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Generalization of a precise L^2 division theorem

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§ Introduction

The purpose of this article is to generalize the following.

Theorem 1 (cf. [O-3]). *Let D be a bounded pseudoconvex domain in $\mathbf{C}^n$ and let $z = (z_1, \dots, z_n)$ be the coordinate of $\mathbf{C}^n$. Then there exists a constant C depending only on the diameter of D such that, for any plurisubharmonic function φ on D and for any holomorphic function f on D satisfying*

$$(1) \qquad \int_D |f(z)|^2 e^{-\varphi(z)} |z|^{-2n}\, d\lambda < \infty$$

there exists a vector valued holomorphic function $g = (g_1, \dots, g_n)$ on D satisfying

$$(2) \qquad f(z) = \sum_{i=1}^{n} z_i g_i(z)$$

with

$$(3) \qquad \int_D |g(z)|^2 e^{-\varphi(z)} |z|^{-2n+2}\, d\lambda \le C \int_D |f(z)|^2 e^{-\varphi(z)} |z|^{-2n}\, d\lambda.$$

Here $d\lambda$ denotes the Lebesgue measure.

We generalize this in order to establish an understanding that the measure $e^{-\varphi}|z|^{-2n}\, d\lambda$ in (1) consists of three parts, i.e. $e^{-\varphi(z)}$ for any plurisubharmonic function φ, $|z|^{-2}$ as the quotient fiber metric associated to the morphism $g \mapsto \sum z_i g_i$, and $|z|^{-2n+2}\, d\lambda$ as the residue of a volume form on $(D \setminus \{0\}) \times \mathbf{P}^{n-1}$ with respect to the embedding of $D \setminus \{0\}$ by $z \mapsto (z, [z])$, where $[z] = (z_1 : \cdots : z_n)$.

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In our generalized circumstance there will be given a complex manifold M and a surjective morphism $\gamma : E \to Q$, where E and Q are holomorphic vector bundles over M.

It was first asked by H. Skoda [S-2] to find an L^2 surjectivity condition for the morphism induced from γ. More precisely speaking, by specifying a C^∞ volume form dV_M on M, a C^∞ fiber metric h_E of E and the fiber metric h_Q of Q induced from h_E via γ, a surjectivity criterion was looked for with respect to the induced morphism

$$\gamma_* : A^2(M, E) \longrightarrow A^2(M, Q)$$

where $A^2(M, \cdot\,)$ $(= A^2(M, \cdot\,, dV_M))$ denotes the space of L^2 holomorphic sections and $\gamma_*(g) := \gamma \circ g$.

Here we shall relax the L^2 condition by considering another volume form dV'_M on M and ask for a surjectivity condition for the induced operator

$$\gamma_* : A^2(M, E, dV_M) \longrightarrow A^2(M, Q, dV'_M)$$

where γ_* is only defined as a map from a linear subspace of $A^2(M,E,dV_M)$.

To state our main result, let us introduce some notation.

Let $Q^\vee$, $E^\vee$ denote the duals of Q, E, let $\gamma^\vee : Q^\vee \to E^\vee$ be the dual of γ, and let

$$P(Q^\vee) = \coprod_{x \in M} P(Q_x^\vee), \quad P(E^\vee) = \coprod_{x \in M} P(E_x^\vee),$$

where $P(Q_x^\vee) = \{\mathbf{C}v \mid v \in Q_x^\vee \setminus \{0\}\}$ and $P(E_x^\vee) = \{\mathbf{C}w \mid w \in E_x^\vee \setminus \{0\}\}$. We shall indentify $P(Q^\vee)$ as a complex submainfold of $P(E^\vee)$ via $\gamma^\vee$.

Let us define a line bundle $L(E^\vee)$ over $P(E^\vee)$ by

$$L(E^\vee) = \coprod_{\xi \in P(E^\vee)} L(E^\vee)_\xi$$

where $L(E^\vee)_\xi = \xi$. Then $L(E^\vee)^\vee$ is, as a holomorphic line bundle over $P(E^\vee)$, naturally indentified with

$$\coprod_{x,\xi} E_x / \operatorname{Ker} \xi \quad (x \in M, \xi \in P(E_x^\vee))$$

where $\operatorname{Ker} \xi := \operatorname{Ker} \alpha$ for any $\alpha \in E_x^\vee$ with $\xi = \mathbf{C}\alpha$. The line bundle $(\gamma^\vee)^* L(E^\vee)^\vee$ over $P(Q^\vee)$ will be naturally indentified with

$$\coprod_{x,\xi} Q_x / \operatorname{Ker} \xi \quad (x \in M, \xi \in P(Q_x^\vee))$$

and denoted simply by $L(E^\vee)^\vee|P(Q^\vee)$.

