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Existence of weak solutions to the three-dimensional steady compressible Navier–Stokes equations for any specific heat ratio $\gamma > 1$

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Abstract.

In this paper we present the recent existence results from [14], [15] on weak solutions to the the steady Navier–Stokes equations for three-dimensional compressible isentropic flows with large data for any specific heat ratio $\gamma > 1$. The existence is proved in the framework of the weak convergence method due to Lions [16] by establishing a new a priori potential estimate of both pressure and kinetic energy (in a Morrey space) and using a bootstrap argument. The results presented in the current paper extend the existence of weak solutions in [9] from $\gamma > 4/3$ to $\gamma > 1$.

§1. Introduction

The steady isentropic compressible Navier–Stokes equations, which describe conservation of the mass and momentum of an isentropic flow, can be written as follows.

$$\operatorname{div}(\rho\mathbf{u}) = 0, \tag{1}$$

$$-\mu\triangle\mathbf{u} - \tilde{\mu}\nabla\operatorname{div}\mathbf{u} + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P = \rho\mathbf{f} + \mathbf{g}. \tag{2}$$

Here $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity and ρ is the density, the viscosity constants μ and $\tilde{\mu}$ satisfy $\mu > 0$, $\tilde{\mu} = \mu + \lambda$ with $\lambda + 2\mu/3 \geq 0$, the pressure P for the isentropic flow is given by

$$P(\rho) = a\rho^\gamma$$

with a being a positive constant and $\gamma > 1$ being the specific heat ratio, $\mathbf{f} = (f_1, f_2, f_3)$ and $\mathbf{g} = (g_1, g_2, g_3)$ are the external forces. We shall

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consider the system (1), (2) in a bounded domain $\Omega \subset \mathbb{R}^3$, and for simplicity, we assume that

$$\mathbf{f},\ \mathbf{g} \in L^\infty(\Omega).$$

Moreover, the total mass is prescribed:

$$\int_\Omega \rho dx = M > 0. \tag{3}$$

In the last decades, the well-posedness of the equations (1), (2) for large $\mathbf{f}$ and $\mathbf{g}$ has been investigated by a number of researchers. In 1998, under the assumption that $\gamma > 1$ in $\mathbb{R}^2$ and $\gamma > 5/3$ in $\mathbb{R}^3$, Lions [16] first proved the existence of weak solutions to different boundary problems for (1), (2). Roughly speaking, the condition on γ comes from the integrability of the density ρ in L^p. The higher integrability of ρ has, the smaller γ can be allowed. If $\mathbf{f}$ is potential and $\mathbf{g} = 0$, then weak solutions are shown to exist for any $\gamma > 3/2$, see [19]. Then, Frehse, Goj and Steinhauer, Plotnikov and Sokolowsk obtained an improved integrability bound for the density by deriving a new weighted estimate of the pressure in [6], [20], where the authors assumed a priori the L^1-boundedness of $\rho\mathbf{u}^2$ which, unfortunately, was not shown to hold. Recently, by combining the L^∞-estimate of $\triangle^{-1}P$ with the (usual) energy and density bounds, Březina and Novotný [3] were able to show the existence of weak solutions to the spatially periodic problem for any $\gamma > (3+\sqrt{41})/8$ when $\mathbf{f}$ is potential, or for any $\gamma > 1.53$ when $\mathbf{f} \in L^\infty$, without assuming the boundedness of $\rho\mathbf{u}^2$ in L^1. More recently, Frehse, Steinhauer and Weigant [9] established the existence of weak solutions to the Dirichlet problem in three dimensions for any $\gamma > 4/3$ in the framework of [3]. Also, the existence of a weak solution to (1), (2) with different boundary conditions was obtained in the two-dimensional isothermal case ($\gamma = 1$) [7], [8].

