf). Let Z be a projective K3-surface, let $\widetilde{X} := Z\times Z$ with projections $p_1, p_2 : \widetilde{X} \to Z$, and let $\phi \in \mathrm{Aut}_{\mathscr{O}}(\widetilde{X})$ be the automorphism which interchanges the two factors. The quotient $X := \widetilde{X}/\langle\phi\rangle = \mathrm{Sym}^2(Z)$ is then a projective holomorphic-symplectic variety with trivial canonical bundle and rational Gorenstein singularities. The quotient map $\pi : \widetilde{X} \to X$ is finite and étale in codimension one. Let h be any ample polarisation on X. The tangent sheaf $\mathscr{T}_X$ of X is obviously not strongly stable.

However, we claim that $\mathscr{T}_X$ is h-stable. Indeed, suppose that there exists a non-trivial h-stable subsheaf $0 \subsetneq \mathscr{S} \subsetneq \mathscr{T}_X$ with slope zero that destabilises $\mathscr{T}_X$. Then, the reflexive pull-back $\widetilde{\mathscr{S}} := \pi^{[*]}(\mathscr{S})$ is $\pi^*(h)$-polystable, see [HL97, Lem. 3.2.3], and injects into $\mathscr{T}_{\widetilde{X}} = p_1^*(\mathscr{T}_Z) \oplus p_2^*(\mathscr{T}_Z)$. Since neither of the two sheaves $p_j^*\mathscr{T}_Z$ is stable under the action of ϕ, clearly $\widetilde{\mathscr{S}}$ is not one of these. More is true: looking at the maps to the two summands of $\mathscr{T}_{\widetilde{X}}$ and using that morphisms between stable sheaves with the same slope are either trivial or isomorphic, we see that

$\mathscr{S}$ has to be stable of rank two, and isomorphic to both $p_1^* \mathscr{T}_Z$ and $p_2^* \mathscr{T}_Z$. This is absurd, as restriction to p_1-fibers shows that the sheaves $p_1^* \mathscr{T}_Z$ and $p_2^* \mathscr{T}_Z$ are in fact not isomorphic.

8.B.2. *Non-degeneracy of differential forms* If X is a canonical variety with numerically trivial canonical class, stability of the tangent bundle has strong implications for the geometry of differential forms X. This section is concerned with degeneracy properties. Conjectural consequences for the structure of the exterior algebra of forms are discussed in the subsequent Section 8.B.3.

Non-degeneracy of differential forms will be measured using the following definition.

Definition 8.7 (Contraction of a reflexive form, degeneracy subsheaf). Let X be a normal complex variety, let $0 < q \leq \dim X$ be any number and $\sigma \in H^0\big(X, \Omega_X^{[q]}\big)$ any reflexive form. The *contraction map of* σ is the unique sheaf morphism

$$i_\sigma : \mathscr{T}_X \to \Omega_X^{[q-1]}$$

whose restriction to X_{reg} is given by $\vec{u} \mapsto \sigma(\vec{u}, \cdot)$. Let $S_\sigma := \ker(i_\sigma)$ be the kernel of i_σ. We call $S_\sigma \subseteq \mathscr{T}_X$ the *degeneracy subsheaf* of the reflexive form σ. If $S_\sigma = 0$, we say that σ is *generically non-degenerate.*

The main result of the present section asserts that in our setup, forms never degenerate. This can be seen as first evidence for the conjectural classification of the stable pieces into "Calabi-Yau" and "irreducible holomorphic-symplectic" which we discuss later in Section 8.B.4 below.

Proposition 8.8 (Non-degeneracy of forms on canonical varieties with stable $\mathscr{T}_X$)**.** *Let X be a normal n-dimensional projective variety X having at worst canonical singularities, $n > 1$. Assume that the canonical divisor K_X is numerically trivial, and that the tangent sheaf $\mathscr{T}_X$ is stable with respect to some ample polarisation. If σ is any non-zero reflexive form on X, then σ is generically non-degenerate, in the sense of Definition 8.7.*

Proof. We argue by contradiction and assume that there exists a reflexive q-form σ whose degeneracy subsheaf does not vanish, $S_\sigma \neq 0$. Consider the exact sequence

$$0 \to S_\sigma \to \mathscr{T}_X \xrightarrow{i(\sigma)} \underbrace{\operatorname{Image} i(\sigma)}_{=:\mathscr{E} \subseteq \Omega_X^{[q-1]}} \to 0. \tag{8.8.1}$$

Recalling from Proposition 5.7 that $\mathscr{T}_X$ is stable with respect to any ample polarisation, we choose an ample Cartier divisor H on X, a sufficiently large number m, and let $(D_j)_{1 \le j \le n-1} \in |mH|$ be general elements. Consider the corresponding general complete intersection curve $C := D_1 \cap \cdots \cap D_{n-1} \subsetneq X$, which avoids the singular locus of X.

