

Weighted projective lines associated to regular systems of weights of dual type

Atsushi Takahashi

Abstract.

We associate to a regular system of weights a weighted projective line over an algebraically closed field of characteristic zero in two different ways. One is defined as a quotient stack via a hypersurface singularity for a regular system of weights and the other is defined via the signature of the same regular system of weights.

The main result in this paper is that if a regular system of weights is of dual type then these two weighted projective lines have equivalent abelian categories of coherent sheaves. As a corollary, we can show that the triangulated categories of the graded singularity associated to a regular system of weights has a full exceptional collection, which is expected from homological mirror symmetries.

The main theorem of this paper will be generalized to more general one, to the case when a regular system of weights is of genus zero, which will be given in [5]. Since we need more detailed study of regular systems of weights and some knowledge of algebraic geometry of Deligne–Mumford stacks there, the author write a part of the result in this paper to which another simple proof based on the idea by Geigle–Lenzing [2] can be applied.

§1. Introduction

Let $W := (a_1, a_2, a_3; h)$ be a tuple of four positive integers. If it satisfies a certain combinatorial condition, it is called a regular system of weights. Let k be an algebraically closed field of characteristic zero. The condition is equivalent to the condition that for a polynomial in $k[x, y, z]$

$$f_W(x, y, z) = \sum_{a_1 i_1 + a_2 i_2 + a_3 i_3 = h} c_{i_1 i_2 i_3} x^{i_1} y^{i_2} z^{i_3}, \quad c_{i_1 i_2 i_3} \in k,$$

Received September 23, 2008.

2000 *Mathematics Subject Classification.* Primary 14J33; Secondary 53D37, 32S25.

with generic coefficients $c_{i_1 i_2 i_3}$, $\text{Spec}(k[x, y, z]/(f_W))$ has at most an isolated singularity only at the origin [7]. Since f_W is weighted homogeneous, one naturally associates to W the following quotient stack:

$$\mathcal{C}_{f_W} := [\text{Spec}(k[x, y, z]/(f_W)) \setminus \{0\} / k^*],$$

where we set $k^* := \text{Spec}(k\mathbb{Z})$. \mathcal{C}_{f_W} is a Deligne–Mumford stack regarded as a smooth projective curve $\text{Proj}(k[x, y, z]/(f_W))$ with a finite number of isotropic points on it.

On the other hand, to the signature $A_W = (\alpha_1, \dots, \alpha_r)$ of W , a combinatorically defined multi-set which can be identified with the multi-set of orders of isotropy group at isotropic points on $\text{Proj}(k[x, y, z]/(f_W))$ [8], one can associate the algebra following Geigle–Lenzing [1]

$$R_{A_W, \lambda} := k[X_1, \dots, X_r] / I_\lambda,$$

where I_λ is an ideal generated by $r - 2$ homogeneous polynomials

$$X_1^{\alpha_1} + X_2^{\alpha_2} + X_3^{\alpha_3}, X_1^{\alpha_1} + \lambda_i X_2^{\alpha_2} + X_i^{\alpha_i}, \quad \lambda_i \in k \setminus \{0, 1\}, \quad i = 4, \dots, r.$$

Since $R_{A_W, \lambda}$ is graded with respect to an abelian group

$$L(A_W) := \bigoplus_{i=1}^r \mathbb{Z}\vec{X}_i / \left(\alpha_i \vec{X}_i - \alpha_j \vec{X}_j; 1 \leq i < j \leq r \right),$$

one can consider the quotient stack

$$\mathcal{C}_{A_W, \lambda} := [\text{Spec}(R_{A_W, \lambda}) \setminus \{0\} / \text{Spec}(kL(A_W))].$$

$\mathcal{C}_{A_W, \lambda}$ is a Deligne–Mumford stack whose underlying quotient scheme is a smooth projective line (it is easy to see that $R_{A_W, \lambda} \supset k[X_1^{\alpha_1}, X_2^{\alpha_2}]$ as a sub-ring). Properties of categories $\text{coh}(\mathcal{C}_{A_W, \lambda})$ and $D^b \text{coh}(\mathcal{C}_{A_W, \lambda})$ are extensively studied by Geigle–Lenzing [1].

It is a very natural and interesting problem to compare \mathcal{C}_{f_W} with $\mathcal{C}_{A_W, \lambda}$ as algebraic stacks. We have the following result.

Theorem 1.1 (Main Theorem 5.1). *Let W be a regular system of weights of dual type. Then, there exists a \mathbb{Z} -graded sub-ring R_W of $R_{A_W, \lambda}$ which induces equivalences of abelian categories:*

$$(1.1) \quad \text{coh}(\mathcal{C}_{f_W}) \simeq \text{coh}(\mathcal{C}_W) \simeq \text{coh}(\mathcal{C}_{A_W, \lambda}),$$

where $\mathcal{C}_W := [\text{Spec}(R_W) \setminus \{0\} / k^*]$.

If W is a regular system of weights of dual type then we have $r = 3$ and hence the parameter λ does not appear in $R_{A_W, \lambda}$. Note also that as an abelian category $\text{coh}(\mathcal{C}_{f_W})$ is independent of the choice of the polynomial f_W .

One of motivations of this work is to study a qualitative structure of the triangulated category $D_{S_g}^{gr}(R_W) := D^b(\text{gr-}R_W)/D^b(\text{grproj-}R_W)$. In particular, we have been interested in finding a “nice” or “special” full strongly exceptional collection of $D_{S_g}^{gr}(R_W)$ by using graded matrix factorizations. So far, we have succeeded to obtain it in [12] for A_I -type, in [3] for ADE-type and in [4] for the case $\epsilon_W = -1$.

