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# Hyperplane arrangements, local system homology and iterated integrals

**Toshitake Kohno**

**Abstract.**

We review some aspects of the homology of a local system on the complement of a hyperplane arrangement. We describe a relationship between linear representations of the braid groups due to R. Lawrence, D. Krammer and S. Bigelow and the holonomy representations of the KZ connection. We explain a method to describe such representations by the iterated integrals of logarithmic 1-forms.

## §1. Introduction

Let $\mathcal{A}$ be an arrangement of hyperplanes in the complex vector space $\mathbf{C}^n$. We denote by $M(\mathcal{A})$ the complement of the union of hyperplanes in $\mathcal{A}$. The purpose of this article is to review developments concerning the homology of a local system on the complement $M(\mathcal{A})$ and its applications.

In the case of a complexification of a real arrangement M. Salvetti [17] constructed a complex which is homotopy equivalent to the complement $M(\mathcal{A})$. First, we briefly review the Salvetti complex and explain how this construction is used to compute the homology of locally finite chains of $M(\mathcal{A})$ with local system coefficients. If a local system $\mathcal{L}$ is generic, then we have a vanishing of homology $H_k(M(\mathcal{A}), \mathcal{L}) \cong 0$ unless $k = n$. We describe such vanishing theorem and a relation to the homology of locally finite chains.

As a typical example we deal with the case of a discriminantal arrangement. This arrangement has an important application to the flat

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connection due to V. G. Knizhnik and A. B. Zamolodchikov [10], which we shall call the KZ connection. It was shown by V. Schechtman and A. Varchenko [18] that the horizontal sections of the KZ connection are expressed by means of hypergeometric integrals over the complement of a discriminantal arrangement. Based on this result we describe a relationship between linear representations of braid groups developed by R. Lawrence, D. Krammer and S. Bigelow and the holonomy representations of the KZ connection. We also investigate a description of the space of conformal blocks in conformal field theory on the Riemann sphere in terms of homology of a local system over the complement of a discriminantal arrangement.

One of the ways to express the holonomy representation of the KZ connection is to use the iterated integrals of logarithmic 1-forms. In the case of the braid arrangement such iterated integral is a prototype of the Kontsevich integral for knots developed in [14]. We review basic facts about such iterated integrals of logarithmic 1-forms in a general situation of the complement of a hyperplane arrangement. We give a criterion so that iterated integrals of logarithmic 1-forms depend only on the homotopy class of loops.

The paper is organized in the following way. In Section 1 we recall basic facts about the homology of a local system on the complement of a hyperplane arrangement. In Section 2 we deal with discriminantal arrangements and describe a relation to the KZ connection. Section 3 is devoted to the iterated integrals of logarithmic 1-forms and holonomy representations of fundamental groups.

## §2. Homology of local systems

Let $\mathcal{A} = \{H_1, \cdots, H_\ell\}$ be an arrangement of affine hyperplanes in the complex vector space $\mathbf{C}^n$. We consider the complement

$$M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H.$$

Let us assume that the hyperplanes $H_1, \cdots, H_\ell$ are defined over $\mathbf{R}$. In this case $H \in \mathcal{A}$ is regarded as a complexification of the real hyperplane $H_{\mathbf{R}}$ in $V_{\mathbf{R}} = \mathbf{R}^n$. The complement $V_{\mathbf{R}} \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbf{R}}$ consists of finitely many connected components called chambers.

The above real hyperplane arrangement $\{H_{\mathbf{R}}\}_{H \in \mathcal{A}}$ determines a natural stratification $S$ of $\mathbf{R}^n$, whose stratum is called a facet. For facets $E$ and $F$ we shall say that $E > F$ if and only if $\overline{E} \supset F$ holds. For an increasing sequence of facets $F_{j_0} < \cdots < F_{j_p}$ we take a point $v_{j_k}$ in each facet $F_{j_k}$, $0 \leq k \leq p$, and consider the simplex spanned by the vertices $v_{j_k}$, $0 \le k \le p$. This simplex defined for $F_{j_0} < \cdots < F_{j_p}$ is denoted by

$$\sigma(F_{j_0} < \cdots < F_{j_p}).$$

For a facet $F$ the dual cell is defined by

$$D(F) = \bigcup \sigma(F^i < F^{i-1} < \cdots < F^0)$$

where the union is for all the increasing sequences of facets $F^i < F^{i-1} < \cdots < F^0$ with $F^i = F$ and $\operatorname{codim} F^j = j$.

Let $\pi : M(\mathcal{A}) \longrightarrow \mathbf{R}^n$ be the projection corresponding to the real part. A facet decomposition of the complexified complement $M(\mathcal{A})$ is given by

$$\bigcup_F \pi^{-1}(F).$$

The associated dual complex is called the Salvetti complex $S(\mathcal{A})$, which is an $n$ dimensional CW complex. It was shown by M. Salvetti [17] that the inclusion

$$S(\mathcal{A}) \longrightarrow M(\mathcal{A})$$

is a homotopy equivalence.

Let $\mathcal{L}$ be a complex rank one local system over $M(\mathcal{A})$ associated with a representation of the fundamental group

$$\rho : \pi_1(M(\mathcal{A}), x_0) \longrightarrow \mathbf{C}^*.$$

We shall investigate the homology of $M(\mathcal{A})$ with coefficients in the local system $\mathcal{L}$. For our purpose the homology of locally finite chains $H_*^{lf}(M(\mathcal{A}), \mathcal{L})$ also plays an important role. It was shown by Z. Chen [6] that the complex associated with the facet decomposition $\bigcup_F \pi^{-1}(F)$ of $M(\mathcal{A})$ can be used to compute the homology of locally finite chains $H_*^{lf}(M(\mathcal{A}), \mathcal{L})$.