Since K_X is torsion, the Kodaira-dimension of X is zero, $\kappa(X) = 0$. As X has only canonical singularities, this implies that X is not covered by rational curves. Miyaoka's Generic Semipositivity Theorem 5.1 therefore asserts that the vector bundle $\mathscr{T}_X|_C \cong \big(\Omega_X^{[n-1]} \otimes \omega_X^*\big)\big|_C$ is nef. This has two consequences in our setup. On the one hand, since $\mathscr{E}$ is a quotient of $\mathscr{T}_X$, it follows that $\mathscr{E}|_C$ and $\det \mathscr{E}|_C$ are nef. On the other hand, since $\mathscr{E} \subseteq \Omega_X^{[q-1]}$ by definition, its dual $\mathscr{E}^*|_C$ is a quotient of $\wedge^{q-1} \mathscr{T}_X|_C$, and it follows that $\mathscr{E}^*|_C$ and $\det \mathscr{E}^*|_C$ are likewise nef. Consequently, we obtain $\det \mathscr{E}|_C \equiv 0$. The exact Sequence (8.8.1) then implies that S_σ destabilises $\mathscr{T}_X$. This contradicts the assumed stability of $\mathscr{T}_X$, and finishes the proof of Proposition 8.8. Q.E.D.

Corollary 8.9 (Reflexive two-forms on canonical varieties with stable $\mathscr{T}_X$, I)**.** *In the setup of Proposition 8.8,* $h^0\big(X, \Omega_X^{[2]}\big) \le 1$.

Proof. We argue by contradiction and assume that there are two linearly independent forms $\sigma_1, \sigma_2 \in H^0\big(\widetilde{X}, \Omega_{\widetilde{X}}^{[2]}\big)$. Since both forms are non-degenerate by Proposition 8.8, they induce linearly independent isomorphisms $\phi_\bullet : \mathscr{T}_{\widetilde{X}} \to \Omega_{\widetilde{X}}^{[1]}$. The composition $\phi_1^{-1} \circ \phi_2$ is thus a non-trivial automorphism of $\mathscr{T}_{\widetilde{X}}$. We obtain that the stable sheaf $\mathscr{T}_{\widetilde{X}}$ is not simple, contradicting [HL97, Cor. 1.2.8] and thereby finishing the proof of Corollary 8.9. Q.E.D.

Corollary 8.10 (Reflexive two-forms on canonical varieties with stable $\mathscr{T}_X$, II)**.** *In the setup of Proposition 8.8, if there exists a non-trivial reflexive two-form* $\sigma \in H^0\big(X, \Omega_X^{[2]}\big)$*, then* σ *is a complex-symplectic form on the smooth part of* X*. In particular,* $\dim X$ *is even,* ω_X *is trivial, and* X *has only rational Gorenstein singularities.*

Proof. Proposition 8.8 implies that the non-degeneracy subsheaf S_σ vanishes. For general points $x \in X_{\mathrm{reg}}$, this implies that $\sigma|_x$ is a non-degenerate, and hence symplectic, form on the vector space $T_X|_x$. This already shows that the dimension of X is even, say $\dim X = 2k$. If $\tau \in H^0\big(X, \omega_X\big)$ is the section induced by $\wedge^k \sigma$, then τ does not vanish at x.

To prove that σ is a complex-symplectic form on the smooth part of X, we need to show that non-degeneracy holds at arbitrary points of X_{reg}. If not, there exists a point $y \in X_{\mathrm{reg}}$ such that $\sigma|_y$ is a degenerate

2-form on the vector space $T_X|_y$. The form τ will therefore vanish at y, showing that K_X can be represented by a non-trivial, effective $\mathbb{Q}$-Cartier divisor, contradicting the assumption that K_X is numerically trivial.

The remaining assertions of Corollary 8.10 follow immediately. Q.E.D.

8.B.3. *Exterior algebras of differential forms on the strongly stable pieces* The algebra of differential forms on irreducible holomorphic-symplectic and Calabi-Yau manifolds has a rather simple structure cf. [Bea83, Props. 1 and 4]. In order to characterise the strongly stable pieces in the singular case one would need a similar description which we formulate as the following problem.