Let $\sigma : P(E^\vee)^\sim \to P(E^\vee)$ be the monoidal transform of $P(E^\vee)$ along $P(Q^\vee)$. For simplicity we put

$$\Sigma = \sigma^{-1}(P(Q^\vee)).$$

Let $p = \operatorname{rank} E$ and $q = \operatorname{rank} Q$. Then the canonical bundles $K_{P(E^\vee)^\sim}$ and $K_{P(E^\vee)}$ are related by a canonical isomorphism

$$K_{P(E^\vee)^\sim} \simeq \sigma^* K_{P(E^\vee)} \otimes [\Sigma]^{p-q-1}.$$

Here Σ denotes the line bundle associated to the divisor Σ. Hence a volume form $dV_{P(E^\vee)^\sim}$ on $P(E^\vee)^\sim$ is induced from dV_M, h_E and a fiber metric of $[\Sigma]$. There is a canonical fiber metric of $[\Sigma]$ induced from h_E, but we shall not stick to it for the sake of generality.

For any Hermitian line bundle L, its curvature form is denoted by Θ_L. For simplicity, the curvature form of the volume form, as a fiber metric of the anticanonical bundle $K_\bullet^\vee$, is denoted by $\mathrm{Ric}_\bullet$.

In this situation, a generalization of Theorem 1 is

Theorem 2. *Suppose that the following are satisfied.*

1. *There exists a closed subset $A \subset M$ such that*

(1.a) *$M \setminus A$ is a Stein mainfold*

and

(1.b) *For any point $x \in A$ and for any neighborhood $U \ni x$, all the L^2 holomorphic finctions on $U \setminus A$ extend holomorphically to U.*

2. *$[\Sigma]$ admits a fiber metric such that*

(2.a) *There exists a bounded canonical section, say s, of $[\Sigma]$.*

(2.b) *There exists a constant R_1 such that $dV_M \le R_1(\varpi\circ\sigma)_* dV_{P(E^\vee)^\sim}$, where ϖ denotes the projection from $P(E^\vee)$ to M.*

(2.c) *There exists a positive number ε_0 such that*

$$\sqrt{-1}(\sigma^*\Theta_{L(E^\vee)^\vee} + \sigma^* \mathrm{Ric}_{P(E^\vee)} - (p-q+\varepsilon)\Theta_{[\Sigma]}) \ge 0 \quad \textit{for all } \varepsilon \in [0, \varepsilon_0].$$

Then the operator $\gamma_* : A^2(M, E, dV_M) \to A^2(M, Q, dV'_M)$ admits a bounded right inverse if there exists a constant R_2 such that

$$R_2 dV'_M \ge (\pi \circ \sigma)_* dV_\Sigma.$$

Here π denotes the projection from $P(Q^\vee)$ to M and dV_Σ denotes the volume form on Σ induced from $dV_{P(E^\vee)^\sim}$ and the fiber metric of $[\Sigma]$.

Corollary 3. *Let D be a pseudoconvex domain in $\mathbf{C}^n$, let $h_1, \ldots, h_p$ be bounded holomorphic functions on D, whose first order derivatives are also bounded, let φ be a plurisubharmonic function on D and let f be a holomorphic function on D satisfying*

$$\|f\|^2 := \int_D |f|^2 e^{-\varphi}|h|^{-2} \bigwedge^n \sqrt{-1}\partial\bar{\partial}(|z|^2 + \log |h|^2) < \infty$$

where $h = (h_1, \ldots, h_p)$. Then there exist holomorphic functions $g_1, \ldots, g_p$ on D such that $f = \sum_{i=1}^p g_i h_i$ and

$$\int_D |g|^2 e^{-\varphi} d\lambda \leq C\|f\|^2.$$

Here C is a constant depending only on h. Moreover, if the Ricci curvature of $\bigwedge^n \sqrt{-1}\partial\bar{\partial}(|z|^2 + \log |h|^2)$ is semipositive, then there exist holomorphic functions $l_1, \ldots, l_p$ on D such that $f = \sum_{i=1}^p l_i h_i$ and

$$\int_D |l|^2 e^{-\varphi} \bigwedge^n \sqrt{-1}\partial\bar{\partial}(|z|^2 + \log |h|^2) \leq C'\|f\|^2$$

where C' is a constant depending only on h.