It is in general a very difficult problem, however, by the main theorem in this paper combining with a result by Orlov [6], one has a slightly weaker statement for *any* regular system of weights of dual type:

Corollary 1.2 (Corollary 6.3). *Let W be a regular system of weights of dual type. The triangulated category $D_{S_g}^{gr}(R_W)$ has a full exceptional collection $(E_1, \dots, E_{\mu_{W^*}})$.*

This is expected from the homological mirror symmetry conjecture that predicts the existence of a triangulated equivalence $D_{S_g}^{gr}(R_W) \simeq D^b\text{Fuk}(f_{W^*}) \simeq D^b\text{Fuk}^{\rightarrow}(\{\gamma_{\bullet}\})$ where $\text{Fuk}^{\rightarrow}(\{\gamma_{\bullet}\})$ is the directed Fukaya category associated to a distinguished basis of vanishing graded Lagrangian sub-manifolds $\{\gamma_{\bullet}\} = \{\gamma_1, \dots, \gamma_{\mu}\}$ in the Milnor fiber of the dual regular system of weights (see Section 7 for details).

We give here an outline of the paper. Section 2 introduces the definition of a regular system of weights and several invariants of it given in [7] and [8]. Section 3 explains the notion of the topological mirror symmetry and the duality of regular systems of weights defined in [11] and [8]. Most of results in this paper rely on the classification of regular systems of weights of dual type and several data given explicitly by them. In Section 4, after preparing some definitions and reviewing a construction of weighted projective line by Geigle–Lenzing [1], we characterize a special sub-ring R_W of their homogeneous coordinate ring R_{A_W} of a weighted projective line, which will play a key role in our story.

Section 5 gives the main theorem of this paper. Its proof uses the data of invariants of regular systems of weights of dual type, which we gave in Appendix. We shall give an application of this result in Section 6. Section 7 explains the homological mirror symmetry conjecture of hypersurface singularities and one of our motivations of the paper. In particular, it is discussed that the bounded derived category of coherent sheaves on a weighted projective line associated to a regular system of weights of dual type is expected to be triangulated equivalent to the

derived category of a directed Fukaya category associated to a cusp singularity associated to the dual signature.

The main theorem of this paper will be generalized to more general one, to the case when a regular system of weights W is of genus zero, i.e., when $g(\text{Proj}(R_W)) = 0$, which will be given in [5].

Acknowledgments. The author would like to thank Hiroshige Kajiuura and Kyoji Saito for valuable discussion. This work was partly supported by Grant-in Aid for Scientific Research grant numbers 17740036 from the Ministry of Education, Culture, Sports, Science and Technology, Japan.

§2. Regular systems of weights

In this paper, we denote by k an algebraically closed field of characteristic zero.

Definition 2.1 ([7]). Let $a_i, i = 1, 2, 3$ and h be positive integers.

- (i) We call a tuple of integers $W := (a_1, a_2, a_3; h)$ a *regular system of weights* if $\text{gcd}(a_1, a_2, a_3; h) = 1$ and a rational function:

$$(2.1) \quad \chi_W(T) := \prod_{i=1}^3 \frac{1 - T^{h-a_i}}{1 - T^{a_i}},$$

is a polynomial in T .

- (ii) The integers a_i are called *weights* of W and h is called the *Coxeter number* of W .

The next proposition relates combinatorics of regular systems of weights with geometries of hypersurface singularities.

Proposition 2.2 ([7]). *The following conditions are equivalent:*

- (i) $W = (a_1, a_2, a_3; h)$ is a regular system of weights.
- (ii) There is at least one weighted homogeneous polynomial in $k[x, y, z]$

$$f(x, y, z) = \sum_{a_1 i_1 + a_2 i_2 + a_3 i_3 = h} c_{i_1 i_2 i_3} x^{i_1} y^{i_2} z^{i_3}, \quad c_{i_1 i_2 i_3} \in k,$$

such that the hypersurface $\{(x, y, z) \in k^3 \mid f(x, y, z) = 0\}$ has at most an isolated singularity only at the origin.

- (iii) There is a non-empty dense subset of

$$\{g(x, y, z) \in k[x, y, z] \mid g(x, y, z) = \sum_{a_1 i_1 + a_2 i_2 + a_3 i_3 = h} c_{i_1 i_2 i_3} x^{i_1} y^{i_2} z^{i_3}, c_{i_1 i_2 i_3} \in k\}$$

such that any polynomial $f(x, y, z)$ belonging to that set defines the hypersurface $\{(x, y, z) \in k^3 \mid f(x, y, z) = 0\}$ which has at most an isolated singularity only at the origin.

Remark 2.3. We shall see later that one can sometimes choose in (iii) a “canonical” polynomial, which will be one of keys to prove the main theorem in this paper.

Definition 2.4 ([7]). Let W be a regular system of weights.

(i) The positive integer μ_W defined by

$$(2.2) \quad \mu_W := \prod_{i=1}^3 \frac{h - a_i}{a_i}$$

is called the *rank* or the *Milnor number* of the regular system of weights W .

(ii) The integer ϵ_W defined by

$$(2.3) \quad \epsilon_W := \left(\sum_{i=1}^3 a_i \right) - h$$

is called the *minimal exponent* or the *Gorenstein parameter* of W .

(iii) There are finite number of integers $m_1 < m_2 \leq \dots \leq m_{\mu_W - 1} < m_{\mu_W}$ such that

$$T^{m_1} + \dots + T^{m_{\mu_W}} = T^{\epsilon_W} \chi_W(T).$$

Each integer m_i is called an *exponent* of W , which satisfies the property

$$(2.4) \quad m_i + m_{\mu_W - i + 1} = h, \quad i = 1, \dots, \mu_W.$$

Define more “geometric” invariants for regular systems of weights, whose meanings will be clear in later sections.

Definition 2.5 ([7][8]). Let $W = (a_1, a_2, a_3; h)$ be a regular system of weights.

(i) The *genus* of W is a non-negative integer g_W defined as the number of exponents of W equal to 0.