We briefly summarize basic properties of the above homology groups. For a complex arrangement $\mathcal{A}$ choose a smooth compactification $i : M(\mathcal{A}) \longrightarrow X$ with normal crossing divisors. We shall say that the local system $\mathcal{L}$ is generic if and only if there is an isomorphism

$$i_* \mathcal{L} \cong i_! \mathcal{L}$$

where $i_*$ is the direct image and $i_!$ is the extension by 0. This means that the monodromy of $\mathcal{L}$ along any divisor at infinity is not equal to 1. The following theorem was shown in [12].

**Theorem 2.1.** *If the local system $\mathcal{L}$ is generic in the above sense, then there is an isomorphism*

$$H_*(M(\mathcal{A}), \mathcal{L}) \cong H_*^{lf}(M(\mathcal{A}), \mathcal{L}).$$

*We have $H_k(M(\mathcal{A}), \mathcal{L}) = 0$ for any $k \neq n$.*

*Proof.* In general we have isomorphisms

$$H^*(X, i_*\mathcal{L}) \cong H^*(M(\mathcal{A}), \mathcal{L}), \quad H^*(X, i_!\mathcal{L}) \cong H_c^*(M(\mathcal{A}), \mathcal{L})$$

where $H_c$ denotes cohomology with compact supports.

There are Poincaré duality isomorphisms:

$$H_k^{lf}(M(\mathcal{A}), \mathcal{L}) \cong H^{2n-k}(M(\mathcal{A}), \mathcal{L})$$
$$H_k(M(\mathcal{A}), \mathcal{L}) \cong H_c^{2n-k}(M(\mathcal{A}), \mathcal{L}).$$

By the hypothesis $i_*\mathcal{L} \cong i_!\mathcal{L}$ we obtain an isomorphism

$$H_k^{lf}(M(\mathcal{A}), \mathcal{L}) \cong H_k(M(\mathcal{A}), \mathcal{L}).$$

It follows from the above Poincaré duality isomorpshims and the fact that $M(\mathcal{A})$ has a homotopy type of a CW complex of dimension $n$ we have

$$H_k^{lf}(M(\mathcal{A}), \mathcal{L}) \cong 0, \quad k < n$$
$$H_k(M(\mathcal{A}), \mathcal{L}) \cong 0, \quad k > n.$$

Therefore we obtain $H_k(M(\mathcal{A}), \mathcal{L}) = 0$ for any $k \neq n$. Q.E.D.

In the case of a complexified real arrangement, the $\mathbf{C}$-vector space $H_n^{lf}(M(\mathcal{A}), \mathcal{L})$ is spanned by bounded chambers.

## §3. KZ connections and hypergeometric integrals

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $\{I_\mu\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan–Killing form. We set $\Omega = \sum_\mu I_\mu \otimes I_\mu$. Let $r_i : \mathfrak{g} \to \mathrm{End}(V_i)$, $1 \leq i \leq n$, be representations of the Lie algebra $\mathfrak{g}$. We denote by $\Omega_{ij}$ the action of $\Omega$ on the $i$-th and $j$-th components of the tensor product $V_1 \otimes \cdots \otimes V_n$. By using the fact that the Casimir element $c = \sum_\mu I_\mu \cdot I_\mu$ lies in the center of the universal enveloping algebra $U\mathfrak{g}$, it can be shown that the relations:

$$[\Omega_{ik}, \Omega_{ij} + \Omega_{jk}] = 0, \quad (i, j, k \ \text{ distinct}),$$
$$[\Omega_{ij}, \Omega_{k\ell}], \quad (i, j, k, \ell \ \text{ distinct})$$

hold. We define the Knizhnik–Zamolodchikov (KZ) connection as the 1-form

$$\omega = \frac{1}{\kappa} \sum_{1 \le i < j \le n} \Omega_{ij} d \log(z_i - z_j)$$

with values in $\mathrm{End}(V_1 \otimes \cdots \otimes V_n)$ for a non-zero complex parameter $\kappa$.

We set $\omega_{ij} = d \log(z_i - z_j)$, $1 \le i, j \le n$. It follows from the above quadratic relations among $\Omega_{ij}$ together with Arnold's relation

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{k\ell} + \omega_{k\ell} \wedge \omega_{ij} = 0$$

that $\omega \wedge \omega = 0$ holds. This implies that $\omega$ defines a flat connection for a trivial vector bundle over the configuration space

$$X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j \text{ if } i \neq j\}$$

with fiber $V_1 \otimes \cdots \otimes V_n$.