Obviously the latter part of Corollary 3 contains Theorem 1.

Corollary 4. *Let D, h and φ be as above. Then, for any holomorphic function f on D satisfying*

$$\int_D |f|^2 e^{-\varphi}|h|^{-2k-2}|dh|^{2k} d\lambda$$

where $k = \inf(n, p-1)$, there exist holomorphic functions $g_1, \ldots, g_p$ such that $f = \sum_{i=1}^p g_i h_i$ and

$$\int_D |g|^2 e^{-\varphi} d\lambda \leq C'' \int_D |f|^2 e^{-\varphi}|h|^{-2k-2}|dh|^{2k} d\lambda$$

where C'' is a constant depending only on h.

The paper is organized as follows. In Section 1 we briefly review the L^2 extension theorem for the reader's convenience. Theorem 2 will be proved in Section 2. In Section 3, we shall recall Skoda's L^2 division theorem and its consequence which is weaker than Theorem 1. We dare to do this because we want to show by a counterexample that a naïve improvement of Skoda's theorem, from which Theorem 1 would follow immediately, is false. This may well mean that our formulation of a generalized L^2 division theorem gives a new insight into the division properties of holomorphic functions.

§1. Preliminaries – L^2 extension theorem

Let N be a complex mainfold of dimension m and let $F \to N$ be a holomorphic line bundle with a C^∞ fiber metric h_F. (The symbols M, n, E, h_E are reserved for the division theory.)

Let $S \subset N$ be a closed complex submainfold of codimension one, and let $[S]$ be the holomorphic line bundle defined by a system of transition functions $e_{\alpha\beta} = s_\alpha / s_\beta$, where s_α are local defining functions of S associated to some open covering of N. Any holomorphic section s of $[S]$ is called a canonical section if $S = s^{-1}(0)$ and $ds|S$ is nowhere zero. Once for all we fix a C^∞ fiber metric b of $[S]$ and a canonical section $s = \{s_\alpha\}$ with $s_\alpha = e_{\alpha\beta} s_\beta$.

Given any C^∞ volume form dV_N on N, a volume form $dV_{N,b}$ on S is induced from dV_N, s and b via the canonical isomorphism

$$(K_M \otimes [S])|S \simeq K_S$$

which is given by

$$\frac{\omega \wedge ds_\alpha}{s_\alpha} \longmapsto \omega|S.$$

One may write on S

$$dV_{N,b} = \frac{dV_N}{\sqrt{-1} b_\alpha ds_\alpha \wedge d\overline{s}_\alpha}.$$

Here the fiber metric b is represented by a system of positive C^∞ functions b_α satisfying $b_\alpha = |e_{\beta\alpha}|^2 b_\beta$. More explicitly writing, let x be any point of S and let $(z_1, \ldots, z_n)$ be a holomorphic local coordinate around x such that $z_n = s_\alpha$ for some α around x, and such that

$$dV_N = \sqrt{-1}^n dz_1 \wedge d\overline{z}_1 \wedge \ldots dz_n \wedge d\overline{z}_n$$

holds at x. Then, identifying $(z_1, \ldots z_{n-1})$ with a local coordinate of S around x, we have

$$dV_{N,b} = \sqrt{-1}^{n-1} b_\alpha^{-1} dz_1 \wedge d\overline{z}_1 \wedge \ldots dz_{n-1} \wedge d\overline{z}_{n-1}$$

at x.

Besides the induced volume form $dV_{N,b}$, there is a volume form associated to the function $\log |s|^2$, which turned out to be more natural in the L^2 extension theory. In general, given any continuous function $\psi : N \to \mathbf{R} \cup \{-\infty\}$ such that $\psi - \log |s|^2$ is bounded near every point of S, we define a positive Radon measure $dV_N[\psi]$ on S by

$$\int_S f dV_N[\psi] = \overline{\lim_{t\to\infty}} \frac{1}{\pi} \int_{\psi^{-1}((-t-1,-t))} f e^{-\psi} dV_N.$$

Here f runs through compactly supported nonnegative continuous fuction on N.

However it is easy to see that

$$(\dagger) \qquad dV_N[\log|s|^2] = \frac{dV_N}{\sqrt{-1}b_\alpha ds_\alpha \wedge d\overline{s}_\alpha} = dV_{N,b},$$

whose verification is left to the reader.

