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Analytic polyhedra with non-compact automorphism group

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Abstract.

The main theme of this article concerns the characterization problem of analytic polyhedra in $\mathbf{C}^n$ with non-compact automorphism group. In particular, we include a proof that every bounded convex analytic polyhedron in $\mathbf{C}^n$ is biholomorphic to the product of a Kobayashi hyperbolic convex cone and a bounded convex domain. Several related recent developments are also introduced.

§1. Introduction

The study of the automorphism groups of domains in $\mathbf{C}^n$ is one of the traditional themes in the research of analytic functions in several complex variables. By an automorphism we mean a biholomorphic self-mapping of the given domain. They form naturally a topological group, endowed with the law of composition and the compact-open topology.

This paper concerns the important special collection of domains that are called the analytic polyhedra. An *analytic polyhedron* is a bounded domain Ω in $\mathbf{C}^n$ which admits holomorphic functions $f_1, \dots, f_N$ defined on an open neighborhood U of the closure of Ω such that Ω is defined by the set of inequalities

$$|f_1(z)| < 1, \dots, |f_N(z)| < 1.$$

The main interest of this article is in the characterization problem of analytic polyhedra which possess non-compact automorphism groups. Notice that this line of research is resonant with the widely known theorems of Wong [13], Rosay [12], Bedford and Pinchuk [1], Greene and Krantz [6], Kim [7], Fu and Wong [5] and others. Here, we present an account of recent developments on the characterization problem of analytic polyhedra with non-compact automorphism groups.

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§2. The Case of Convex Polyhedral Domains

Note that the boundary of an analytic polyhedron is Levi flat wherever the boundary is smooth. Thus, the class of analytic polyhedron is a subset of the collection of polyhedral domains defined as follows.

We call a bounded domain D in $\mathbf{C}^n$ *polyhedral*, if it admits the real valued smooth functions $\rho_1, \ldots, \rho_N$ defined in an open neighborhood U of the closure of D satisfying:

(1) D is defined by the inequalities $\rho_1(z) < 0, \ldots, \rho_N(z) < 0$.
(2) The boundary of D is defined by the relations $\rho_{i_1} = \ldots = \rho_{i_k} = 0$ for a non-empty collection of indices $\{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}$.
(3) Each surface defined by $\rho_j = 0$ in U is C^∞ smooth Levi flat, for $j = 1, \ldots, k$.

Notice that the analytic polyhedra are polyhedral domains. Even if the choices for the defining system $\rho_1, \ldots, \rho_N$ are not in general unique for a polyhedral domain, they are essentially unique in almost all practical situations.

The typical generic subclass is also commonly considered; we call a polyhedral domain *normal*, if the only singularities in the boundary are produced by a complex normal crossing singularities. Now we introduce the following theorem, followed by a simpler and descriptive proof.

Theorem 2.1 (Kim [7]). *Let D be a convex normal polyhedral domain in $\mathbf{C}^n$. If the automorphism group $Aut(D)$ is non-compact, then D is biholomorphic to the product of the unit open disc and a convex domain in $\mathbf{C}^{n-1}$.*

Corollary 2.2. *A convex normal polyhedral domain in $\mathbf{C}^2$ possesses a non-compact automorphism group if, and only if, it is biholomorphic to the bidisc.*

Proof. We present here a proof of Theorem 2.1, which is simpler and in fact more general in its implication than the one originally presented in [7]. Since the automorphism group is non-compact, we have a sequence $\varphi_j \in \mathrm{Aut}\,(D)$, a point $q \in D$ and a boundary point $p \in \partial D$ such that

$$\lim_{j \to \infty} \varphi_j(q) = p.$$

Now, let $\rho_1, \ldots, \rho_m$ be a minimal set of defining functions for D. Then without loss of generality we may assume that

$$\rho_1(p) = \ldots = \rho_k(p) = 0 \text{ and } \rho_{k+1}(p) < 0, \ldots, \rho_m(p) < 0,$$

and that the gradient vectors $\nabla\rho_1(p), \ldots, \nabla\rho_k(p)$ are linearly independent over $\mathbf{C}$. Thus in particular, we have $1 \leq k \leq n$. Now, consider

$$\Sigma_\ell = \{z \mid \rho_\ell(z) = 0\}$$

for each $\ell = 1, \ldots, m$. This is a Levi flat surface defined in an open neighborhood of $\overline{D}$, and hence is foliated by smooth complex analytic varieties of complex dimension $n-1$. But then, due to convexity, the analytic varieties contained in Σ_ℓ are in fact a linear subvariety. (Convexity and the maximum principle imply that the variety, say $V \subset \Sigma_\ell$ is contained in the real affine linear subspace, say $\tilde{V}$ of $\mathbf{C}^n$ of real codimension one. Then, being a complex subvariety of $\tilde{V}$ of real codimension one, V itself is a linear subvariety, linearly biholomorphic to a domain in $\mathbf{C}^{n-1}$.) Now let V_ℓ be the maximal (with respect to the inclusion) varieties though p in Σ_ℓ for each $\ell = 1, \ldots, m$. Then the maximal analytic variety in ∂D passing through p is in fact

$$X = V_1 \cap \cdots \cap V_k.$$

The linear independency condition implies that $\dim_{\mathbf{C}} X = n - k$.

