Advanced Studies in Pure Mathematics 56, 2009
Singularities — Niigata–Toyama 2007
pp. 39–47

On the Fatou–Julia decomposition of transversally holomorphic foliations of complex codimension one

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Abstract.

The Fatou–Julia decomposition of complex codimension-one foliations is given by Ghys, Gomez-Mont and Saludes. The Julia sets are expected to play a role of minimal sets of real codimension-one foliations. For example, it is known that the Godbillon–Vey class is trivial if the Julia set is empty. In this paper, we propose another decomposition obtained in a slightly different way and announce some results.

§1. Introduction

The Fatou–Julia decomposition is significant in the study of complex dynamical systems. Such a decomposition will be also interesting in the study of transversally holomorphic foliations, in particular if the complex codimension is equal to one. For example, the Julia sets are expected to play the role of minimal sets of real codimension-one foliations. We begin with briefly recalling Duminy's theorem which is one of the most important results for real codimension-one foliations [4].

Theorem 1.1 (Duminy). *Let M be a closed manifold and let $\mathcal{F}$ be a real codimension-one foliation of M. If the Godbillon–Vey class of $\mathcal{F}$ is non-trivial, then $\mathcal{F}$ admits a resilient leaf. In addition, there is a minimal set which is not a compact leaf.*

We will explain notions in the statement. A closed, non-empty subset of M is called a minimal set if it is the union of leaves of $\mathcal{F}$ and

Received January 22, 2008.
2000 *Mathematics Subject Classification.* Primary 57R30; Secondary 58H05, 37F75.
Key words and phrases. Holomorphic foliations, Fatou set, Julia set, Riemannian foliations.
The author is partially supported by Grant-in Aid for Scientific research (No. 19684001).

minimal with respect to inclusions. Minimal sets of real codimension-one foliations are classified into three types by looking the transversal direction, namely, a minimal set looks like either a point, an interval or a Cantor set. A leaf of $\mathcal{F}$ is resilient if it accumulates on itself by a non-trivial holonomy of the leaf. Finally, the Godbillon–Vey class is a secondary characteristic class of foliations. According to Duminy's theorem, the non-triviality of the Godbillon–Vey class implies that the foliation has rich dynamics.

The theory of secondary classes of foliations is well-developed also for foliations of codimension greater then one, for example, the Godbillon–Vey class is defined and an analogous result to Duminy's theorem is known [7]. For transversally holomorphic foliations of complex codimension one, a weak analogy of Duminy's theorem is also known for the Julia sets of Ghys, Gomez-Mont and Saludes, namely, if a certain characteristic class is non-trivial, then the Julia set is non-empty (Theorem 2.4). In addition, it is known that the foliations restricted to Fatou sets are transversally Hermitian, namely, it admits transversal invariant Hermitian metrics, so that the dynamics of foliations on the Fatou sets are simple. These are some of reasons for which the Julia set is expected to play the role of minimal sets. From this point of view, however, the Julia set seems too large. Indeed, there are foliations of which the Julia sets are the whole manifolds which are at the same time transversally Hermitian [5]. The above-mentioned characteristic class is trivial in such examples, so its non-triviality and the non-vacancy of the Julia set is not strongly related. Moreover, these examples show that the Julia sets can be non-empty although the dynamics of the foliations are not quite complicated. These facts suggest that there remain some parts to be removed in the Julia sets. Taking these considerations into account, we will propose another decomposition in this paper. It can be shown that if a foliation is transversally Hermitian, then the Julia set is empty. Moreover, a version on Duminy's theorem is still valid in the same form for the Julia sets of Ghys, Gomez-Mont and Saludes (Theorem 2.13). The dynamics of foliations in the Julia sets are not yet clear but the study seems easier than minimal sets. Indeed, it is possible to show that the dynamics in the Julia sets are rich except some trivial cases. We refer to [2] for the details including proofs of results.

This paper is based on a talk which the author gave in the Niigata Workshop on Complex Geometry and Singularities held during August 20–24, 2007 at Niigata. The author expresses his gratitude to the organizers.

§2. Fatou–Julia decomposition

We begin with some examples of transversally holomorphic foliations.