Let $A^2(N, F, h_F, dV_N)$ (resp. $A^2(S, F, h_F, dV_N[\log|s|^2])$) be the Hilbert space of L^2 holomorphic sections of F over N (resp. over S) with respect to (h_F, dV_N) (resp. w.r.t. $(h_F, dV_N[\log|s|^2])$).

Theorem 1.1. *Let N, dV_N, F, h_F, S, b and s be as above, and assume that the following are satisfied.*

(1.1) *N contains a Stein open subset N' such that*

(1.1.a) *N' intersects with every connected component of S and*

(1.1.b) *For any point $x \in N \setminus N'$ and for any neighborhood $U \ni x$, all the L^2 holomorphic functions on $U \cap N'$ extend holomorphically to U.*

(1.2) $\sup_N |s| < \infty$.

(1.3) *There exists a positive number ε_0 such that*

$$\sqrt{-1}(\Theta_F + \mathrm{Ric}_N - (1+\varepsilon)\Theta_{[S]}) \geq 0 \quad \text{for all } \varepsilon \in [0, \varepsilon_0].$$

Then there exists a bounded linear operator I from $A^2(S,F,h_F,dV_N[\log[s]^2])$ to $A^2(N,F,h_F,dV_N)$ such that $I(f)|S = f$ for any $f \in A^2(S,F,h_F,dV_N[\log[s]^2])$. Here the norm of I is bounded by a constant dependly only on $\sup_N |s|$ and ε_0.

This result is essentially contained in [O-2, Theorem 4]. Nevertheless we want to prove it here because the curvature assumption (1.3) is somewhat weaker than that of [O-2].

Let us recall first a basic L^2 existence theorem for the $\overline{\partial}$-equation whose proof is contained in [O-2].

Theorem 1.2. *Let (N, g) be a complete Kähler manifold of dimension m, let η be a bounded positive C^∞ function on N and let c be a positive continuous function on $(0, \infty)$ such that $c(\eta)$ is bounded. Let*

(F, h_F) be a Hermitian holomorphic line bundle over N whose curvature form Θ_F satisfies

$$\kappa := \sqrt{-1}(\eta\Theta_F - \partial\overline{\partial}\eta - c(\eta)^{-1}\partial\eta \wedge \overline{\partial}\eta) \geq 0.$$

Then, for any positive integer q and for any $\overline{\partial}$-closed locally square integrable F-valued (m, q) form u on N satisfying $((\kappa\Lambda_g)^{-1}u, u) < \infty$, there exists a square integrable F-valued $(m, q-1)$ form v such that

$$\overline{\partial}(\sqrt{\eta + c(\eta)}v) = u \quad \textit{and} \quad \|v\|^2 \leq ((\kappa\Lambda_g)^{-1}u, u).$$

Here Λ_g denotes the adjoint of $u \mapsto$ (the fundamental form of g) $\wedge u$.

The proof of Theorem 1.2 is a straightforward application of Hahn-Banach's theorem. (We note that the boundedness assumption on η and $c(\eta)$ was missing in [O-2]. See alse [O-1].)

Proof of Theorem 1.1. By (1.1) it suffices to prove that, for any relatively compact Stein open subset $\Omega \subset N$ with C^2 strongly pseudoconvex boundary, there exists a bounded linear operator

$$I_\Omega : A^2(S, F, h_F, dV_N[\log|s|^2]) \longrightarrow A^2(\Omega, F, h_F, dV_N)$$

such that $I_\Omega(f)|S \cap \Omega = f|S \cap \Omega$ for any $f \in A^2(S, F, h_F, dV_N[\log|s|^2])$ and that $\|I_\Omega\|$ is bounded by a constant that depends only on $\sup_N |s|^2$ and ε_0.

Once for all we fix such Ω and f. Then, by extending f to a neighborhood of $\overline{\Omega \cap S}$ as a holomorphic section of F, say $\tilde{f}$, we consider a C^∞ extension of f to $\overline{\Omega}$ of the form

$$\tilde{f}_t = \chi(\log|s|^2 + t + 2)\tilde{f} \quad (t \gg 1)$$

where χ is a C^∞ function $\mathbf{R}$ satisfying $\chi(x) = 1$ for $x < 1$ and $\chi(x) = 0$ for $x > 2$.

By solving the equation $\overline{\partial}v_t = \overline{\partial}\tilde{f}_t/s$ on Ω with an L^2 norm estimate and by taking a weak limit of $\tilde{f}_t - sv_t$ on Ω, we shall obtain a holomorphic extension of f with a required L^2 norm bound.