The fundamental group of the configuration space $X_n$ is the pure braid group on $n$ strings denoted by $P_n$. As the holonomy of the connection $\omega$ we have a one-parameter family of linear representations of the pure braid group

$$\theta_\kappa : P_n \to \mathrm{GL}(V_1 \otimes \cdots \otimes V_n).$$

The symmetric group $S_n$ acts on $X_n$ by permutations of coordinates. We denote the quotient space $X_n/S_n$ by $Y_n$. The fundamental group of $Y_n$ is the braid group on $n$ strings denoted by $B_n$. In the case $V_1 = \cdots = V_n = V$, the symmetric group $S_n$ acts diagonally on the trivial vector bundle over $X_n$ with fiber $V^{\otimes n}$ and the connection $\omega$ is invariant by this action. Thus we have a one-parameter family of linear representations of the braid group

$$\theta_\kappa : B_n \to \mathrm{GL}(V^{\otimes n}).$$

Following V. Schechtman and A. Varchenko [18], we shall express the horizontal sections of the KZ connection $\omega$ in terms of hypergeometric integrals. Let

$$\pi : X_{m+n} \longrightarrow X_n$$

be the projection map defined by

$$(z_1, \cdots, z_n, t_1, \cdots, t_m) \mapsto (z_1, \cdots, z_n).$$

We denote by $X_{n,m}$ a fiber of $\pi$, which is the complement of a discriminantal arrangement.

In this article we deal with the case $\mathfrak{g} = sl_2(\mathbf{C})$. As a complex vector space the Lie algebra $sl_2(\mathbf{C})$ has a basis $H, E$ and $F$ satisfying the relations:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

For a complex number $\lambda$ we denote by $M_\lambda$ the Verma module of $sl_2(\mathbf{C})$ with highest weight $\lambda$. Namely, there is a non-zero vector $v \in M_\lambda$ called the highest weight vector satisfying

$$Hv = \lambda v, \ Ev = 0$$

and $M_\lambda$ is spanned by $F^j v$, $j \geq 0$.

We shall consider the tensor product $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$ of the Verma modules of $sl_2(\mathbf{C})$. We set $\lambda = \lambda_1 + \cdots + \lambda_n$. For a non-negative integer $m$ we define the space of weight vectors with weight $\lambda - 2m$ by

$$W[\lambda - 2m] = \{x \in M_{\lambda_1} \otimes \cdots M_{\lambda_n} \ ; \ Hx = (\lambda - 2m)x\}$$

and consider the space of coinvariants defined by

$$N[\lambda - 2m] = W[\lambda - 2m]/F \cdot W[\lambda - 2m + 2].$$

The KZ connection $\omega$ commutes with the diagonal action of $\mathfrak{g}$ on $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$, hence it acts on the space of coinvariants $N[\lambda - 2m]$.

For parameters $\kappa$ and $\lambda$ we consider the multi-valued function

$$\Phi = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{\kappa}} \prod_{1 \leq i \leq m, 1 \leq \ell \leq n} (t_i - z_\ell)^{-\frac{\lambda_\ell}{\kappa}} \prod_{1 \leq i < j \leq m} (t_i - t_j)^{\frac{2}{\kappa}}$$

defined over $X_{n+m}$. Let $\mathcal{L}$ denote the local system associated to the multi-valued function $\Phi$. The symmetric group $S_m$ acts on $X_{n,m}$ by permutations of the coordinates $(t_1, \cdots, t_m)$. The function $\Phi$ is invariant by the action of $S_m$. We put $Y_{n,m} = X_{n,m}/S_m$. The local system $\mathcal{L}$ over $X_{n,m}$ defines a local system on $Y_{n,m}$, which we denote by the same letter $\mathcal{L}$. The dual local system is denoted by $\mathcal{L}^*$.

Let us choose the parameters $\kappa$ and $\lambda_1, \cdots, \lambda_m$ so that the associated local system $\mathcal{L}$ is generic in the sense of the previous section. Under this condition it follows from the vanishing theorem in the previous section that we have

$$H_k(X_{n,m}, \mathcal{L}^*) \cong 0, \quad k \neq m$$

and there is an isomorphism

$$H_m(X_{n,m}, \mathcal{L}^*) \cong H_m^{lf}(X_{n,m}, \mathcal{L}^*).$$

The twisted de Rham complex $(\Omega^*(X_{n,m}), \nabla)$ is a complex with differential $\nabla : \Omega^j(X_{n,m}) \to \Omega^{j+1}(X_{n,m})$ defined by

$$\nabla\omega = d\omega + d\log\Phi \wedge \omega,$$

for $\omega \in \Omega^j(X_{n,m})$. There is a pairing between the homology of the local system $\mathcal{L}^*$ and the cohomology of the twisted de Rham complex

$$H_m(X_{n,m}, \mathcal{L}^*) \times H^m(\Omega^*(X_{n,m}), \nabla) \to \mathbf{C}$$

defined by

$$(c, \omega) \mapsto \int_c \Phi\omega.$$

Such integrals are called hypergeometric integrals. We refer the reader to [16] for a detailed treatment of hypergeometric integrals in the more general situation of hyperplane arrangements.

We define a map

$$\rho : W[\lambda - 2m] \to \Omega^m(X_{n,m})$$

given by

$$\rho(w) = R_w(t, z)dt_1 \wedge \cdots \wedge dt_m$$

using the rational function $R_w(t, z)$ for $w \in W[\lambda - 2m]$ defined in the following manner. Let $v_k \in M_{\lambda_k}$, $1 \le k \le n$, be the highest weight vectors and we set $v = v_1 \otimes \cdots \otimes v_n$. For an $n$-tuple of non-negative integers $J = (j_1, \cdots, j_n)$ we set

$$F^J v = F^{j_1}v_1 \otimes \cdots \otimes F^{j_n}v_n.$$

The weight space $W[\lambda - 2m]$ has a basis $F^J v$ for each $J$ with $j_1 + \cdots + j_n = m$. For the sequence of integers $(i_1, \cdots, i_m) = (\underbrace{1, \cdots, 1}_{j_1}, \cdots, \underbrace{n, \cdots, n}_{j_n})$ we set

$$\eta_J(z, t) = \frac{1}{(t_1 - z_{i_1}) \cdots (t_m - z_{i_m})}.$$

We define the rational function $R_J(z, t)$ by

$$R_J(z, t) = \frac{1}{j_1! \cdots j_n!} \sum_{\sigma \in S_m} \eta_J(z_1, \cdots, z_n; t_{\sigma(1)}, \cdots, t_{\sigma(m)}).$$

