Advanced Studies in Pure Mathematics 57, 2010
Probabilistic Approach to Geometry
pp. 119–128

Non-Markov property of certain eigenvalue processes analogous to Dyson's model

Ryoki Fukushima, Atsushi Tanida and Kouji Yano

Abstract.

It is proven that the eigenvalue process of Dyson's random matrix process of size two becomes non-Markov if the common coefficient $1/\sqrt{2}$ in the non-diagonal entries is replaced by a different positive number.

§1. Introduction

Dyson [3] has introduced the matrix-valued stochastic process

$$\Xi(t) = \begin{pmatrix} B_{1,1}(t) & \frac{1}{\sqrt{2}}B_{1,2}(t) & \cdots & \frac{1}{\sqrt{2}}B_{1,N}(t) \\ \frac{1}{\sqrt{2}}\overline{B_{1,2}(t)} & B_{2,2}(t) & \cdots & \frac{1}{\sqrt{2}}B_{2,N}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2}}\overline{B_{1,N}(t)} & \frac{1}{\sqrt{2}}\overline{B_{2,N}(t)} & \cdots & B_{N,N}(t) \end{pmatrix}$$

to model the dynamics of particles with the Coulomb type interactions, where $B_{i,i}$'s are real Brownian motions and $B_{i,j}$'s for $i < j$ are complex Brownian motions all of which are mutually independent. He proved that the eigenvalue processes $\lambda_1, \ldots, \lambda_N$ satisfy the (system of) stochastic differential equations

$$d\lambda_i(t) = d\beta_i(t) + \frac{\beta}{2}\sum_{j \neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)} dt$$

with $\beta = 2$. It has been proven later that if the complex Brownian motions are replaced by real or quaternion Brownian motions, the

Received January 14, 2009.
Revised March 5, 2009.
2000 *Mathematics Subject Classification.* 15A52, 60-06, 60J65, 60J99.
Key words and phrases. Non-Markov property, random matrix, eigenvalue process, Dyson's model, beta-ensembles.
The research of the third author was supported by KAKENHI (20740060).

eigenvalue processes satisfy similar stochastic differential equations with $\beta = 1$ or 4, respectively. (See [1, 4] for discussions based on the stochastic analysis.) These processes are now called Dyson's Brownian motion models for GOE, GUE, and GSE when $\beta = 1, 2$, and 4, respectively. In any case, it is remarkable that the process $\Lambda = (\lambda_1, \dots, \lambda_N)$ is Markov.

We may ask the following question: "Does the process Λ remain Markov if we replace the common coefficient $1/\sqrt{2}$ by a different positive number?" In this paper, we give the *negative* answer to this question when the matrix size $N = 2$.

Let $c \geq 0$ and $\delta > 0$. Consider the 2×2-matrix-valued process

$$\Xi^{c,\delta}(t) = \begin{pmatrix} B_1(t) & \sqrt{c/2}\,\xi^\delta(t) \\ \sqrt{c/2}\,\xi^\delta(t) & B_2(t) \end{pmatrix}, \tag{1.1}$$

where B_1 and B_2 are two independent standard Brownian motions and ξ^δ is a Bessel process of dimension δ starting from 0 which is independent of B_1 and B_2. We see in Lemma 2.2 that $\Xi^{c,\delta}$ with $\delta = 1, 2$, or 4 is unitarily equivalent in law to

$$\widetilde{\Xi}^{c,\delta}(t) = \begin{pmatrix} B_1(t) & \sqrt{c/2}\,B_3(t) \\ \sqrt{c/2}\,\overline{B_3(t)} & B_2(t) \end{pmatrix} \tag{1.2}$$

with B_3 a real, complex, or quaternion Brownian motion independent of B_1 and B_2, respectively. Let $\lambda_1(t)$ and $\lambda_2(t)$ for $t \geq 0$ denote the eigenvalues of the Hermitian matrix $\Xi^{c,\delta}(t)$ such that $\lambda_1(t) \geq \lambda_2(t)$. Define the two-dimensional process $\Lambda^{c,\delta} = (\lambda_1, \lambda_2)$.

When $c = 0$, $\lambda_1(t)$ and $\lambda_2(t)$ are nothing but the order statistics of $B_1(t)$ and $B_2(t)$, that is, $\lambda_1(t) = \max\{B_1(t), B_2(t)\}$ and $\lambda_2(t) = \min\{B_1(t), B_2(t)\}$. Hence it is obvious that the process $\Lambda^{0,\delta}$ is Markov.