For that we regard $\overline{\partial}\tilde{f}_t/s$ as a $K_N^\vee \otimes F \otimes [S]^\vee$-valued $(m, 1)$ form on Ω, and apply Theorem 1.2 for any complete Kähler metric on Ω. Note that Ω carries a complete Kähler metric because Ω is Stein (cf. [G]). Multiplying s by a constant if necessary, we may assume that $\sup_N \log|s| < -1$. Then we put $\Psi = \log|s|^2$, $\Phi = \log(|s|^2 + e^{-t})$ and

$$\eta = \frac{1}{\min(\varepsilon_0, 1)} + \log(|s|^2 + e^{-t}) + \log(-\log(|s|^2 + e^{-t})).$$

By a straightforward computation we obtain

$$\partial\overline{\partial}\Phi = e^{-\Phi}|s|^2\partial\overline{\partial}\Psi + e^{-2\Phi-t}|s|^2\partial|s|^2 \wedge \overline{\partial}|s|^2 \quad \text{on } \Omega \setminus S$$

and

$$-\partial\overline{\partial}\eta = \left(1 - \frac{1}{\Phi}\right)^2 \partial\overline{\partial}\Phi + \Phi^{-2}\partial\Phi \wedge \overline{\partial}\Phi.$$

Let us choose t_0 so that $\Phi < -2$ if $t > t_0$. Then, for all $t > t_0$ we have

$$\begin{aligned}
&\sqrt{-1}(\Phi^{-2}\partial\Phi \wedge \overline{\partial}\Phi - \eta^{-3}\partial\eta \wedge \overline{\partial}\eta) \\
&= \sqrt{-1}\left(\Phi^{-2}\partial\Phi \wedge \overline{\partial}\Phi - \frac{1}{(\Phi + \log(-\Phi))^3}\left(1 - \frac{1}{\Phi}\right)^2 \partial\Phi \wedge \overline{\partial}\Phi\right) \\
&\geq \sqrt{-1}(\Phi^{-2} - \Phi^{-3})\partial\Phi \wedge \overline{\partial}\Phi \geq \frac{\sqrt{-1}}{8}\partial\Phi \wedge \overline{\partial}\Phi.
\end{aligned}$$

Therefore if we put

$$\kappa = \sqrt{-1}(\eta\Theta_{F\otimes K_N^\vee\otimes[S]^\vee} - \partial\overline{\partial}\eta - \eta^{-3}\partial\eta \wedge \overline{\partial}\eta)$$

and $\varepsilon_1 = \min(\varepsilon_0, 1)$, on $\Omega \setminus S$ we have

$$\begin{aligned}
\kappa &\geq \frac{1}{\varepsilon_1}\Theta_{F\otimes K_N^\vee\otimes[S]^\vee} + \left(1 - \frac{1}{\Phi}\right)^2 \partial\overline{\partial}\Phi + \frac{\sqrt{-1}}{8}\partial\Phi \wedge \overline{\partial}\Phi \\
&\geq \frac{1}{\varepsilon_1}(\Theta_{F\otimes K_N^\vee\otimes[S]^\vee} + \varepsilon_1 e^{-\Phi}|s|^2\partial\overline{\partial}\Psi) + \frac{\sqrt{-1}}{8}\partial\Phi \wedge \overline{\partial}\Phi \\
&\geq \frac{1}{\varepsilon_1}(\Theta_F + \mathrm{Ric}_N - (1 + \varepsilon_1 e^{-\Phi}|s|^2)\Theta_{[S]}) + \frac{\sqrt{-1}}{8}\partial\Phi \wedge \overline{\partial}\Phi.
\end{aligned}$$

Since $e^{-\Phi}|s|^2 < 1$, the first term in the last inequality is semipositive by assumption. Therefore we obtain

$$\kappa \geq \frac{\sqrt{-1}}{8}\partial\Phi \wedge \overline{\partial}\Phi \quad \text{on } \Omega.$$

Hence, for any Hermitian metric g on Ω we obtain

$$\left((\kappa\Lambda_g)^{-1}\left(\frac{\overline{\partial}\tilde{f}_t}{s}\right), \frac{\overline{\partial}\tilde{f}_t}{s}\right) \leq C_0\|f\|^2, \quad \text{for } t \gg 1.$$

Here the L^2 norm $\|f\|$ of f is with respect to h_F and $dV_N[\log|s|^2]$, the inner product on the left hand side is with respect to h_F, dV_N and g, and C_0 depends only on $\sup|\chi'|$.

