Advanced Studies in Pure Mathematics 69, 2016
Development of Moduli Theory — Kyoto 2013
pp. 287–302

Divisors on Burniat surfaces

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Abstract.

In this short note, we extend the results of [Alexeev-Orlov, 2012] about Picard groups of Burniat surfaces with $K^2 = 6$ to the cases of $2 \le K^2 \le 5$. We also compute the semigroup of effective divisors on Burniat surfaces with $K^2 = 6$. Finally, we construct an exceptional collection on a nonnormal semistable degeneration of a 1-parameter family of Burniat surfaces with $K^2 = 6$.

Dedicated to Prof. Shigeru Mukai on the occasion of his 60th birthday

Contents

§ Introduction

This note strengthens and extends several geometric results of the paper [AO12], joint with Dmitri Orlov, in which we constructed exceptional sequences of maximal possible length on Burniat surfaces with $K^2 = 6$. The construction was based on certain results about the Picard group and effective divisors on Burniat surfaces.

Here, we extend the results about Picard group to Burniat surfaces with $2 \le K^2 \le 5$. We also establish a complete description of the

Received November 24, 2013.
Revised February 23, 2014.
2010 *Mathematics Subject Classification.* 14J29, 14J10, 18E30.
Key words and phrases. Burniat surfaces, derived categories, exceptional collections.

semigroup of effective $\mathbb{Z}$-divisors on Burniat surfaces with $K_X^2 = 6$. (For the construction of exceptional sequences in [AO12] only a small portion of this description was needed.)

Finally, we construct an exceptional collection on a nonnormal semistable degeneration of a 1-parameter family of Burniat surfaces with $K^2 = 6$.

§1. Definition of Burniat surfaces

In this paper, Burniat surfaces will be certain smooth surfaces of general type with $q = p_g = 0$ and $2 \le K^2 \le 6$ with big and nef canonical class K which were defined by Peters in [Pet77] following Burniat. They are Galois $\mathbb{Z}_2^2$-covers of (weak) del Pezzo surfaces with $2 \le K^2 \le 6$ ramified in certain special configurations of curves.

Recall from [Par91] that a $\mathbb{Z}_2^2$-cover $\pi\colon X \to Y$ with smooth and projective X and Y is determined by three branch divisors $\bar A, \bar B, \bar C$ and three invertible sheaves L_1, L_2, L_3 on the base Y satisfying fundamental relations $L_2 \otimes L_3 \simeq L_1(\bar A)$, $L_3 \otimes L_1 \simeq L_2(\bar B)$, $L_1 \otimes L_2 \simeq L_3(\bar C)$. These relations imply that $L_1^2 \simeq \mathcal{O}_Y(\bar B + \bar C)$, $L_2^2 \simeq \mathcal{O}_Y(\bar C + \bar A)$, $L_3^2 \simeq \mathcal{O}_Y(\bar A + \bar B)$.

One has $X = \operatorname{Spec}_Y \mathcal{A}$, where the $\mathcal{O}_Y$-algebra $\mathcal{A}$ is $\mathcal{O}_Y \oplus \oplus_{i=1}^3 L_i^{-1}$. The multiplication is determined by three sections in

$$\operatorname{Hom}(L_i^{-1} \otimes L_j^{-1}, L_k^{-1}) = H^0(L_i \otimes L_j \otimes L_i^{-1}),$$

where $\{i,j,k\}$ is a permutation of $\{1,2,3\}$, i.e. by sections of the sheaves $\mathcal{O}_Y(\bar A)$, $\mathcal{O}_Y(\bar B)$, $\mathcal{O}_Y(\bar C)$ vanishing on $\bar A$, $\bar B$, $\bar C$.

Burniat surfaces with $K^2 = 6$ are defined by taking Y to be the del Pezzo surface of degree 6, i.e. the blowup of $\mathbb{P}^2$ in three noncollinear points, and the divisors $\bar A = \sum_{i=0}^3 \bar A_i$, $\bar B = \sum_{i=0}^3 \bar B_i$, $\bar C = \sum_{i=0}^3 \bar C_i$ to be the ones shown in red, blue, and black in the central picture of Figure 1 below.

The divisors $\bar A_i, \bar B_i, \bar C_i$ for $i = 0, 3$ are the (-1)-curves, and those for $i = 1, 2$ are 0-curves, fibers of rulings $\operatorname{Bl}_3 \mathbb{P}^2 \to \mathbb{P}^1$. The del Pezzo surface also has two contractions to $\mathbb{P}^2$ related by a quadratic transformation, and the images of the divisors form a special line configuration on either $\mathbb{P}^2$. We denote the fibers of the three rulings f_1, f_2, f_3 and the preimages of the hyperplanes from $\mathbb{P}^2$'s by h_1, h_2.

Burniat surfaces with $K^2 = 6 - k$, $1 \le k \le 4$ are obtained by considering a special configuration in Figure 1 for which some k triples of curves, one from each group $\{\bar A_1, \bar A_2\}$, $\{\bar B_1, \bar B_2\}$, $\{\bar C_1, \bar C_2\}$, meet at common points P_s. The corresponding Burniat surface is the $\mathbb{Z}_2^2$-cover of the blowup of $\operatorname{Bl}_3 \mathbb{P}^2$ at these points.

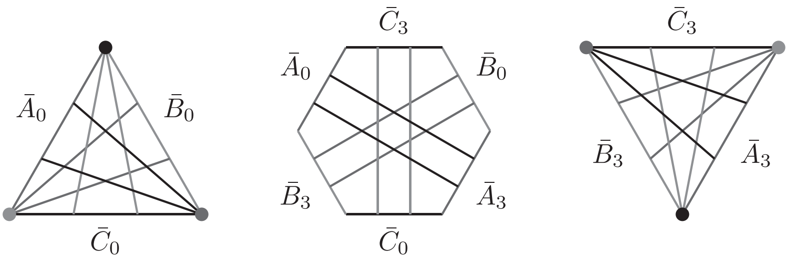

Fig. 1. Burniat configuration on $\mathrm{Bl}_3\,\mathbb{P}^2$

Up to symmetry, there are the following cases, see [BC11]:

(1) $K^2 = 5$: $P_1 = \bar{A}_1\bar{B}_1\bar{C}_1$ (our shortcut notation for $\bar{A}_1 \cap \bar{B}_1 \cap \bar{C}_1$).
(2) $K^2 = 4$, nodal case: $P_1 = \bar{A}_1\bar{B}_1\bar{C}_1$, $P_2 = \bar{A}_1\bar{B}_2\bar{C}_2$.
(3) $K^2 = 4$, non-nodal case: $P_1 = \bar{A}_1\bar{B}_1\bar{C}_1$, $P_2 = \bar{A}_2\bar{B}_2\bar{C}_2$.
(4) $K^2 = 3$: $P_1 = \bar{A}_1\bar{B}_1\bar{C}_2$, $P_2 = \bar{A}_1\bar{B}_2\bar{C}_1$, $P_3 = \bar{A}_2\bar{B}_1\bar{C}_1$.
(5) $K^2 = 2$: $P_1 = \bar{A}_1\bar{B}_1\bar{C}_1$, $P_2 = \bar{A}_1\bar{B}_2\bar{C}_2$, $P_3 = \bar{A}_2\bar{B}_1\bar{C}_2$, $P_4 = \bar{A}_2\bar{B}_2\bar{C}_1$.