In this paper, we shall present recent existence results from [14], [15] which are inspired by the works [9], [3] and extend the existence in [9] from $\gamma > 4/3$ to $\gamma > 1$. Roughly speaking, the basic idea in our proof is to employ a careful bootstrap argument to obtain the higher integrability of the density which eventually relaxes the restriction on γ in [9]. We point out that quite recently, using the idea in [14], Jesslé and Novotný [11] showed the existence of weak solutions to (1), (2) with slip (or Navier) boundary conditions for any $\gamma > 1$. As indicated in [11], however, their result does not imply any improvement with respect to [9] in the case of the Dirichlet boundary conditions.

We mention that for a three-dimensional model of steady compressible heat-conducting flows (i.e., the steady compressible Navier–Stokes–Fourier system), Mucha, Novotný, Pokorný [17], [18] recently studied the existence of weak solutions under some assumptions on the pressure and heat-conductivity, which unfortunately exclude the case of polytropic idea gases. For the corresponding non-steady system (to (1), (2)) with large initial data, Lions [16] first proved the global existence of weak solutions in the case of $\gamma \geq 3n/(n+2)$ ($n = 2, 3$: dimension). His result has been improved and generalized recently in [5], [12], [13] and among others, where the condition $\gamma > 3/2$ is required in three dimensions for general initial data.

This paper is organized as follows. In Section 1 we investigate the case that solutions are spatially periodic, while at the end of the paper, we give a remark on the Dirichlet boundary value problem.

§2. Spatially periodic solutions

In this section, we consider the case of spatially periodic solutions to (1), (2), namely, $(\rho, \mathbf{u})$ is periodic in each x_i with period 2π for all $1 \leq i \leq 3$. For this purpose, we assume that $\mathbf{f}$ is periodic in each x_i with period 2π for $1 \leq i \leq 3$, and $\mathbf{g} = 0$ without loss of generality. For simplicity, throughout this section, we denote by Ω the periodic cell $(-\pi, \pi)^3$.

In general, there could be no solution for arbitrary $\mathbf{f}$, since for a (smooth) solution, which is periodic in x with period 2π, $\mathbf{f}$ has to satisfy the necessary condition:

$$\int_\Omega \rho f_i dx = 0 \qquad \text{for } 1 \leq i \leq 3. \tag{4}$$

However, if we consider $\mathbf{f}$ with symmetry

$$f_i(x) = -f_i(Y_i(x)) \text{ and } f_i(x) = f_i(Y_j(x)), \text{ if } i \neq j, \; i, j = 1, 2, 3, \tag{5}$$

where

$$Y_i(\cdots, x_i, \cdots) = (\cdots, -x_i, \cdots),$$

then $\mathbf{u}$ will have the same symmetry and ρ with the symmetry

$$\rho(x) = \rho(Y_i(x)) \qquad \text{for } i = 1, 2, 3, \tag{6}$$

and the condition (4) is satisfied automatically. Moreover, $\mathbf{u}$ satisfies

$$\int_\Omega u_i(x) dx = 0 \qquad \text{for all } 1 \leq i \leq 3.$$

We now introduce some notations (see [1]). Define

$$\mathcal{D}(\mathbb{R}^3) = \quad \{\phi \in C^\infty(\mathbb{R}^3),\ \phi \text{ is periodic in } x_i \text{ of period } 2\pi \text{ for all } 1 \le i \le 3\}$$

and

$$\mathcal{D}(\Omega) = \{\phi(x) \mid \exists\, \tilde{\phi}(x) \in \mathcal{D}(\mathbb{R}^3),\ \text{s.t. } \phi(x) = \tilde{\phi}(x),\ \text{for } x \in \Omega\}.$$

By $\mathcal{D}'(\mathbb{R}^3)$ (resp. $\mathcal{D}'(\Omega)$), we denote the dual space of $\mathcal{D}(\mathbb{R}^3)$ (resp. $\mathcal{D}(\Omega)$). For example, $\mathcal{D}'(\mathbb{R}^3)$ is the space of periodic distributions in $\mathbb{R}^3$ (dual to $\mathcal{D}(\mathbb{R}^3)$). We also introduce the spaces of symmetric functions: $(W^{k,p}_{\text{sym}}(\Omega))^3$ denotes the space of vector functions in $W^{k,p}(\Omega)$ which possess the symmetry (5), while $L^p_{\text{sym}}(\Omega)$ stands for the space of functions in $L^p(\Omega)$ with symmetry (6). $B_R(a) := \{x \in \mathbb{R}^3 :\ |x-a| < R\}$ denotes the open ball centered at a with radius R.

