Advanced Studies in Pure Mathematics 44, 2006
Potential Theory in Matsue
pp. 155–166

Singular directions of meromorphic solutions of some non-autonomous Schröder equations

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Abstract.

Let $s = |s|e^{2\pi\lambda i}$ be a complex constant satisfying $|s| > 1$ and $\lambda \notin \mathbb{Q}$. We show that for a transcendental meromorphic solution $f(z)$ of some non-autonomous Schröder equation $f(sz) = R(z, f(z))$, any direction is a Borel direction.

§1. Introduction

Let $R(z, w)$ be a rational function in z and w of $\deg_w R(z, w)$ at least 2, and let $s \in \mathbb{C}$ be a constant of modulus bigger than 1. This note is devoted to investigate singular directions of meromorphic solutions of functional equations of the form

$$f(sz) = R(z, f(z)), \quad d = \deg_w[R(z, w)] \geq 2. \tag{1.1}$$

In this note "meromorphic" means "meromorphic in the complex plane $\mathbb{C}$", and we assume that the reader is familiar with the Nevanlinna theory, see e.g., [1], [4]. By a simple transformation, we can assume that $R(0, 0) = 0$. In order to state an existence theorem of a meromorphic solution for (1.1), we write

$$R(z, w) = \sum_{n+m \geq 1} a_{n,m} z^n w^m.$$

Received March 25, 2005.
Revised June 9, 2005.
2000 *Mathematics Subject Classification.* Primary 30D35; Secondary 39B32.
Key words and phrases. Schröder equation, Nevanlinna theory, Borel direction Value distribution theory in angular domains .
Grant-in-Aid for Scientific Research, Japan Society for the Promotion of Science, (C) (1) (No. 16540202).

Under the assumption that either

$$a_{1,0} \neq 0 \quad \text{and} \quad s^n \neq a_{0,1} \quad \text{for all } n \in \mathbb{N}, \tag{1.2}$$

or

$$a_{1,0} = 0 \quad \text{and} \quad s = a_{0,1}, \tag{1.3}$$

the equation (1.1) admits the unique meromorphic solution $f(z) \not\equiv 0$ with $f(0) = 0$ for the case (1.2), and also the unique solution $f(z)$ with $f(0) = 0, f'(0) = 1$, for the case (1.3). For the proof, see [8, p.153]. When $f(z)$ is transcendental, the order of growth $\rho = \rho(f)$ is given by $\rho = \log d / \log |s|$, $d = \deg_w R(z, w)$, and there holds

$$K_1 r^\rho < T(r, f) < K_2 r^\rho, \tag{1.4}$$

with some positive constants K_1, K_2, where $T(r, f)$ is the Nevanlinna characteristic of $f(z)$, see [8, p.159].

Let $\mathfrak{d}_\omega = \{z = re^{i\omega},\ r > 0\}$ be a ray and $\Omega(\omega, \alpha), \alpha \in (0, \pi)$, be a sector $\Omega(\omega, \alpha) = \{z\ ;\ |\arg[z] - \omega| < \alpha\}$. When ω is fixed, we write for $\Omega(\omega, \alpha)$ simply as Ω_α. Further we define $\Omega_\alpha^{(r)} = \Omega_\alpha \cap \{|z| < r\}$.

Let $f(z)$ be a transcendental meromorphic function of order $\rho > 0$. Let $\mathfrak{d}_\omega$ be fixed. For any $a \in \mathbb{C} \cup \{\infty\}$, write zeros of $f(z) - a$ in $\Omega_\alpha = \Omega(\omega, \alpha)$ as $z_n^*(a, \Omega_\alpha)$, $n = 0, 1, \cdots$, multiple zeros counted only once. On the other hand, zeros of $f(z) - a$, counted with multiplicity, are denoted as $z_n(a, \Omega_\alpha)$. We say $\mathfrak{d}_\omega$ to be a *Borel direction of divergence type in the sense of Tsuji* (resp. *in the sense of Valiron*), for $f(z)$ [6, p.274] (resp. [7]), if for any $a \in \mathbb{C}$, with at most two possible exception(s),

$$\sum_{n=0}^{\infty} \frac{1}{|z_n^*(a, \Omega_\alpha)|^\rho} = \infty \quad \text{for any } \alpha > 0,$$

$$\left(\text{resp.} \quad \sum_{n=0}^{\infty} \frac{1}{|z_n(a, \Omega_\alpha)|^\rho} = \infty \quad \text{for any } \alpha > 0\right).$$