Problem 8.11 (Forms on varieties with strongly stable tangent bundle)**.** *Let X be a normal projective variety of dimension $n > 1$ with $\omega_X \cong \mathscr{O}_X$, having at worst canonical singularities. Assume that the tangent sheaf $\mathscr{T}_X$ is strongly stable. Then show that the following holds.*

(8.11.1) *For all odd numbers $q \neq n$, we have $H^0\big(\widetilde{X}, \Omega^{[q]}_{\widetilde{X}}\big) = 0$ for all finite covers $f : \widetilde{X} \to X$, étale in codimension one.*

(8.11.2) *If there exists a finite cover $g : X' \to X$, étale in codimension one, and an even number $0 < q < n$ such that $H^0\big(X', \Omega^{[q]}_{X'}\big) \neq 0$, then there exists a reflexive 2-form $\sigma' \in H^0\big(X', \Omega^{[2]}_{X'}\big)$, symplectic on the smooth locus X'_{reg}, such that for any finite cover $f : \widetilde{X} \to X'$, étale in codimension one, the exterior algebra of global reflexive forms on $\widetilde{X}$ is generated by $f^*(\sigma')$. In other words,*

$$\bigoplus_p H^0\Big(\widetilde{X}, \Omega^{[p]}_{\widetilde{X}}\Big) = \mathbb{C}\big[f^*(\sigma)\big].$$

Remark 8.12. Notice that the assumptions on the strong stability of $\mathscr{T}_X$ and on the dimension of X automatically imply $\widetilde{q}(X) = 0$.

As we will discuss in more detail in the subsequent Section 8.B.4, a positive solution to Problem 8.11 leads to a characterisation of canonical varieties with trivial canonical class and strongly stable tangent bundle as singular analogues of Calabi-Yau or irreducible holomorphic symplectic manifolds. There are a number of cases where Problem 8.11 can be solved. We conclude Section 8.B.3 with two propositions that provide evidence by discussing the case where X is smooth, or of dimension ≤ 5, respectively.

Proposition 8.13. *The claims of Problem 8.11 hold if X is smooth.*

Proof. Note that on a smooth variety X the sheaves Ω^p_X and $\Omega^{[p]}_X$ coincide and that any finite cover of X that is étale in codimension one is actually étale by purity of the branch locus.

Let now X be a smooth projective variety of dimension n with $\omega_X \cong \mathscr{O}_X$. Assume that the tangent bundle $\mathscr{T}_X$ is strongly stable. As noticed in Remark 8.12, this implies that $\widetilde{q}(X) = 0$. Consequently, the fundamental group of X is finite by [Bea83, Thm. 2(2)]. Let $\widehat{X} \to X$ be the universal cover. Since $\mathscr{T}_X$ is strongly stable, $\mathscr{T}_{\widehat{X}}$ is stable with respect to any polarisation, and $\widehat{X}$ is hence irreducible in the sense of the Beauville-Bogomolov decomposition Theorem 1.1. As $\widetilde{q}(X) = 0$, the manifold $\widehat{X}$ is therefore either Calabi-Yau or irreducible holomorphic-symplectic.

In order to show (8.11.1), pulling back forms from any étale cover $\widetilde{X} \to X$ to the universal cover $\widehat{X}$ if necessary, it suffices to note that both in the Calabi–Yau and in the irreducible holomorphic-symplectic case, $\widehat{X}$ does not support differential forms of odd degree $p < n$ by [Bea83, Props. 1 and 3].

To show (8.11.2), let $X' \to X$ be any étale cover, and assume that there exists a non-vanishing form such that that $\sigma' \in H^0\big(X', \Omega^{[q]}_{X'}\big)$ for some even number $0 < q < n$. Pulling back σ' to the universal cover $\widehat{X}$, we see that $\widehat{X}$ cannot be Calabi–Yau and is therefore irreducible holomorphic-symplectic, say with symplectic form $\widehat{\sigma}$. Consequently, [Bea83, Prop. 3] implies that the algebra of differential forms on $\widehat{X}$ is generated by $\widehat{\sigma}$. Hence, in order to establish the claim it therefore suffices to show that $\widehat{X}$ is biholomorphic to X and therefore also to X'. In other words, we need to show that X is already simply-connected. This is done in Lemma 8.14 below. Q.E.D.

We are grateful to Keiji Oguiso for pointing us towards [OS11] and for explaining the following observation to us.

Lemma 8.14. *Let X be a projective manifold whose universal cover is an irreducible holomorphic-symplectic manifold. If the canonical bundle of X is trivial, $\omega_X \cong \mathscr{O}_X$, then X is simply-connected, and therefore itself irreducible holomorphic-symplectic.*

Proof. The assumptions on X imply that X is an *Enriques manifold* in the sense of Oguiso and Schröer [OS11], see also [BNWS11]. Since the canonical bundle of X is trivial, and since the universal cover of X is irreducible holomorphic-symplectic, the fundamental group of X is finite, cf. [Bea83, Thm. 2(2)]. Let d denote the degree of the universal covering map $\widehat{X} \to X$, and set $\dim X = \dim \widehat{X} = n = 2k$. It then follows from [OS11, Prop. 2.4] that $d \mid (k+1)$. Moreover, since X is assumed to have trivial canonical bundle, [OS11, Prop. 2.6] implies that additionally $d \mid k$. Consequently, we have $d = 1$, which proves the claim. Q.E.D.