Therefore, choosing g to be a complete Kähler metric on Ω, we may apply Theorem 1.2 and obtain a square integrable $F \otimes K_N^\vee \otimes [S]^\vee$-valued $(m, 0)$ form w satisfying

$$\overline{\partial}(\sqrt{\eta + \eta^3} w) = u$$

and

$$\|w\|^2 \leq C_0 \|f\|^2.$$

Clearly $\sup_N |s\sqrt{\eta + \eta^3}| \leq C_1$, where C_1 depends only in $\sup_N |s|$ and ε_0.

Therefore $\sqrt{\eta + \eta^3} w$ $(= \sqrt{\eta_t + \eta_t^3} w_t)$ is a wanted solution to the $\overline{\partial}$-equation $\overline{\partial} v_t = \overline{\partial} \tilde{f}_t / s$.

§2. Proof of Theorem 2

Let the notation be as in Theorem 2 and let ϖ be the projection from $P(E^\vee)$ to M. Then we have a canonical commutative diagram

$$\begin{array}{ccccc} L(E^\vee)^\vee & \longleftarrow & \varpi^* E & \longrightarrow & E \\ & \searrow & \downarrow & & \downarrow \\ & & P(E^\vee) & \longrightarrow & M \end{array}$$

to which an isomorphism

$$\begin{aligned} A^2(M, E, dV_M) \xrightarrow{\sim} & A^2(P(E^\vee), L(E^\vee)^\vee) \\ & (= A^2(P(E^\vee), L(E^\vee)^\vee, \varpi^* dV_M \wedge dV_{FS})) \end{aligned}$$

is associated, which is an isometry up to multiplication by the volume of $\mathbf{P}^{p-1}$. Here dV_{FS} denotes the Fubini-Study volume form on the fibers of $P(E^\vee)$. Identifying $L(E^\vee)^\vee | P(E^\vee)$ with $L(Q^\vee)^\vee$ as in the introduction we have a commutative diagram

$$\begin{array}{ccc} A^2(M, E, dV_M) & \xrightarrow{\sim} & A^2(P(E^\vee), L(E^\vee)^\vee) \\ \downarrow \gamma_* & & \downarrow \rho \\ A^2(M, Q, dV'_M) & \xrightarrow{\sim} & A^2(P(Q^\vee), L(Q^\vee)^\vee) \end{array}$$

where ρ denotes the natural restriction operator.

Now suppose that (1.a)–(2.c) and $R_2 dV'_M \geq (\pi \circ \sigma)_* (dV_\Sigma / |ds|^2)$ are satisfied. Then, to prove the existence of the right inverse of γ_*, it suffices to prove that the restriction operator

$$\tilde{\rho} : A^2(P(E^\vee)^\sim, \sigma^* L(E^\vee)^\vee) \longrightarrow A^2(\Sigma, \sigma^* L(E^\vee)^\vee, dV_\Sigma / |ds|^2)$$

admits a bounded right inverse. For that we shall verify the conditions (1.1)–(1.3) of Theorem 1.1 for $N = P(E^\vee)^\sim$ and $S = \Sigma$.

(1.1): Since $M \setminus A$ is Stein and $\varpi^{-1}(M \setminus A)$ is a $\mathbf{P}^{p-1}$-bundle over $M \setminus A$, $\varpi^{-1}(M \setminus A)$ admits a positive line bundle, and therefore so is $\sigma^{-1}(\varpi^{-1}(M\setminus A))$, too. Hence $\sigma^{-1}(\varpi^{-1}(M\setminus A))$ contains as ample effective divisor Z which intersects with every component of Σ transversally. One may then put $N' = Z^c$.

(1.2) follows from (2.a). (1.3) follows from (2.c) because $\mathrm{Ric}_{P(E^\vee)^\sim} = \sigma^* \mathrm{Ric}_{P(E^\vee)} - (p-q-1)\Theta_{[\Sigma]}$ by the definition of the volume form $dV_{P(E^\vee)^\sim}$.