In the following, we call a *Borel direction of divergence type* simply as a *Borel direction.*

Obviously, if c is a Borel exceptional value in the sense of Valiron, then c is so in the sense of Tsuji, too, but the converse is not true. We write s as

$$s = |s|e^{2\pi\lambda i}, \quad |s| > 1, \quad \lambda \in [0, 1). \tag{1.5}$$

In the autonomous case, i.e., $R(z, w)$ does not contain z, we proved [2] that for a meromorphic solution $g(z)$ of the equation $g(sz) = R(g(z))$, any direction $\mathfrak{d}_\omega$ is a Borel direction in the sense of both Valiron as well as Tsuji, supposed $\lambda \notin \mathbb{Q}$. Further, a Borel exceptional value c, if any, must be a Picard exceptional value, i.e. $g(z) \neq c$ for any $z \in \mathbb{C}$.

§2. Non-autonomaus Schröder equations and a main result

In order to consider the non-autonomous case where $R(z,w)$ contains z, we need to make some provisions. Write $R(z,w)$ in (1.1)

$$R(z,w) = \frac{P(z,w)}{Q(z,w)},$$
$$P(z,w) = \sum_{j=0}^{p} a_j(z)w^j, \quad Q(z,w) = \sum_{k=0}^{q} b_k(z)w^k,$$

where $a_j(z), b_k(z)$ are polynomials. We have $d = \max(p,q) \geq 2$.

Proposition 1. By some linear transformation

$$L[w] = \frac{\alpha w + \beta}{\gamma w + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}, \quad \alpha\delta - \beta\gamma \neq 0, \tag{2.1}$$

the equation (1.1) can be reduced to the following form

$$L[f(sz)] = R^\circ(z, L[f(z)]), \qquad R^\circ(z,w) = \frac{P^\circ(z,w)}{Q^\circ(z,w)},$$
$$P^\circ(z,w) = \sum_{j=0}^{d} a_j^\circ(z)w^j, \quad Q^\circ(z,w) = \sum_{k=0}^{d} b_k^\circ(z)w^k,$$

in which we have

$$\begin{aligned} \deg_w[P^\circ(z,w)] = \deg_w[Q^\circ(z,w)] = d, \\ \deg[a_j^\circ(z)] = \deg[b_k^\circ(z)] = D. \end{aligned} \tag{2.2}$$

We remark that the conditions (2.2) are satisfied with any quadruple $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $\alpha\delta - \beta\gamma \neq 0$ for other than a finite number of exception.

Proof of Proposition 1 Let

$$f(sz) = \frac{\sum_{j=0}^{p} a_j(z)f(z)^j}{\sum_{k=0}^{q} b_k(z)f(z)^k}, \quad \max(p,q) = d.$$

Put $f(z) = f_1(z) + \alpha$. Then

$$f_1(sz) = \frac{\sum_{j=0}^{d} a_j^{[1]}(z)f_1(z)^j}{\sum_{k=0}^{q} b_k^{[1]}(z)f_1(z)^k},$$

where

$$a_j^{[1]}(z) = \begin{cases} \sum_{m=j}^{p} \binom{m}{j} \alpha^{m-j} a_m(z) - \alpha \sum_{m=j}^{\min(q,j)} \binom{m}{j} \alpha^{m-j} b_m(z), \\ \qquad \text{for } j \le p, \text{ when } p \le d, \\ -\alpha \left\{ \sum_{m=j}^{d} \binom{m}{j} \alpha^{m-j} b_m(z) \right\}, \quad \text{for } j > p, \\ \qquad \text{for } j > p, \text{ when } p < q = d, \end{cases}$$

$$b_k^{[1]}(z) = \sum_{m=k}^{q} \binom{m}{k} \alpha^{m-k} b_m(z), \quad \text{for } k \le q.$$

Hence, except a finite number of values α, we have

$$\deg[a_0^{[1]}(z)] = \max_{j,k}(\deg[a_j^{[1]}(z)], \deg[b_k^{[1]}(z)]),$$
$$\deg[b_0^{[1]}(z)] = \max_k \deg[b_k^{[1]}(z))].$$