Notation 1.1. We generally denote the divisors upstairs by D and the divisors downstairs by $\bar{D}$ for the reasons which will become clear from Lemmas 2.1, 3.1. We denote $Y = \mathrm{Bl}_3\,\mathbb{P}^2$ and $\epsilon\colon Y' \to Y$ is the blowup map at the points P_s. The exceptional divisors are denoted by $\bar{E}_s$.

The curves $\bar{A}_i, \bar{B}_i, \bar{C}_i$ are the curves on Y, the curves $\bar{A}'_i, \bar{B}'_i, \bar{C}'_i$ are their strict preimages under ϵ. (So that $\epsilon^*(\bar{A}_1) = \bar{A}'_1 + E_1$ in the case (1), etc.) The divisors A'_i, B'_i, C'_i, E_s are the curves (with reduced structure) which are the preimages of the latter curves and $\bar{E}_s$ under $\pi'\colon X' \to Y'$. The surface X' is the Burniat surface with $K^2 = 6 - k$.

The building data for the $\mathbb{Z}_2^2$-cover $\pi'\colon X' \to Y'$ consists of three divisors $A' = \sum \bar{A}'_i$, $B' = \sum \bar{B}'_i$, $C' = \sum \bar{C}'_i$. It does *not* include the exceptional divisors $\bar{E}_s$, they are not in the ramification locus.

One has $\pi'^*(\bar{A}'_i) = 2A'_i$, $\pi'^*(\bar{B}'_i) = 2B'_i$, $\pi'^*(\bar{C}'_i) = 2C'_i$, and $\pi'^*(\bar{E}_s) = E_s$.

For the canonical class, one has $2K_{X'} = \pi^*(-K_{Y'})$. Indeed, from Hurwitz formula $2K_{X'} = \pi^*(2K_{Y'} + R')$, where $R' = A' + B' + C'$. Therefore, the above identity is equivalent to $R' = -3K_{Y'}$. This holds on $Y = \mathrm{Bl}_3\,\mathbb{P}^2$, and

$$R' = \epsilon^* R - 3\sum \bar{E}_s = \epsilon^*(-3K_Y) - 3\sum \bar{E}_s = -3K_{Y'}.$$

For the surfaces with $K^2 = 6, 5$ and 4 (non-nodal case), $-K_Y$ and K_X are ample. For the remaining cases, including $K^2 = 2, 3$, the divisors $-K_Y$ and K_X are big, nef, but not ample. Each of the curves $\bar{L}_j$ (among $\bar{A}_i, \bar{B}_i, \bar{C}_i$) through two of the points P_s is a (-2)-curve (a $\mathbb{P}^1$ with square -2) on the surface Y. (For example, for the nodal case with $K^2 = 4$ $\bar{L}_1 = \bar{A}_1$ is such a line). Its preimage, a curve L_j on X, is also a (-2)-curve. One has $-K_Y \bar{L}_j = K_X L_j = 0$, and the curve L_j is contracted to a node on the canonical model of X.

Note that both of the cases with $K^2 = 2$ and 3 are nodal.

§2. Picard group of Burniat surfaces with $K^2 = 6$

In this section, we recall two results of [AO12].

Lemma 2.1 ([AO12], Lemma 1)**.** *The homomorphism $\bar{D} \mapsto \frac{1}{2}\pi^*(\bar{D})$ defines an isomorphism of integral lattices $\frac{1}{2}\pi^*\colon \operatorname{Pic} Y \to \operatorname{Pic} X/\operatorname{Tors}$. Under this isomorphism, one has $\frac{1}{2}\pi^*(-K_Y) = K_X$.*

This lemma allows one to identify $\mathbb{Z}$-divisors $\bar{D}$ on the del Pezzo surface Y with classes of $\mathbb{Z}$-divisors D on X up to torsion, equivalently up to numerical equivalence. This identification preserves the intersection form.

The curves A_0, B_0, C_0 are elliptic curves (and so are the curves $A_3 \simeq A_0$, etc.). Moreover, each of them comes with a canonical choice of an origin, denoted P_{00}, which is the point of intersection with the other curves which has a distinct color, different from the other three points. (For example, for A_0 one has $P_{00} = A_0 \cap B_3$.)

On the elliptic curve A_0 one also defines $P_{10} = A_0 \cap C_3$, $P_{01} = A_0 \cap C_1$, $P_{11} = A_0 \cap C_2$. This gives the 4 points in the 2-torsion group $A_0[2]$. We do the same for B_0, C_0 cyclically.

Theorem 2.2. *[*[AO12]*, Theorem 1] One has the following:*

(1) *The homomorphism*

$$\begin{array}{rcl} \phi\colon \operatorname{Pic} X & \to & \mathbb{Z} \times \operatorname{Pic} A_0 \times \operatorname{Pic} B_0 \times \operatorname{Pic} C_0 \\ L & \mapsto & (d(L) = L \cdot K_X,\ L|_{A_0},\ L|_{B_0},\ L|_{C_0}) \end{array}$$

is injective, and the image is the subgroup of index 3 of

$$\mathbb{Z} \times (\mathbb{Z}.P_{00} + A_0[2]) \times (\mathbb{Z}.P_{00} + B_0[2]) \times (\mathbb{Z}.P_{00} + C_0[2]) \simeq \mathbb{Z}^4 \times \mathbb{Z}_2^6.$$

consisting of the elements with $d + a_0^0 + b_0^0 + c_0^0$ divisible by 3. Here, we denote an element of the group $\mathbb{Z}.P_{00} + A_0[2]$ by $(a_0^0\ a_0^1 a_0^2)$, etc., where $a_0^0 = \deg L|_{A_0}$, etc.

(2) *ϕ induces an isomorphism* $\mathrm{Tors}(\operatorname{Pic} X) \to A_0[2] \times B_0[2] \times C_0[2]$.
(3) *The curves* A_i, B_i, C_i, $0 \le i \le 3$, *generate* $\operatorname{Pic} X$.

This theorem provides one with explicit coordinates for the Picard group of a Burniat surface X, convenient for making computations.

§3. Picard group of Burniat surfaces with $2 \le K^2 \le 5$

In this section, we extend the results of the previous section to the cases $2 \le K^2 \le 5$. First, we show that Lemma 2.1 holds verbatim if $3 \le K^2 \le 5$.

