Advanced Studies in Pure Mathematics 42, 2004
Complex Analysis in Several Variables
pp. 85–93

Demailly's 2-jet negativity of certain hyperbolic fibrations

Jawher El Goul

Abstract.

We prove here a weak negativity property on Demailly's 2-jet bundles of hyperbolic (singular) fibrations on hyperbolic curves with some restrictions on the singularities of special fibres.

§1. Introduction

The concept of "k-jet negativity" was introduced by Demailly in [2] as a generalization, to higher jets, of the negativity of holomorphic sectional curvature of Finsler metrics on the tangent bundle. He conjectured that the existence of such a metric of negative curvature (in a weak sense) on a k-jet bundle he constructed should characterize Kobayashi's hyperbolicity for compact manifolds. This notion of negativity, with some appropriate non-degeneracy conditions, implies the hyperbolicity by an Ahlfors-Schwarz type lemma. In our case, we consider this conjecture only for fibrations on a hyperbolic curve with certain conditions on the singularities of special fibres. In fact, our method so far only works up to the 2-jet stage and thus imposes our restrictions on the singularities. The method is carried out as follows. We use some algebro-geometric arguments to obtain sections of the jet tautological bundle. This allows us to construct metrics of negative curvatures with some degeneracy sets. Then we do the same constructions by considering the restriction of bundles on the degeneracy sets of metrics and we continue this process. In this way one obtains a collection of metrics of negative curvatures which we piece-together to get the desired global metric. This is done by Demailly's technique of piecing together plurisubharmonic functions.

The author would like to thank warmly Professor J.P. Demailly for the initiation to this problem and for his constant help and encouragement.

§2. Demailly's k-jet negativity

Let X be a compact complex manifold. For a holomorphic vector bundle E on X, we denote by $P(E)$ the associated projective bundle of lines of E. Recall that a Finsler metric on E is a homogenous continuous function on its total space, smooth outside the zero section. Alternatively we can define a Finsler metric as a hermitian semi-norm on the tautological line bundle $\mathcal{O}_{P(E)}(-1)$ on $P(E)$.

Now we have the following classical theorem in [7]

Theorem 2.1 (Kobayashi 70). *Suppose that T_X admits a Finsler metric of negative holomorphic sectional curvature. Then X is hyperbolic.*

Remark that the ampleness of $T^\star_X$, i.e., the ampleness of $\mathcal{O}_{P(T_X)}(1)$ is equivalent to the existence of a hermitian metric of negative curvature on its dual $\mathcal{O}_{P(T_X)}(-1)$. This implies the hypothesis of Kobayashi's theorem but we actually need negativity only in "some important directions" in this theorem. For this reason, Demailly in [2] introduced the bundle

$$V_1 := ((\pi_1)_\star)^{-1}(\mathcal{O}_{X_1}(-1)) \subset T_{X_1},$$

where $X_1 := P(T_X)$ and $\pi_1 : X_1 \to X$ is the natural projection, and gave the following definition.

Definition 2.2. *We say that X has (or more precisely, can be given a metric of) negative* 1*-jet curvature, if, for some smooth hermitian metric h on $\mathcal{O}_{X_1}(-1)$, there exist $\epsilon > 0$ and a smooth hermitian metric ω such that,*

$$\Theta_h(\mathcal{O}_{X_1}(-1))(\xi) \leq -\epsilon\|\xi\|^2_\omega, \ \forall \xi \in V_1.$$

Remark that if the metric h in the definition above come from an hermitian metric on T_X, then this negativity is equivalent to the negativity of holomorphic sectional curvature of X.