Put $f_1(z) = 1/f_2(z)$. Then

$$f_2(sz) = \frac{\sum_{j=0}^{q} a_j^{[2]}(z) f_2(z)^j}{\sum_{k=0}^{d} b_k^{[2]}(z) f_2(z)^k},$$
$$\deg[b_d^{[2]}(z)] = \max_{j,k}(\deg[a_j^{[2]}(z)], \deg[b_k^{[2]}(z)]).$$

Put $f_2(z) = f_3(z) + \beta$, $f_3(z) = 1/f_4(z)$, and $f_4(z) = f_5(z) + \gamma$, then we obtain (2.2) for $a_j^{[5]}(z), b_k^{[5]}(z)$, except for a finite number of values β, γ. We have thus proved Proposition 1.

Write the coefficients of w^j in $P(z,w)$ and those of w^k in $Q(z,w)$ as

$$a_j(z) = a_D^{(j)} z^D + a_{D-1}^{(j)} z^{D-1} + \cdots + a_0^{(j)},$$
$$b_k(z) = b_D^{(k)} z^D + b_{D-1}^{(k)} z^{D-1} + \cdots + b_0^{(k)},$$

with $a_D^{(j)} \neq 0$ and $b_D^{(k)} \neq 0$ for $0 \le j, k \le d$, and put

$$\text{(2.3)} \qquad \begin{aligned} P_j^*(w) &= a_D^{(j)} w^j + a_D^{(j-1)} w^{j-1} + \cdots + a_D^{(0)}, \ 0 \le j \le d, \\ Q_d^*(w) &= b_D^{(d)} w^d + b_D^{(d-1)} w^{d-1} + \cdots + b_D^{(0)}. \end{aligned}$$

The main result in this note is the following

Theorem 1. Suppose $\lambda \notin \mathbb{Q}$ in (1.5) and $R(z,w)$ in (1.1) satisfies (2.2). Assume that $P_d^*(w)$ and $Q_d^*(w)$ for $R(z,w)$ defined in (2.3) are relatively prime. Let (1.1) have a transcendental meromorphic solution $f(z)$. Then any $\mathfrak{d}_{\omega_0}$ ($\omega_0 \in [0, 2\pi)$) is a Borel direction in the sense of Tsuji for $f(z)$.

Remark 1. We can assume without losing generality that $P_j^*(w)$ and $Q_d^*(w)$ are relatively prime for each j, $1 \leq j \leq d-1$, which can be attained by a suitable choice of $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ in (2.1).

On the contrary to autonomuous case, a Borel exceptional value need not be a Picard exceptional value (see the end of Section 1). Further, Borel exceptional value in the sense of Tsuji may be not exceptional in the sense of Valiron. We can see these in Examples 1 and 2 below.

Example 1. Consider the equation [8, p.158]

$$f_1(sz) = \frac{1+z}{1-z} f_1(z)^2, \quad s = |s|e^{2\pi i\lambda},\ \lambda \in [0,1) \setminus \mathbb{Q},\ |s| > 2. \tag{2.4}$$

If we put $f_1(z) = 1 + h_1(z)$, then

$$h_1(sz) = \frac{2z}{1-z} + 2\frac{1+z}{1-z}h_1(z) + \frac{1+z}{1-z}h_1(z)^2 = 2z + 2h_1(z) + \cdots,$$

and we have that $a_{1,0} = 2 \neq 0$, $a_{0,1} = 2 \neq s^n$ for any $n \in \mathbb{N}$, hence there is the unique solution $h_1(z) \not\equiv 0$, $h_1(0) = 0$. Therefore, there is the unique non-trivial solution for (2.4) which is given by

$$f_1(z) = \prod_{n=1}^{\infty} \left(\frac{1 + \frac{z}{s^n}}{1 - \frac{z}{s^n}} \right)^{2^{n-1}}.$$

Hence $f_1(z)$ has two Borel exceptional values $0, \infty$ in the sense of Tsuji. They are not a Borel exceptional values in the sense of Valiron.

Example 2. Consider also the equation

$$f_2(sz) = (1+z)f_2(z)^2, \quad s = |s|e^{2\pi i\lambda},\ \lambda \in [0,1) \setminus \mathbb{Q},\ |s| > 2. \tag{2.5}$$

As in Example 1, there is the unique solution which is given by

$$f_2(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{s^n} \right)^{2^{n-1}}.$$

$f_2(z)$ has a Borel exceptional value 0 in the sense of Tsuji, which is not exceptional in the sense of Valiron. For $f_2(z)$, ∞ is exceptional in the sense of Valiron (in fact, Picard exceptional).