It turns out that $\rho$ induces a map to the cohomology of the twisted de Rham complex

$$N[\lambda - 2m] \longrightarrow H^m(\Omega^*(X_{n,m}), \nabla).$$

By this construction we obtain a map

$$\phi : H_m(Y_{n,m}, \mathcal{L}^*) \longrightarrow N[\lambda - 2m]^*$$

defined by

$$\langle \phi(c), w \rangle = \int_c \Phi\rho(w).$$

A lot of works have been done on the expression of the solutions of the KZ equation by means of hypergeometric type integrals (see [7] and [18]). According to the formulation due to V. Schechtman and A. Varchenko [18] the integral

$$\int_c \Phi\rho(w)$$

is a horizontal section of the KZ connection with values in $N[\lambda - 2m]$.

**Theorem 3.1.** *Let us suppose that the local system $\mathcal{L}$ is generic. Then the map $\phi$ gives an isomophism*

$$H_m(Y_{n,m}, \mathcal{L}^*) \cong N[\lambda - 2m]^*.$$

*Moreover, the above isomorphism is equivariant with respect to the action of the pure braid group $P_n$.*

As a consequence we obtain that the following two representations of the pure braid groups are equivalent:

(1) Action of $P_n$ on the twisted homology $H_m(Y_{n,m}, \mathcal{L}^*)$,
(2) Holonomy representation of the KZ equation with values in $N[\lambda - 2m]^*$.

In the case $m = 1$, the above representation of the pure braid group $P_n$ is equivalent to the Gassner representation (see [3]). Let us consider the case $\lambda_1 = \cdots = \lambda_n$ and $m = 2$. We have a two-parameter family of linear representations of the braid group $B_n$. Here the parameters correspond to $\lambda$ and $\kappa$. The representations of the braid group $B_n$ on $H_2(Y_{n,2}, \mathcal{L}^*)$ is exactly the one investigated by R. Lawrence, D. Krammer and S. Bigelow, which we shall call the LKB representation. The following theorem relates the LKB representation and the holonomy representation of the KZ connection.

**Theorem 3.2.** *In the case $\lambda_1 = \cdots = \lambda_n$ and $m = 2$, the LKB representation*

$$B_n \to Aut\ H_2(Y_{n,2}, \mathcal{L}^*)$$

*is equivalent to the action of $B_n$ obtained as the holonomy of the KZ connection*

$$B_n \to Aut\ N[\lambda - 4]^*.$$

We take distinct $n+1$ points $p_1, \cdots, p_{n+1} \in \mathbf{C}P^1$ with $p_{n+1} = \infty$ and we associate to these points the highest weights $\lambda_1, \cdots, \lambda_n, \lambda_{n+1}$. Then the space of coinvariants

$$(M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \otimes M^*_{\lambda_{n+1}})/\mathfrak{g}$$

is identifined with

$$N[\lambda_{n+1}] = N[\lambda - 2m]$$

with

$$m = \frac{1}{2}(\lambda_1 + \cdots + \lambda_n - \lambda_{n+1}).$$

Let us briefly discuss a relation between the space of conformal blocks in conformal field theory on the Riemann sphere and the space of coinvariants $N[\lambda - 2m]$.

First, we recall the definition of the space of conformal blocks. We refer the reader to [13] for an introductory treatment of this subject. We put $\kappa = K + 2$ and assume that $K$ is a positive integer. We suppose that the highest weights $\lambda_1, \cdots, \lambda_{n+1}$ associated with the points $p_1, \cdots, p_{n+1} \in \mathbf{C}P^1$ with $p_{n+1} = \infty$ are non-negative integers and satisfy $0 \leq \lambda_1, \cdots, \lambda_{n+1} \leq K$. We denote by $V_{\lambda_1}, \cdots, V_{\lambda_{n+1}}$ the irreducible representations of $\mathfrak{g} = sl_2(\mathbf{C})$ with highest weights $\lambda_1, \cdots, \lambda_{n+1}$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}((\xi)) \oplus \mathbf{C}c$ be the affine Lie algebra, which is the central extension of the loop algebra $\mathfrak{g} \otimes \mathbf{C}((\xi))$. Here $\mathbf{C}((\xi))$ denotes the ring of Laurent series. Let $\mathcal{H}_{\lambda_j}$ be the integrable highest weight module of $\widehat{\mathfrak{g}}$ with highest weight $\lambda_j$.

We denote by $\mathcal{M}_p$ the set of meromorphic functions on $\mathbf{C}P^1$ with poles at most at $p_1, \cdots, p_{n+1}$. Then $\mathfrak{g} \otimes \mathcal{M}_p$ has a structure of a Lie algebra and acts diagonally on the tensor product $\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} \otimes \mathcal{H}^*_{\lambda_{n+1}}$ by means of the Laurent expansions of a meromorphic function at the points $p_1, \cdots, p_{n+1} \in \mathbf{C}P^1$.