We are now able to introduce the notation of a renormalized bounded energy weak solution.

Definition 1. (Renormalized bounded energy weak solution) *We call* $(\rho, \mathbf{u})$ *a renormalized bounded energy weak solution to the spatially periodic problem of the system* (1) *and* (2), *if*
i) $\rho \geq 0$, $\rho \in L^\gamma(\Omega)$, $\mathbf{u} \in H^1(\Omega)$, $\int_\Omega \rho(x)dx = M > 0$.
ii) $(\rho, \mathbf{u})$ *satisfies the energy inequality:*

$$\int_\Omega (\mu|\nabla \mathbf{u}|^2 + \tilde{\mu}|\operatorname{div} \mathbf{u}|^2)dx \leq \int_\Omega (\rho \mathbf{f} + \mathbf{g}) \cdot \mathbf{u} dx.$$

iii) The system (1), (2) *holds in the sense of* $\mathcal{D}'(\Omega)$.
iv) The mass equation (1) *holds in the sense of renormalized solutions, i.e.,*

$$\operatorname{div}[b(\rho)\mathbf{u}] + [b'(\rho)\rho - b(\rho)]\operatorname{div}\mathbf{u} = 0 \quad \textit{in } \mathcal{D}'(\Omega) \tag{7}$$

for any $b \in C^1(\mathbb{R})$, *such that* $b'(z) = 0$ *when* z *is big enough.*

Remark 1. In the periodic case, the periodic cell Ω in Definition 1 actually can be replaced by any cube in $\mathbb{R}^3$ with length 2π.

Thus, the existence theorem for (1), (2) in the spatially periodic case reads as follows.

Theorem 1. *Let* $\gamma > 1$ *and* $\mathbf{f} \in L^\infty(\mathbb{R}^3)$ *satisfy (5). Then, there exists a renormalized bounded energy weak solution* $(\rho, \mathbf{u})$, *satisfying (6) and (5), to the spatially periodic problem of the system (1), (2).*

Roughly speaking, the proof of Theorem 1 is based on the new a priori estimates for the approximate solutions and the weak convergence method in the framework of Lions [16]. The crucial point, compared with [9], [3], is to establish a new higher than L^γ-integrability of the (approximate) density for any $\gamma > 1$ by deriving simultaneous weighted boundedness of both P_δ and $\rho_\delta|\mathbf{u}_\delta|^2$ in a Morrey space. In the following, we give the main steps of the proof.

MAIN STEPS OF THE PROOF:

Step I. Approximate system.

We first work with the standard approximation by introducing an artificial pressure term

$$P_\delta(\rho) := a\rho^\gamma + \delta\rho^6,$$

where $0 < \delta \leq 1$. Here we choose ρ^6 just for technical reason, and in fact we can take ρ^α for any $\alpha \geq 6$ instead of ρ^6. We consider the following approximate problem in Ω:

$$\begin{aligned} &(8) & \operatorname{div}(\rho_\delta \mathbf{u}_\delta) &= 0, \\ &(9) & -\mu\triangle\mathbf{u}_\delta - \tilde{\mu}\nabla\operatorname{div}\mathbf{u}_\delta + \operatorname{div}(\rho_\delta\mathbf{u}_\delta \otimes \mathbf{u}_\delta) + \nabla P_\delta(\rho_\delta) &= \rho_\delta\mathbf{f}. \end{aligned}$$