Lemma 3.1. *Assume* $3 \le K^2 \le 5$. *Then the homomorphism* $\bar{D} \mapsto \frac{1}{2}\pi'^*(\bar{D})$ *defines an isomorphism of integral lattices* $\frac{1}{2}\pi'^*\colon \operatorname{Pic} Y' \to \operatorname{Pic} X'/\operatorname{Tors}$, *and the inverse map is* $\frac{1}{2}\pi'_*$. *Under this isomorphism, one has* $\frac{1}{2}\pi'^*(-K_{Y'}) = K_{X'}$.

Proof. The proof is similar to that of Lemma 2.1. The map $\frac{1}{2}\pi^*$ establishes an isomorphism of $\mathbb{Q}$-vector spaces $(\operatorname{Pic} Y')\otimes\mathbb{Q}$ and $(\operatorname{Pic} X')\otimes\mathbb{Q}$ together with the intersection product because:

(1) Since $h^i(\mathcal{O}_{X'}) = h^i(\mathcal{O}_{Y'}) = 0$ for $i = 1, 2$ and $K^2_{X'} = K^2_{Y'}$, by Noether's formula the two vector spaces have the same dimension.
(2) $\frac{1}{2}\pi'^*\bar{D}_1 \cdot \frac{1}{2}\pi'^*\bar{D}_2 = \frac{1}{4}\pi'^*(\bar{D}_1 \cdot \bar{D}_2) = \bar{D}_1\bar{D}_2$.

A crucial observation is that $\frac{1}{2}\pi'^*$ sends $\operatorname{Pic} Y'$ to integral classes. To see this, it is sufficient to observe that $\operatorname{Pic} Y'$ is generated by divisors $\bar{D}$ which are in the ramification locus and thus for which $D = \frac{1}{2}\pi'^*(\bar{D})$ is integral.

Consider for example the case of $K^2 = 5$. One has $\operatorname{Pic} Y' = \epsilon^*(\operatorname{Pic} Y)\oplus\mathbb{Z}E$. The group $\epsilon^*(\operatorname{Pic} Y)$ is generated by $\bar{A}'_0$, $\bar{B}'_0$, $\bar{C}'_0$, $\bar{A}'_3$, $\bar{B}'_3$, $\bar{C}'_3$. Since $\epsilon^*(\bar{A}_1) = \bar{A}'_1 + \bar{E}_1$, the divisor class $\bar{E}_1$ lies in group spanned by $\bar{A}'_1$ and $\epsilon^*(\operatorname{Pic} Y)$. So we are done.

In the nodal case $K^2 = 4$, $\bar{E}_1$ is spanned by $\bar{B}'_1$ and $\epsilon^*(\operatorname{Pic} Y)$, $\bar{E}_2$ by $\bar{B}'_2$ and $\epsilon^*(\operatorname{Pic} Y)$; exactly the same for the non-nodal case. In the case $K^2 = 3$, $\bar{E}_1$ is spanned by $\bar{C}'_2$ and $\epsilon^*(\operatorname{Pic} Y)$, $\bar{E}_2$ by $\bar{B}'_2$ and $\epsilon^*(\operatorname{Pic} Y)$, $\bar{E}_3$ by $\bar{A}'_2$ and $\epsilon^*(\operatorname{Pic} Y)$.

Therefore, $\frac{1}{2}\pi'^*(\operatorname{Pic} Y')$ is a sublattice of finite index in $\operatorname{Pic} X'/\operatorname{Tors}$. Since the former lattice is unimodular, they must be equal.

One has $\frac{1}{2}\pi'_* \circ \frac{1}{2}\pi'^*(\bar{D}) = \bar{D}$, so the inverse map is $\frac{1}{2}\pi'_*$. Q.E.D.

Remark 3.2. I thank Stephen Coughlan for pointing out that the above proof that $\operatorname{Pic} Y'$ is generated by the divisors in the ramification locus does not work in the $K^2 = 2$ case. In this case, each of the lines $\bar{A}_i, \bar{B}_i, \bar{C}_i$, $i = 1, 2$ contains exactly two of the points P_1, P_2, P_3. What we can see easily is the following: there exists a free abelian group $H \simeq \mathbb{Z}^8$ which can be identified with a subgroup of index 2 in $\operatorname{Pic} Y'$ and a subgroup of index 2 in $\operatorname{Pic} X'/\operatorname{Tors}$.

Consider a $\mathbb{Z}$-divisor (not a divisor class) on Y'

$$\bar{D} = a_0\bar{A}'_0 + \ldots + c_3\bar{C}'_3 + \sum_s e_s\bar{E}_s$$

such that the coefficients e_s of $\bar{E}_s$ are even. Then we can define a canonical lift

$$D = a_0A_0 + \ldots + c_3C_3 + \sum_s \frac{1}{2}e_sE_s,$$

which is a divisor on X', and numerically one has $D = \frac{1}{2}\pi'^*(\bar{D})$. Note that $\bar{D}$ is linearly equivalent to 0 iff D is a torsion.

By Theorem 2.2, for a Burniat surface with $K^2 = 6$, we have an identification

$$V := \operatorname{Tors}\operatorname{Pic} X = A_0[2] \times B_0[2] \times C_0[2] = \mathbb{Z}_2^2 \times \mathbb{Z}_2^2 \times \mathbb{Z}_2^2.$$

It is known (see [BC11]) that for Burniat surfaces with $2 \le K^2 \le 6$ one has $\operatorname{Tors}\operatorname{Pic} X \simeq \mathbb{Z}_2^{K^2}$ with the exception of the case $K^2 = 2$ where $\operatorname{Tors}\operatorname{Pic} X \simeq \mathbb{Z}_2^3$. We would like to establish a convenient presentation for the Picard group and its torsion for these cases which would be similar to the above.

For the above definiiton, recall the standard coordinates on V given at the beginning of Section 2.

Definition 3.3. We define the following vectors, forming a basis in the $\mathbb{Z}_2$-vector space V: $\vec{A}_1 = 00\ 10\ 00$, $\vec{A}_2 = 00\ 11\ 00$, $\vec{B}_1 = 00\ 00\ 10$, $\vec{B}_2 = 00\ 00\ 11$, $\vec{C}_1 = 10\ 00\ 00$, $\vec{C}_2 = 11\ 00\ 00$.

Further, for each point $P_s = A_iB_jC_k$ we define a vector $\vec{P}_s = \vec{A}_i + \vec{B}_j + \vec{C}_k$.

Definition 3.4. We also define the standard bilinear form $V \times V \to \mathbb{Z}_2$: $(x_1, \ldots, x_6) \cdot (y_1, \ldots, y_6) = \sum_{i=1}^6 x_iy_i$.