Now consider the sequence $q_j := \varphi_j(q)$, which we shall call *an automorphism orbit* of q, accumulating at p. Then we change coordinates linearly at q_j, by a linear affine biholomorphism $\Psi_j : \mathbf{C}^n \to \mathbf{C}^n$, so that the new coordinate system $\zeta := \Psi_j(z)$ satisfy:

- $\Psi_j(q_j) = 0$ for each $j = 1, 2, \ldots$.
- $d\Psi_j|_{q_j}(\nabla\rho_\ell(p)) = (0, \ldots, 0, 1, 0, \ldots, 0)$ (the ℓ^{th} component is 1) for $\ell = 1, \ldots, k$.
- $\Psi_j(X) = \{\zeta_1 = \ldots = \zeta_k = 0\} \cap \partial D$.

Then we consider the scaling map $L_j : \mathbf{C}^n \to \mathbf{C}^n$ defined by

$$L_j(\zeta_1, \ldots, \zeta_n) = \left(\frac{\zeta_1}{\lambda_1^{(j)}}, \ldots, \frac{\zeta_k}{\lambda_k^{(j)}}, \zeta_{k+1}, \ldots, \zeta_n\right)$$

where $\lambda_\ell^{(j)}$ is the distance from the origin to $\Psi_j(\Sigma_\ell)$. Then we consider the sequence

$$\omega_j := L_j \circ \Psi_j \circ \varphi_j : D \to \mathbf{C}^n$$

of holomorphic imbedding maps. First notice that

$$\omega_j(D) = L_j \circ \Psi_j(D)$$

since $\varphi_j(D) = D$. Then, the closure of $L_j \circ \Psi_j(X)$ forms a sequence that converges, since it is in fact a sequence of closed convex subsets of $\mathbf{C}^n$. Notice that each member of this sequence is the closure of a convex domain in a complex affine subspace of codimension k, the limit set, say $\check{X}$ is also the closure of the same type. Notice here that Ψ_j converges to a non-degenerate complex affine mapping of $\mathbf{C}^n$. Therefore, the definition of $L_j \circ \Psi_j$ implies that $\check{X}$ has a non-empty $n-k$ complex dimensional interior in $\{\zeta \in \mathbf{C}^n \mid \zeta_1 = \ldots = \zeta_k = 0\}$. We shall denote by $\widehat{X}$ this interior of $\check{X}$.

Finally, we let

$$\begin{aligned}\widehat{D} := \big\{(\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n \mid \\ &\Re\zeta_1 < 1, \ldots, \Re\zeta_k < 1, \text{ and} \\ &\qquad (0, \ldots, 0, \zeta_{k+1}, \ldots, \zeta_n) \in \widehat{X}\big\}.\end{aligned}$$

Notice that $\widehat{D}$ is biholomorphic to $\Delta^k \times \widehat{X}$, where Δ^k denotes the k-dimensional polydisc.

Now, examining the construction so far, one can easily see that for each compact subset $K \subset\subset D$, there exists j_0 such that $\omega_j(K) \subset \widehat{D}$ for every $j > j_0$. Moreover, for any compact subset K' of $\widehat{D}$, one can see that there exists j_1 such that $K' \subset \omega_j(D)$ for every $j > j_1$. Moreover, observe that $\omega_j(q) = 0$ for every j, and that the origin 0 is an interior point of $\widehat{D}$. Altogether, Montel's theorem now implies that both ω_j and ω_j^{-1} form convergent normal families. Then, choosing a subsequence and applying Cartan's generalization of Schwarz's lemma, we can conclude that D is in fact biholomorphic to the domain $\widehat{D}$. This establishes the theorem as claimed. Q.E.D.

Notice that one of the key roles of convexity of the analytic polyhedron in consideration is that the analytic varieties in the boundary are necessarily affine linear subsets of $\mathbf{C}^n$. In fact, it is true that *the normality condition is* not *essential* in the preceding proof. Therefore, with a small modification of the preceding arguments regarding the scaling method part, we arrive at the following slightly more general result.

Theorem 2.3. *Let Ω be a convex analytic polyhedron in $\mathbf{C}^n$. Then, Ω is biholomorphic to the product of a Kobayashi hyperbolic convex cone and a bounded domain if, and only if, the automorphism group $Aut(\Omega)$ is non-compact.*

§3. Recent Developments and Concluding Remarks

In light of preceding arguments, the natural direction to study is obviously on the analytic polyhedra that are not necessarily convex.

In fact, the case of normal analytic polyhedra in complex dimension two admitting a non-compact automorphism group has been analyzed further. We introduce

Theorem 3.1 (Kim-Pagano [10], 2001). *If $\Omega \subset \mathbf{C}^2$ is a normal analytic polyhedron with a non-compact automorphism group, then the holomorphic universal covering space of Ω is biholomorphic to the bidisc.*

While this theorem clarifies the situation without the convexity assumption, one aspect in contrast to consider is that the holomorphic quotients of the bidisc admitting a non-compact automorphism group is usually quite special. It had been conjectured that the deck transformation group acts only on one component of the bidisc resulting that the polyhedron be biholomorphic to the product of the disc and a Riemann surface. This conjecture was well analyzed recently and answered affirmatively by the author in a collaboration with S.G. Krantz and A.F. Spiro.

Theorem 3.2 (Kim/Krantz/Spiro [9]). *Let $\Omega \subset \mathbf{C}^2$ be a normal analytic polyhedron with a non-compact automorphism orbit accumulating at a boundary point $p \in \partial\Omega$. Let V_p denote the maximal analytic variety at p in $\partial\Omega$. Then, Ω is biholomorphic to the product of V_p and the unit open disc in $\mathbf{C}$.*

Since the case of normal analytic polyhedra in $\mathbf{C}^2$ with a noncompact automorphism group has received such a comprehensive result, the direction to progress seems pointing to the general analytic polyhedra without normality assumption.

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