Hence, by Theorem 1.1, the restriction operator from $A^2(P(E^\vee)^\sim, \sigma^* L(E^\vee)^\vee)$ to $A^2(\Sigma, \sigma^* L(E^\vee)^\vee, dV_{P(E^\vee)^\sim}[\log|s|^2])$ admits a bounded right inverse. This completes the proof of Theorem 2 because $dV_{P(E^\vee)^\sim}[\log|s|^2] = dV_\Sigma$ by (†). □

To deduce Corollary 3 from Theorem 2, we put $M = D \setminus h^{-1}(0)$, $E = M \times \mathbf{C}^p$, $Q = M \times \mathbf{C}$ and $\gamma(z,\zeta) = \sum \zeta_i h_i(z)$. Then we may put $A = h_i^{-1}(0)$ for any nonzero h_i. As for the fiber metric of $[\Sigma]$, we may take $|\zeta|^{-2} \sum_{i\neq j} |\zeta_i h_j - \zeta_j h_i|^2$ as the squared length of the canonical section $s = \left\{h_j \frac{\zeta_i}{\zeta_j} - h_i\right\}_{i\neq j}$ where the local expression $h_j \frac{\zeta_i}{\zeta_j} - h_i$ is effective on the completement of the proper transform of the set $\{h_j\zeta_i - h_i\zeta_j = 0\}$ in $\{\zeta_j \neq 0\}$. Clearly $|s|$ is bounded on M, so what remains is to verify (2.c) and the estimates for the volume forms.

For that we notice that

$$dV_\Sigma = \frac{|\zeta|^2 dV_{P(E^\vee)^\sim}}{\sqrt{-1}\left(\sum_{i\neq j} |\zeta_i h_j - \zeta_j h_i|^2\right) d\left(h_l - \frac{\zeta_l}{\zeta_k} h_k\right) \wedge d\left(\overline{h}_l - \frac{\overline{\zeta}_l}{\overline{\zeta}_k}\overline{h}_k\right)}$$

where

$$dV_{P(E^\vee)^\sim} = \frac{|\zeta|^{2p-4}}{\left(\sum_{i\neq j} |\zeta_i h_j - \zeta_j h_i|^2\right)^{p-2}} \bigwedge^{n+p-1} \sigma^*(\sqrt{-1}\partial\overline{\partial}(|z|^2 + \log|\zeta|^2)).$$

From this expression of $dV_{P(E^\vee)^\sim}$ it is easy to see that the curvature condition (2.c) holds true.

To see that the required estimates for $dV_{P(E^\vee)^\sim}$ and dV_Σ hold, we consider an embedding

$$\begin{array}{ccc} D \times \mathbf{P}^{p-1} & \hookrightarrow & D \times \mathbf{C}^p \times \mathbf{P}^{p-1} \\ \Cup & & \Cup \\ (z,\zeta) & \longmapsto & (z, h(z), \zeta) \end{array}$$

and the associated commutative diagram between the blow ups

$$\begin{array}{cccc} \iota: & (D\times \mathbf{P}^{p-1})^{\sim} & \hookrightarrow & D\times(\mathbf{C}^p\times\mathbf{P}^{p-1})^{\sim} \\ & \big\downarrow \sigma_1 & & \big\downarrow \sigma_2 \\ & D\times\mathbf{P}^{p-1} & \hookrightarrow & D\times\mathbf{C}^p\times\mathbf{P}^{p-1}. \end{array}$$

Since $\sup_D |dh| < \infty$ by assumption, there exists a constant C such that

$$(*)\quad C^{-1}dV_{P(E^\vee)^\sim} < \iota^*\left\{\frac{|\zeta|^{2p-4}}{\left(\sum_{i\neq j}|\zeta_i w_j-\zeta_j w_i|^2\right)^{p-2}}\bigwedge^{n+p-1}\sigma_2^*(\sqrt{-1}\partial\overline{\partial}(|z|^2+|w|^2+\log|\zeta|^2))\right\} < CdV_{P(E^\vee)^\sim}$$

where w denotes the coordinate of $\mathbf{C}^p$.

In particular, $dV_{P(E^\vee)^\sim}$ dominates the pull back of a bounded $(n+p-1,n+p-1)$ form on $D\times(\mathbf{C}^p\times\mathbf{P}^{p-1})^\sim$, so that

$$\text{const.}(\varpi\circ\sigma)_*dV_{P(E^\vee)^\sim} \geq \bigwedge^n \sqrt{-1}\partial\overline{\partial}|z|^2.$$

$(*)$ also shows that dV_Σ is quasi-equivalent to the pull back of $\bigwedge^{n+p-2}\omega$ for some smooth positive $(1,1)$ form, say ω, on the exceptional set of σ_2.