Proposition 8.15. *The claims of Problem 8.11 hold if* $\dim X \leq 5$.

Proof. Let X be a projective variety of dimension greater than one, having at worst canonical singularities. Assume that X has a trivial canonical bundle, $\omega_X \cong \mathscr{O}_X$, and a strongly stable tangent sheaf $\mathscr{T}_X$. Again we have $\widetilde{q}(X) = 0$, since $\mathscr{T}_X$ is strongly stable. Fix a finite cover $\widetilde{X} \to X$, étale in codimension one.

If $\dim X = 3$, then Corollary 6.11 immediately implies that $h^0\bigl(\widetilde{X}, \Omega^{[1]}_{\widetilde{X}}\bigr) = h^0\bigl(\widetilde{X}, \Omega^{[2]}_{\widetilde{X}}\bigr) = 0$. Conditions (8.11.1) and (8.11.2) of Problem 8.11 are therefore satisfied.

Now assume that $\dim X = 4$. In this setting, Corollary 6.11 gives that $h^0\bigl(\widetilde{X}, \Omega^{[1]}_{\widetilde{X}}\bigr) = h^0\bigl(\widetilde{X}, \Omega^{[3]}_{\widetilde{X}}\bigr) = 0$. The claims of Problem 8.11 thus follow from Corollary 8.10 and from the fact that $h^0\bigl(\widetilde{X}, \Omega^{[2]}_{\widetilde{X}}\bigr) \leq 1$, as shown in Corollary 8.9.

It remains to consider the case where $\dim X = 5$, where Corollary 6.11 asserts that $h^0\bigl(\widetilde{X}, \Omega^{[1]}_{\widetilde{X}}\bigr) = h^0\bigl(\widetilde{X}, \Omega^{[4]}_{\widetilde{X}}\bigr) = 0$. The claims of Problem 8.11 will follow once we show that $h^0\bigl(\widetilde{X}, \Omega^{[2]}_{\widetilde{X}}\bigr)$ and $h^0\bigl(\widetilde{X}, \Omega^{[3]}_{\widetilde{X}}\bigr)$ vanish as well. For that, recall from item (6.10.2) of Corollary 6.10 that there exists a non-trivial 3-form on $\widetilde{X}$ if and only if there exists a non-trivial reflexive 2-form on $\widetilde{X}$. However, by Corollary 8.10 any non-trivial 2-form would be non-degenerate, forcing $\dim X$ to be even, a contradiction. Q.E.D.

8.B.4. *Calabi-Yau and holomorphic-symplectic varieties* The following definition is motivated by the description of the exterior algebra of Calabi-Yau manifolds and irreducible holomorphic-symplectic manifolds, [Bea83, Props. 1 and 4], and by the discussion of Problem 8.11 in the previous section.

Definition 8.16 (Calabi-Yau and symplectic varieties in the singular case). Let X be a normal projective variety with $\omega_X \cong \mathscr{O}_X$, having at worst canonical singularities.

(8.16.1) We call X *Calabi-Yau* if $H^0\bigl(\widetilde{X}, \Omega^{[q]}_{\widetilde{X}}\bigr) = 0$ for all numbers $0 < q < \dim X$ and all finite covers $\widetilde{X} \to X$, étale in codimension one.

(8.16.2) We call X *irreducible holomorphic-symplectic* if there exists a reflexive 2-form $\sigma \in H^0\bigl(X, \Omega^{[2]}_X\bigr)$ such that σ is everywhere non-degenerate on X_{reg}, and such that for all finite covers $f : \widetilde{X} \to X$, étale in codimension one, the exterior algebra of global reflexive forms is generated by $f^*(\sigma)$.

Remark 8.17 (Augmented irregularity of Calabi-Yau and symplectic varieties)*.* If X is Calabi-Yau or irreducible holomorphic-symplectic in the sense of Definition 8.16, it follows immediately that the augmented irregularity of X vanishes, $\widetilde{q}(X) = 0$.

Remark 8.18 (Definition (8.16.1) for "Calabi-Yau" in the smooth case)*.* By [Bea83, Sect. 3, Prop. 2] the conditions spelled out in (8.16.1) are in the smooth case equivalent to the existence of a Kähler metric with holonomy $SU(m)$. If X is smooth and Calabi-Yau in the sense of Definition 8.16, then X is not necessarily simply-connected, but may have finite fundamental group. If we assume additionally that $\dim X$ is even, then a simple computation with holomorphic Euler characteristics shows that X is in fact simply-connected, cf. [Bea83, Prop. 2 and Rem.].