When $c = 1$, the process (1.1) is a time-dependent version of Dumitriu–Edelman's matrix model for beta-ensembles (cf. [2]) and we see in Lemma 2.1 that the processes $\lambda_1(t)$ and $\lambda_2(t)$ satisfy Dyson's stochastic differential equations with index $\beta = \delta$ given by

$$d\lambda_1(t) = d\beta_1(t) + \frac{\delta}{2(\lambda_1(t) - \lambda_2(t))}dt, \tag{1.3}$$

$$d\lambda_2(t) = d\beta_2(t) + \frac{\delta}{2(\lambda_2(t) - \lambda_1(t))}dt \tag{1.4}$$

for two independent Brownian motions $\beta_1(t)$ and $\beta_2(t)$. In particular, the process $\Lambda^{1,\delta}(t)$ is Markov.

Theorem 1.1. *The process $\Lambda^{c,\delta}$ is Markov if and only if $c \in \{0, 1\}$.*

We prove this theorem by reducing it to the following.

Theorem 1.2. *Let $\delta_1, \delta_2 > 0$. Let X^{δ_1} and Y^{δ_2} be two independent squared Bessel processes starting from 0 of dimension δ_1 and δ_2, respectively. Then the process $Z^c(t) = cX^{\delta_1}(t) + Y^{\delta_2}(t)$ for $c \geq 0$ is Markov if and only if $c \in \{0, 1\}$.*

Theorems 1.1 and 1.2 seem similar to Matsumoto–Ogura's $cM - X$ theorem [6]. Let X be a Brownian motion and set $M(t) = \sup_{0\leq s\leq t} X(s)$. When $c \in \{0, 1, 2\}$, the process $cM - X$ is Markov; indeed, $-X$ is a Brownian motion, $M - X$ is a reflecting Brownian motion by Lévy's theorem (see, e.g., [7, Thm.VI.2.3]), and $2M - X$ is a three-dimensional Bessel process by Pitman's theorem (see, e.g., [7, Thm.VI.3.5]).

Theorem 1.3 ([6]). *The process $cM - X$ is Markov if and only if $c \in \{0, 1, 2\}$.*

§2. Non-Markov property of the eigenvalue processes

Proof of Theorem 1.1 provided Theorem 1.2 is justified. An elementary calculation shows that λ_1 and λ_2 are given by

$$\lambda_1(t) = \frac{1}{2}\left\{B_1(t) + B_2(t) + \sqrt{(B_1(t) - B_2(t))^2 + 2c\xi^\delta(t)^2}\right\},$$
$$\lambda_2(t) = \frac{1}{2}\left\{B_1(t) + B_2(t) - \sqrt{(B_1(t) - B_2(t))^2 + 2c\xi^\delta(t)^2}\right\}.$$

Set $B_3(t) = \{B_1(t) + B_2(t)\}/\sqrt{2}$, $X^1(t) = \{B_1(t) - B_2(t)\}^2/2$ and $Y^\delta(t) = \xi^\delta(t)^2$. Then B_3 is a real Brownian motion, X^1 and Y^δ are squared Bessel processes of dimension 1 and δ, respectively. Moreover, B_3, X^1, and Y^δ are mutually independent. It follows that

$$\lambda_1(t) = \frac{1}{\sqrt{2}}\left\{B_3 + \sqrt{X^1(t) + cY^\delta(t)}\right\},$$
$$\lambda_2(t) = \frac{1}{\sqrt{2}}\left\{B_3 - \sqrt{X^1(t) + cY^\delta(t)}\right\}.$$

It is obvious that the two dimensional process $\Lambda^{c,\delta} = (\lambda_1, \lambda_2)$ is Markov if and only if so is the process $(\lambda_1 + \lambda_2, \lambda_1 - \lambda_2)$. Since

$$\lambda_1 + \lambda_2 = \sqrt{2}B_3, \tag{2.1}$$
$$\lambda_1 - \lambda_2 = \sqrt{2}\sqrt{X^1 + cY^\delta} \tag{2.2}$$

and they are independent, for the process $\Lambda^{c,\delta}$ to be Markov it is necessary and sufficient that the process $X^1 + cY^\delta$ is Markov. This is equivalent to $c = 0$ or 1 by Theorem 1.2. Q.E.D.