Lemma 3.5. *The restriction map* $\rho\colon \operatorname{Tors}\operatorname{Pic}(X') \to A_0[2] \times B_0[2] \times C_0[2]$ *is injective, and the image is identified with the orthogonal complement of the subspace generated by the vectors* $\vec{P}_s$*.*

Proof. The restrictions of the following divisors to V give the subset $B_0[2]$:

$$0,\ A_1-A_2 = 00\,10\,00,\ A_1-A_3-C_0 = 00\,11\,00,\ A_2-A_3-C_0 = 00\,01\,00.$$

Among these, the divisors containing A_1 are precisely those for which the vector $v \in B_0[2] \subset V$ satisfies $v \cdot \vec{A}_1 = 1$. Repeating this verbatim, one has the same results for the divisors $A_2, \dots, C_2$ and vectors $\vec{A}_2, \dots, \vec{C}_2$.

Let $\bar{D}$ be a linear combination of the divisors $\bar{A}_1 - \bar{A}_2$, $\bar{A}_1 - \bar{A}_3 - \bar{C}_0$, $\bar{A}_2 - \bar{A}_3 - \bar{C}_0$, and the corresponding divisors for $C_0[2]$, $A_0[2]$. Define the vector $v(D) \in V$ to be the sum of the corresponding vectors $A_1 - A_2 \in V$, etc.

Now assume that the vector $v(D)$ satisfies the condition $v(D) \cdot \vec{P}_s = 0$ for all the points P_s. Then the coefficients of the exceptional divisors $\bar{E}_s$ in the divisor $\epsilon^*(\bar{D})$ on Y' are even (and one can also easily arrange them to be zero since the important part is working modulo 2). Therefore, a lift of $\epsilon^*(\bar{D})$ to X' is well defined and is a torsion in $\operatorname{Pic}(X')$.

This shows that the image of the homomorphism $\rho\colon \operatorname{Tors}\operatorname{Pic} X' \to V$ contains the space $\langle \vec{P}_s \rangle^{\perp}$. But this space already has the correct dimension. Indeed, for $3 \le K^2 \le 5$ the vectors $\vec{P}_s$ are linearly independent, and for $K^2 = 2$ the vectors $\vec{P}_1 = \vec{A}_1 + \vec{B}_1 + \vec{C}_1$, $\vec{P}_2 = \vec{A}_1 + \vec{B}_2 + \vec{C}_2$, $\vec{P}_3 = \vec{A}_2 + \vec{B}_1 + \vec{C}_2$, $\vec{P}_4 = \vec{A}_2 + \vec{B}_2 + \vec{C}_1$ are linearly dependent (their sum is zero) and span a subspace of dimension 3; thus the orthogonal complement has dimension 3 as well. Therefore, ρ is a bijection of $\operatorname{Tors}\operatorname{Pic}(X')$ onto $\langle \vec{P}_s \rangle^{\perp}$. Q.E.D.

Theorem 3.6. *Let $3 \le K^2 \le 5$. Then one has the following:*

(1) *The homomorphism*

$$\begin{array}{rcl} \phi\colon \operatorname{Pic} X' & \to & \mathbb{Z}^{1+k} \times \operatorname{Pic} A_0' \times \operatorname{Pic} B_0' \times \operatorname{Pic} C_0' \\ L & \mapsto & (d(L) = L \cdot K_{X'},\ L \cdot \frac{1}{2} E_s,\ L|_{A_0'},\ L|_{B_0'},\ L|_{C_0'}) \end{array}$$

is injective, and the image is the subgroup of index $3 \cdot 2^n$ in $\mathbb{Z}^{4+k} \times A_0'[2] \times B_0'[2] \times C_0'[2]$, where $n = 6 - K^2$ for $3 \le K^2 \le 6$ and $n = 3$ for $K^2 = 2$.

(2) *ϕ induces an isomorphism $\operatorname{Tors}(\operatorname{Pic} X') \xrightarrow{\sim} \langle \vec{P}_s \rangle^{\perp} \subset A_0'[2] \times B_0'[2] \times C_0'[2]$.*

(3) *The curves A_i', B_i', C_i', $0 \le i \le 3$, generate $\operatorname{Pic} X'$.*

Proof. (2) is (3.5) and (1) follows from it. For (3), note that $\operatorname{Pic} X' / \operatorname{Tors} = \operatorname{Pic} Y'$ is generated by the divisors A_i', B_i', C_i' and that the proof of the previous theorem shows that $\operatorname{Tors}\operatorname{Pic} X'$ is generated by certain linear combinations of these divisors. Q.E.D.

§4. Effective divisors on Burniat surfaces with $K^2 = 6$

Since $\frac{1}{2}\pi^*$ and $\frac{1}{2}\pi_*$ provide isomorphisms between the $\mathbb{Q}$-vector spaces $(\operatorname{Pic} Y) \otimes \mathbb{Q}$ and $(\operatorname{Pic} X) \otimes \mathbb{Q}$, it is obvious that the cones of effective $\mathbb{Q}$- or $\mathbb{R}$-divisors on X and Y are naturally identified. In this section, we would like to prove the following description of the semigroup of effective $\mathbb{Z}$-divisors:

Theorem 4.1. *The curves A_i, B_i, C_i, $0 \le i \le 3$, generate the semigroup of effective $\mathbb{Z}$-divisors on Burniat surface X.*

We start with several preparatory lemmas.

Lemma 4.2. *The semigroup of effective $\mathbb{Z}$-divisors on Y is generated by the (-1)-curves $\bar{A}_0, \bar{B}_0, \bar{C}_0, \bar{A}_3, \bar{B}_3, \bar{C}_3$.*

Proof. Since $-K_Y$ is ample, the Mori-Kleiman cone $NE_1(Y)$ of effective curves in $(\operatorname{Pic} Y) \otimes \mathbb{Q}$ is generated by extremal rays, i.e. the (-1)-curves $\bar{A}_0, \bar{B}_0, \bar{C}_0, \bar{A}_3, \bar{B}_3, \bar{C}_3$. We claim that moreover the semigroup of integral points in $NE_1(Y)$ is generated by these points, i.e. the polytope $Q = NE_1(Y) \cap \{C \mid -K_Y C = 1\}$ is totally generating. The vertices of this polytope in $\mathbb{R}^3$ are $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1)$, $(0,1,1)$, $(1,0,1)$, $(1,1,0)$, and the lattice $\operatorname{Pic} Y = \mathbb{Z}^4$ is generated by them. It is a prism over a triangular base, and it is totally generating because it can be split into 3 elementary simplices. Q.E.D.

Lemma 4.3. *The semigroup of nef $\mathbb{Z}$-divisors on Y is generated by f_1, f_2, f_3, h_1, and h_2.*

Proof. Again, for the $\mathbb{Q}$-divisors this is obvious by MMP: a divisor $\bar{D}$ is nef iff $\bar{D}\bar{F} \ge 0$ for $\bar{F} \in \{\bar{A}_0, \bar{B}_0, \bar{C}_0, \bar{A}_3, \bar{B}_3, \bar{C}_3\}$, and the extremal nef $\bar{D}$ divisors correspond to contractions $Y \to Y'$ with $\operatorname{rk}\operatorname{Pic} Y' = 1$. Another proof: the extremal nef divisors correspond to the faces of the triangular prism from the proof of Lemma 4.2, and there are 5 of them: 3 sides, top, and the bottom.