The space of conformal blocks is defined as the space of coinvariants

$$\mathcal{H}(p, \lambda) = (\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} \otimes \mathcal{H}^*_{\lambda_{n+1}})/(\mathfrak{g} \otimes \mathcal{M}_p).$$

It turns out that there is a surjective map

$$(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \otimes V^*_{\lambda_{n+1}})/\mathfrak{g} \longrightarrow \mathcal{H}(p, \lambda)$$

and the kernel is described by some algebraic equations coming from the definition of the space of conformal blocks. It was shown by B. Feigin, V. Schechtman and A. Varchenko [8] that these algebraic equations correspond to exact forms in the twisted de Rham cohomology $\Omega^*(Y_{n,m})$ by the map $\rho : W[\lambda - 2m] \to \Omega^m(Y_{n,m})$ and $\rho$ induces a map

$$\mathcal{H}(p, \lambda) \to H^m(\Omega^*(Y_{n,m}), \nabla).$$

Therefore we obtain a map

$$\phi : H_m(Y_{n,m}, \mathcal{L}^*) \to \mathcal{H}(p, \lambda)^*$$

defined by

$$\langle \phi(c), w \rangle = \int_c \Phi \rho(w).$$

It was shown by A. Varchenko [20] that the above map $\phi$ is surjective.

It is a basic result in conformal field theory that the spaces of conformal blocks form a vector bundle over the configuration space $X_n$ equipped with the flat KZ connection. Therefore, the pure braid group $P_n$ acts on the space of conformal blocks $\mathcal{H}(p, \lambda)^*$ by means of the holonomy of this connection. The map $\phi$ is equivariant with respect to the action of the pure braid group $P_n$.

The map $\phi : H_m(Y_{n,m}, \mathcal{L}^*) \to \mathcal{H}(p, \lambda)^*$ is not in general injective. Our local system $\mathcal{L}$ is no longer generic and there is a subtle point about the structure of the homology $H_m(Y_{n,m}, \mathcal{L}^*)$.

Let us consider the natural map

$$\alpha : H_m(Y_{n,m}, \mathcal{L}^*) \to H_m^{lf}(Y_{n,m}, \mathcal{L}^*)$$

and put $\mathrm{Im}(\alpha) = H_m^{lf}(Y_{n,m}, \mathcal{L}^*)_{reg}$. We call $H_m^{lf}(Y_{n,m}, \mathcal{L}^*)_{reg}$ the space of regularizable cycles.

We describe the relation between the space of conformal blocks and the space of regularizable cycles in the case $n = 2$. This corresponds to the case of 3 points on the Riemann sphere. We fix $K$ a positive integer and suppose that the highest weights $\lambda_1, \lambda_2, \lambda_3$ are non-negative integers satisfying $0 \leq \lambda_1, \lambda_2, \lambda_3 \leq K$.

It follows from a result due to R. Silvotti [19] that $\phi$ induces an isomorphism

$$H_m^{lf}(Y_{2,m}, \mathcal{L}^*)_{reg} \cong \mathcal{H}(p_1, p_2, p_3; \lambda_1, \lambda_2, \lambda_3)^*.$$

The above homology group $H_m^{lf}(Y_{2,m}, \mathcal{L}^*)_{reg}$ is isomorphic to $\mathbf{C}$ if the quantum Clebsch–Gordan condition

$$\begin{aligned} &|\lambda_1 - \lambda_2| \leq \lambda_3 \leq \lambda_1 + \lambda_2 \\ &\lambda_1 + \lambda_2 + \lambda_3 \in 2\mathbf{Z} \\ &\lambda_1 + \lambda_2 + \lambda_3 \leq 2K \end{aligned}$$

is satisfied and is isomorphic to 0 otherwise.

## §4. Iterated integrals of logarithmic forms

In this section we give a description of the holonomy representation of the fundamental group of the complement $M(\mathcal{A})$ of a hyperplane arrangement. We refer the reader to [13] for a more detailed account of this subject.

First, we briefly recall basic definitions for iterated integrals. Let $M$ be a smooth manifold and $\omega_1, \cdots, \omega_k$ be differential forms on $M$. We fix a base point $x_0 \in M$ and we denote by $\Omega M$ the loop space of $M$ based at $x_0$. Namely, $\Omega M$ is the space of piecewise smooth maps $\gamma : I \to M$ such that $\gamma(0) = \gamma(1) = x_0$. Let $\Delta_k$ denote the Euclidean simplex defined by

$$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbf{R}^k \ ; \ 0 \le t_1 \le \cdots \le t_k \le 1\}.$$

There is an evaluation map

$$\varphi : \Delta_k \times \Omega M \to \underbrace{M \times \cdots \times M}_{k}$$

defined by $\varphi(t_1, \cdots, t_k; \gamma) = (\gamma(t_1), \cdots, \gamma(t_k))$. Let $\pi_i : \underbrace{M \times \cdots \times M}_{k} \to M$ be the projection on the $i$-th factor and put

$$\omega_1 \times \cdots \times \omega_k = \pi_1^* \omega_1 \wedge \cdots \wedge \pi_k^* \omega_k.$$

We define the iterated integral $\int \omega_1 \cdots \omega_k$ by the expression

$$\int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

which is the integration along the fiber with respect to the projection $p : \Delta_k \times \Omega M \to \Omega M$.

The iterated integral $\int \omega_1 \cdots \omega_k$ is considered to be a differential form on the loop space $\Omega M$ with degree $p_1 + \cdots + p_k - k$, where $p_j$ is the degree of the differential form $\omega_j$.