(ii) Set $m(a_i, a_j : h) := \#\{(u, v) \in (\mathbb{Z}_{\geq 0})^2 \mid a_i u + a_j v = h\}$. Consider the following multi-set of positive integers

$$A'_W := \{a_i \mid h/a_i \notin \mathbb{Z}, i = 1, 2, 3\} \coprod \{\gcd(a_i, a_j)^{(m(a_i, a_j : h) - 1)} \mid 1 \leq i < j \leq 3\},$$

where we mean by $\gcd(a_i, a_j)^{(m(a_i, a_j; h) - 1)}$ that the integer $\gcd(a_i, a_j)$ appears $m(a_i, a_j; h) - 1$ times in A'_W . Let A_W be a subset of A'_W consisting of integers greater than 1 such that $A'_W = A_W \prod \{1^s\}$ for some $s \in \mathbb{Z}_{\geq 0}$.

- (iii) The pair $(g_W; A_W)$ is called the *signature* of W . If g_W is 0, then we often omit g_W and call A_W the signature of W .

Remark 2.6. Usually, we shall write A_W as $(\alpha_1, \dots, \alpha_r)$ so that $2 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_r$ for some $r \in \mathbb{Z}_{\geq 0}$.

Definition 2.7 ([8]). Let W be a regular system of weights.

- (i) The polynomial

$$(2.5) \quad \varphi_W(\lambda) := \prod_{i=1}^{\mu_W} (\lambda - e[\frac{m_i}{h}]), \quad e[*] = \exp(2\pi\sqrt{-1}*),$$

is called the *characteristic polynomial* of W . $\varphi_W(\lambda)$ is a cyclotomic polynomial.

- (ii) $\varphi_W(\lambda)$ has a unique expression

$$(2.6) \quad \varphi_W(\lambda) = \prod_{i \in \mathbb{Z}_{>0}} (\lambda^i - 1)^{e_W(i)}$$

for some integers $e_W(i)$. A set of integers $M(W) := \{i \mid e_W(i) \neq 0\}$ is called the *classifying poset* of W since $M(W)$ is a partially ordered set (poset) with respect to the division relation on integers.

- (iii) The *dual characteristic polynomial* $\varphi_W^*(\lambda)$ of W is a polynomial

$$(2.7) \quad \varphi_W^*(\lambda) := \prod_{i \in M(W)} (\lambda^i - 1)^{-e_W(h/i)}.$$

§3. Duality of regular systems of weights

In this subsection, we recall the notion of the duality of regular systems of weights discussed in [8] and [11].

Definition 3.1. Let W be a regular system of weights and let G be a finite abelian subgroup of $GL(3, k)$, whose elements are of the form $\text{diag}(e[\omega_1\alpha_1], e[\omega_2\alpha_2], e[\omega_3\alpha_3])$ where $\omega_i := a_i/h$ and $\alpha_i \in \mathbb{Z}$.

For a pair (W, G) , set

$$(3.1) \quad \chi(W, G)(y, \bar{y}) := \frac{(-1)^3}{|G|} \sum_{\alpha \in G} \chi_\alpha(W, G)(y, \bar{y}),$$

(3.2)

$$\chi_\alpha(W, G)(y, \bar{y}) := \sum_{\beta \in G} \prod_{\omega_i \alpha_i \notin \mathbb{Z}} (y\bar{y})^{\frac{1-2\omega_i}{2}} \left(\frac{y}{\bar{y}}\right)^{-\omega_i \alpha_i + [\omega_i \alpha_i] + \frac{1}{2}} \\ \times \prod_{\omega_i \alpha_i \in \mathbb{Z}} e^{\left[\omega_i \beta_i + \frac{1}{2}\right]} \frac{1 - e^{[(1 - \omega_i \beta_i)]} (y\bar{y})^{1-\omega_i}}{1 - e^{[\omega_i \beta_i]} (y\bar{y})^{\omega_i}},$$

where $[\omega_i \alpha_i]$ denotes the greatest integer smaller than $\omega_i \alpha_i$. We call $\chi(W, G)(y, \bar{y})$ the orbifoldized Poincaré polynomial of (W, G) .

Remark 3.2. Note that if we put $T^h = y\bar{y}$, we have

$$\chi_W(T) = \chi(W, \{1\})(y, \bar{y}).$$

By this orbifoldized Poincaré polynomial $\chi(W, G)$, one can define the following notion of the duality among pairs (W, G) :

Definition 3.3 (Topological Mirror Symmetry). Let (W, G) and (W^*, G^*) be pairs as in Definition 3.1. A pair (W^*, G^*) is called *topological mirror dual* to a pair (W, G) , if

$$(3.3) \quad \chi(W^*, G^*)(y, \bar{y}) = (-1)^3 \bar{y}^{\hat{c}_W} \chi(W, G)(y, \bar{y}^{-1}),$$

where $\hat{c}_W := 1 - 2\frac{\epsilon_W}{h}$.

Let W be a regular system of weights. We shall call the group generated by $\text{diag}(e[\omega_1], e[\omega_2], e[\omega_3])$ the *principal discrete group* for W and denote it by G_W^0 . As a special case of topological mirror symmetry, we define the following duality of regular systems of weights:

Definition 3.4 ([11]). Let W and W^* be regular systems of weights. W^* is said to be *dual* to W if

$$(3.4) \quad \chi(W^*, \{1\})(y, \bar{y}) = (-1)^3 \bar{y}^{\hat{c}_W} \chi(W, G_W^0)(y, \bar{y}^{-1}).$$

Proposition 3.5 ([11]). *Let W and W^* be regular systems of weights. W^* is dual to W if and only if W^* is $*$ -dual to W in the sense of K. Saito [8], more precisely, if and only if*

$$(3.5) \quad \varphi_{W^*}(\lambda) = \prod_{i \in M(W^*)} (\lambda^i - 1)^{e_{W^*}(i)} = \prod_{i \in M(W)} (\lambda^i - 1)^{-e_W(h/i)} = \varphi_W^*(\lambda),$$

and $W^* = (lm - l + 1, mk - m + 1, kl - k + 1; klm + 1)$ when $W = (lm - m + 1, mk - k + 1, kl - l + 1; klm + 1)$.