Remark 8.19 (Definition (8.16.2) for "irreducible symplectic" in the smooth case)*.* If X is smooth and irreducible holomorphic-symplectic in the sense of Definition 8.16, then X is simply-connected. In fact, even without the condition on the algebra of differential forms on étale covers, if X is a holomorphic-symplectic manifold of complex dimension $2n$ such that

$$H^{k,0}(X) \cong \begin{cases} \mathbb{C} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd,} \end{cases}$$

then X is simply-connected, that is, X is an irreducible holomorphic-symplectic manifold, see [HNW11, Prop. A.1].

Assuming Problem 8.11 can be solved, the following two propositions provide a classification of the strongly stable pieces in the conjectural version of the Beauville-Bogomolov decomposition for the singular case.

Proposition 8.20 (Characterisation of strongly stable pieces, I)**.** *Let X be Calabi-Yau or irreducible holomorphic-symplectic in the sense of Definition 8.16. Then $\mathscr{T}_X$ is strongly stable in the sense of Definition 7.2.*

Proof. Let X be Calabi-Yau or irreducible symplectic. We argue by contradiction and assume that there exists a finite cover $g : \widetilde{X} \to X$, étale in codimension one, and ample Cartier divisors $\widetilde{H}_1, \ldots \widetilde{H}_{n-1}$ on $\widetilde{X}$ such that the tangent sheaf $\mathscr{T}_{\widetilde{X}}$ is not stable with respect to the $\widetilde{H}_i$. In this setting, Theorem 7.1 asserts that there exists a further finite cover $h : \widehat{X} \to \widetilde{X}$ and a proper decomposition

$$\mathscr{T}_{\widehat{X}} \cong \mathscr{E} \oplus \mathscr{F}. \tag{8.20.1}$$

with $\det \mathscr{E} \cong \mathscr{O}_{\widehat{X}}$. Setting $r := \operatorname{rank} \mathscr{E}$, the splitting (8.20.1) immediately gives an embedding $\mathscr{O}_{\widehat{X}} \cong \det \mathscr{E}^* \hookrightarrow \Omega^{[r]}_{\widehat{X}}$, and an associated form $\tau \in H^0\big(X, \Omega^{[r]}_{\widehat{X}}\big)$. Since $0 < r < \dim X$, it follows that X cannot be Calabi-Yau.

Since $\mathscr{F}$ is contained in the degeneracy subsheaf S_τ, as introduced in Definition 8.7, it is clear that τ cannot be a wedge-power of the pullback of any symplectic form on X. This rules out that X is irreducible holomorphic-symplectic in the sense of Definition 8.16. We obtain a contradiction, which finishes the proof of Proposition 8.20. Q.E.D.

A positive solution to Problem 8.11 would immediately give a partial converse to Proposition 8.20.

Proposition 8.21 (Characterisation of strongly stable pieces, II). *Let X be a normal projective variety with $\omega_X = \mathscr{O}_X$ having at worst canonical singularities. Assume that $\mathscr{T}_X$ is strongly stable. If the assertions of Problem 8.11 hold, then either*

(8.21.1) *the semistable sheaf $\wedge^{[2]} \mathscr{T}_X$ is strongly stable, and X is Calabi-Yau, or*

(8.21.2) *there exists a finite cover $\widetilde{X} \to X$, étale in codimension one, such that the sheaf $\wedge^{[2]} \mathscr{T}_{\widetilde{X}}$ is not $\widetilde{H}$-stable for some polarisation $\widetilde{H}$ on $\widetilde{X}$, and $\widetilde{X}$ is irreducible holomorphic-symplectic.* Q.E.D.

Remark 8.22. In the second case of the previous proposition one would of course rather like X itself to be irreducible holomorphic-symplectic. This is in fact true if X is additionally assumed to be smooth: If X is not Calabi–Yau, then the universal cover $\widetilde{X}$ of X is irreducible holomorphic-symplectic. Since additionally the canonical bundle ω_X is assumed to be trivial, Lemma 8.14 implies that X itself is irreducible holomorphic-symplectic.

8.C. Fundamental groups of varieties with trivial canonical class

A Kähler manifold X with trivial canonical class and vanishing augmented irregularity $\widetilde{q}(X)$ has finite fundamental group, see [Bea83, Thm. 1]. We believe that the same should hold for projective varieties, in our singular setting. We show that this is true, at least under the assumption that $\chi(X, \mathscr{O}_X) \neq 0$.