Now let $\bar{D} \in \operatorname{Pic} Y$ be a nonnegative linear combination $\bar{D} = \sum a_i f_i + b_j h_j$ with $a_i, b_j \in \mathbb{Q}$ and let us assume that $a_1 > 0$ (resp. $b_1 > 0$). Since the intersections of f_1 (resp. h_1) with the curves F above are 0 or 1, it follows that $\bar{D} - f_1$ (resp. $\bar{D} - h_1$) is also nef. We finish by induction on $\deg \bar{D} = -K_Y \bar{D}$.

Q.E.D.

We write the divisors $\bar{D}$ in $\operatorname{Pic} Y$ using the symmetric coordinates

$$(d; a_0^0, b_0^0, c_0^0; a_3^0, b_3^0, c_3^0), \text{ where } d = \bar{D}(-K_Y),\ a_0^0 = \bar{D}\bar{A}_0, \dots, c_3^0 = \bar{D}\bar{C}_3.$$

Note that, as in Theorem 2.2, $\operatorname{Pic} Y$ and can be described either as the subgroup of $\mathbb{Z}^4$ with coordinates $(d; a_0^0, b_0^0, c_0^0)$ satisfying the congruence $3|(d + a_0^0 + b_0^0 + c_0^0)$, or as the subgroup of $\mathbb{Z}^4$ with coordinates $(d; a_3^0, b_3^0, c_3^0)$ satisfying the congruence $3|(d + a_3^0 + b_3^0 + c_3^0)$.

Lemma 4.4. *The function* $p_a(\bar{D}) = \dfrac{\bar{D}(\bar{D} + K_Y)}{2} + 1$ *on the set of nef* $\mathbb{Z}$*-divisors on* Y *is strictly positive, with the exception of the following divisors, up to symmetry:*

(1) $(2n; n, 0, 0; n, 0, 0)$ *for* $n \geq 1$*, one has* $p_a = -(n-1)$

(2) $(2n; n-1, 1, 0; n-1, 1, 0)$ *for* $n \geq 1$*, one has* $p_a = 0$.

(3) $(2n+1; n, 1, 1; n-1, 0, 0)$ *and* $(2n+1; n-1, 0, 0; n, 1, 1)$ *for* $n \geq 1$, $p_a = 0$.

(4) $(6; 2, 2, 2; 0, 0, 0)$ *and* $(6; 0, 0, 0; 2, 2, 2)$, $p_a = 0$.

The divisors in (1) *are in the linear system* $|nf_i|$*, where* f_i *is a fiber of one of the three rulings* $Y \to \mathbb{P}^1$*. The divisors in* (2) *and* (3) *are obtained from these by adding a section. The divisors in* (4) *belong to the linear systems* $|2h_1|$ *and* $|2h_2|$*.*

Proof. Let $\bar{D}$ be a nef $\mathbb{Z}$-divisor. By Lemma 4.3, we can write $\bar{D} = \sum n_i f_i + m_j h_j$ with $n_i, m_j \in \mathbb{Z}_{\geq 0}$. Let us say $n_1 > 0$. If $\bar{D} = n_1 f_1$ then $p_a(\bar{D}) = -(n_1 - 1)$. Otherwise, $n_1 f_1 + g \leq \bar{D}$, where $g = f_j$, $j \neq 1$, or $g = h_j$. Then using the elementary formula $p_a(\bar{D}_1 + \bar{D}_2) = p_a(\bar{D}_1) + p_a(\bar{D}_2) + \bar{D}_1\bar{D}_2 - 1$, we see that $p_a(n_1 f_1 + g) = 0$. Continuing this by induction and adding more f_j's and h_j's, one easily obtains that $p_a(\bar{D}) > 0$ with the only exceptions listed above. Starting with $m_1 h_1$ instead of $n_1 f_1$ works the same. Q.E.D.

Corollary 4.5. *The function* $\chi(D) = \dfrac{D(D - K_X)}{2} + 1$ *on the set of nef* $\mathbb{Z}$*-divisors on* Y *is strictly positive, with the same exceptions as above.*

Proof. Indeed, since $\chi(\mathcal{O}_X) = 1$, one has $\chi(D) = p_a(\bar{D})$. Q.E.D.

Lemma 4.6. *Assume that* $\bar{D} \neq 0$ *is a nef divisor on* X *with* $p_a(\bar{D}) > 0$*. Then the divisor* $\bar{D} + K_Y$ *is effective.*

Proof. One has $\chi(\bar{D} + K_Y) = \dfrac{(\bar{D} + K_Y)\bar{D}}{2} + 1 = p_a(\bar{D}) > 0$. Since $h^2(\bar{D} + K_Y) = h^0(-\bar{D}) = 0$, this implies that $h^0(\bar{D}) > 0$. Q.E.D.

Definition 4.7. We say that an effective divisor D on X is *in minimal form* if $DF \geq 0$ for the elliptic curves $F \in \{A_0, B_0, C_0, A_3, B_3, C_3\}$, and for the curves among those that satisfy $DF = 0$, one has $D|_F = 0$ in $F[2]$.

If either of these conditions fails then $D-F$ must also be effective since F is then in the base locus of $|D|$. A minimal form is obtained by repeating this procedure until it stops or one obtains a divisor of negative degree, in which case D obviously was not effective. We do not claim that a minimal form is unique.

Proof of Thm. 4.1. Let D be an effective divisor on X. We have to show that it belongs to the semigroup $\mathcal{S} = \langle A_i, B_i, C_i,\ 0 \le i \le 3\rangle$.

Step 1: One can assume that D is in minimal form. Obviously.

Step 2.: The statement is true for $d \le 6$. There are finitely many cases here to check. We checked them using a computer script. For each of the divisors, putting it in minimal form makes it obvious that it is either in $\mathcal{S}$ or it is not effective because it has negative degree, with the exception of the following three divisors, in the notations of Theorem 2.2: $(3;1\ 10\ 1\ 10\ 1\ 10)$, $(3;0\ 00\ 0\ 00\ 0\ 00)$, $(3;1\ 00\ 1\ 00\ 1\ 00)$. The first two divisors are not effective by [AO12, Lemma 5]. The third one is not effective because it is K_X and $h^0(K_X) = p_g(X) = 0$.

Step 3: The statement is true for nef divisors of degree $d \ge 7$ which are not *the exceptions listed in Lemma 4.4.*

One has $K_X(K_X - D) < 0$, so $h^0(K_X - D) = 0$ and the condition $\chi(D) > 0$ implies that D is effective. We are going to show that D is in the semigroup $\mathcal{S}$.