Remark 3.6. If W^* is dual to W , then W^* is uniquely determined by W and $(W^*)^* = W$. Therefore, one sees that these two equivalent notions of duality define equivalence relations.

Definition 3.7. A regular system of weights W is called of *dual type* if W has the dual regular system of weights W^* .

Proposition 3.8 ([8]). *Let W be a regular system of weights W of dual type. Then, the signature $(g_W; A_W)$ of W is of the form $(0; \alpha_1, \alpha_2, \alpha_3)$, i.e., $g_W = 0$ and $r = 3$.*

Remark 3.9. Regular systems of weights of dual type are classified into five types with respect to the classifying poset $M(W)$. See Appendix for details of their data, i.e., weights, dual weights, signatures, and so on.

§4. Weighted projective lines associated to A_W

Definition 4.1 ([1]). Let $W = (a_1, a_2, a_3; h)$ be a regular system of weights of genus 0 and let $A_W = (\alpha_1, \dots, \alpha_r)$ be its signature.

- (i) Consider an abelian group generated by r -letters \vec{X}_i ($i = 1, \dots, r$),

$$(4.1) \quad L(A_W) := \bigoplus_{i=1}^r \mathbb{Z}\vec{X}_i / \left(\alpha_i \vec{X}_i - \alpha_j \vec{X}_j; 1 \leq i < j \leq r \right).$$

It is an ordered group with $L(A_W)_+ := \sum_{i=1}^r \mathbb{Z}_{>0} \vec{X}_i$ as its positive elements.

- (ii) An element $\omega_{A_W} := (r - 2) \cdot \vec{c} - \sum_{i=1}^r \vec{X}_i$ is called the *dualizing element* of $L(A_W)$, where $\vec{c} := \alpha_1 \vec{X}_1 = \dots = \alpha_r \vec{X}_r$.
- (iii) Let $\alpha := \text{lcm}(\alpha_1, \dots, \alpha_r)$. The *degree map* is a group homomorphism $\text{deg} : L(A_W) \rightarrow \mathbb{Z}$ defined on generators by $\text{deg}(\vec{X}_i) := \alpha / \alpha_i$. The degree map deg is an epimorphism and $\text{deg}(\vec{X}) = 0$ if $\alpha \cdot \vec{X} = 0$. Note that $\text{deg}(\vec{c}) = \alpha$.
- (iv) Define an $L(A_W)$ -graded k -algebra by

$$(4.2) \quad R_{A_W, \lambda} := k[X_1, \dots, X_r] / I_\lambda,$$

where I_λ is an ideal generated by $r - 2$ homogeneous polynomials

$$(4.3) \quad X_1^{\alpha_1} + X_2^{\alpha_2} + X_3^{\alpha_3}, X_1^{\alpha_1} + \lambda_i X_2^{\alpha_2} + X_i^{\alpha_i}, \quad \lambda_i \in k \setminus \{0, 1\}, \quad i = 4, \dots, r.$$

$R_{A_W, \lambda}$ is of non-negatively graded.

If $r = 3$, then we shall write $R_{A_W, \lambda}$ as R_{A_W} for simplicity.

Denote by $\text{mod}_{L(A_W)\text{-}R_{A_W, \lambda}}$ the abelian category of finitely generated $L(A_W)$ -graded $R_{A_W, \lambda}$ -modules and denote by $\text{tor}_{L(A_W)\text{-}R_{A_W, \lambda}}$ the

full subcategory of $\text{mod}_{L(A_W)}\text{-}R_{A_W,\lambda}$ whose objects are finite dimensional $L(A_W)$ -graded $R_{A_W,\lambda}$ -modules.

Definition 4.2. Define an algebraic stack $\mathcal{C}_{A_W,\lambda}$ by

$$(4.4) \quad \mathcal{C}_{A_W,\lambda} := [\text{Spec}(R_{A_W,\lambda}) \setminus \{0\} / \text{Spec}(k \cdot L(A_W))].$$

The abelian category $\text{coh}(\mathcal{C}_{A_W,\lambda})$ of coherent sheaves on the algebraic stack $\mathcal{C}_{A_W,\lambda}$ is equivalent to $\text{mod}_{L(A_W)}\text{-}R_{A_W,\lambda} / \text{tor}_{L(A_W)}\text{-}R_{A_W,\lambda}$.

If $r = 3$, we shall denote $\mathcal{C}_{A_W,\lambda}$ by \mathcal{C}_{A_W} since λ does not appear. Note that if W is a regular system of weights of dual type then $r = 3$.

$L(A_W)$ has a special subgroup isomorphic to \mathbb{Z} as we shall see below, which will be identified with the Lie algebra of k^* acting on the hypersurface singularity defined by W .

Definition 4.3. Let $W = (a_1, a_2, a_3; h)$ be a regular system of weights of dual type. Choose a triple $\vec{l}_W := (\vec{l}_1, \vec{l}_2, \vec{l}_3)$ of elements in $L(A_W)_+$ as follows (see also Appendix):

Type of W	\vec{l}_1	\vec{l}_2	\vec{l}_3
I	\vec{X}_1	\vec{X}_2	\vec{X}_3
II	$\vec{X}_1 + \vec{X}_3$	$p_1 \vec{X}_3$	\vec{X}_2
III	$\vec{X}_1 + \vec{X}_2 + \vec{X}_3$	$p_1 \vec{X}_3$	$p_1 \vec{X}_2$
IV	$\frac{p_3}{p_2} \vec{X}_2 + \vec{X}_3$	$p_1 \vec{X}_3$	$p_1 \vec{X}_2$
V	$\vec{X}_2 + l \vec{X}_3$	$\vec{X}_1 + k \vec{X}_2$	$\vec{X}_3 + m \vec{X}_1$

We shall call \vec{l}_W the *principal generators* of W .