Proposition 8.23 (Fundamental groups of canonical varieties with $K_X \equiv 0$, I). *Let X be a normal projective variety with at worst canonical* *singularities. If K_X is torsion and if $\chi(X, \mathscr{O}_X) \neq 0$, then $\pi_1(X)$ is finite, of cardinality*

$$|\pi_1(X)| \leq \frac{2^{n-1}}{|\chi(X, \mathscr{O}_X)|}.$$

Remark 8.24. If X is smooth and K_X is torsion, then then classical Beauville-Bogomolov Decomposition Theorem 1.1 together with Proposition 8.23 shows that $\chi(X, \mathscr{O}_X) \neq 0$ implies $\widetilde{q}(X) = 0$.

Proof of Proposition 8.23. Set $n := \dim X$. Let $\pi : \widetilde{X} \to X$ be a strong resolution of singularities. Recalling from [KM98, Thm. 5.22] that X has rational singularities, we obtain that $\chi(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = \chi(X, \mathscr{O}_X) \neq 0$. Consider the invariant

$$\kappa^+\big(\widetilde{X}\big) := \max\Big\{\kappa\big(\det \mathscr{F}\big) \;\big|\; \mathscr{F} \subseteq \Omega^p_{\widetilde{X}} \text{ a coherent subsheaf, for some } p\Big\}.$$

We are going to show that $\kappa^+\big(\widetilde{X}\big) = 0$. Using the assumption that $\chi(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) \neq 0$, Campana has then shown in [Cam95, Cor. 5.3] that $\pi_1(\widetilde{X})$ is finite, of cardinality at most $2^{n-1} \cdot |\chi(\widetilde{X}, \mathscr{O}_{\widetilde{X}})|^{-1}$. Since the natural map $\pi_1\big(\widetilde{X}\big) \to \pi_1(X)$ is isomorphic by [Tak03, Thm. 1.1], this implies that $\pi_1(X)$ is likewise finite of the same cardinality.

So let $0 \leq p \leq n$ be any number and let $\mathscr{F} \subseteq \Omega^p_{\widetilde{X}}$ be a coherent subsheaf. As a subsheaf of a torsion-free sheaf, $\mathscr{F}$ is itself torsion-free, and therefore locally free in codimension one. Next, let $C \subset X$ be a general complete intersection curve. Recall that the strong resolution map π is isomorphic along C, and denote the preimage curve by $\widetilde{C} := \pi^{-1}(C)$. The restricted sheaves $\Omega^p_{\widetilde{X}}\big|_{\widetilde{C}}$ and $\mathscr{F}|_{\widetilde{C}}$ are then both locally free.

Since K_X is torsion, the Kodaira-dimension of X is zero, $\kappa(X) = 0$. As X has only canonical singularities, this shows that X is not covered by rational curves. Miyaoka's Generic Semipositivity Theorem 5.1 therefore implies that $\Omega^q_{\widetilde{X}}|_{\widetilde{C}}$ is nef for all q. Better still, we have $\deg \Omega^n_{\widetilde{X}}|_{\widetilde{C}} = 0$, so that

$$\big(\Omega^p_{\widetilde{X}}|_{\widetilde{C}}\big)^* \cong \wedge^p \mathscr{T}_X|_{\widetilde{C}} \cong \operatorname{Hom}\Big(\Omega^n_{\widetilde{X}}|_{\widetilde{C}},\, \Omega^{n-p}_{\widetilde{X}}|_{\widetilde{C}}\Big) \cong \big(\Omega^n_{\widetilde{X}}|_{\widetilde{C}}\big)^* \otimes \Omega^{n-p}_{\widetilde{X}}|_{\widetilde{C}}$$

is likewise as a nef vector bundle on the curve $\widetilde{C}$. Its quotient $\mathscr{F}^*|_{\widetilde{C}}$ is then nef as well. In summary, we obtain that $c_1(\mathscr{F}) \cdot \widetilde{C} \leq 0$. Since the curves $\widetilde{C}$ are moving, this implies $\kappa(\det \mathscr{F}) \leq 0$, and therefore $\kappa^+\big(\widetilde{X}\big) \leq 0$. Since $\kappa\big(\widetilde{X}\big) = 0$, we obtain $\kappa^+\big(\widetilde{X}\big) = 0$, as claimed. This finishes the proof of Proposition 8.23. Q.E.D.

Corollary 8.25 (Fundamental groups of canonical varieties with $K_X \equiv 0$, II)**.** *Let X be a normal projective variety with at worst canonical singularities. Assume that* $\dim X \leq 4$*, and that the canonical divisor K_X is numerically trivial. Then $\pi_1(X)$ is almost Abelian, that is, $\pi_1(X)$ contains an Abelian subgroup of finite index.*

Proof. Recall from [Kol95, 4.17.3] that the statement of Corollary 8.25 is well-known if $\dim X \leq 3$. We will therefore assume for the remainder of the proof that X is of dimension four.

