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On Nevanlinna theory for holomorphic curves in Abelian varieties

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Abstract.

We give some observations and results on Nevanlinna theory for holomorphic curves in algebraic varieties.

§1. Intersection theory and Nevanlinna theory

In this note, we consider Nevanlinna theory as non-compact, transcendental intersection theory. First we begin with an algebraic intersection theory. Let X be a smooth, projective algebraic variety and let $D \subset X$ be an effective reduced divisor. Let C be a smooth, projective curve and let S be a finite set of points on C, which will be fixed for the following discussion. Let $f : C \to X$ be an algebraic map such that $f(C) \not\subset \operatorname{supp} D$. Then we have

$$\sum_{x \in C \setminus S} \operatorname{ord}_x f^*D + \sum_{x \in S} \operatorname{ord}_x f^*D = \int_C f^*(c_1(D)). \tag{1}$$

The left hand side of (1) is a sum of local intersection numbers between $f(C)$ and D, while the right hand side is a cohomological invariant which only depend on f and $\mathcal{O}(D)$.

There is a kind of intersection theory for a holomorphic map $f : \mathbb{C} \to X$ which may be transcendental. This is called Nevanlinna's First Main Theorem. We want to count a intersection number between $f(\mathbb{C})$ and D. Since this number is infinite in general, we use an exhaustion $\mathbb{C} = \cup_{r>0}\{z \in \mathbb{C}; |z| < r\}$. We define the counting function as

$$N(r, f, D) = \int_1^r \left(\sum_{|z|<t} \operatorname{ord}_z f^*D \right) \frac{dt}{t}.$$

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As in the first term of the left hand side of (1), the counting function counts intersection numbers just on the non-compact part $\mathbb{C}$. Hence we need to count intersection number on the boundary of $\mathbb{C}$. This is the following proximity function which corresponds to the second term of the left hand side of (1). Let $L(D)$ be the associated line bundle for D. Let $||\cdot||$ be a Hermitian metric of $L(D)$ and let s_D be a section of $L(D)$ such that D is the zero divisor for s_D. Then we define the proximity function of D by

$$m(r, f, D) = \int_0^{2\pi} \log \frac{1}{||s_D \circ f(re^{i\theta})||} \frac{d\theta}{2\pi}.$$

We define an analogue of degree of f with respect to a line bundle L on X as

$$T(r, f, L) = \int_1^r \frac{dt}{t} \int_{\mathbb{C}(t)} f^* c_1(L) + O(1) \quad (r \to \infty),$$

which is called the order function. We define the height function of D by $T(r, f, D) = T(r, f, L(D)) + O(1)$. Then the First Main Theorem in Nevanlinna theory is

$$N(r, f, D) + m(r, f, D) = T(r, f, D) + O(1), \tag{2}$$

which is an analogue of (1). Here the left hand side depends on external geometry of $f(\mathbb{C})$ and D in X, while the right hand side only depend on $f(\mathbb{C})$ and a cohomology class of D.

§2. Conjectures

Our Problem is the following;

What happen if we don't count intersection multiplicity in (1) or (2) ?

Of course, we can't obtain an equality any more, but we hope that there is some inequality. We motivate this estimate by the following heuristic and optimal observation for an algebraic map $f : C \to X$. Let $\mathcal{M}_f$ be the connected component of the moduli space of f.
(i) For a generic $f_0 \in \mathcal{M}_f$, we have $\deg f_0^* D = \deg(f_0^* D)_{\mathrm{red}}$. This is because $f_0(C)$ and D would intersect transversely.
(ii) For an integer $k \geq 0$, put

$$\mathcal{M}_f^k = \{f \in \mathcal{M}_f; \deg f^* D - \deg(f^* D)_{\mathrm{red}} \geq k\}.$$

Then $\mathcal{M}_f^k$ is a Zariski closed subset of $\mathcal{M}_f$ and form a sequence

$$\mathcal{M}_f = \mathcal{M}_f^0 \supset \mathcal{M}_f^1 \supset \mathcal{M}_f^2 \supset \cdots .$$

(iii) We hope that $\operatorname{codim}(\mathcal{M}_f^{k+1}, \mathcal{M}_f^k) \geq 1$ in general.
(iv) Hence for $k = \dim \mathcal{M}_f + \epsilon$, we have "$\mathcal{M}_f^k = \emptyset$".
(v) We hope that $\dim \mathcal{M}_f = -\deg f^* K_X + \epsilon$ for the canonical line bundle K_X.
(vi) We have $\deg(f^*D)_{\mathrm{red}} = \deg_{C \setminus S}(f^*D)_{\mathrm{red}} + O(1)$ where $O(1)$ is a bounded term independent to f. This is because

$$\#S < \infty. \tag{3}$$

Hence we hope that the following conjecture is true (cf. [7]).