Lemma 4.4. Let $W = (a_1, a_2, a_3; h)$ be a regular system of weights of dual type and let $\vec{l}_W = (\vec{l}_1, \vec{l}_2, \vec{l}_3)$ be its principal generators. There exists a unique element $\vec{\omega}_W \in L(A_W)$ such that

- (i) $-\epsilon_W \cdot \vec{\omega}_W = \vec{\omega}_{A_W}$
- (ii) $a_i \cdot \vec{\omega}_W = \vec{l}_i$ for $i = 1, 2, 3$.

Proof. By direct calculations, one sees that

$$(4.5) \quad a_i \cdot \vec{\omega}_{A_W} = -\epsilon_W \cdot \vec{l}_i \quad \text{and} \quad a_i \cdot \vec{l}_j = a_j \cdot \vec{l}_i, \quad \text{for all } i, j = 1, 2, 3.$$

Note also that there exists $k_1, k_2, k_3 \in \mathbb{Z}$ such that $\sum_{i=1}^3 k_i a_i = 1$ since $\text{gcd}(a_1, a_2, a_3) = 1$.

(Existence) Set $\vec{\omega}_W := \sum_{i=1}^3 k_i \cdot \vec{l}_i$. Then

$$-\epsilon_W \cdot \vec{\omega}_W = \sum_{i=1}^3 k_i (-\epsilon_W \cdot \vec{l}_i) = \sum_{i=1}^3 k_i (a_i \cdot \vec{\omega}_{A_W}) = \left(\sum_{i=1}^3 k_i a_i \right) \cdot \vec{\omega}_{A_W} = \vec{\omega}_{A_W}.$$

Moreover,

$$\begin{aligned} a_1 \cdot \vec{\omega}_W &= a_1 k_1 \cdot \vec{l}_1 + a_1 k_2 \cdot \vec{l}_2 + a_1 k_3 \cdot \vec{l}_3 \\ &= (1 - k_2 a_2 - k_3 a_3) \cdot \vec{l}_1 + a_1 k_2 \cdot \vec{l}_2 + a_1 k_3 \cdot \vec{l}_3 \\ &= \vec{l}_1 + k_2 (a_1 \cdot \vec{l}_2 - a_2 \cdot \vec{l}_1) + k_3 (a_1 \cdot \vec{l}_3 - a_3 \cdot \vec{l}_1) \\ &= \vec{l}_1. \end{aligned}$$

In a similar way, it is shown that $a_2 \cdot \vec{\omega}_W = \vec{l}_2$ and $a_3 \cdot \vec{\omega}_W = \vec{l}_3$.

(Uniqueness) Suppose that an element $\vec{\omega}'_W \in L(A_W)$ satisfies the two conditions (i) and (ii) above. Then,

$$\vec{\omega}'_W = \left(\sum_{i=1}^3 k_i a_i \right) \cdot \vec{\omega}'_W = \sum_{i=1}^3 k_i (a_i \cdot \vec{\omega}'_W) = \sum_{i=1}^3 k_i \cdot \vec{l}_i = \vec{\omega}_W.$$

Q.E.D.

Remark 4.5. Let $W = (a_1, a_2, a_3; h)$ be a regular system of weights of genus 0 and let $A_W = (\alpha_1, \dots, \alpha_r)$ be its signature. Then one has

$$\begin{aligned} \deg(\vec{\omega}_W) &= -\frac{\deg(\omega_{A_W})}{\epsilon_W} = -\frac{(r-2) \cdot \deg(\vec{c}) - \sum_{i=1}^r \deg(\vec{X}_i)}{\epsilon_W} \\ (4.6) \quad &= \frac{h}{a_1 a_2 a_3} \cdot \alpha, \end{aligned}$$

where $\alpha := \text{lcm}(\alpha_1, \dots, \alpha_r)$.

This element $\vec{\omega}_W \in L(A_W)$ leads to the following definition:

Definition 4.6. Let W be a regular system of weights of dual type and let $\vec{\omega}_W \in L(A_W)$ be the element given by Lemma 4.4.

- (i) The subgroup $L_W := \mathbb{Z}\vec{\omega}_W$ of $L(A_W)$ is called the *principal lattice* of W , which we shall often identify with \mathbb{Z} .
- (ii) Define an L_W -graded k -algebra R_W by

$$R_W := \bigoplus_{d \in \mathbb{Z}_{\geq 0}} R_{W,d}, \quad R_{W,d} := R_{A_W, d \cdot \vec{\omega}_W}.$$

Denote by $\text{gr-}R_W$ the abelian category of finitely generated L_W -graded R_W -modules and by $\text{tor-}R_W$ the full abelian subcategory of $\text{gr-}R_W$ whose objects are finite dimensional L_W -graded R_W -modules.

Definition 4.7. Define an algebraic stack \mathcal{C}_W by

$$(4.7) \quad \mathcal{C}_W := [\text{Spec}(R_W) \setminus \{0\} / \text{Spec}(kL_W)].$$

The abelian category $\text{coh}(\mathcal{C}_W)$ of coherent sheaves on the algebraic stack \mathcal{C}_W is equivalent to $\text{gr-}R_W / \text{tor-}R_W$.

Our main purpose is to compare \mathcal{C}_{A_W} with \mathcal{C}_W as algebraic stacks, or equivalently, $\text{coh}(\mathcal{C}_{A_W}) = \text{mod}_{L(A_W)\text{-}R_{A_W}} / \text{tor}_{L(A_W)\text{-}R_{A_W}}$ with $\text{coh}(\mathcal{C}_W) = \text{gr-}R_W / \text{tor-}R_W$ as abelian categories, and relate them to the hyper-surface singularity with k^* -action defined by W .