Let $f : \widetilde{X} \to X$ be the index-one cover associated with K_X. As we have noted before, f is étale in codimension one, $\widetilde{X}$ has canonical singularities, and $\omega_{\widetilde{X}} = \mathscr{O}_{\widetilde{X}}$, cf. [KM98, 5.19 and 5.20]. The image of the natural map $\pi_1(\widetilde{X}) \to \pi_1(X)$ has finite index in $\pi_1(X)$, cf. [Kol95, Prop. 2.10(2)] and [Cam91, Prop. 1.3]. Replacing X by $\widetilde{X}$, if necessary, we may therefore assume without loss of generality that ω_X is trivial. Passing to a further cover, Corollary 3.6 even allows to assume that X is of the form $X = A \times Z$, where A is an Abelian variety, and Y is normal projective variety with at worst canonical singularities, with trivial canonical class and vanishing augmented irregularity, $\omega_Z \cong \mathscr{O}_Z$ and $\widetilde{q}(Z) = 0$.

If $\dim Z \leq 3$, then [Kol95, 4.17.3] asserts that $\pi_1(Z)$ is almost Abelian. Since $\pi_1(A)$ is Abelian, this finishes the proof.

It remains to consider that case where $\dim Z = 4$, that is, where $X = Z$ and $\widetilde{q}(Z) = 0$. In this case, we finish proof by showing that the fundamental group of X is finite. Recall from Corollary 6.11 that X does not carry any reflexive 1-form or 3-forms. Using Proposition 6.9 to relate $H^p\big(X, \mathscr{O}_X\big)$ with the space of reflexive p-forms we see that $\chi(X, \mathcal{O}_X) > 0$, and we conclude by Proposition 8.23 that $\pi_1(X)$ is finite, thus finishing the proof of Corollary 8.25. Q.E.D.

Remark 8.26 (Fundamental groups of smooth 4-folds with $\kappa = 0$). If X is a smooth projective 4-fold with $\kappa(X) = 0$ admitting a good minimal model X', then $\pi_1(X) = \pi_1(X')$ by [Tak03, Thm. 1.1] or [Kol93, Thm. 7.8.1]. Consequently, $\pi_1(X)$ is almost Abelian.

Assuming that the claims of Problem 8.11 hold, the following corollary complements the results obtained in Section 6, and in particular the results of Corollary 6.10.

Corollary 8.27 (Fundamental groups of even-dim. X with $\mathscr{T}_X$ strongly stable)**.** *Let X be a normal projective variety with $\omega_X \cong \mathscr{O}_X$ having at worst canonical singularities. Suppose furthermore that* $\dim X$ *is even and that $\mathscr{T}_X$ is strongly stable. If Problem 8.11 has a positive solution, then $\chi(X, \mathscr{O}_X) > 0$ and $\pi_1(X)$ is finite.* *Q.E.D.*

References

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, Journal of the AMS **23** (2010), 405–468, DOI:10.1090/S0894-0347-09-00649-3. MR 2601039 (2011f:14023)

[Bea83] Arnaud Beauville, *Variétés Kählériennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), no. 4, 755–782 (1984). MR 730926 (86c:32030)

[BNWS11] Samuel Boissière, Marc Nieper-Wißkirchen, and Alessandra Sarti, *Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties*, J. Math. Pures Appl. (9). **95** (2011), no. 5. 553–563. DOI:10.1016/j.matpur.2010.12.003. MR 2786223

[Bog74] Fedor A. Bogomolov, *Kähler manifolds with trivial canonical class*, Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 11–21. MR 0338459 (49 #3223)

[Cam91] Frédéric Campana, *On twistor spaces of the class $\mathscr{C}$*, J. Differential Geom. **33** (1991), no. 2, 541–549. MR 1094468 (92g:32059)

[Cam95] ———, *Fundamental group and positivity of cotangent bundles of compact Kähler manifolds*, J. Algebraic Geom. **4** (1995), no. 3, 487–502. MR 1325789 (96f:32054)

[CP11] Frédéric Campana and Thomas Peternell, *Geometric stability of the cotangent bundle and the universal cover of a projective manifold*, Bull. Soc. Math. France **139** (2011), no. 1, 41–74.

[Dem02] Jean-Pierre Demailly, *On the Frobenius integrability of certain holomorphic p-forms*, Complex geometry (Göttingen, 2000), Springer, Berlin, 2002, pp. 93–98. MR 1922099 (2003f:32029)

[Dem09] ———, *Complex Analytic and Differential Geometry*, September 2009, OpenContent Book.
URL: http://www-fourier.ujf-grenoble.fr/∼demailly/books.html.