Example 2.1. Let $X = \lambda z \frac{\partial}{\partial z} + \mu w \frac{\partial}{\partial w}$ be a holomorphic vector field on $\mathbb{C}^2$. Suppose that $\lambda/\mu \notin \mathbb{R}$ and denote by $\mathcal{F}$ the singular foliation of $\mathbb{C}^2$ of which the leaves are the integral curves of X. The foliation $\mathcal{F}$ is transversal to S^3 in the sense $TS^3 + T_{\mathbb{R}}\mathcal{F} = T_{\mathbb{R}}\mathbb{C}^2$ on S^3, where $T_{\mathbb{R}}\mathbb{C}^2$ denotes the real tangent bundle of $\mathbb{C}^2$ and $T_{\mathbb{R}}\mathcal{F}$ denotes the real tangent bundle formed by vectors tangent to the leaves of $\mathcal{F}$. It follows that the connected components of the leaves of $\mathcal{F}$ and S^3 form a transversally holomorphic foliation of S^3. Another foliation can be obtained from $\mathcal{F}$. The foliation $\mathcal{F}$ can be extended to a singular foliation of $\mathbb{C}P^2$ with singularities $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$. After removing small balls around these singularities and taking the double, a transversally holomorphic foliation is obtained.

Example 2.2 (Ghys, Gomez-Mont and Saludes [5]). It is known that there is a cocompact lattice $\Gamma \subset \mathrm{PSL}(2;\mathbb{R})^n$, $n \geq 2$, such that the projection to the first factor is dense in $\mathrm{PSL}(2;\mathbb{R})$. Let Γ_1 be the image of Γ. Let $\mathcal{F}$ be the foliation of $\mathbb{H}^n/\Gamma$ induced by the foliation of $\mathbb{H}^n$ with leaves $\{p\} \times \mathbb{H}^{n-1}$, where $\mathbb{H}$ denotes the upper half space. It is possible to show that the transversal structure is given by the action of Γ_1 on $\mathbb{H}$. Hence the Poincaré metric is preserved and any continuous vector field on $\mathbb{H}$ invariant under the action is trivial.

Definition 2.3. A complex codimension-one transversally holomorphic foliation of M is a decomposition of M into immersed submanifolds L_λ such that there exists an atlas $\{U_\alpha\}$ of M with the following properties:

(1) each U_α is homeomorphic to $V_\alpha \times D_\alpha$, where $V_\alpha \subset \mathbb{R}^p$ and D_α is an open disc in $\mathbb{C}$,

(2) the connected components of $L_\lambda \cap U_\alpha$ are of the form $V_\alpha \times \{z\}$, $z \in D_\alpha$, and

(3) the transition functions are of the form $(x, z) \mapsto (\varphi(x, z, \bar{z}), \gamma(z))$, where γ is a biholomorphic diffeomorphism onto its image.

Such an atlas is called a foliation atlas.

The Fatou–Julia decomposition is firstly introduced by Ghys, Gomez-Mont and Saludes [5]. The decomposition is given according to the existence of certain vector fields (Definition 2.6). The Fatou set is simple in a certain sense while the Julia set is complicated. Indeed, the following theorem holds [5], [1].

Theorem 2.4. (1) *The foliation restricted to each Fatou component admits a Hermitian metric on the complex normal bundle invariant under holonomies.*
(2) *If the Godbillon–Vey class is non-trivial, then the Julia set is non-empty.*

The above theorem implies that a weak analogue of Duminy's theorem holds by replacing the minimal sets by the Julia sets in the sense of Ghys, Gomez-Mont and Saludes.

We recall some notions in order to introduce the Fatou–Julia decomposition in the sense of Ghys, Gomez-Mont and Saludes. First, the complex normal bundle, denoted by $\nu^{1,0}$, of the foliation makes a sense because holonomies preserve the transversal complex structure. In terms of foliation atlas as above, $\nu^{1,0}$ is locally spanned by $(1,0)$-vectors tangent to D_α. $\nu^{1,0}$ is also characterized by the property $\nu^{1,0} \oplus \overline{\nu^{1,0}} = (TM/T\mathcal{F}) \otimes \mathbb{C}$, where $\overline{\nu^{1,0}}$ is the complex conjugate of $\nu^{1,0}$. Sections of $\nu^{1,0}$ are called vector fields transversally of type $(1,0)$. The set of continuous vector fields transversally of type $(1,0)$ and constant along the leaves is denoted by $C_\mathcal{F}(\nu^{1,0})$.