Consider the divisor $D - K_X$ which modulo torsion is identified with the divisor $\bar{D} + K_Y$ on Y. By Lemmas 4.6 and 4.2, $\bar{D} + K_Y$ is a positive $\mathbb{Z}$-combination of $\bar{A}_0, \bar{B}_0, \bar{C}_0, \bar{A}_3, \bar{B}_3, \bar{C}_3$. This means that

$$D = K_X + (\text{a positive combination of } A_0, B_0, C_0, A_3, B_3, C_3) + (\text{torsion } \nu)$$

A direct computer check shows that for any torsion ν the divisor $K_X + F + \nu$ is in $\mathcal{S}$ for a single curve $F \in \{A_0, B_0, C_0, A_3, B_3, C_3\}$. (In fact, for any $\nu \ne 0$ the divisor $K_X + \nu$ is already in $\mathcal{S}$.) Thus,

$$D - (\text{a nonnegative combination of } A_0, B_0, C_0, A_3, B_3, C_3) \in \mathcal{S} \implies D \in \mathcal{S}.$$

Step 4: The statement is true for nef divisors in minimal form of degree $d \ge 7$ which are *the exceptions listed in Lemma 4.4.*

We claim that any such divisor is in $\mathcal{S}$, and in particular is effective. For $d = 7, 8$ this is again a direct computer check. For $d \ge 9$, the claim is true by induction, as follows: If D is of exceptional type (1,2, or 3) of Lemma 4.4 then $D - C_1$ has degree $d' = d - 2$ and is of the same exceptional type. This concludes the proof. Q.E.D.

Remark 4.8. Note that we proved a little more than what Theorem 4.1 says. We also proved that every divisor D in minimal form and of degree ≥ 7 is effective and is in the semigroup $\mathcal{S}$.

Remark 4.9. For Burniat surfaces with $2 \leq K^2 \leq 5$, a natural question to ask is whether the semigroup of effective $\mathbb{Z}$-divisors is generated by the preimages of the (-1)- and (-2) curves on Y'. These include the divisors A'_i, B'_i, C'_i and E_s but in some cases there are other curves, too.

§5. Exceptional collections on degenerate Burniat surfaces

Degenerations of Burniat surfaces with $K_X^2 = 6$ were described in [AP09]. Here, we will concentrate on one particularly nice degeneration depicted in Figure 2.

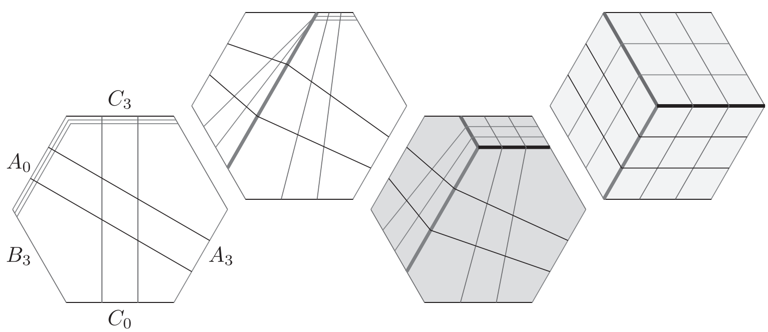

Fig. 2. One-parameter degeneration of Burniat surfaces

It is described as follows. One begins with a one-parameter family $f\colon (Y \times \mathbb{A}^1, \sum_{i=0}^3 \bar{A}_i + \bar{B}_i + \bar{C}_i) \to \mathbb{A}^1$ of del Pezzo surfaces, in which the curves degenerate in the central fiber $f^{-1}(0)$ to a configuration shown in the left panel. The surface $\mathcal{Y}$ is obtained from $Y \times \mathbb{A}^1$ by two blowups in the central fiber, along the smooth centers $\bar{A}_0$ and then (the strict preimage of) $\bar{C}_3$. The resulting 3-fold $\mathcal{Y}$ is smooth, the central fiber $\mathcal{Y}_0 = \mathrm{Bl}_3\,\mathbb{P}^2 \cup \mathrm{Bl}_2\,\mathbb{P}^2 \cup (\mathbb{P}^1 \times \mathbb{P}^1)$ is reduced and has normal crossings. This central fiber is shown in the third panel.

The log canonical divisor $K_{\mathcal{Y}} + \frac{1}{2}\sum_{i=0}^3(\bar{A}_i + \bar{B}_i + \bar{C}_i)$ is relatively big and nef over $\mathbb{A}^1$. It is a relatively minimal model. The relative canonical model $\mathcal{Y}^{\mathrm{can}}$ is obtained from $\mathcal{Y}$ by contracting three curves. The 3-fold $\mathcal{Y}^{\mathrm{can}}$ is singular at three points and not $\mathbb{Q}$-factorial. Its central fiber $\mathcal{Y}_0^{\mathrm{can}}$ is shown in the last, fourth panel.

The 3-folds $\pi\colon \mathcal{X} \to \mathcal{Y}$ and $\pi^{\mathrm{can}}\colon \mathcal{X}^{\mathrm{can}} \to \mathcal{Y}^{\mathrm{can}}$ are the corresponding $\mathbb{Z}_2^2$-Galois covers. The 3-fold $\mathcal{X}$ is smooth, and its central fiber $\mathcal{X}_0$ is reduced and has normal crossings. It is a relatively minimal model: $K_{\mathcal{X}}$ is relatively big and nef.

The 3-fold $\mathcal{X}^{\mathrm{can}}$ is obtained from $\mathcal{X}$ by contracting three curves. Its canonical divisor $K_{\mathcal{X}^{\mathrm{can}}}$ is relatively ample. It is a relative canonical model. We note that $\mathcal{X}$ is one of the 6 relative minimal models $\mathcal{X}^{(k)}$, $k = 1, \dots, 6$, that are related by flops.

Let $U \subset \mathbb{A}^1$ be the open subset containing 0 and all $t \neq 0$ for which the fiber $\mathcal{X}_t$ is smooth, and let $\mathcal{X}_U = \mathcal{X} \times_{\mathbb{A}^1} U$. The aim of this section is to prove the following:

Theorem 5.1. *Then there exists a sequence of line bundles $\mathcal{L}_1, \dots, \mathcal{L}_6$ on $\mathcal{X}_U$ whose restrictions to any fiber (including the nonnormal semistable fiber $\mathcal{X}_0$) form an exceptional collection of line bundles.*

Remark 5.2. It seems to be considerably harder to construct an exceptional collection on the surface $\mathcal{X}_0^{\mathrm{can}}$, the special fiber in a singular 3-fold $\mathcal{X}^{\mathrm{can}}$. And perhaps looking for one is not the right thing to do. A well-known result is that different smooth minimal models $\mathcal{X}^{(k)}$ related by flops have equivalent derived categories. In the same vein, in our situation the central fibers $\mathcal{X}_0^{(k)}$, which are reduced reducible semistable varieties, should have the same derived categories. The collection we construct works the same way for any of them.