Lemma 2.1. *For $c = 1$ and $\delta > 0$, consider the 2×2-matrix-valued process $\Xi^{1,\delta}$ defined by* (1.1)*. Then the corresponding eigenvalue processes satisfy the stochastic differential equations* (1.3)–(1.4).

Proof. Set $\widetilde{\lambda} = (\lambda_1 - \lambda_2)/\sqrt{2}$. Then, by (2.2) for $c = 1$ and by Shiga–Watanabe's theorem (see, e.g., [7, Thm.XI.1.2]), we see that the process $\widetilde{\lambda}$ is a Bessel process of dimension $1 + \delta$. Hence we have

$$d\widetilde{\lambda}(t) = dB_4(t) + \frac{\delta}{2}\frac{1}{\widetilde{\lambda}(t)}dt, \tag{2.3}$$

where B_4 is a real Brownian motion independent of B_3. If we set $\beta_1 = (B_3 + B_4)/\sqrt{2}$ and $\beta_2 = (B_3 - B_4)/\sqrt{2}$, then β_1 and β_2 are two independent real Brownian motions. Therefore, combining (2.3) with (2.1), we conclude that (1.3)–(1.4) hold. Q.E.D.

Lemma 2.2. *Let $c > 0$, $\delta = 1, 2$, or 4, and $\Xi^{c,\delta}$ and $\widetilde{\Xi}^{c,\delta}$ be the matrix-valued processes defined by* (1.1) *and* (1.2)*, respectively. Then, there exists a unitary matrix-valued process $U_\delta(t)$ such that*

$$\left(\Xi^{c,\delta}(t)\right)_{t\geq 0} \stackrel{\text{law}}{=} \left(U_\delta(t)\widetilde{\Xi}^{c,\delta}(t)U_\delta^*(t)\right)_{t\geq 0}.$$

In particular, eigenvalue processes associated with $\Xi^{c,\delta}$ and $\widetilde{\Xi}^{c,\delta}$ have the same law.

Proof. We define

$$U_\delta(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{B_3(t)}{|B_3(t)|} \end{pmatrix} 1_{\{B_3(t)\neq 0\}} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} 1_{\{B_3(t)=0\}}$$

by using B_3 in (1.2). Then we have

$$U_\delta(t)\widetilde{\Xi}^{c,\delta}(t)U_\delta^*(t) = \begin{pmatrix} B_1(t) & \sqrt{c/2}\,|B_3(t)| \\ \sqrt{c/2}\,|B_3(t)| & B_2(t) \end{pmatrix},$$

which shows the desired result since $|B_3| \stackrel{\text{law}}{=} \xi^\delta$. Q.E.D.

§3. Transition probability density of squared Bessel processes

In this section, we recall some basic asymptotic estimates on the transition probability density $p_t^\delta(x, y)$ of squared Bessel processes of dimension δ which we shall use later. We first note that it has an expression

$$p_t^\delta(x, y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{(\delta-2)/4} \exp\left(-\frac{x+y}{2t}\right) I_{(\delta-2)/2}\left(\frac{\sqrt{xy}}{t}\right) \tag{3.1}$$

for $x, y > 0$, where I_ν stands for the modified Bessel function of index ν (see, e.g., [7, Cor.XI.1.4]). Now let us recall the following two asymptotic estimates on the modified Bessel function (see, e.g., Sect. 5.16.4 of [5]):

$$I_\nu(x) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \quad \text{as } x \downarrow 0, \tag{3.2}$$

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{as } x \uparrow \infty. \tag{3.3}$$

Here, $f(x) \sim g(x)$ means $f(x)/g(x) \to 1$ in the subsequently indicated limit.