Notation 2.5. Let $\mathcal{C}_\mathcal{F}(\nu^{1,0})$ be the subset of $C_\mathcal{F}(\nu^{1,0})$ which consists of vector fields X of which the distributional derivative is locally of class L^2 and $\bar{\partial}X \in \mathcal{L}^\infty_\mathcal{F}(\nu^{1,0} \otimes (\overline{\nu^{1,0}})^*)$, namely, $\bar{\partial}X$ induces a Beltrami coefficient invariant under the holonomies.

The Fatou sets in the sense of Ghys, Gomez-Mont and Saludes are defined as follows [5].

Definition 2.6. Let M be a closed manifold and let $\mathcal{F}$ be a complex codimension-one transversally holomorphic foliation of M. $x \in M$ belongs to the Fatou set if $X(x) \neq 0$ for some $X \in \mathcal{C}_\mathcal{F}(\nu^{1,0})$. The Julia set is by definition the complement of the Fatou set. The Fatou set and the Julia set are denoted by $F_{\mathrm{GGS}}(\mathcal{F})$ and $J_{\mathrm{GGS}}(\mathcal{F})$, respectively.

It is clear that the Fatou set is open and the union of leaves of $\mathcal{F}$. The structure of Fatou sets is well-understood. See the original paper [5] for details.

Some Julia sets in the above sense are quite complicated as expected.

Example 2.7. In Example 2.2, $J_{\mathrm{GGS}}(\mathcal{F})$ is the whole manifold. Indeed, continuous invariant vector fields corresponds to continuous vector fields on $\mathbb{H}$ invariant under the action of Γ_1. Such vector fields should be invariant under $\mathrm{PSL}(2;\mathbb{R})$ because Γ_1 is dense, therefore trivial.

On the other hand, some of them have simple dynamics.

Example 2.8. Let f_θ be the rotation of $\mathbb{C}P^1$ around an axis and let $\mathcal{F}_\theta$ be the suspension of f_θ over S^1, namely, we equip $\mathbb{R} \times \mathbb{C}P^1$ with the foliation with leaves $\mathbb{R} \times \{z\}$ and identify $(0, z)$ with $(-1, f_\theta(z))$. If f_θ is not the identity map, then $J_{\mathrm{GGS}}(\mathcal{F}_\theta) = S^1 \times \{p_0\} \cup S^1 \times \{p_\infty\}$, where p_0 and p_∞ are the intersection of the axis and $\mathbb{C}P^1$.

The Fatou–Julia decomposition is usually defined in terms of normal families. It is also possible for foliations as follows. Let $\{U_i\}_{i \in I}$ be a foliation atlas, where I is an index set. We may assume I is finite. Choose $x_i \in V_i$, where $U_i \cong V_i \times D_i$, and set $T_i = \{x_i\} \times D_i$ and $T = \cup T_i$. We denote by γ_{ji} the biholomorphic diffeomorphism induced from the transition function from U_i to U_j. It is a mapping from an open subset of T_i to T_j. Let Γ be the pseudogroup generated by $\{\gamma_{ji}\}_{i,j \in I}$. Roughly speaking, elements of Γ are obtained by composing a finite number of γ_{ji}, $i, j \in I$, possibly by considering restrictions. Namely, if $\gamma \in \Gamma$ is already defined and if the range of γ meets the domain of γ_{ji}, then a suitable restriction of γ can be composed with γ_{ji} to produce another mapping. Thus obtained mappings, together with mappings obtained by some other operations, are also elements of Γ. Elements of Γ correspond to parallel translations along the leaves of $\mathcal{F}$, and Γ is called the holonomy pseudogroup of $\mathcal{F}$. In order to distinguish from holonomies along loops on leaves, we call elements of Γ local holonomies. Let D_i' be an open disc slightly smaller than D_i and assume that the closure of D_i' is contained in D_i. Let $T_i' = \{x_i\} \times D_i'$, then the closure of T_i' is also contained in T_i. We may assume that $T' = \cup T_i'$ meets all leaves of $\mathcal{F}$. The restriction of Γ to T' is denoted by Γ'.