Notation 5.3. On the surface $\mathcal{X}_0$, we have 12 Cartier divisors A_i, B_i, C_i, $i = 0, 1, 2, 3$. The "internal" divisors A_i, B_i, C_i, $i = 1, 2$ have two irreducible components each. Of the 6 "boundary" divisors, A_0, A_3, C_0 are irreducible, and $B_0 = B_0' + B_0''$, $B_3 = B_3' + B_3''$, $C_3 = C_3' + C_3''$ are reducible.

Our notation for the latter divisors is as follows: the curve C_3' is a smooth elliptic curve (on the bottom surface $(\mathcal{Y})_0$ the corresponding curve has 4 ramification points), and the curve C_3'' is isomorphic to $\mathbb{P}^1$ (on the bottom surface the corresponding curve has 2 ramification points).

For consistency of notation, we also set $A_0' = A_0$, $A_3' = A_3$, $C_0' = C_0$.

Definition 5.4. Let $\psi = \psi_{C_3}\colon C_3 \to C_3'$ be the projection which is an isomorphism on the component C_3' and collapses the component C_3'' to a point.

We have natural norm map $\psi_* = (\psi_{C_3})_*\colon \operatorname{Pic} C_3 \to \operatorname{Pic} C_3'$. Indeed, every line bundle on the reducible curve C_3 can be represented as a Cartier divisor $\mathcal{O}_{C_3}(\sum n_i P_i)$, where P_i are nonsingular points. Then we define

$$\psi_*\big(\mathcal{O}_{C_3}(\sum n_i P_i)\big) = \mathcal{O}_{C'_3}(\sum n_i \psi(P_i)).$$

Since the dual graph of the curve C_3 is a tree, one has $\operatorname{Pic}^0 C_3 = \operatorname{Pic}^0 C'_3$ and $\operatorname{Pic} C_3 = \operatorname{Pic}^0 C'_3 \oplus \mathbb{Z}^2$.

We also have similar morphisms ψ_{B_0}, ψ_{B_3} and norm maps for the other two reducible curves.

Definition 5.5. We define a map $\phi_{C_3}\colon \operatorname{Pic}\mathcal{X}_0 \to \operatorname{Pic} C'_3$ as the composition of the restriction to C_3 and the norm map $\psi_*\colon C_3 \to C'_3$. We also have similar morphisms ϕ_{B_0}, ϕ_{B_3} for the other two reducible curves. For the irreducible curves A_0, A_3, C_0 the corresponding maps are simply the restriction maps on Picard groups.

For the following Lemma, compare Theorem 2.2 above.

Lemma 5.6. *Consider the map*

$$\phi_0\colon \operatorname{Pic}\mathcal{X}_0 \to \mathbb{Z} \oplus \operatorname{Pic} A'_0 \oplus \operatorname{Pic} B'_0 \oplus \operatorname{Pic} C'_0$$

defined as $D \mapsto D\cdot K_{\mathcal{X}_0}$ in the first component and the maps ϕ_{A_0}, ϕ_{B_0}, ϕ_{C_0} in the other components. Then the images of the Cartier divisors A_i, B_i, C_i, $i = 0, 1, 2, 3$ are exactly the same as for a smooth Burniat surface $\mathcal{X}_t$, $t \neq 0$.

Proof. Immediate check. Q.E.D.

Definition 5.7. We will denote this image by $\operatorname{im}\phi_0$. One has $\operatorname{im}\phi_0 \simeq \mathbb{Z}^4 \oplus \mathbb{Z}_2^6$. We emphasize that $\operatorname{im}\phi_0 = \operatorname{im}\phi_t = \operatorname{Pic}\mathcal{X}_t$, where $\mathcal{X}_t$ is a smooth Burniat surface.

Lemma 5.8. *Let D be an effective Cartier divisor D on the surface $\mathcal{X}_0$. Suppose that $D\cdot A_i < 0$ for $i = 0$ or $i = 3$. Then the Cartier divisor $D - A_i$ is also effective. (Similarly for B_i, C_i.)*

Proof. For an irreducible divisor this is immediate, so let us do it for the divisor $C_3 = C'_3 + C''_3$ which spans two irreducible components, say X', X'' of the surface $\mathcal{X}_0 = X' \cup X'' \cup X'''$. Let $D' = D|_{X'}$, $D'' = D|_{X''}$, $D''' = D|_{X'''}$. Then

$$D\cdot C_3 = (D'\cdot C'_3)_{X'} + (D''\cdot C''_3)_{X''},$$

where the right-hand intersections are computed on the smooth irreducible surfaces. One has $(C'_3)^2_{X'} = 0$ and $(C''_3)^2_{X''} = -1$. Therefore, $(D'\cdot C'_3)_{X'} \geq 0$. Thus, $D\cdot C_3 < 0$ implies that $(D''\cdot C''_3)_{X''} < 0$. Then C''_3 must be in the base locus of the linear system $|D''|$ on the smooth surface X''. Let $n > 0$ be the multiplicity of C_3'' in D''. Then the divisor $D'' - nC_3''$ is effective and does not contain C_3''.

By what we just proved, D must contain nC_3''. Thus, it passes through the point $P = C_3' \cap C_3''$ and the multiplicity of the curve $(D')_{X'}$ at P is $\geq n$, since D is a Cartier divisor. Suppose that D does not contain the curve C_3'. Then $(D' \cdot C_3')_{X'} \geq n$, and

$$D \cdot C_3 = (D' \cdot C_3')_{X'} + (D'' \cdot C_3'')_{X''} \geq n + (-n) = 0,$$

which provides a contradiction. We conclude that D contains C_3' as well, and so $D - C_3$ is effective. Q.E.D.

Lemma 5.9. *Let D be an effective Cartier divisor D on the surface $\mathcal{X}_0$. Suppose that $D \cdot A_i = 0$ for $i = 0, 3$ but $\phi_{A_i}(D) \neq 0$ in $\operatorname{Pic} A_i$. Then the Cartier divisor $D - A_i$ is also effective. (Similarly for B_i, C_i.)*

Proof. We use the same notations as in the proof of the previous lemma. Since D' is effective, one has $(D' \cdot C_3')_{X'} \geq 0$.

If $(D'' \cdot C_3'')_{X''} < 0$ then, as in the above proof let $n > 0$ be the multiplicity of C_3'' in D''. Then either D' contains C_3' (and so D contains C_3 as claimed) or: $(D'' \cdot C_3'')_{X''} = -n$, $(D' \cdot C_3')_{X'} = n$, $D'' - nC_3''$ is disjoint from C_3'' and D' intersects C_3' only at the unique point $P = C_3' \cap C_3''$. But then $\phi_{C_3}(D) = 0$ in $\operatorname{Pic} C_3'$, a contradiction.