Conjecture 1. Let L be an ample line bundle on X and let $\epsilon > 0$. Then there exists a proper Zariski closed subset $\Lambda = \Lambda(X, D, L, \epsilon) \subsetneqq X$ such that

$$\deg f^* K_X(D) \leq \deg_{C \setminus S}(f^*D)_{\mathrm{red}} + \epsilon \deg f^* L + O_\epsilon(1)$$

for all algebraic map $f : C \to X$ with $f(C) \not\subset \Lambda$. Here $O_\epsilon(1)$ is a bounded term independent to f but dependent on ϵ and L.

For a closed subvariety Z of X with $\operatorname{codim}(Z, X) \geq 2$, we put

$$\mathcal{M}_f^k = \{f \in \mathcal{M}_f ; \deg f^* Z \geq k\},$$

and the same observation makes us to hope

Conjecture 2. There exists a proper Zariski closed subset $\Xi = \Xi(X, Z, L, \epsilon) \subsetneqq X$ such that

$$\deg f^* Z \leq -\deg f^* K_X + \epsilon \deg f^* L + O_\epsilon(1)$$

for all algebraic map $f : C \to X$ with $f(C) \not\subset \Xi$.

There are counterparts in Nevanlinna theory for the above conjectures (cf. [1]). Define the truncated counting function by

$$N^{(1)}(r, f, D) = \int_1^r \left(\sum_{|z|<t} \min(\operatorname{ord}_z f^* D, 1) \right) \frac{dt}{t}.$$

Conjecture 3. There exists a proper Zariski closed subset $\Lambda = \Lambda(X, D, L, \epsilon) \subsetneqq X$ such that

$$T(r, f, K_X(D)) \leq N^{(1)}(r, f, D) + \epsilon T(r, f, L) \ ||$$

for all holomorphic map $f : \mathbb{C} \to X$ with $f(\mathbb{C}) \not\subset \Lambda$.

Conjecture 4. There exists a proper Zariski closed subset $\Xi = \Xi(X, Z, L, \epsilon) \subsetneqq X$ such that

$$N(r, f, Z) \leq -T(r, f, K_X) + \epsilon T(r, f, L) \ ||$$

for all holomorphic map $f : \mathbb{C} \to X$ with $f(\mathbb{C}) \not\subset \Xi$.

Here the symbol $||$ means that the inequality holds for $r > 0$ outside a set of finite linear measure. In the above, conjectures 3 and 4 correspond to those of 1 and 2, respectively.

Remark. (1) The counterpart for inequality (3) in Nevanlinna theory is Nevanlinna's lemma on logarithmic derivatives for a meromorphic function φ, i.e., $m(r, \varphi'/\varphi, \infty) < O(\log(rT(r, \varphi, \infty))) \ ||$. To see this, we note that

$$\#S < \infty \iff \sum_{x \in S} \mathrm{ord}_x(\partial\varphi/\varphi)^*(\infty) < O(1) \quad \text{for all } \varphi \in \mathbb{C}(C),$$

where ∂ is a vector field on C and $O(1)$ is a constant independent of φ.

(2) To be precise, we need the condition that D is simple normal crossing in the above conjectures (cf. [6]).

§3. The case for curves

When $\dim X = 1$, we have the natural morphism between logarithmic 1-forms $f^*\Omega^1_X(\log D) \to \Omega^1_C(\log(f^*D)_{\mathrm{red}})$ for algebraic map $f : C \to X$. Hence by taking degrees and using (3), we obtain Conjecture 1 in this case. For the holomorphic case $f : \mathbb{C} \to X$, the following result is classical (R. Nevanlinna, L. Ahlfors).

Theorem 1. *Suppose* $\dim X = 1$. *Then Conjecture 3 is true.*

Suppose $g(X) \geq 2$. Since we have $N^{(1)}(r, f, D) \leq T(r, f, D)$, Theorem 1 implies the inequality $T(r, f, K_X) \leq O(1)||$. But since K_X is ample, this inequality implies that f is constant. Hence we have

Corollary 1. *Suppose* $g(X) \geq 2$. *Then all holomorphic map* $f : \mathbb{C} \to X$ *is a constant map.*

The higher dimensional version of this corollary is the following conjecture (cf. [1]).