According to [3], there is at least a weak solution $(\rho_\delta, \mathbf{u}_\delta)$ to the problem (8), (9) with the following properties ($\overline{\gamma} = \max(\gamma, 6)$):

$$\begin{aligned} &\rho_\delta \in L^{2\overline{\gamma}}_{\mathrm{sym}}(\Omega), \quad \mathbf{u}_\delta \in (W^{1,2}_{\mathrm{sym}}(\Omega))^3, \quad \int_\Omega \rho_\delta dx = M; \\ &\operatorname{div}[b(\rho_\delta)\mathbf{u}_\delta] + \big[b'(\rho_\delta)\rho_\delta - b(\rho_\delta)\big]\operatorname{div}\mathbf{u}_\delta = 0 \quad \text{in } \mathcal{D}'(\Omega); \\ &\int_\Omega \big[\mu|\nabla\mathbf{u}_\delta|^2 + \tilde{\mu}|\operatorname{div}\mathbf{u}_\delta|^2\big]dx \leq \int_\Omega \rho_\delta\mathbf{f}\cdot\mathbf{u}_\delta dx, \end{aligned} \tag{10}$$

where b is the same as in (7).

Denote

$$A = \|P_\delta|\mathbf{u}_\delta|^2 + \rho_\delta^\beta|\mathbf{u}_\delta|^{2+2\beta}\|_{L^1}, \qquad 0 < \beta < 1, \tag{11}$$

where and in what follows, $\|\cdot\|_{L^p} := \|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{H^m} := \|\cdot\|_{H^m(\Omega)}$, etc.

Our next goal is to bound A for a suitable β (sufficiently close to 1) by a bootstrap argument, the boundedness of A will lead to the desired uniform-in-δ estimates which will be used in passing to the limit as $\delta \to 0$ to get a weak solution of the system (1), (2). To this end, we start with the following potential estimate which can also be understood as an estimate in a Morrey space.

Step II. A potential estimate

For $x_0 \in \overline{\Omega}$, we define $\phi = (\phi^1, \phi^2, \phi^3)$ with

$$\phi^i(x) = \frac{(x - x_0)^i}{|x - x_0|^\beta}\eta(|x - x_0|) \text{ in } b(x_0, \pi), \quad i = 1, 2, 3, \quad x = (x^1, x^2, x^3),$$

where $0 < \beta \le 1$, $b(x_0, \pi) = \{x \in \mathbb{R}^3 : |x^i - x_0^i| < \pi, i = 1, 2, 3\}$ is a periodic cell, and $\eta \in C_0^\infty(\mathbb{R})$ is a cut-off function satisfying $0 \le \eta(t) \le 1$, $|D\eta| \le 2$, $\eta(t) = 1$ if $|t| \le 1$ and $\eta(t) = 0$ if $|t| \ge 2$.

If we extend ϕ to $\mathbb{R}^3$ periodically in x_i with period 2π for all $1 \le i \le 3$, then $\phi \in H^1_{\text{loc}}(\mathbb{R}^3)$ can be a test function. We thus test (9) with this ϕ to deduce, after a careful but straightforward calculation, that

Lemma 1. *Let $(\rho_\delta, \mathbf{u}_\delta)$ be the solutions of the approximate problem* (8), (9). *Then the following estimate holds.*

$$\int_{B_1(x_0)} \frac{P_\delta + (\rho_\delta|\mathbf{u}_\delta|^2)^\beta}{|x - x_0|}dx \le C\big(1 + \|P_\delta\|_{L^1} + \|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^1} + \|\mathbf{u}_\delta\|_{H^1}\big)$$

for all $\beta \in (0, 1)$ and $x_0 \in \overline{\Omega}$, where the constant C depends only on $\|\mathbf{f}\|_{L^\infty}$, μ, $\tilde{\mu}, M, \gamma$ and β, but not on x_0 and δ.

Step III. Estimate of A.

Let $\Omega' \supset\supset \Omega$ be a domain and E be a bounded linear extension operator from $W^{1,p}(\Omega)$ into $W_0^{1,p}(\Omega')$, such that $Eu = u$ in Ω (see, for example, [10, Theorem 7.25])

Since P_δ and $\mathbf{u}_\delta$ are periodic in x_i with period 2π for all $1 \le i \le 3$, we can get from Lemma 1 that

$$\int_{\Omega'} \frac{P_\delta + (\rho_\delta|\mathbf{u}_\delta|^2)^\beta}{|x - x_0|}dx \le C(1 + \|P_\delta\|_{L^1} + \|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^1} + \|\mathbf{u}_\delta\|_{H^1}) \tag{12}$$

for any $0 < \beta < 1$ and $x_0 \in \overline{\Omega'}$, where the constant C is independent of δ and x_0.