It is possible to consider normal families related to foliations by using the holonomy pseudogroups.

Definition 2.9. (1) A connected open set $U \subset T'$ is a Fatou neighborhood if the germ of any element of Γ' defined on a neighborhood of a point in U can be extended to an element of Γ defined on U.

(2) The union of Fatou neighborhoods is by definition the Fatou set F' of T'.

(3) The Fatou set F of T is by definition the Γ-orbit of F' in T:

$$F = \{x \in T \mid x = \gamma x' \text{ for some } x' \in F' \text{ and } \gamma \in \Gamma\}.$$

The Julia set of T is the complement of the Fatou set of T.

(4) The union of leaves which pass through points of F is the Fatou set of $\mathcal{F}$ and denoted by $F(\mathcal{F})$, and the union of leaves which pass through points of J is the Julia set of $\mathcal{F}$ and denoted by $J(\mathcal{F})$.

(5) The connected components of $F(\mathcal{F})$ and $J(\mathcal{F})$ are called the Fatou components and the Julia components.

Remark 2.10. The Fatou–Julia decomposition is defined in terms of compactly generated pseudogroups in [2]. The definition of Julia components are different from [5].

Let U be a Fatou neighborhood and let Γ_U be the subset of Γ obtained by extending the germs as above, then Γ_U is a normal family by virtue of Montel's theorem. This is what we expected.

We can show the following.

Lemma 2.11. (1) *The Fatou set of T is independent of the choice of T'.*

(2) *The Fatou set of $\mathcal{F}$ is independent of the choice of the foliation atlas.*

Moreover, we have the following.

Proposition 2.12. $F_{\mathrm{GGS}}(\mathcal{F}) \subset F(\mathcal{F})$. *The inclusion can be strict.*

Sketch of the proof. Let $x \in T' \cap F_{\mathrm{GGS}}(\mathcal{F})$. There is a non-trivial continuous vector field X on M with $X(x) \neq 0$ which belongs to $\mathcal{C}_{\mathcal{F}}(\nu^{1,0})$. Since X is constant along the leaves, X induces a continuous vector field on T which is of type $(1,0)$. We denote the induced vector field again by X, then X is invariant under Γ. Integration of X induces a one-parameter family $\phi\colon D \times T' \to T$, where D is a small open disc in $\mathbb{C}$. As X is continuous and the closure of T' is compact, ϕ is well-defined if D is small enough. We may also assume that $U = \phi(D, x)$ is contained in T', however, $\phi(D, y)$ need not be contained in T' if $y \neq x$ so that we cannot always replace the target with T'. Let $y \in U$ and assume that $\gamma \in \Gamma'$ is defined on a neighborhood of y. Since X preserves $\mathcal{F}$, γ is in fact defined on U. Indeed, if $y = \phi(t, x)$, where $t \in D$, then $\gamma(y) = \phi(t, \gamma x)$. This means that $x \in T' \cap F(\mathcal{F})$ admits a neighborhood U such that the germ of any element of Γ' at a point U extends an element of Γ defined on U. Q.E.D.

The structure of Fatou sets in the sense of Ghys, Gomez-Mont and Saludes is well-studied, where existence of non-trivial vector field invariant under the holonomy of $\mathcal{F}$ is extensively used. Although we do not have such a vector field any more, the Fatou sets can be described in a quite similar way. Instead of presenting the description, we will sketch the proof of the following fact.

Theorem 2.13. (1) *The foliation restricted to the Fatou set admits a Hermitian metric on $\nu^{1,0}$ invariant under the holonomy.*

(2) *If $\mathcal{F}$ is transversally Hermitian, then $J(\mathcal{F})$ is empty.*
(3) *The Godbillon–Vey class is trivial if $J(\mathcal{F})$ is empty.*