§5. Main theorem

Theorem 5.1. *Let W be a regular system of weights of dual type.*

- (i) *There exist homogeneous elements $x, y, z \in R_W$ of degree a_1, a_2, a_3 with respect to L_W such that $R_W \simeq k[x, y, z] / (f_W)$ for some weighted homogeneous polynomial $f_W \in k[x, y, z]$ of degree h which defines at most an isolated singularity only at the origin. In other words, R_W is the graded ring of functions on the isolated hypersurface singularity defined by W .*
- (ii) *There exists an equivalence of abelian categories:*

$$(5.1) \quad \text{mod}_{L(A_W)\text{-}R_{A_W}} / \text{tor}_{L(A_W)\text{-}R_{A_W}} \simeq \text{gr-}R_W / \text{tor-}R_W .$$

Proof. Proofs for both of two statements are generalizations of those given by Geigle–Lenzing [2] for the case $\epsilon_W = 1$, which can be applied to the general cases here.

- (i) The statement is shown by direct calculations using the classification of regular systems of weights of dual type. Generators x, y, z and their unique relation $f_W(x, y, z)$ are listed in Appendix below.
- (ii) One sees that the functor

$$(5.2) \quad F : \text{mod}_{L(A_W)\text{-}R_{A_W}} \rightarrow \text{gr-}R_W, \quad M \mapsto M|_{L_W},$$

defined by the natural inclusion $R_W \subset R_{A_W}$ is exact and essentially surjective since by Kan extension there exists the right adjoint G of F such that $F \circ G(M') \simeq M'$ for any $M' \in \text{gr-}R_W$. Therefore, what has to be shown is only that any

module $N \in \text{tor-}R_W$ is an essential image of a module $N' \in \text{tor-}R_{A_W}$, however, this also follows from Proposition 1.3 in [1].

Q.E.D.

§6. Application

Corollary 6.1. *Let W be a regular system of weights of dual type with $\epsilon_W < 0$. The triangulated category $D_{Sg}^{gr}(R_W) := D^b(\text{gr-}R_W)/D^b(\text{grproj-}R_W)$ has a full exceptional collection $(E_1, \dots, E_{\mu_{W^*}})$.*

Proof. Orlov shows (Theorem 2.5 in [6]) that there exists a semi-orthogonal decomposition

$$(6.1) \quad D_{Sg}^{gr}(R_W) \simeq \langle R_W/\mathfrak{m}_W(a), \dots, R_W/\mathfrak{m}_W(a + \epsilon_W + 1), D^b \text{coh}(\mathcal{C}_W) \rangle$$

for any $a \in \mathbb{Z}$ where we denote by (\cdot) the auto-equivalence induced by the grading shift functor on $\text{gr-}R_W$ and by \mathfrak{m}_W the unique graded maximal ideal in R_W . Combining our main theorem with the Proposition 4.1 in [1], the statement follows. Q.E.D.

Remark 6.2. There is an isomorphism of functors $(h) \simeq [2]$ on the triangulated category $D_{Sg}^{gr}(R_W)$ where $[1]$ is the translation functor.

Corollary 6.3. *Let W be a regular system of weights of dual type. The triangulated category $D_{Sg}^{gr}(R_W)$ has a full exceptional collection $(E_1, \dots, E_{\mu_{W^*}})$.*

Proof. Since W is of dual type, it is of ADE type or $\epsilon_W < 0$. Since a regular system of weights of type ADE has a full strongly exceptional collection [3], it is the direct consequence of the previous corollary. Q.E.D.

§7. Conjectures

In this section, we shall assume that $k = \mathbb{C}$.

Definition 7.1. Let W be a regular system of weights of dual type.

- (i) Set $B_W := A_{W^*}$ and call it the dual signature of W .
- (ii) Let $B_W = (\beta_1, \beta_2, \beta_3)$ be the dual signature of W . Denote by f_{B_W} the defining polynomial of the cusp singularity, namely, $f_{B_W} := x_1^{\beta_1} + x_2^{\beta_2} + x_3^{\beta_3} + x_1x_2x_3$.

Remark 7.2. For singularities associated to regular system of weights with $\epsilon_W = -1$, i.e., Arnold’s 14 exceptional singularities, B_W is called the Gabrielov number.

Kontsevich observed that the origin of mirror symmetry is a symmetry between complex geometry and symplectic geometry. In particular, he conjectures that the algebraically constructed triangulated category in complex side should be triangulated equivalent to the geometrically constructed one in symplectic side, and that mirror phenomena should be explained from this triangulated equivalence.

Let f be a polynomial which defines an isolated singularity only at the origin $0 \in \mathbb{C}^n$. Consider a polynomial f as a holomorphic map $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ and choose a *Morsification* $\{f_t\}_{0 \leq t < 1}$ of f , i.e., a smooth family of polynomials with $f_0 = f$ and such that critical points of f_t ($t \neq 0$) are isolated and non-degenerate. For very small $0 < t \ll \delta \ll \epsilon \ll 1$, put

$$(7.1) \quad X := \{x \in \mathbb{C}^n \mid \|x\| \leq \epsilon, |f_t| \leq \delta\},$$

$$(7.2) \quad D := \{y \in \mathbb{C} \mid |y| \leq \delta\}.$$

Then, after suitable modifications near the boundary, the holomorphic map $f_t : X \rightarrow D$ defines an *exact Lefschetz fibration* and the relative holomorphic $n - 1$ -form $dx_1 \wedge \cdots \wedge dx_n / df_t$ defines a *relative Maslov map* which enables one to define *graded Lagrangian sub-manifolds*.

Theorem 7.3 (Seidel [9] [10]). *There exists an A_∞ -category $\text{Fuk}(f)$ called the Fukaya category of the exact Lefschetz fibration such that*

$$(7.3) \quad D^b\text{Fuk}(f) \simeq D^b\text{Fuk}^{-1}(\{\gamma_\bullet\}),$$

as a triangulated category, where $\text{Fuk}^{-1}(\{\gamma_\bullet\})$ is the directed Fukaya category of a distinguished basis of vanishing graded Lagrangian sub-manifolds $\{\gamma_\bullet\} := \{\gamma_1, \dots, \gamma_\mu\}$ in the Milnor fiber $X_y = f_t^{-1}(y)$. In particular, $D^b\text{Fuk}(f)$ is independent of all choices and is an invariant of f .