As a differential form on the loop space $d \int \omega_1 \cdots \omega_k$ is expressed as

$$\sum_{j=1}^{k} (-1)^{\nu_{j-1}+1} \int \omega_1 \cdots \omega_{j-1} d\omega_j \ \omega_{j+1} \cdots \omega_k$$

$$+ \sum_{j=1}^{k-1} (-1)^{\nu_j+1} \int \omega_1 \cdots \omega_{j-1} (\omega_j \wedge \omega_{j+1}) \omega_{j+2} \cdots \omega_k$$

where $\nu_j = p_1 + \cdots + p_j - j$.

Let $\mathcal{A} = \{H_1, \cdots, H_\ell\}$ be a hyperplane arrangement in $\mathbf{C}^n$ and we consider the complement

$$M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H.$$

Let $f_j$ be a linear form defining $H_j$ and we set

$$\omega_j = d \log f_j, \quad 1 \leq j \leq \ell.$$

We denote by $A = \bigoplus_{q \geq 0} A^q$ the Orlik–Solomon algebra of the hyperplane arrangement $\mathcal{A}$, which is isomorphic to the algebra over $\mathbf{C}$ generated by 1 and the logarithmic forms $\omega_j$, $1 \leq j \leq \ell$. It is known from work of E. Brieskorn [4] that there is an isomorphism

$$A \cong H^*(M(\mathcal{A}); \mathbf{C}).$$

For more details about the Orlik–Solomon algebra we refer the reader to Orlik–Terao [15].

We define the reduced complex $\overline{A}$ by shifting the degrees by one as

$$\overline{A}^q = \begin{cases} 0, & q < 0 \\ A^{q+1}, & q \geq 0. \end{cases}$$

The reduced bar complex $\overline{B}^*(A)$ is the tensor algebra

$$\overline{B}^*(A) = \bigoplus_{k \geq 0} \left( \bigotimes^k \overline{A} \right)$$

generated by $\overline{A}$. Then $\overline{B}^*(A)$ has a natural structure of a graded algebra and we introduce the coboundary operator by

$$\begin{aligned} & d(\varphi_1 \otimes \cdots \otimes \varphi_k) \\ & = \sum_{j=1}^{k-1} (-1)^{\nu_j + 1} \varphi_1 \otimes \cdots \otimes (\varphi_j \wedge \varphi_{j+1}) \otimes \cdots \otimes \varphi_k \end{aligned}$$

where $\varphi_j \in \overline{A}^{q_j}$ and we wet $\nu_j = q_1 + \cdots + q_j$. We define the iterated integral map $\mathcal{I}$ from the reduced bar complex to the space of differential forms on the loop space by

$$\mathcal{I}(\varphi_1 \otimes \cdots \otimes \varphi_k) = \int \varphi_1 \cdots \varphi_k.$$

We define the filtration on the reduced bar complex by

$$\mathcal{F}^{-k}(\overline{B}^*(A)) = \bigoplus_{\ell \le k}(\bigotimes^{\ell} \overline{A}).$$

Let $J$ be the augmentation ideal of the group algebra of the fundamental group of $M = M(\mathcal{A})$. There is a pairing map

$$H^0(\overline{B}^*(A)) \times \mathbf{Z}\pi_1(M, \mathbf{x}_0) \longrightarrow \mathbf{C}$$

given by iterated integrals of logarithmic 1-forms. There is an induced filtration $\mathcal{F}^{-k}$ on $H^0(\overline{B}^*(A))$. By the basic theorem on de Rham fundamental groupd due to K. T. Chen [5] and the fact that the inclusion map from the Orlik–Solomon algebra to the de Rham complex of $M$ induces an isomorphism on cohomology we obtain the following theorem.

**Theorem 4.1.** [13] *For the reduced bar complex of the Orlik–Solomon algebra the iterated integral map gives an isomorphism*

$$\mathcal{F}^{-k}H^0(\overline{B}^*(A)) \cong \mathrm{Hom}(\mathbf{Z}\pi_1(M, \mathbf{x}_0)/J^{k+1}, \mathbf{C}).$$

Let $\mathcal{L}(X_1, \cdots, X_\ell)$ be the free Lie algebra whose generators are in one to one correspondence with the hyperplanes in $\mathcal{A}$. We define the holonomy Lie algebra by

$$\mathfrak{h}(M) = \mathcal{L}(X_1, \cdots, X_\ell)/\mathfrak{a}$$

where $\mathfrak{a}$ is the ideal generate by

$$[X_{j_p}, X_{j_1} + \cdots + X_{j_k}], \quad 1 \le p < k$$

for the maximal family of hyperplanes $\{H_{j_1}, \cdots, H_{j_k}\}$ such that

$$\mathrm{codim}_{\mathbf{C}}(H_{j_1} \cap \cdots \cap H_{j_k}) = 2.$$

We put

$$\omega = \sum_{j=1}^{m} \omega_j X_j.$$

Then we have the universal holonomy map

$$\Theta_0 : \pi_1(M, \mathbf{x}_0) \longrightarrow \mathbf{C}\langle\langle X_1, \cdots, X_m \rangle\rangle/\mathfrak{a}$$

defined by

$$\Theta_0(\gamma) = 1 + \sum_{k=1}^{\infty} \int_\gamma \underbrace{\omega \cdots \omega}_{k}.$$

This induces an isomorphism

$$\mathbf{C}\widehat{\pi}_1(M, \mathbf{x}_0) \cong \mathbf{C}\langle\langle X_1, \cdots, X_m\rangle\rangle/\mathfrak{a}.$$

By taking the primitive part, we have an isomorphism between the nilpotent completion of the fundamental group and the completed holonomy Lie algebra over $\mathbf{C}$.