Using (3.2) in (3.1), we can derive

$$p_t^\delta(0+, y) = \frac{y^{(\delta/2)-1}}{(2t)^{\delta/2}\Gamma(\delta/2)} \exp\left(-\frac{y}{2t}\right) \tag{3.4}$$

for $t, y > 0$ and

$$\begin{aligned} \lim_{y \to 0+} y^{1-\delta/2} p_t^\delta(x, y) &= x^{1-\delta/2} p_t^\delta(0+, x) \\ &= \frac{1}{(2t)^{\delta/2}\Gamma(\delta/2)} \exp\left(-\frac{x}{2t}\right) \end{aligned} \tag{3.5}$$

for $t, x > 0$. On the other hand (3.3) together with (3.1) yields

$$p_t^\delta(x, y) \sim \frac{1}{2t\sqrt{2\pi}} \frac{y^{(\delta-3)/4}}{x^{(\delta-1)/4}} \exp\left(-\frac{x + y - 2\sqrt{xy}}{2t}\right) \tag{3.6}$$

as $\sqrt{xy} \to \infty$.

§4. Non-Markov property of weighted sums of two independent squared Bessel processes

For the proof of Theorem 1.2, we may restrict ourselves to $0 < c < 1$; otherwise consider Z^c/c instead. We prove that Z^c is non-Markov by checking that the conditional law

$$P\left(Z^c(2) \in dz_3 \mid Z^c(\varepsilon) = z_1,\ Z^c(1) = z_2\right) \quad \text{for } 0 < \varepsilon < 1 \tag{4.1}$$

does depend on (ε, z_1). This conditional law has the density

$$P\left(Z^c(2) \in dz_3 \mid Z^c(\varepsilon) = z_1,\ Z^c(1) = z_2\right) = \frac{q(z_2, z_3; \varepsilon, z_1)}{q(z_2; \varepsilon, z_1)} dz_3,$$

where $q(z_2, z_3; \varepsilon, z_1)$ and $q(z_2; \varepsilon, z_1)$ are the densities of the joint laws of $(Z^c(\varepsilon), Z^c(1), Z^c(2))$ and $(Z^c(\varepsilon), Z^c(1))$, respectively. Thus it suffices to prove that the fraction $q(z_2, z_3; \varepsilon, z_1)/q(z_2; \varepsilon, z_1)$ depends on (ε, z_1).

To this end, we shall use the integral expression

$$q(z_2, z_3; \varepsilon, z_1) = \int_0^{z_1} dx_1 \int_0^{z_2} dx_2 \int_0^{z_3} dx_3 A_{1,1} A_{1,2} A_{1,3},$$
$$q(z_2; \varepsilon, z_1) = \int_0^{z_1} dx_1 \int_0^{z_2} dx_2 A_{1,1} A_{1,2},$$

where

$$\begin{aligned} A_{1,1} =& p_\varepsilon^{\delta_1}(0+, x_1) p_\varepsilon^{\delta_2}(0+, z_1 - cx_1), \\ A_{1,2} =& p_{1-\varepsilon}^{\delta_1}(x_1, x_2) p_{1-\varepsilon}^{\delta_2}(z_1 - cx_1, z_2 - cx_2), \\ A_{1,3} =& p_1^{\delta_1}(x_2, x_3) p_1^{\delta_2}(z_2 - cx_2, z_3 - cx_3). \end{aligned}$$

We divide the proof into several steps. First of all, we prove

Lemma 4.1. *Let $f(\lambda, \cdot)$ for $\lambda > 0$ be a bounded measurable function on $(0,1)$. Suppose that $f(\lambda, x/\lambda)$ converges to a constant $f(\infty, 0)$ for any $x \in (0,1)$ as $\lambda \to \infty$. Let $\phi \in C^1((0,1))$ and suppose that $\phi(0+) = a \in \mathbb{R}$, $\phi'(0+) = b > 0$ and $\phi'(x) > 0$ for $x \in (0,1)$. Let $\nu > 0$. Then*

$$\int_0^1 e^{-\lambda\phi(x)} f(\lambda, x) x^{\nu-1} dx \sim f(\infty, 0) \frac{\Gamma(\nu)}{b^\nu} \lambda^{-\nu} e^{-a\lambda} \quad \text{as } \lambda \to \infty. \tag{4.2}$$

Proof. Changing variables to $u = \lambda x$, we find that the left hand side of (4.2) equals

$$\lambda^{-\nu} e^{-a\lambda} \int_0^\lambda e^{-\lambda\{\phi(u/\lambda) - a\}} f(\lambda, u/\lambda) du.$$

Note that $\lambda\{\phi(u/\lambda) - a\} \geq Ku$ for $u \in (0, \lambda)$ and $\lambda > 0$ where $K = \inf_{x \in (0,1)} \{\phi(x) - \phi(0+)\}/x > 0$. Hence we see that

$$\lim_{\lambda \to \infty} \int_0^\lambda e^{-\lambda\{\phi(u/\lambda) - a\}} f(\lambda, u/\lambda) du = f(\infty, 0) \int_0^\infty e^{-bu} u^{\nu-1} du$$

by the dominated convergence theorem. Q.E.D.