We observe primeness in the examples above. For the equation (2.4) we have, putting $g_1(z) = 1/(f_1(z) - 1)$,

$$g_1(sz) = \frac{(-1 + \frac{1}{z})g_1(z)^2}{2g_1(z)^2 + 2(1 + \frac{1}{z})g_1(z) + (1 + \frac{1}{z})}.$$

We have $P_2^*(w) = -w^2$ and $Q_2^*(w) = 2w^2 + 2w + 1$ which are relatively prime.

For the equation (2.5), we see that $P_2^*(w)$ and $Q_2^*(w)$ are not relatively prime, and the assumption in Theorem 1 is not satisfied. But if we put $g_2(z) = 1/(f_2(z) - 1)$, then we get

$$zg_2(sz) = \frac{g_2(z)^2}{g_2(z)^2 + 2(1 + \frac{1}{z})g_2(z) + (1 + \frac{1}{z})} = R_2(z, g_2(z)).$$

For $R_2(z, w)$, we have that $P_2^*(w) = w^2$ and $Q_2^*(w) = w^2 + 2w + 1$ are relatively prime. Hence the arguments in the proof of Theorem 1 stated in Sections 4, 5 can be applied to $R_2(z, g_2(z))$. But we do not know whether for $m \in \mathbb{Z}$ with some $K > 0$,

$$T(r; \Omega_\alpha; z^m g_2(z)) \leq KT(r; \Omega_\alpha; g_2(z)) + O((\log r)^2)$$

holds or not. Therefore, our Theorem 1 can not be applied to (2.5).

§3. Characteristic functions in a sector

Following Tsuji [6, p.272], we define the sectorial characteristic of a meromorphic function $w(z)$. Fix $\omega \in [0, 2\pi)$. With $\Omega_{\alpha_0} = \Omega(\omega, \alpha_0)$ and $\Omega_{\alpha_0}^{(r)}$ as in Section 1, we define

$$S(r; \Omega_{\alpha_0}; w) = \frac{1}{\pi} \iint_{\Omega_{\alpha_0}^{(r)}} \left(\frac{|w'(te^{i\theta})|}{1 + |w(te^{i\theta})|^2} \right)^2 t\,dt\,d\theta,$$

$$T(r; \Omega_{\alpha_0}; w) = \int_0^r \frac{S(t; \Omega_{\alpha_0}; w)}{t} dt.$$

Let $\overline{n}(r, b; \Omega_\alpha; w), \Omega_\alpha = \Omega(\omega, \alpha)$, be the number of zeros of $w(z) - b$ contained in $\Omega_\alpha^{(r)}$, multiple zeros counted only once, and put

$$\overline{N}(r, b; \Omega_\alpha; w) = \int_1^r \frac{\overline{n}(t, b; \Omega_\alpha; w)}{t} dt.$$

Then by [6, p.272, Theorem VII.3], we have with any $\alpha > \alpha_0$,

$$T(r; \Omega_{\alpha_0}; w) \leq 3 \sum_{i=1}^{3} \overline{N}(2r, b_i; \Omega_\alpha; w) + O((\log r)^2). \tag{3.1}$$

We note that (3.1) is generalized by Toda [5].

§4. A preliminary Lemma

Let $R(z, w)$ is a rational function in w whose coefficients are rational functions. Suppose that $R(z, w)$ satisfies the condition in Theorem 1.

Lemma 1. Write $\Omega(\omega, \alpha)$ as Ω_α. We have for a constant K

$$T(r; \Omega_\alpha; R(z, f(z))) \leq KT(r; \Omega_\alpha; f(z)) + O((\log r)^2).$$