Let

$$\pi_1(M) = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_k \supset \cdots$$

be the lower central series defined by

$$\Gamma_{k+1} = [\Gamma_1, \Gamma_k], \quad k \geq 1.$$

Then there is an isomorphism of graded Lie algebras

$$\bigoplus_{k\geq 1} [\Gamma_k/\Gamma_{k+1}] \otimes \mathbf{C} \cong \mathfrak{h}(M).$$

Because of the integrability of the connection

$$\omega = \sum_{j=1}^{m} \omega_j X_j$$

we have the following linear representations of the fundamental group.

**Proposition 4.1.** *For any representation of the holonomy Lie algebra $r : \mathfrak{h}(M) \to \mathrm{End}(V)$ there is a linear representation of the fundamental group*

$$\pi_1(M, \mathbf{x}_0) \longrightarrow \mathrm{GL}(V)$$

*obtained by substituting the representation $r$ to the universal holonomy homomorphism.*

We denote by $C^\infty(\widetilde{M})$ the space of smooth functions on the universal covering of $M$. We have a map

$$\mathcal{F}^{-k} H^0(\overline{B}^*(A)) \longrightarrow C^\infty(\widetilde{M}).$$

by the iterated integral for paths wth fixed starting point and we denote its image by $F_k(M)$. There is an increasing filtration of functions

$$\mathbf{C} = F_0(M) \subset F_1(M) \subset \cdots \subset F_k(M) \subset \cdots$$

called hyperlogarithms and we have

$$dF_{k+1}(M) \subset F_k(M) \otimes A^1, \quad k = 0, 1, 2, \cdots .$$

We set $H_+(M) = \bigoplus_{p>0} H_p(M;\mathbf{C})$ and equip the tensor algebra $TH_+(M)$ with a structure of a graded algebra such that each non-zero element of $H_p(M;\mathbf{C})$ has degree $p-1$. Let $\{\varphi_j\}$ be a basis of the Orlik–Solomon algebra and $\{Z_j\}$ be its dual basis. We introduce a differential $\delta : TH_+(M)_p \longrightarrow TH_+(M)_{p-1}$ as the dual of the cup product homomorphism. More explicitly, we define the differential $\delta$ by

$$\delta Z_k = -\sum_k (-1)^{p_i} c_{ij}^k [Z_i, Z_j]$$

when we have the eqality

$$\varphi_i \wedge \varphi_j = \sum_k c_{ij}^k \varphi_k$$

holds in the Orlik–Solomon algebra. The reduced bar complex $\overline{B}^*(A)$ is considered to be the dual complex of $TH_+(M)$ with the differential $\delta$.

**Theorem 4.2.** [13] *Let $\mathcal{A} = \{H_1, \cdots, H_\ell\}$ be a hyperplane arrangement. Then linear combinations of iterated integrals of the logarithmic forms $\omega_j$, $1 \leq j \leq \ell$, on the complement of the hyperplane arrangement*

$$\sum_{j_1 \cdots j_k} a_{j_1 \cdots j_k} \int_\gamma \omega_{j_1} \cdots \omega_{j_k}, \quad a_{j_1 \cdots j_k} \in \mathbf{C}$$

*depends only on the homotopy class of a loop $\gamma$ if and only if the correspondence $X_{j_1} \cdots X_{j_k} \mapsto a_{j_1 \cdots j_k}$ defines a linear map $U\mathfrak{h}(M) \otimes \mathbf{C} \longrightarrow \mathbf{C}$.*

*Proof.* We consider the above iterated integral as a function on the loop space $\Omega M$. It will be enough to show that the condition

$$\sum_{j_1 \cdots j_k} a_{j_1 \cdots j_k} \, d \int_\gamma \omega_{j_1} \cdots \omega_{j_k} = 0$$

holds if and only if the correspondence $X_{j_1} \cdots X_{j_k} \mapsto a_{j_1 \cdots j_k}$ defines a linear map $U\mathfrak{h}(M) \otimes \mathbf{C} \longrightarrow \mathbf{C}$. The above condition is equivalent to the equality

$$\sum_{j_1 \cdots j_k} a_{j_1 \cdots j_k} \, d(\omega_{j_1} \otimes \cdots \otimes \omega_{j_k}) = 0$$

in the reduced bar complex $\overline{B}^*(A)$. By means of the duality between $\overline{B}^*(A)$ and $TH_+(M)$ this equality is equivalent to the condition

$$\sum_{j_1 \cdots j_k} \langle a_{j_1 \cdots j_k} \omega_{j_1} \otimes \cdots \otimes \omega_{j_k}, \partial Z \rangle = 0$$

for any $Z \in TH_+(M)_1$. By writing down explicitly the above condition we obtain the desired statement. Q.E.D.

Finally, we describe a correspondence between unipotent representations of the fundamental group of the complement of a hyperplane arrangement and nilpotent connections. The following fact was first obtained by K. Aomoto [1] (see also the work due to R. Hain [9]).