Second, we take the limit as $\varepsilon \to 0$.

Lemma 4.2.

$$\lim_{\varepsilon \to 0+} \frac{q(z_2, z_3; \varepsilon, z_1)}{q(z_2; \varepsilon, z_1)} = \frac{q(z_2, z_3; z_1)}{q(z_2; z_1)}$$

with

$$q(z_2, z_3; z_1) = \int_0^{z_2} dx_2 \int_0^{z_3} dx_3 A_{2,1} A_{2,2}, \quad q(z_2; z_1) = \int_0^{z_2} dx_2 A_{2,1},$$

where $A_{2,2} = A_{1,3}$ *and*

$$A_{2,1} = A_{1,2}\Big|_{\varepsilon\to 0+,\, x_1\to 0+} = p_1^{\delta_1}(0+, x_2)p_1^{\delta_2}(z_1, z_2 - cx_2).$$

Proof. We know that

$$A_{1,1} = \frac{(x_1)^{(\delta_1/2)-1}(z_1 - cx_1)^{(\delta_2/2)-1}}{(2\varepsilon)^{(\delta_1+\delta_2)/2}\Gamma(\delta_1/2)\Gamma(\delta_2/2)} \exp\left(-\frac{1}{2\varepsilon}\{z_1 + (1-c)x_1\}\right)$$

from (3.4). Now we can rewrite $q(z_2, z_3; \varepsilon, z_1)/q(z_2; \varepsilon, z_1)$ as F_1/G_1 with

$$F_1 = \int_0^{z_1} A_{1,4}(\varepsilon, x_1)x_1^{(\delta_1/2)-1}e^{-(\widetilde{c}/\varepsilon)x_1}dx_1, \tag{4.3}$$

$$G_1 = \int_0^{z_1} A_{1,5}(\varepsilon, x_1)x_1^{(\delta_1/2)-1}e^{-(\widetilde{c}/\varepsilon)x_1}dx_1, \tag{4.4}$$

where $\widetilde{c} = (1-c)/2$ and

$$\begin{aligned} A_{1,4}(\varepsilon, x_1) =&(z_1 - cx_1)^{(\delta_2/2)-1}\int_0^{z_2} dx_2 \int_0^{z_3} dx_3 A_{1,2}A_{1,3}, \\ A_{1,5}(\varepsilon, x_1) =&(z_1 - cx_1)^{(\delta_2/2)-1}\int_0^{z_2} dx_2 A_{1,2}. \end{aligned}$$

Using Lemma 4.1 in the integrals (4.3) and (4.4), we have

$$\begin{aligned} F_1 \sim&\varepsilon^{\delta_1/2}\Gamma(\delta_1/2)\widetilde{c}^{-\delta_1/2}A_{1,4}(0,0), \\ G_1 \sim&\varepsilon^{\delta_1/2}\Gamma(\delta_1/2)\widetilde{c}^{-\delta_1/2}A_{1,5}(0,0) \end{aligned}$$

as $\varepsilon \to 0+$. Here we have used the fact that $A_{1,4}(\varepsilon, x_1)$ and $A_{1,5}(\varepsilon, x_1)$ are continuous in $\varepsilon \in [0,\infty)$ and $x_1 \in [0, z_1]$. Therefore, F_1/G_1 approaches to $A_{1,4}(0,0)/A_{1,5}(0,0) = q(z_2, z_3; z_1)/q(z_2; z_1)$. Q.E.D.

Third, we study the asymptotic behavior of the numerator $q(z_2, z_3; z_1)$ as $z_3 \to 0+$.