Let h be the unique weak solution of the elliptic problem:

$$\triangle h = P_\delta + (\rho_\delta|\mathbf{u}_\delta|^2)^\beta \ge 0 \text{ in } \Omega'; \quad h = 0 \text{ on } \partial\Omega'.$$

Then by the classical theory for elliptic equations and (12), we have

$$\begin{aligned} \|h\|_{L^\infty(\Omega')} &\le C \sup_{x_0 \in \overline{\Omega'}} \int_{\Omega'} \frac{P_\delta + (\rho_\delta|\mathbf{u}_\delta|^2)^\beta}{|x - x_0|}dx \\ &\le C(1 + \|P_\delta\|_{L^1} + \|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^1} + \|\mathbf{u}_\delta\|_{H^1}). \qquad (13) \end{aligned}$$

Since $\mathbf{u}_\delta \in H^1(\Omega)$, $E\mathbf{u}_\delta \in H_0^1(\Omega')$. Now, we consider

$$\begin{aligned} A' : &= \int_{\Omega'} [P_\delta + (\rho_\delta |\mathbf{u}_\delta|^2)^\beta] |E\mathbf{u}_\delta|^2 dx = \int_{\Omega'} \triangle h |E\mathbf{u}_\delta|^2 dx \\ &\leq C\|E\mathbf{u}_\delta\|_{H_0^1(\Omega')} \| |E\mathbf{u}_\delta| |\nabla h| \|_{L^2(\Omega')}, \end{aligned} \tag{14}$$

where, by integrating by parts, one infers that

$$\begin{aligned} & \| |E\mathbf{u}_\delta| \, |\nabla h| \|_{L^2(\Omega')}^2 \\ & \leq C \int_{\Omega'} (|h| \, |\triangle h| \, |E\mathbf{u}_\delta|^2 + |h| \, |\nabla h| \, |E\mathbf{u}_\delta| \, |\nabla \mathbf{u}_\delta|) dx \\ & \leq C\|h\|_{L^\infty(\Omega')} (A' + \| |E\mathbf{u}_\delta| \, |\nabla h| \|_{L^2(\Omega')} \|E\mathbf{u}_\delta\|_{H_0^1(\Omega')}). \end{aligned} \tag{15}$$

Thus, the inequalities (14) and (15) imply that

$$A' \leq C\|E\mathbf{u}_\delta\|_{H_0^1(\Omega')}^2 \|h\|_{L^\infty(\Omega')} \leq C\|\mathbf{u}_\delta\|_{H_0^1}^2 \|h\|_{L^\infty(\Omega')},$$

which, by combining with (13) and recalling $A \leq A'$, proves that

Lemma 2. *Let A be defined by* (11)*, then we have*

$$A \leq C\|\mathbf{u}_\delta\|_{H^1}^2 (1 + \|P_\delta\|_{L^1} + \|\rho_\delta |\mathbf{u}_\delta|^2\|_{L^1} + \|\mathbf{u}_\delta\|_{H^1}), \tag{16}$$

where the constant C depends on $\|\mathbf{f}\|_{L^\infty}, \mu, \tilde{\mu}, M, \gamma$ and β, but not on δ.

Remark 2. We point out here that Lemma 2 can be also obtained by using the arguments in [3].

Step IV. Boundedness of $\mathbf{u}_\delta$ in H^1 and P_δ in L^s (for some $s > 1$).