Proof of Lemma 1 Let $a(z)$ be a rational function satisfying $a(z) \to M \neq 0, \infty$ as $z \to \infty$. Then $|M|/2 \leq |a(z)| \leq 2|M|$ for $|z| \geq r_0$ with sufficiently large r_0, and we have

$$\begin{aligned}
\frac{|(af)'|}{1+|af|^2} &\leq \frac{|a'f|}{1+|af|^2} + \frac{|af'|}{1+|af|^2} \\
&\leq \frac{1}{2} \cdot \frac{|a'|}{|a|} + \frac{|af^2|}{1+|af|^2} \cdot \frac{|f'|}{1+|f|^2} + \frac{|a|}{1+|af|^2} \cdot \frac{|f'|}{1+|f|^2} \\
&\leq \frac{1}{2} \cdot \frac{|a'|}{|a|} + \frac{|af|^2}{1+|af|^2} \cdot \frac{1}{|a|} \cdot \frac{|f'|}{1+|f|^2} + 2|M| \frac{|f'|}{1+|f|^2} \\
&\leq \frac{1}{2} \cdot \frac{|a'|}{|a|} + 2(\frac{1}{|M|} + |M|) \cdot \frac{|f'|}{1+|f|^2}.
\end{aligned}$$

Hence we get

$$(4.1)\quad T(r; \Omega_\alpha; a(z)f(z)) \leq 8(|M|^{-1} + |M|)^2 T(r; \Omega_\alpha; f(z)) + O((\log r)^2).$$

Note that, if $a(z) = M$ a constant, then $O((\log r)^2)$ in (4.1) can be omitted.

We have for $c \in \mathbb{C}$

$$(4.2)\qquad K_1(c)T(r; \Omega_\alpha; f) \leq T(r; \Omega_\alpha; f - c) \leq K_2(c)T(r; \Omega_\alpha; f),$$

where $K_j(c)$, $j = 1, 2$, are constants depending on c. In fact, (4.2) is trivial when $c = 0$. Suppose $c \neq 0$. If $|f(z)| \leq 2|c|$,

$$\frac{1}{1+9|c|^2} \leq \frac{1+|f(z)|^2}{1+|f(z)-c|^2} \leq 1 + 4|c|^2,$$

and if $|f(z)| > 2|c|$, we obtain from $|c/f(z)| < 1/2$,

$$\frac{4}{9} \leq \frac{1+|f(z)|^2}{1+|f(z)-c|^2} = \frac{1+1/|f(z)|^2}{|1-c/f(z)|^2 + 1/|f(z)|^2} \leq 4.$$

When $c(z)$ is a rational function with $c(z) \to M \in \mathbb{C}$ as $z \to \infty$, by the similar calculation as above we infer that

$$(4.3) \qquad \begin{aligned} K_1(c)T(r;\Omega_\alpha;f) &\le T(r;\Omega_\alpha;f-c) + O((\log r)^2) \\ &\le K_2(c)T(r;\Omega_\alpha;f) + O((\log r)^2). \end{aligned}$$

In fact, $|c(z)| \le M_1$ with some $M_1 > |M|$, for large $|z|$. Since we have

$$\frac{|f'-c'|}{1+|f-c|^2} \le \frac{1+|c|^2}{1+|f-c|^2}\frac{|c'|}{1+|c|^2} + \frac{1+|f|^2}{1+|f-c|^2}\frac{|f'|}{1+|f|^2}$$

and

$$\frac{1+|c|^2}{1+|f-c|^2} \le 1+4M_1^2, \quad \frac{1+|f|^2}{1+|f-c|^2} \le \max\left(1+4M_1^2, 4+\frac{1}{M_1^2}\right),$$

we obtain the second inequality in (4.3). Thus for a meromorphic function $g(z)$, we see $T(r;\Omega_\alpha;g+c) + O((\log r)^2) \le \tilde{K}_2(c)T(r;\Omega_\alpha;g) + O((\log r)^2)$ with a constant $\tilde{K}_2(c)$. Set $g(z) = f(z) - c(z)$ in this inequality, we get the first inequality in (4.3).

By (4.1) and (4.2), we see that, with a constant K_L

$$T(r;\Omega_\alpha;f) \le K_L T(r;\Omega_\alpha;L[f]), \qquad L[w] = \frac{\alpha w+\beta}{\gamma w+\delta},$$

where $\alpha,\beta,\gamma,\delta \in \mathbb{C}$, $\alpha\delta - \beta\gamma \ne 0$.