Lemma 4.3.

$$\lim_{z_3\to 0+} z_3^{1-(\delta_1+\delta_2)/2}q(z_2, z_3; z_1) = C_1\widetilde{q}(z_2; z_1)$$

with

$$C_1 = \int_0^1 u^{(\delta_1/2)-1}(1-cu)^{(\delta_2/2)-1}du, \quad \widetilde{q}(z_2; z_1) = \int_0^{z_2} dx_2 A_{3,1}A_{3,2},$$

where $A_{3,1} = A_{2,1}$ *and*

$$A_{3,2} = (x_2)^{1-\delta_1/2}(z_2 - cx_2)^{1-\delta_2/2}p_1^{\delta_1}(0+, x_2)p_1^{\delta_2}(0+, z_2 - cx_2).$$

Proof. Recall that

$$q(z_2, z_3; z_1) = \int_0^{z_3} dx_3 A_{2,3}(z_3, x_3), \tag{4.5}$$

where

$$A_{2,3}(z_3, x_3) = \int_0^{z_2} dx_2 A_{3,1} p_1^{\delta_1}(x_2, x_3) p_1^{\delta_2}(z_2 - cx_2, z_3 - cx_3).$$

Here we note that $A_{3,1}$ does not depend on z_3 nor x_3. If we take $x_3 = z_3 u$ for $0 < u < 1$, we have

$$A_{2,3}(z_3, z_3 u) = \int_0^{z_2} dx_2 A_{3,1} p_1^{\delta_1}(x_2, z_3 u) p_1^{\delta_2}(z_2 - cx_2, z_3(1 - cu)).$$

Using (3.5), we have, as $z_3 \to 0+$,

$$z_3^{2-(\delta_1+\delta_2)/2} A_{2,3}(z_3, z_3 u) \to u^{(\delta_1/2)-1}(1 - cu)^{(\delta_2/2)-1} \int_0^{z_2} dx_2 A_{3,1} A_{3,2}.$$

Changing variables to $u = x_3/z_3$ in the integral (4.5), we obtain

$$z_3^{1-(\delta_1+\delta_2)/2} q(z_2, z_3; z_1) = z_3^{2-(\delta_1+\delta_2)/2} \int_0^1 du A_{2,3}(z_3, z_3 u),$$

which converges to $C_1 \widetilde{q}(z_2; z_1)$ as $z_3 \to 0+$. Q.E.D.

Fourth, we study the asymptotic behaviors of $\widetilde{q}(z_2; z_1)$ and $q(z_2; z_1)$ as $z_2 \to \infty$. Recall that

$$\begin{aligned}
\widetilde{q}(z_2; z_1) &= \int_0^{z_2} dx_2 A_{3,1} A_{3,2} \\
&= \int_0^{z_2} dx_2\, x_2^{1-\delta_1/2} (z_2 - cx_2)^{1-\delta_2/2} p_1^{\delta_1}(0+, x_2) \\
&\quad \times p_1^{\delta_2}(z_1, z_2 - cx_2) p_1^{\delta_1}(0+, x_2) p_1^{\delta_2}(0+, z_2 - cx_2) \\
&= z_2^{3-(\delta_1+\delta_2)/2} \int_0^1 du\, u^{1-\delta_1/2}(1 - cu)^{1-\delta_2/2} p_1^{\delta_1}(0+, z_2 u) \\
&\quad \times p_1^{\delta_2}(z_1, z_2(1 - cu)) p_1^{\delta_1}(0+, z_2 u) p_1^{\delta_2}(0+, z_2(1 - cu))
\end{aligned}$$

and that

$$q(z_2; z_1) = z_2 \int_0^1 du\, p_1^{\delta_1}(0+, z_2 u) p_1^{\delta_2}(z_1, z_2(1 - cu)).$$

Lemma 4.4. *Let $r > 0$. Then*

$$\frac{\widetilde{q}(z_2; z_2 r)}{q(z_2; z_2 r)} \sim C_2 D(r)^{-\delta_1/2} e^{-z_2/2} \quad \text{as } z_2 \to \infty, \tag{4.6}$$

where C_2 is some positive constant depending only on δ_1 and δ_2 and

$$D(r) = 1 + \frac{1-c}{1-c+\sqrt{rc}}.$$

Proof. If we express $\widetilde{q}(z_2; z_2 r)$ as

$$r^{(1-\delta_2)/4} z_2^{(\delta_1-1)/2} \int_0^1 f_1(z_2, u) e^{-z_2 \phi_1(u)} u^{\delta_1/2-1} du$$