Conjecture 5. Let X be a projective variety of general type. Then there exists a proper Zariski closed subset $Y \subsetneq X$ such that the image of all non-constant holomorphic map $f : \mathbb{C} \to X$ is contained in Y.

A remarkable fact is that Theorem 1 for $X = \mathbb{P}^1$ implies Corollary 1. Suppose $g(X) \geq 2$ and let $\pi : X \to \mathbb{P}^1$ be a ramified covering. Let $E' \subset X$ be the ramification divisor of π and put $D = \operatorname{supp} \pi_*(E')$, $E = \operatorname{supp} \pi^* D$. Then we have an equality $\pi^* K_{\mathbb{P}^1}(D) = K_X(E)$. Hence for a holomorphic map $f : \mathbb{C} \to X$, we apply Theorem 1 to $\pi \circ f$ and we have

$$\begin{aligned} T(r, f, K_X(E)) &= T(r, f, \pi^* K_{\mathbb{P}^1}(D)) = T(r, \pi \circ f, K_{\mathbb{P}^1}(D)) \\ &\leq N^{(1)}(r, \pi \circ f, D) = N^{(1)}(r, f, E) \leq T(r, f, E) \end{aligned}$$

modulo small term $\epsilon T(r, f, L)\,||$. Hence $T(r, f, K_X) \leq \epsilon T(r, f, L)\,||$ for all $\epsilon > 0$, which implies Corollary 1.

Remark. This argument is quite general. And it also works in the higher dimensional case: Conjecture 3 for X implies Conjecture 5 for X' which is a ramified covering of X.

§4. The case of Abelian varieties

In the higher dimensional case, the conjectures in section 2 seem to be difficult problem. But when X is an Abelian variety, we have interesting results. (cf. [2],[3],[4],[5],[8],[9])

Theorem 2. *Let X be an Abelian variety. Then Conjectures 3 and 4 are true.*

Remark This theorem holds without any restriction for the singularities of D.

As corollaries to this theorem, we have

Corollary 2. *Let X be a projective variety with irregularity condition* $\dim H^0(X, \Omega_X^1) \geq \dim X$. *Then Conjecture 5 is true for X.*

The case $\dim H^0(X, \Omega_X^1) > \dim X$ is famous Bloch-Ochiai's Theorem and our new part is the case $\dim H^0(X, \Omega_X^1) = \dim X$. To prove this case, we use the albanese map $X \to \operatorname{Alb}(X)$ which is a generically finite map, the argument for the remark in section 3 and the above Theorem 2.

The following Corollary is a unicity theorem for elliptic curves. Though there is a higher dimensional version for general Abelian varieties, we just present an one dimensional case for the sake of simplicity.

Corollary 3. *Let E_1, E_2 be elliptic curves and let $O_i \in E_i$ $(i = 1, 2)$ be the points of identities. Let $f_i : \mathbb{C} \to E_i$ $(i = 1, 2)$ be non-constant holomorphic maps such that* $\mathrm{supp}\, f_1^*(O_1) = \mathrm{supp}\, f_2^*(O_2)$*. Then there exists an isomorphism $\alpha : E_1 \to E_2$ such that $f_2 = \alpha \circ f_1$.*

The idea of the proof of this corollary is the following. Consider the holomorphic map $f_1 \times f_2 : \mathbb{C} \to E_1 \times E_2$ and suppose that the image $f_1 \times f_2(\mathbb{C})$ is Zariski dense in $E_1 \times E_2$. Then since $\mathrm{codim}(O_1 \times O_2, E_1 \times E_2) \geq 2$, Theorem 2 implies that $N^{(1)}(r, f_1 \times f_2, O_1 \times O_2)$ is very small term. On the other hand the assumption $\mathrm{supp}\, f_1^*(O_1) = \mathrm{supp}\, f_2^*(O_2)$ implies that $N^{(1)}(r, f_1 \times f_2, O_1 \times O_2) = N^{(1)}(r, f_1, O_1)$ but this right hand side is a big term. These give a contradiction, hence $f_1 \times f_2(\mathbb{C})$ is not Zariski dense in $E_1 \times E_2$. By Bloch-Ochiai's theorem, $f_1 \times f_2(\mathbb{C})$ is contained in some elliptic curve $F \subset E_1 \times E_2$ and this F gives the graph of α.

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