If $(D'' \cdot C_3'')_{X''} = 0$ but $D'' - nC_3''$ is effective for some $n > 0$, the same argument gives $DC_3 > 0$, so we get an even easier contradiction.

Finally, assume that $(D' \cdot C_3')_{X'} = (D'' \cdot C_3'')_{X''} = 0$ and D'' does not contain C_3''. By assumption, we have $D' \cdot C_3' = 0$ but $D'|_{C_3'} \neq 0$ in $\operatorname{Pic} C_3'$. This implies that $D' - C_3'$ is effective and that D contains the point $P = C_3' \cap C_3''$. But then $(D'' \cdot C_3'')_{X''} > 0$. Contradiction. Q.E.D.

The following lemma is the precise analogue of [AO12, Lemma 5] (Lemma 4.5 in the arXiv version).

Lemma 5.10. *Let $F \in \operatorname{Pic} \mathcal{X}_0$ be an invertible sheaf such that*

$$\operatorname{im} \phi_0(F) = (3; 1\ 10, 1\ 10, 1\ 10) \in \mathbb{Z} \oplus \operatorname{Pic} A_0 \oplus \operatorname{Pic} B_0 \oplus C_0$$

Then $h^0(\mathcal{X}_0, F) = 0$.

Proof. The proof of [AO12, Lemma 5], used verbatim together with the above Lemmas 5.8, 5.9 works. Crucially, the three "corners" $A_0 \cap C_3$, $B_0 \cap A_3$, $C_0 \cap B_3$ are smooth points on $\mathcal{X}_0$. Q.E.D.

Proof of Thm. 5.1. We define the sheaves $\mathcal{L}_1, \dots, \mathcal{L}_6$ by the same linear combinations of the Cartier divisors $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i$ as in the smooth case [AO12, Rem.2] (Remark 4.4 in the arXiv version), namely:

$$\begin{aligned}
\mathcal{L}_1 &= \mathcal{O}_{\mathcal{X}}(\mathcal{A}_3 + \mathcal{B}_0 + \mathcal{C}_0 + \mathcal{A}_1 - \mathcal{A}_2),\\
\mathcal{L}_2 &= \mathcal{O}_{\mathcal{X}}(\mathcal{A}_0 + \mathcal{B}_3 + \mathcal{C}_3 + \mathcal{A}_2 - \mathcal{A}_1),\\
\mathcal{L}_3 &= \mathcal{O}_{\mathcal{X}}(\mathcal{C}_2 + \mathcal{A}_2 - \mathcal{C}_0 - \mathcal{A}_3),\\
\mathcal{L}_4 &= \mathcal{O}_{\mathcal{X}}(\mathcal{B}_2 + \mathcal{C}_2 - \mathcal{B}_0 - \mathcal{C}_3),\\
\mathcal{L}_5 &= \mathcal{O}_{\mathcal{X}}(\mathcal{A}_2 + \mathcal{B}_2 - \mathcal{A}_0 - \mathcal{B}_3),\\
\mathcal{L}_6 &= \mathcal{O}_{\mathcal{X}}.
\end{aligned}$$

By [AO12], for every $t \neq 0$ they restrict to the invertible sheaves $L_1, \ldots, L_6 \in \operatorname{im} \phi_t = \operatorname{Pic} \mathcal{X}_t$ on a smooth Burniat surface which form an exceptional sequence. By Lemma 5.6, the images of $\mathcal{L}_i|_{\mathcal{X}_0} \in \operatorname{Pic} \mathcal{X}_0$ under the map

$$\phi_0 \colon \operatorname{Pic} \mathcal{X}_0 \twoheadrightarrow \operatorname{im} \phi_0 = \operatorname{im} \phi_t = \operatorname{Pic} \mathcal{X}_t, \quad t \neq 0.$$

are also $L_1, \ldots, L_6$. We claim that $\mathcal{L}_i|_{\mathcal{X}_0}$ also form an exceptional collection.

Indeed, the proof in [AO12] of the fact that $L_1, \ldots, L_6$ is an exceptional collection on a smooth Burniat surface $\mathcal{X}_t$ ($t \neq 0$) consists of showing that for $i < j$ one has

(1) $\chi(L_i \otimes L_j^{-1}) = 0$,
(2) $h^0(L_i \otimes L_j^{-1}) = 0$, and
(3) $h^0(K_{\mathcal{X}_t} \otimes L_i^{-1} \otimes L_j) = 0$.

The properties (2) and (3) are checked by repeatedly applying (the analogues of) Lemmas 5.8, 5.9, 5.10 until $D \cdot K_{\mathcal{X}_t} < 0$ (in which case D is obviously not effective).

In our case, one has $\chi(\mathcal{X}_0, \mathcal{L}_i|_{\mathcal{X}_0} \otimes \mathcal{L}_j|_{\mathcal{X}_0}^{-1}) = \chi(\mathcal{X}_t, \mathcal{L}_i|_{\mathcal{X}_t} \otimes \mathcal{L}_j|_{\mathcal{X}_t}^{-1}) = 0$ by flatness. Since we proved that Lemmas 5.8, 5.9, 5.10 hold for the surface $\mathcal{X}_0$, and since the Cartier divisor $K_{\mathcal{X}_0}$ is nef, the same exact proof for vanishing of h^0 goes through unchanged. Q.E.D.

Remark 5.11. The semiorthogonal complement $\mathcal{A}_t$ of the full triangulated category generated by the sheaves $\langle \mathcal{L}_1, \ldots, \mathcal{L}_6 \rangle|_{\mathcal{X}_t}$ is the quite mysterious "quasiphantom". A viable way to understand it could be to understand the degenerate quasiphantom $\mathcal{A}_0 = \langle \mathcal{L}_1, \ldots, \mathcal{L}_6 \rangle|_{\mathcal{X}_t}^{\perp}$ on the semistable degeneration $\mathcal{X}_0$ first. The irreducible components of $\mathcal{X}_0$ are three bielliptic surfaces and they are glued nicely. Then one could try to understand $\mathcal{A}_t$ as a deformation of $\mathcal{A}_0$.

References

[AO12] Valery Alexeev and Dmitri Orlov, *Derived categories of Burniat surfaces and exceptional collections*, Math. Annalen, to appear (2012), arXiv:1208.4348.

[AP09] Valery Alexeev and Rita Pardini, *Explicit compactifications of moduli spaces of Campedelli and Burniat surfaces*, Preprint (2009), 26pp, arXiv:0901.4431.

[BC11] I. Bauer and F. Catanese, *Burniat surfaces I: fundamental groups and moduli of primary Burniat surfaces*, Classification of algebraic varieties, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, pp. 49–76.

[Par91] Rita Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math. **417** (1991), 191–213.

[Pet77] C. A. M. Peters, *On certain examples of surfaces with $p_g = 0$ due to Burniat*, Nagoya Math. J. **66** (1977), 109–119.

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