**Theorem 4.3.** *Let*

$$\rho : \pi_1(M, \mathbf{x}_0) \longrightarrow \mathrm{GL}(V)$$

*be a unipotent representation of the fundamental group of the complement of a hyperplane arrangement. Then there exists an integrable connection*

$$\omega = \sum_{j=1}^{\ell} A_j \omega_j, \quad A_j \in \mathrm{End}(V)$$

*such that each $A_j$ is nilpotent and the monodromy representation of $\omega$ coincides with $\rho$.*

*Proof.* For a unipotent representation $\rho : \pi_1(M, \mathbf{x}_0) \longrightarrow \mathrm{GL}(V)$ there exists a sufficiently large integer $k$ such that $\rho$ induces a homomorphism

$$\widetilde{\rho} : \mathbf{C}\pi_1(M, \mathbf{x}_0)/J^{k+1} \longrightarrow \mathrm{End}(V).$$

On the other hand the universal holonomy homomorphism of the connection $\sum_{j=1}^{\ell} \omega_j X_j$ induces an isomorphism

$$\theta : \mathbf{C}\pi_1(M, \mathbf{x}_0)/J^{k+1} \cong \mathbf{C}\langle\langle X_1, \cdots, X_\ell \rangle\rangle/(\mathfrak{a} + \widehat{J}^{k+1})$$

where $\widehat{J}$ denotes the completed augmentation ideal. We define a homomorphism

$$\alpha : \mathbf{C}\langle\langle X_1, \cdots, X_\ell \rangle\rangle/(\mathfrak{a} + \widehat{J}^{k+1}) \longrightarrow \mathrm{End}(V)$$

by $\alpha = \widetilde{\rho} \circ \theta^{-1}$ and put $A_j = \alpha(X_j)$, $1 \leq j \leq \ell$. Here $A_j^{k+1} = 0$ holds and $A_j$ is a nilpotent endomorphism. Since $\widetilde{\rho} = \alpha \circ \theta$ the representation $\rho$ is obtained from the universal holonomy homomorphism by the substitution $X_j = A_j$. Q.E.D.

## References

[1] K. Aomoto, Fonctions hyperlogarithmiques et groupes de monodromie unipotents, J. Fac. Sci. Univ. Tokyo, **25** (1978), 149–156.

[2] S. Bigelow, The Lawrence–Krammer representation, In: Topology and Geometry of Manifolds, Athens, GA, 2001, Proc. Sympos. Pure Math., **71**, Amer. Math. Soc., Providence, RI, 2003, pp. 51–68.

[3] J. Birman, Braids, Links, and Mapping Class Groups, Ann. of Math. Stud., **82**, Princeton Univ. Press, 1974.

[4] E. Brieskorn, Sur les groupes de tresses [d'après V. I. Arnol'd], In: Séminaire Bourbaki 1971/72 Exposé 401, Lecture Notes in Math., **317**, Springer-Verlag, 1973, pp. 21–44.

[5] K. -T. Chen, Iterated path integrals, Bull. of Amer. Math. Soc., **83** (1977), 831–879.

[6] Z. Chen, Stratifications of complementary spaces of hyperplane arrangements and homology groups with local coefficients, Thesis, the University of Tokyo, 2005.

[7] E. Date, M Jimbo, A. Matsuo and T. Miwa, Hypergeometric type integrals and the $sl(2, \mathbf{C})$ Knizhnik-Zamolodchikov equations, Intern. J. Modern Phys., **B4** (1990), 1049–1057.

[8] B. Feigin, V. Schechtman and A. Varchenko, On algebraic equations satisfied by hyergeometric correlators in WZW models. I, Comm. Math. Phys., **163** (1994), 173–184.

[9] R. M. Hain, On a generalization of Hilbert's 21st problem, Ann. Sci. Ecole Norm. Sup., **19** (1986), 609–627.

[10] V. G. Knizhnik and A. B. Zamolodchikov, Current algebra and Wess–Zumino models in two dimensions, Nuclear Phys. B, **247** (1984), 83–103.

[11] T. Kohno, Homology of a local system on the complement of hyperplanes, Proc. Japan Acad. Ser. A, **62** (1986), 144–147.

[12] T. Kohno, Bar complex of the Orlik–Solomon algebra, Topology Appl., **118** (2002), 147–157.

[13] T. Kohno, Conformal Field Theory and Topology, Transl. Math. Monogr., **210**, Amer. Math. Soc., 2002.

[14] M. Kontsevich, Vassiliev's knot invariants, Adv. Soviet Math., **16** (1993), 137–150 .

[15] P. Orlik and H. Terao, Arrangements of Hyperplanes, Springer-Verlag, 1992.

[16] P. Orlik and H. Terao, Arrangements and Hypergeometric Integrals, MSJ Memoirs, **9**, Math. Soc. Japan, 2001; 2nd ed., 2007.

[17] M. Salvetti, Topology of the complement of real hyperplanes in $C^N$. Invent. Math., **88** (1987), 603–618.

[18] V. Schechtman and A. Varchenko, Hypergeometric solutions of the Knizhnik–Zamolodchikov equation, Lett. in Math. Phys., **20** (1990), 93–102.

[19] R. Silvotti, Local systems on the complement of hyperplanes and fusion rules in conformal field theory, Internat. Math. Res. Notices, **1994**, no. 3, 111 ff., approx. 17 pp. (electronic).

[20] A. Varchenko, Multidimensional Hypergeometric Functions and Representation Theory of Lie Algebras and Quantum Groups, Adv. Ser. Math. Phys., **21**, World Scientific, 1995.

Toshitake Kohno
*IPMU, Graduate School of Mathematical Sciences, the University of Tokyo, Tokyo, 153-8914, Japan*
*E-mail address*: kohno@ms.u-tokyo.ac.jp