We expect that the topological mirror symmetry for isolated hypersurface singularities can also be “categorified”. The homological mirror symmetry principle leads the following conjecture:

Conjecture 7.4 (Homological mirror symmetry for weighted projective lines). *Let W be a regular system of weights of dual type. There should exist a triangulated equivalence*

$$(7.4) \quad D^b\text{coh}(\mathcal{C}_W) \simeq D^b\text{Fuk}(f_{B_W}).$$

Remark 7.5. The above conjecture can be verified at the level of the Grothendieck group. More precisely, one has the following isomorphism as lattices

$$(7.5) \quad (K_0(D^b \text{coh}(\mathcal{C}_W)), \chi + {}^t\chi) \simeq (H_2(f_{B_W}^{-1}(y), \mathbb{Z}), -I).$$

We expect that this conjecture follows from the following one since semi-orthogonal decompositions of triangulated categories in algebraic geometry side should correspond to operations “blowing down” which are mirror dual to unfoldings of singularities in symplectic geometry side.

Conjecture 7.6 (HMS for regular systems of weights of dual type). *Let W be a regular system of weights of dual type with $\epsilon_W < 0$.*

(i) *There should exist a triangulated equivalence*

$$(7.6) \quad D_{Sg}^{gr}(R_W) \simeq D^b \text{Fuk}(f_{W^*}).$$

(ii) *$D^b \text{Fuk}(f_{W^*})$ has the following semi-orthogonal decomposition*

$$(7.7) \quad D^b \text{Fuk}(f_{W^*}) \simeq \langle \mathcal{L}(a), \dots, \mathcal{L}(a + \epsilon_W + 1), D^b \text{Fuk}(f_{B_W}) \rangle,$$

for any $a \in \mathbb{Z}$, where \mathcal{L} is an object of $D^b \text{Fuk}(f_{W^})$ and (1) is the auto-equivalence induced by the shift of the gradings of Lagrangian sub-manifolds by $2/h$ such that $(h) = [2]$ where $[1]$ is the translation functor on $D^b \text{Fuk}(f_{W^*})$.*

Remark 7.7. We can define an abelian group L_{f_W} for R_W called the *maximal grading* of R_W in a similar way to Definition 4.1 by introducing the degree vectors for x_i and letting all monomials appearing in f_W have the same degree vectors. As a result, R_W becomes L_{f_W} -graded and hence we can consider the triangulated category $D_{Sg}^{L_{f_W}}(R_W)$. It is expected that if f_W is of the form listed in Appendix (without any conditions for their exponents) then $(W, \text{Spec}(kL_W))$ is topological mirror dual to $(W^*, \{1\})$ with the same W^* in the list. In this case, we can formulate the homological mirror symmetry conjecture as a triangulated equivalence $D_{Sg}^{L_{f_W}}(R_W) \simeq D^b \text{Fuk}(f_{W^*})$. The above conjecture is the special case for regular systems of weights with $L_W = \mathbb{Z}$ (see Definition 3.4).

This homological mirror symmetry conjecture is one of motivations of our study of the triangulated category $D_{Sg}^{gr}(R_W)$, from which we can expect that $D_{Sg}^{gr}(R_W)$ has a full strongly exceptional collection (see [12]). Indeed, we prove this for several important classes of regular systems of weights in [3] (for W corresponding to ADE singularities), [4] (for W

with $\epsilon_W = -1$ including Arnold's 14 exceptional singularities) and [13] (for W of Type I, II and III), where we also checked the homological mirror symmetry conjecture at the level of Grothendieck group.

§8. Appendix

The followings are data of weights, dual weights, signatures, generators and relations of R_W and of R_{W^*} , for regular systems of weights of dual type.

8.1. Type I

$$(8.1) \quad W = W^* = (p_2 p_3, p_3 p_1, p_1 p_2; p_1 p_2 p_3), \quad (p_i, p_j) = 1, i = 1, 2, 3.$$

$$(8.2) \quad A_W = A_{W^*}^* = (p_1, p_2, p_3).$$

$$(8.3) \quad \vec{l}_W = (\vec{X}_1, \vec{X}_2, \vec{X}_3).$$

$$(8.4) \quad x = X_1, \quad y = X_2, \quad z = X_3.$$

$$(8.5) \quad f_W(x, y, z) = x^{p_1} + y^{p_2} + z^{p_3}.$$

8.2. Type II

$$(8.6) \quad W = (p_3, \frac{p_1 p_3}{p_2}, (p_2 - 1)p_1; p_1 p_3),$$

$$(8.7) \quad W^* = (p_3, p_1 p_2, (\frac{p_3}{p_2} - 1)p_1; p_1 p_3),$$

where $p_2 \neq p_3, p_2 | p_3, (p_1, p_3) = 1, (p_2 - 1, p_3) = 1$ and $(p_3/p_2 - 1, p_3) = 1$.

$$(8.8) \quad A_W = (p_1, \frac{p_3}{p_2}, (p_2 - 1)p_1), \quad A_{W^*} = (p_1, p_2, (\frac{p_3}{p_2} - 1)p_1).$$

$$(8.9) \quad \vec{l}_W = (\vec{X}_1 + \vec{X}_3, p_1 \vec{X}_3, \vec{X}_2).$$

$$(8.10) \quad x = X_1 X_3, \quad y = X_3^{p_1}, \quad z = X_2.$$

$$(8.11) \quad f_W(x, y, z) = x^{p_1} + y^{p_2} + yz^{\frac{p_3}{p_2}}.$$

8.3. Type III

$$(8.12) \quad W = W^* = (p_2, p_1 q_2, p_1 q_3; p_1 p_2),$$

where $(p_1, p_2) = 1$, $p_2 + 1 = (q_2 + 1)(q_3 + 1)$ and $(q_2, q_3) = 1$.