Sketch of the proof. In order to show 1), we first construct an invariant Hermitian metric on $\nu^{1,0}$ which is locally Lipschitz continuous. It is based on the fact that Γ_U is a normal family. Noticing that T can be regarded as a subset of $\mathbb{C}$, fix any Hermitian metric on T. Given a point $x \in F$ and a vector v at x, set $\|v\|^L = \sup\limits_{\gamma} \|\gamma_* v\|_{\gamma x}$, where γ runs through elements of Γ defined on a neighborhood of x and $\|\cdot\|$ is the norm with respect to the fixed metric. It is easy to see that $\|\cdot\|^L$ is a norm invariant under Γ and it induces a Hermitian metric. It is however not necessarily continuous so that we need to carefully choose the initial metric in order to make this naive idea to work. The existence of an invariant Hermitian metric locally Lipschitz continuous implies that the elements of Γ and their uniform limits are determined by 1-jets. Then by applying classical theorems of H. Cartan [3], it can be shown that the closure of Γ is locally given by local analytic actions of Lie transformation groups. By following the arguments in [5], [6] and [8], it turns out the Fatou sets are described in a similar way to the Fatou sets in the sense of Ghys, Gomez-Mont and Saludes. As a corollary, an invariant metric of class C^ω can be found. This completes the proof of 1).

2) follows easily from the definition of the Fatou set. We remark here that the transversal Hermitian metric need only to be of class C^0. A precise form of 3), which we do not state here, can be shown by defining residues at the Julia sets. It is a repetition of the construction of a residue for the Julia sets in the sense of Ghys, Gomez-Mont and Saludes, which is done in [1]. The method is essentially due to Bott and Heitsch. By using the fact that the foliation admits an invariant Hermitian metric on $F(\mathcal{F})$, a connection which is adapted both to the foliation and to the metric can be found on $F(\mathcal{F})$. This connection can be extended to the whole manifold after modifying it on a neighborhood of $J(\mathcal{F})$. If the Godbillon–Vey class is calculated by using this connection, then we can find a representative of the Godbillon–Vey class supported on a neighborhood of $J(\mathcal{F})$. If $J(\mathcal{F})$ is empty, the representative is identically zero. Q.E.D.

It is possible to show that the Julia set contains repelling local holonomies converging to a leaf. Here the trivial case where the leaf contains a hyperbolic holonomy is allowed. Combined with Theorem 2.13, a weak version of Duminy's theorem is obtained in the following form.

Theorem 2.14. *Let M be a closed manifold and let $\mathcal{F}$ be a complex codimension-one transversally holomorphic foliation. If $J(\mathcal{F})$ is* *non-empty, then there exist repelling local holonomies converging to a leaf. Here the trivial case where the leaf contains a hyperbolic holonomy is allowed.*

In order to obtain further results, it is necessary to study the properties of the Julia sets. Many fundamental properties remain unclear. For example, it is known that the Julia set in the sense of Ghys, Gomez-Mont and Saludes can have non-empty interior without being the whole manifold. On the other hand, it seems unknown if such an example exists for the Julia set in the sense of Definition 2.9.

§3. Review of Examples

Example 3.1. In Example 2.1, $J_{\mathrm{GGS}}(\mathcal{F}) = J(\mathcal{F})$ and they consist of three compact leaves derived from separatrices of the singularities of $\mathcal{F}$. After blowing up these singularities, there always remain the same type of singularities but the number of singularities is increased. Hence by removing neighborhoods of singularities and taking the double, a foliation of which the number of Julia components is greater than three can be obtained. In fact, without posing any restriction on the ambient manifold, we can find a foliation of which the number of Julia components is any positive integer. On the other hand, we do not know if there is any restriction on the number of Julia components if the ambient manifold is fixed.

Example 3.2. In Example 2.2, $J_{\mathrm{GGS}}(\mathcal{F}) = M$ and $J(\mathcal{F}) = \varnothing$.

Example 3.3. In Example 2.8, $J_{\mathrm{GGS}}(\mathcal{F}) = S \times \{p_0\} \cup S \times \{p_\infty\}$ and $J(\mathcal{F}) = \varnothing$.

Example 3.4. Let $\Gamma \subset \mathrm{PSL}(2;\mathbb{C})$ be a Kleinian group and let $\mathcal{F}$ be the suspension of Γ over $\mathbb{H}/\Gamma$. If Γ is torsion-free, then $J(\mathcal{F})$ and $J_{\mathrm{GGS}}(\mathcal{F})$ are the same and in fact the suspension of the limit set of Γ. On the other hand, if Γ contains torsion elements, then $J(\mathcal{F})$ is again the suspension of Λ but $J_{\mathrm{GGS}}(\mathcal{F})$ is the suspension of the union of Λ with the fixed points of torsion elements.

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