We have for $\ell \ge 2$,

$$\frac{|(f(z)^\ell)'|}{1+|f(z)^\ell|^2} = \ell\frac{|f(z)^{\ell-1}|+|f(z)^{\ell+1}|}{1+|f(z)^\ell|^2}\frac{|f'(z)|}{1+|f(z)|^2} \le \ell\frac{|f'(z)|}{1+|f(z)|^2},$$

since $x^{\ell-1}+x^{\ell+1}-1-x^{2\ell} = -(1-x^{\ell-1})(1-x^{\ell+1}) \le 0$ for $x \ge 0$, and hence

$$T(r;\Omega_\alpha;f(z)^\ell) \le \ell^2 T(r;\Omega_\alpha;f(z)).$$

For $R(z,w) = P(z,w)/Q(z,w)$, define $P_j^*(w)$ and $Q_d^*(w)$ as in (2.3). We assume that $P_d^*(w), Q_d^*(w)$ are relatively prime, following Theorem 1. Then, as stated in Remark 1, we can assume that $P_j^*(w), Q_d^*(w)$ are relatively prime, without losing generality. Of course we assume (2.2). Write $P(z,w) = a_d(z)P_1(z,w), Q(z,w) = b_d(z)Q_1(z,w)$ and $R_1(z,w) = P_1(z,w)/Q_1(z,w)$. Note that

$$P_1(z,w) = w^d + \sum_{j=0}^{d-1} a^{[j]}(z)w^j, \quad a^{[j]}(z) = \frac{a_j(z)}{a_d(z)} = a_0^{[j]} + \sum_{n=1}^{\infty}\frac{a_n^{[j]}}{z^n},$$

$$Q_1(z,w) = w^d + \sum_{j=0}^{d-1} b^{[j]}(z)w^j, \quad b^{[j]}(z) = \frac{b_j(z)}{b_d(z)} = b_0^{[j]} + \sum_{n=1}^{\infty}\frac{b_n^{[j]}}{z^n},$$

with $a_0^{[j]} \neq 0$ and $b_0^{[j]} \neq 0$. Since $\lim_{z\to\infty} a(z) \neq 0, \infty$, where $a(z) = a_d(z)/b_d(z)$, we have by (4.1) for a constant K_a

$$T(r;\Omega_\alpha;R(z,f(z))) \leq K_a T(r;\Omega_\alpha;R_1(z,f(z))) + O((\log r)^2).$$

Hence we may treat $P(z,w) = P_1(z,w)$, $Q(z,w) = Q(z,w)$. We write $R_1(z,f(z)) = w_1 + w_2$, where

$$w_1 = \frac{\sum_{j=0}^{d-1} a^{[j]}(z)f(z)^j}{Q_1(z,f(z))}, \quad w_2 = \frac{f(z)^d}{Q_1(z,f(z))}.$$

We will show that, with some constant K_1,

$$(4.4)\quad T(r;\Omega_\alpha;w_1+w_2) \leq K_1\{T(r;\Omega_\alpha;w_1)+T(r;\Omega_\alpha;w_2)\}+O((\log r)^2).$$

In fact, we have

$$\frac{|(w_1+w_2)'|}{1+|w_1+w_2|^2} \leq \frac{1+|w_1|^2}{1+|w_1+w_2|^2}\frac{|w_1'|}{1+|w_1|^2} + \frac{1+|w_2|^2}{1+|w_1+w_2|^2}\frac{|w_2'|}{1+|w_2|^2}.$$

If either $|w_1| \geq 2|w_2|$ or $|w_2| \geq 2|w_1|$, then

$$\frac{1+|w_1|^2}{1+|w_1+w_2|^2} \leq 4 \text{ (or } \leq 1), \quad \frac{1+|w_2|^2}{1+|w_1+w_2|^2} \leq 1 \text{ (or } \leq 4).$$

If $2|w_1| > |w_2| > (1/2)|w_1|$, then

$$|f(z)|^2 \leq 2\left(|a^{[d-1]}(z)||f(z)^{d-1}| + \cdots + |a^{[0]}(z)|\right)$$

and $a^{[j]}(z), 0 \leq j \leq d-1$, are bounded as $z \to \infty$. Hence $f(z)$ must be bounded. Since $P_d^*(w), Q_d^*(w)$ are relatively prime by the assumption, $|P_d^*(w)|^2+|Q_d^*(w)|^2 \geq K' > 0$ with a constant K'. Hence $|P_1(z,f(z))|^2 + |Q_1(z,f(z))|^2 \geq K^*$ with a constant $K^* > 0$, if $|z|$ is sufficiently large. Thus we have