Clearly

$$\sigma_{2*}\omega \leq \text{const.}\sqrt{-1}\partial\overline{\partial}(|z|^2+|w|^2+\log|\zeta|^2)$$

in the sense of current, so that

$$\begin{aligned}(\varpi\circ\sigma)_*dV_\Sigma &\leq \text{const.}\bigwedge^n\sqrt{-1}\partial\overline{\partial}(|z|^2+|h(z)|^2+\log|h(z)|^2)\\ &\leq \text{const.}\bigwedge^n\sqrt{-1}\partial\overline{\partial}(|z|^2+\log|h(z)|^2).\end{aligned}$$

The first part of Corollary 3 follows from this by regarding $e^{-\varphi}$ as an increasing limit of smooth fiber metrices of E whose curvature forms are semipositive. To obtain the latter part we have only to set $dV_M = \bigwedge^n\sqrt{-1}\partial\overline{\partial}(|z|^2+\log|h|^2)$. □

Corollary 4 follows imediately from Corollary 3.

§3. A note on Skoda's division theorem

It might be worthwhile to compare our results with the following which are due to Skoda [S-2] (see also [D]).

Theorem 3.1. *Let M be a complex manifold of dimension n admitting a Kähler metric and a plurisubharmonic exhaustion function of class C^2, let E be a holomorphic Hermitian vector bundle of rank p over M whose curvature form is semipositive in the sense of Griffiths, and let $\gamma : E \to Q$ be a surjective morphism to a holomorphic vector bundle Q of rank q. Then, for any holomorphic Hermitian line bundle L whose curvature form satisfies*

$$(S) \qquad \sqrt{-1}(\Theta_L - \Theta_{\det E} - k\Theta_{\det Q}) \geq 0$$

for some $k > \inf(n, p-q)$, the induced linear map

$$\gamma_* : A^2(M, E \otimes K_M \otimes L) \longrightarrow A^2(M, Q \otimes K_M \otimes L)$$

is surjective.

Corollary 3.2. *Let D be a pseudoconvex domain in $\mathbf{C}^n$, let $h_1,\ldots,h_p$ be holomorphic functions on D, and let $k = \inf(n, p-1)$. Then, for any positive number ε, there exists a constant C_ε such that, for any plurisubharmonic function φ on D and for any holomorphic function f on D satisfying*

$$\int_D |f|^2 e^{-\varphi} |h|^{-2k-2-\varepsilon} d\lambda < \infty$$

there exist holomorphic functions $g_1, \ldots g_p$ such that $f = \sum_{i=1}^p g_i h_i$ and

$$\int_D |g|^2 e^{-\varphi} |h|^{-2k-\varepsilon} d\lambda \leq C_\varepsilon \int_D |f|^2 e^{-\varphi} |h|^{-2k-2-\varepsilon} d\lambda.$$

There are two points to be noted here. One point is that Corollary 3.2 is not contained in Corollary 3 because we had to assume the boundedness of h and its first derivative. The other point is that one cannot drop the above ε by weaking the inequality $k > \inf(n, p-q)$ in the hypothesis to $k \geq \inf(n, p-q)$, as the following counterexample shows.

Let $\mathcal{O}(k)$ denote the holomorphic line bundle of degree k over $\mathbf{P}^1$ ($\mathcal{O} := \mathcal{O}(0)$).

Define a morphism $\iota : \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}(1)$ by $\iota(z, \zeta) = (z, (z\zeta, (z+1)\zeta))$, and let $0 \to \mathcal{O} \xrightarrow{\iota} \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathcal{O}(2) \to 0$ be the associated exact sequence. Tensoring $\mathcal{O}(-1)$ to this we have

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \oplus \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

Letting $M = \mathbf{P}^1$, $E = \mathcal{O} \oplus \mathcal{O}$, $Q = \mathcal{O}(1)$, $L = \mathcal{O}(1)$ and $k = \inf(n, p - q) = 1$, we have

$$\deg L = \deg(\det E) - k \deg(\det Q) = 1 - 0 - 1 = 0.$$

Hence (S) is satisfied, but

$$A^2(M, K_M \otimes E \otimes L) = H^0(\mathbf{P}^1, \mathcal{O}(-1) \oplus \mathcal{O}(-1)) = \{0\}$$

and

$$A^2(M, K_M \otimes Q \otimes L) = H^0(\mathbf{P}^1, \mathcal{O}) \neq \{0\}.$$

Therefore γ_* is not surjective!

Open Question. *Establish a general L^2 division theory that unifies Theorem 2 and Theorem 3.1.*

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