To close the estimate for A, we have to bound the terms on the right-hand side of (16). To this end, we use the energy inequality (10) to obtain

$$\mu \int_\Omega |\nabla \mathbf{u}_\delta|^2 dx + \tilde{\mu} \int_\Omega |\mathrm{div} \mathbf{u}_\delta|^2 dx \leq \int_\Omega \rho_\delta f \cdot \mathbf{u}_\delta dx \leq C\|\rho_\delta \mathbf{u}_\delta\|_{L^1}, \tag{17}$$

where the right-hand side can be bounded as follows, using Hölder's and Sobolev's inequalities, and recalling $\int_\Omega \rho_\delta = M$.

$$\begin{aligned} \|\rho_\delta \mathbf{u}_\delta\|_{L^1(\Omega)} &= \int_\Omega (P_\delta \mathbf{u}_\delta^2)^{\frac{1-\beta}{2(\gamma\beta+\gamma-2\beta)}} (\rho_\delta^\beta \mathbf{u}_\delta^{2\beta+2})^{\frac{\gamma-1}{2(\gamma\beta+\gamma-2\beta)}} \rho_\delta^{\frac{2\gamma\beta+\gamma-3\beta}{2(\gamma\beta+\gamma-2\beta)}} \\ &\leq CA^{\frac{\gamma-\beta}{2(\gamma\beta+\gamma-2\beta)}}, \end{aligned}$$

which together with (17) and Poincaré's inequality results in

$$\|\mathbf{u}_\delta\|_{H^1} \leq CA^{\frac{\gamma-\beta}{4(\gamma\beta+\gamma-2\beta)}}. \tag{18}$$

Let ω_δ be a solution of the problem

$$\operatorname{div} \omega_\delta = \mathbf{f}_\delta \text{ in } \Omega, \qquad \omega = 0 \text{ on } \partial\Omega,$$

where

$$\mathbf{f}_\delta = P_\delta^{s-1} - \frac{1}{|\Omega|}\int_\Omega P_\delta^{s-1} dx \quad \text{ with } 1 < s \leq \beta + 1 - \beta/\gamma$$

satisfying $\int_\Omega \mathbf{f}_\delta(x)dx = 0$. Then, from a lemma due to Bogovskij [2] we get

$$\|\omega_\delta\|_{W^{1,\frac{s}{s-1}}} \leq C\|\mathbf{f}_\delta\|_{L^{\frac{s}{s-1}}} \leq C(s,\Omega)\|P_\delta\|_s^{s-1}. \tag{19}$$

Now, we use the function ω_δ to test the momentum equation (9) to obtain by employing (19) and a direct computation similar to Lemma 2.3 in [6] that

$$\|P_\delta\|_{L^s}^s \leq C(1 + \|\mathbf{u}_\delta\|_{W^{1,2}}^s + \|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^s}^s), \tag{20}$$

where the last term can be bounded as follows, using Hölder's and Sobolev's inequalities, and recalling $1 < s \leq \beta + 1 - \beta/\gamma$.

$$\begin{aligned} \|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^s}^s &\leq C\|P_\delta|\mathbf{u}_\delta|^2\|_{L^1}^{\frac{2s-\beta-1}{\gamma\beta+\gamma-2\beta}} \|\rho_\delta^\beta|\mathbf{u}_\delta|^{2\beta+2}\|_{L^1}^{\frac{\gamma s+1-2s}{\gamma\beta+\gamma-2\beta}} \\ &\leq CA^{\frac{\gamma s-\beta}{\gamma\beta+\gamma-2\beta}}, \end{aligned} \tag{21}$$

which, together (20) and (18), gives

$$\|P_\delta\|_{L^s(\Omega)}^s \leq C(1 + A^{\frac{s(\gamma-\beta)}{4(\gamma\beta+\gamma-2\beta)}} + A^{\frac{\gamma s-\beta}{\gamma\beta+\gamma-2\beta}}) \leq C(1 + A^{\frac{\gamma s-\beta}{\gamma\beta+\gamma-2\beta}}).$$

The above inequality and (18) implies thus

Lemma 3. *We have*

$$\|\mathbf{u}_\delta\|_{H^1} \leq CA^{\frac{\gamma-\beta}{4(\gamma\beta+\gamma-2\beta)}}; \qquad \|P_\delta\|_{L^s}^s \leq C(1 + A^{\frac{\gamma s-\beta}{\gamma\beta+\gamma-2\beta}})$$

for $s \in (1, \beta + 1 - \beta/\gamma]$, where the constant C depends only on $\|\mathbf{f}\|_{L^\infty}$, μ, λ, M, γ and Ω.