using

$$\phi_1(u) = b_1 u + \sqrt{r}\left\{1 - \sqrt{1-cu}\right\} + a_1$$

with $b_1 = 1 - c$ and $a_1 = (\sqrt{r} - 1)^2/2 + 1/2$, then $f_1(z_2, \cdot)$ turns out to be a bounded continuous function such that $f_1(z_2, u/z_2)$ converges to a constant depending only on δ_1 and δ_2 as $z_2 \to \infty$, by (3.6). Since ϕ_1 and f_1 satisfies the assumptions, we can use Lemma 4.1 and hence we obtain

$$\widetilde{q}(z_2; z_2 r) \sim C_{2,1} r^{(1-\delta_2)/4} \phi_1'(0+)^{-\delta_1/2} z_2^{-1/2} e^{-a_1 z_2} \quad \text{as } z_2 \to \infty \tag{4.7}$$

with some constant $C_{2,1}$ depending only on δ_1 and δ_2.

We also have a similar expression

$$r^{(1-\delta_2)/4} z_2^{(\delta_1-1)/2} \int_0^1 f_2(z_2, u) e^{-z_2 \phi_2(u)} u^{\delta_1/2-1} du$$

for $q(z_2; z_2 r)$ using

$$\phi_2(u) = b_2 u + \sqrt{r}\left\{1 - \sqrt{1-cu}\right\} + a_2$$

with $b_2 = (1-c)/2$ and $a_2 = (\sqrt{r} - 1)^2/2$ and a function $f_2(z_2, \cdot)$ as before. Thus the same argument yields

$$q(z_2; z_2 r) \sim C_{2,2} r^{(1-\delta_2)/4} \phi_2'(0+)^{-\delta_1/2} z_2^{-1/2} e^{-a_2 z_2} \quad \text{as } z_2 \to \infty \tag{4.8}$$

with some constant $C_{2,2}$ depending only on δ_1 and δ_2.

Using (4.7) and (4.8) together with $\phi_1'(0+) = b_1 + \sqrt{r}c/2$ and $\phi_2'(0+) = b_2 + \sqrt{r}c/2$, we obtain (4.6). Q.E.D.

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $0 < c < 1$. We combine Lemmas 4.2, 4.3 and 4.4 to obtain

$$\lim_{z_2 \to \infty} e^{z_2/2} \lim_{z_3 \to 0+} z_3^{1-(\delta_1+\delta_2)/2} \lim_{\varepsilon \to 0+} \frac{q(z_2, z_3; \varepsilon, z_2 r)}{q(z_2; \varepsilon, z_2 r)} = C_3 D(r)^{-\delta_1/2}$$

for some constant C_3 which depends only on δ_1, δ_2 and c. Therefore we conclude that the conditional probability (4.1) does depend on (ε, z_1), which proves that Z^c is non-Markov. Q.E.D.

Acknowledgments. The authors thank Professors Hideki Tanemura and Makoto Katori for helpful discussions. They also thank Professor Tomoyuki Shirai for drawing their attention to [2].

References

[1] M.-F. Bru, Wishart processes, J. Theoret. Probab., **4** (1991), 725–751.
[2] I. Dumitriu and A. Edelman, Matrix models for beta ensembles, J. Math. Phys., **43** (2002), 5830–5847.
[3] F. J. Dyson, A Brownian-motion model for the eigenvalues of a random matrix, J. Mathematical Phys., **3** (1962), 1191–1198.
[4] M. Katori and H. Tanemura, Symmetry of matrix-valued stochastic processes and noncolliding diffusion particle systems, J. Math. Phys., **45** (2004), 3058–3085.
[5] N. N. Lebedev, Special Functions and Their Applications, Revised ed., Dover Publications Inc., New York, 1972.
[6] H. Matsumoto and Y. Ogura, Markov or non-Markov property of $cM - X$ processes, J. Math. Soc. Japan, **56** (2004), 519–540.
[7] D. Revuz and M. Yor, Continuous martingales and Brownian motion, Third ed., Springer-Verlag, Berlin, 1999.

Ryoki Fukushima
Department of Mathematics, Kyoto University
Kyoto 606-8502, Japan

Atsushi Tanida
Department of Mathematics, Kyoto University
Kyoto 606-8502, Japan

Kouji Yano
Graduate School of Science, Kobe University
Kobe 657-8501, Japan

E-mail address: `fukusima@math.kyoto-u.ac.jp`
`tanida@math.kyoto-u.ac.jp`
`kyano@math.kobe-u.ac.jp`