$$\frac{1+|w_1|^2}{1+|w_1+w_2|^2} \leq \sqrt{K_1/2}, \quad \frac{1+|w_2|^2}{1+|w_1+w_2|^2} \leq \sqrt{K_1/2}$$

with some K_1, for $|z| \geq r_0$ if r_0 is large, which shows (4.4). Next, write

$$P_2(z,w) = a^{[d-1]}(z)w^{d-1} + P_3(z,w), \quad P_3(z,w) = \sum_{j=0}^{d-2} a^{[j]}(z)w^j$$

and $w_1 = P_3(z,w)/Q_1(z,w)$ and $w_2 = a^{[d-1]}(z)w^{d-1}/Q_1(z,w)$. Since $P^*_{d-1}(w)$ and $Q^*_d(w)$ are relatively prime, we obtain (4.7) as above. Applying these arguments repeatedly, we have

$$\begin{aligned}
T(r;\Omega_\alpha; R(z,f(z))) &\le K_2 \sum_{j=0}^{d} T(r;\Omega_\alpha; \frac{f(z)^j}{Q_1(z,f(z))}) + O((\log r)^2) \\
&\le K_2 \sum_{j=0}^{d} T(r;\Omega_\alpha; \frac{Q_1(z,f(z))}{f(z)^j}) + O((\log r)^2) \\
&\le K_3 \sum_{0\le j,k\le d} T(r;\Omega_\alpha; f(z)^{|k-j|}) + O((\log r)^2) \\
&\le KT(r;\Omega_\alpha; f(z)) + O((\log r)^2).
\end{aligned}$$

with some constants K_2, K_3, K, making use of (4.3). We have thus proved Lemma 1.

§5. Proof of Theorem 1

Let $T(r,f)$ be the characteristic function of $f(z)$ in the sense of Shimizu–Ahlfors. As in [6, p.274], we see from (1.4) that there is $\omega^* \in [0, 2\pi)$ such that

$$\int^{\infty} \frac{T(r;\Omega(\omega^*,\alpha_0); f)}{r^{\rho+1}} dr = \infty,$$

for any $\alpha_0 \in (0,\pi)$. Define $R^0(z,w) = w$ and

$$R^m(z,w) = R(s^{m-1}z, R^{m-1}(z,w)) \quad \text{for} \quad m \ge 1.$$

Then we have

$$f(s^m z) = R(s^{m-1}z, f(s^{m-1}z)) = R^m(z, f(z)).$$

It is not difficult to see that $R^m(z,w)$ satisfies (2.2) from the assumption for $R(z,w)$, and also see that $P^*_{d^m}(w)$ and $Q^*_{d^m}(w)$ corresponding to $R^m(z,w)$ are relatively prime. We can assume that $P^*_j(w)$ and $Q^*_{d^m}(w)$ corresponding to $R^m(z,w)$ are relatively prime, for each $j < d^m$ by a suitable linear transformation, if necessary.

Take $\omega_0 \in [0,2\pi)$ and $\alpha \in (0,\pi)$ arbitrarily. Let $m \in \mathbb{N}$ be so large that $\alpha_0 = |\omega_0 + 2\pi m\lambda - \omega^*| < \alpha/8$, mod 2π, see e.g., [3]. Then we have

$$\int^{\infty} \frac{T(r;\Omega(\omega_0,\alpha/2);R^m(z,f(z)))}{r^{\rho+1}}dr$$
$$= \int^{\infty} \frac{T(r;\Omega(\omega_0,\alpha/2);f(s^m z))}{r^{\rho+1}}dr$$
$$\geq C\int^{\infty} \frac{T(|s|^m r;\Omega(\omega^*,\alpha_0);f(z))}{r^{\rho+1}}dr = \infty.$$

for some positive constant C. Thus by Lemma 1

$$\int^{\infty} \frac{T(r;\Omega(\omega_0,\alpha/2);f(z))}{r^{\rho+1}}dr = \infty.$$

By means of Tsuji's result (3.1), for any distinct three values $b_i \in \mathbb{C} \cup \{\infty\}, 1 \leq i \leq 3$,

$$\sum_{i=1}^{3} \int^{\infty} \frac{\overline{N}(2r,b_i;\Omega(\omega_0,\alpha);f(z))}{r^{\rho+1}}dr = \infty,$$

which implies our assertion.

Acknowledgement. We thank the referee for careful reading of the manuscript and for valuable suggestions.

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