[Fle84] Hubert Flenner, *Restrictions of semistable bundles on projective varieties*, Comment. Math. Helv. **59** (1984), no. 4, 635–650. MR 780080 (86m:14014)

[GKK10] Daniel Greb, Stefan Kebekus, and Sándor J Kovács, *Extension theorems for differential forms, and Bogomolov-Sommese vanishing on log canonical varieties*, Compos. Math. **146** (2010), 193–219, DOI:10.1112/S0010437X09004321. MR 2581247 (2011c:14054). A slightly extended version is available as arXiv:0808.3647.

[GKKP11] Daniel Greb, Stefan Kebekus, Sándor Kovács, and Thomas Peternell, *Differential forms on log canonical spaces*, Publ. Math. IHES. **114** (2011), no. 1, 87–169. DOI:10.1007/s10240-011-0036-0. MR 2854859. An extended version with additional graphics is available as arXiv:1003.2913.

[HL97] Daniel Huybrechts and Manfred Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997. MR 1450870 (98g:14012)

[HNW11] Daniel Huybrechts and Marc Nieper-Wisskirchen, *Remarks on derived equivalences of Ricci-flat manifolds*, Math. Z. **267** (2011), no. 3-4, 939–963, DOI:10.1007/s00209-009-0655-z. MR 2776067

[Hör07] Andreas Höring, *Uniruled varieties with split tangent bundle*, Math. Z. **256** (2007), no. 3, 465–479, DOI:10.1007/s00209-006-0072-5. MR 2299565 (2008f:32024)

[Kaw85] Yujiro Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. **363** (1985), 1–46, DOI:10.1515/crll.1985.363.1. MR 814013 (87a:14013)

[Kaw08] ———, *Flops connect minimal models*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 419–423, DOI:10.2977/prims/1210167332. MR 2426353 (2009d:14011)

[Keb13] Stefan Kebekus, *Differential forms on singular spaces, the minimal model program, and hyperbolicity of moduli stacks*, in: Handbook of Moduli, Volume II, editors Gavril Farkas and Ian Morrison. Number 26 in Advanced Lectures in Mathematics, International Press, Beijing, 2013. Also available as arXiv:1107.4239.

[Kle66] Steven L. Kleiman, *Toward a numerical theory of ampleness*, Ann. of Math. (2) **84** (1966), 293–344. MR 0206009 (34 #5834)

[KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR 2000b:14018

[Kob87] Shoshichi Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, vol. 15, Iwanami Shoten and Princeton University Press, Princeton, NJ, 1987, Kanô Memorial Lectures, 5. MR 909698 (89e:53100)

[Kol93] János Kollár, *Shafarevich maps and plurigenera of algebraic varieties*, Invent. Math. **113** (1993), no. 1, 177–215, DOI:10.1007/BF01244307. MR 1223229 (94m:14018)

[Kol95] ———, *Shafarevich maps and automorphic forms*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 1995. MR 1341589 (96i:14016)

[Miy87a] Yoichi Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 449–476. MR 89k:14022

[Miy87b] ———, *Deformations of a morphism along a foliation and applications*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 245–268. MR 927960 (89e:14011)

[MR82] Vikram B. Mehta and Annamalai Ramanathan, *Semistable sheaves on projective varieties and their restriction to curves*, Math. Ann. **258** (1981/82), no. 3, 213–224. MR 83f:14013

[OS11] Keiji Oguiso and Stefan Schröer, *Enriques manifolds*, J. Reine Angew. Math. **661** (2011), 215–235, DOI:10.1515/CRELLE.2011.077.

[OSS80] Christian Okonek, Michael Schneider, and Heinz Spindler, *Vector bundles on complex projective spaces*, Progress in Mathematics, vol. 3, Birkhäuser Boston, Mass., 1980. MR 561910 (81b:14001)

[Pet94] Thomas Peternell, *Minimal varieties with trivial canonical classes. I*, Math. Z. **217** (1994), no. 3, 377–405, DOI:10.1007/BF02571950. MR 1306667 (96f:14041)

[Rei87] Miles Reid, *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 345–414. MR 927963 (89b:14016)

[Ros68] Hugo Rossi, *Picard variety of an isolated singular point*, Rice Univ. Studies **54** (1968), no. 4, 63–73. MR 0244517 (39 #5831)

[Tak03] Shigeharu Takayama, *Local simple connectedness of resolutions of log-terminal singularities*, Internat. J. Math. **14** (2003), no. 8, 825–836, DOI:10.1142/S0129167X0300196X. MR 2013147 (2004m:14023)

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