$$(8.13) \quad A_W = A_{W^*} = (p_1, p_1 q_2, p_1 q_3).$$

$$(8.14) \quad \vec{l}_W = (\vec{X}_1 + \vec{X}_2 + \vec{X}_3, p_1 \vec{X}_3, p_1 \vec{X}_2).$$

$$(8.15) \quad x = X_1 X_2 X_3, \quad y = X_3^{p_1}, \quad z = X_2^{p_1}.$$

$$(8.16) \quad f_W(x, y, z) = x^{p_1} + y^{q_3+1} z + y z^{q_2+1}.$$

8.4. Type IV

$$(8.17) \quad W = \left(\frac{p_3}{p_1}, (p_1 - 1) \frac{p_3}{p_2}, p_2 - p_1 + 1; p_3 \right),$$

$$(8.18) \quad W^* = \left(p_2, \left(\frac{p_3}{p_2} - 1 \right) p_1, \frac{p_3}{p_1} - \frac{p_3}{p_2} + 1; p_3 \right),$$

where $p_1 \neq p_2 \neq p_3$, $p_1 | p_2$, $p_2 | p_3$, $(p_1 - 1, p_2) = 1$, $(p_2 - p_1 + 1, p_3) = 1$, $(p_3/p_2 - 1, p_3/p_1) = 1$ and $(p_3/p_1 - p_3/p_2 + 1, p_3) = 1$.

$$(8.19) \quad A_W = \left(\frac{p_3}{p_2}, (p_1 - 1) \frac{p_3}{p_2}, p_2 - p_1 + 1 \right), \quad A_{W^*} = \left(p_1, \left(\frac{p_3}{p_2} - 1 \right) p_1, \frac{p_3}{p_1} - \frac{p_3}{p_2} + 1 \right).$$

$$(8.20) \quad \vec{l}_W = \left(\frac{p_3}{p_2} \vec{X}_2 + \vec{X}_3, p_1 \vec{X}_3, \vec{X}_1 + \vec{X}_2 \right).$$

$$(8.21) \quad x = X_2^{\frac{p_3}{p_2}} X_3, \quad y = X_3^{p_1}, \quad z = X_1 X_2.$$

$$(8.22) \quad f_W(x, y, z) = x^{p_1} + x y^{\frac{p_2}{p_1}} + y z^{\frac{p_3}{p_2}}.$$

8.5. Type V

$$(8.23) \quad W = (lm - m + 1, mk - k + 1, kl - l + 1; klm + 1),$$

$$(8.24) \quad W^* = (lm - l + 1, mk - m + 1, kl - k + 1; klm + 1).$$

where $(lm - m + 1, klm + 1) = 1$, $(mk - k + 1, klm + 1) = 1$, and $(kl - l + 1, klm + 1) = 1$.

$$(8.25) \quad \begin{aligned} A_W &= (lm - m + 1, mk - k + 1, kl - l + 1), \\ A_{W^*} &= (lm - l + 1, mk - m + 1, kl - k + 1). \end{aligned}$$

$$(8.26) \quad \vec{l}_W = (\vec{X}_2 + l\vec{X}_3, \vec{X}_1 + k\vec{X}_2, \vec{X}_3 + m\vec{X}_1).$$

$$(8.27) \quad x = X_2X_3^l, \quad y = X_1X_2^k, \quad z = X_3X_1^m.$$

$$(8.28) \quad f_W(x, y, z) = zx^k + xy^m + yz^l.$$

References

- [1] W. Geigle and H. Lenzing, A class of weighted projective curves arising in representation theory of finite-dimensional algebras, In: *Singularities, Representation of Algebras, and Vector Bundles*, Lambrecht, 1985, *Lecture Notes in Math.*, **1273**, Springer-Verlag, Berlin, 1987, pp. 9–34.
- [2] W. Geigle and H. Lenzing, Perpendicular categories with applications to representations and sheaves, *J. Algebra*, **144** (1991), 273–343.
- [3] H. Kajiura, K. Saito and A. Takahashi, Matrix factorizations and representations of quivers II: type ADE case, *Adv. in Math.*, **211** (2007), 327–362; arXiv:math/0511155[math.AG].
- [4] H. Kajiura, K. Saito and A. Takahashi, Triangulated categories of matrix factorizations for regular systems of weights with $\varepsilon = -1$, arXiv: 0708.0210.
- [5] H. Kajiura, K. Saito and A. Takahashi, Weighted projective lines associated to regular systems of weights, work in progress.
- [6] D. Orlov, Derived categories of coherent sheaves and triangulated categories of singularities, arXiv:math/0503632[math.AG].
- [7] K. Saito, Regular systems of weights and associated singularities, In: *Complex Analytic Singularities*, *Adv. Stud. Pure Math.*, **8**, North-Holland, Amsterdam, 1986, pp. 479–526.

- [8] K. Saito, Duality for regular systems of weights, *Asian. J. Math.*, **2** (1998), 983–1048.
- [9] P. Seidel, More about vanishing cycles and mutation, symplectic geometry and mirror symmetry, In: *Proceedings of the 4th KIAS Annual International Conference*, World Scientific, 2001, pp. 429–465.
- [10] P. Seidel, Fukaya categories and Picard–Lefschetz theory, to appear in the *ETH Lecture Notes series of the European Math. Soc.*
- [11] A. Takahashi, K. Saito’s duality for regular weight systems and duality for orbifoldized Poincaré polynomials, *Commun. Math. Phys.*, **205** (1999), 571–586.
- [12] A. Takahashi, Matrix factorizations and representations of quivers I, arXiv: math/0506347[math.AG].
- [13] A. Takahashi, Homological mirror symmetry for isolated hypersurface singularities in dimension one, in preparation.

Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560-0043
Japan

E-mail address: takahashi@math.sci.osaka-u.ac.jp