Step V. Uniform-in-δ a priori estimates.

Noting that Lemma 3 holds for any $s \in (1, \beta + 1 - \beta/\gamma]$, we write $s = 1 + \epsilon$, where ϵ will be chosen small enough later on, and use (16), Hölder's inequality, Lemma 3 and (21) to infer that

$$\begin{aligned} A &\leq CA^{\frac{\gamma-\beta}{2(\gamma\beta+\gamma-2\beta)}}(1 + A^{\frac{\gamma-\beta}{4(\gamma\beta+\gamma-2\beta)}} + A^{\frac{\gamma s-\beta}{(\gamma\beta+\gamma-2\beta)}\cdot\frac{1}{1+\epsilon}}) \\ &\leq C(1 + A^{\frac{3(\gamma-\beta)}{2(\gamma\beta+\gamma-2\beta)}+O(\epsilon)}). \end{aligned} \tag{22}$$

Now, recalling $\gamma > 1$, we choose $\beta \in (0,1)$ sufficiently close to 1, such that $\gamma/(2\gamma-1) < \beta$, i.e.,

$$\frac{3(\gamma-\beta)}{2(\gamma\beta+\gamma-2\beta)} < 1 \quad \Rightarrow \quad \frac{3(\gamma-\beta)}{2(\gamma\beta+\gamma-2\beta)} + O(\epsilon) < 1,$$

provided that ϵ is chosen small enough. Therefore, we conclude by (22) that $A \leq C$, which immediately implies the following uniform estimate:

Lemma 4. *There is a number $\sigma > 1$, such that*

$$A + \|\mathbf{u}_\delta\|_{H^1} + \|P_\delta\|_{L^\sigma} + \|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^\sigma} + \|\rho_\delta\mathbf{u}_\delta\|_{L^\sigma} \leq C,$$

where the constant C depends only on $\|\mathbf{f}\|_{L^\infty}$, μ, $\tilde{\mu}$, M and γ (but not on δ). Moreover,

$$\begin{aligned}\delta\int_\Omega \rho_\delta^6 dx &\leq C\delta^{\frac{\gamma(\sigma-1)}{6+\gamma(\sigma-1)}}\left(\int_\Omega \delta\rho_\delta^{6+\gamma(\sigma-1)}dx\right)^{\frac{6}{6+\gamma(\sigma-1)}} \\ &\leq C\delta^{\frac{\gamma(\sigma-1)}{6+\gamma(\sigma-1)}}\left(\int_\Omega P_\delta^\sigma dx\right)^{\frac{6}{6+\gamma(\sigma-1)}} \to 0 \text{ as } \sigma \to 0.\end{aligned}$$

Step VI. Limit as $\delta \to 0$.

Having had the a priori estimates Lemma 4, we can in general follow the framework of the weak convergence method due to Lions [16] (also see [5]) to take to the limit as $\delta \to 0$ for the approximate problem (8) and (9) to obtain a weak solution of (1), (2) for any $\gamma > 1$. However, we could not directly use the arguments in [16], since we just have $\rho_\delta \in L^{\gamma\sigma}(\Omega)$ with $\sigma > 1$ being very close to 1 when γ is close to 1, while in [16] $\rho_\delta \in L^p(\Omega)$ $(p > 5/3)$ is required. Fortunately, this difficulty can be circumvented by exploiting the estimates established in Lemma 4 and a simple lemma on the weak convergence of product of two functional sequences [14, Lemma 3.1], and consequently getting the weak compactness of the effective viscous flux. Then, by the standard procedure of the weak convergence method (see [16, 4, 5]) we obtain a spatially periodic weak solution to (1), (2). This completes the proof of Theorem 1.

Remark 3. Very recently, Plotnikov and Weigant [21] established the existence for the Dirichlet boundary value problem for any $\gamma > 1$ by using elaborate weighted estimates up to boundary. Now, the existence in the isothermal case $\gamma = 1$ is left open only.

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