Now, we iterate the construction $(X, T_X) \to (X_1, V_1)$ to

$$(X_1, V_1) \to (X_2 := P(V_1), V_2 := ((\pi_2)_\star)^{-1}(\mathcal{O}_{X_2}(-1)) \subset T_{X_1}),$$

where $\pi_2 : X_2 \to X_1$ is the natural projection and $\mathcal{O}_{X_2}(-1)$ is the tautological line bundle associated to V_1. We obtain a tower

$$X_k \to X_{k-1} \to \ldots \to X_2 \to X_1 \to X,$$

with the important propriety that every holomorphic germ $f : (\mathbb{C}, 0) \to X$ can be lifted to a germ $f_{[k]} : (\mathbb{C}, 0) \to X_k$ with $f'_{[k]}(0) \in V_k$. Such a holomorphic germ of curve is said to be regular if $f'(0) \neq 0$. We define two sets contained in X_k:

- X_k^{reg} := the set of liftings of regular germs of curves which is an open set in X_k.
- $X_k^{\text{sing}} := X_k \setminus X_k^{\text{reg}}$ called the set of singular jets of curves.

If we let $D_j := P(TX_{j-1}/X_{j-2}) \subset X_j$ then it was proved in [2] that $X_k^{\text{sing}} = \cup_{j=2}^{k} \pi_{k,j}^{-1}(D_j)$, where $\pi_{k,j} : X_k \to X_j$ is the projection map.

We can define now the negativity of Demailly's k-jets for $k \geq 2$.

Definition 2.3. *Let h_k be a metric on $\mathcal{O}_{X_k}(-1)$ (possibly singular with L^1_{loc} weight). We say that h_k has negative curvature in the sense of Demailly if there exist $\epsilon > 0$ and ω_k a smooth metric on X_k such that,*

$$\Theta_{h_k}(\mathcal{O}_{X_k}(-1))(\xi) \leq -\epsilon \|\xi\|_{\omega_k^2}, \ \forall \xi \in V_k.$$

Remark that for $k \geq 2$, $\mathcal{O}_{X_k}(1)$ is not relatively ample with respect to $X_k \to X$. Hence we need to allow singularities in the metric h_k in the above definition. This notion of negativity implies Kobayashi's hyperbolicity as stated in the following theorem in [2].

Theorem 2.4 (Demailly 95). *If X has a k-jet metric h_k with negative curvature in the sense of Demailly, then every entire non-constant curve $f : \mathbb{C} \to X$ has an image $f_{[k]}(\mathbb{C}) \subset \Sigma_{h_k}$, where Σ_{h_k} is the degeneracy set of h_k. In particular, if $\Sigma_{h_k} \subset X_k^{\text{sing}}$ (in this case we say that X has nondegenerate negative Demailly's k-jet curvature), then X is Kobayashi hyperbolic.*

Now we have the following conjecture this paper is concerned with.

Conjecture 2.5. *A compact complex manifold X is hyperbolic if and only if X has nondegenerate negative Demailly's k-jet curvature for k sufficiently large.*

Using a hyperbolic surface fibred over a hyperbolic base, J.-P. Demailly showed in [2] that for each $k_0 > 0$ there exists a surface which has not nondegenerate negative Demailly's k_0-jet curvature. Consequently the sought jet metric can not be absolutely bounded.

We will now study Conjecture 2.5 for a fibered surface. In the following X will be a compact complex hyperbolic surface fibred over a hyperbolic base $X \to B$. In other words the genera of all components of fibres and of the base B are at least 2. When the fibres are all smooth, we can easily construct a hermitian metric of negative holomorphic sectional curvature and then X has a nondegenerate negative Demailly's

1-jet curvature. Therefore, in the sequel, we will consider fibrations which have at least one singular fibre and nonisotrivial, i.e., not locally trivial outside singular fibres.

§3. Almost ampleness on k-jets

In this section we prove an algebraic statement for our fibration, which, in the general case of a projective general type manifold, would imply the celebrated Green-Griffiths conjecture on degeneration of entire curves and provides an important step toward the resolution of Conjecture 2.5. We begin with the following definition introduced by S. Lu in [8] in the 1-jet case (the terminology comes from Miyaoka's almost everywhere ampleness in [9]).

Definition 3.1. *Let d be an integer with $1 \leq d \leq \dim X_k$. We say that $T^\star_X$ is almost ample on k-jets in all dimensions $\geq d$, if the restriction $\mathcal{O}_{X_k}(1)_{|Y}$ is big for every subvariety $Y \subset X_k$ such that*

$$\dim Y \geq d \quad \text{and} \quad \dim \pi_k(Y) \geq \inf\{d, \dim X\},$$

where $\pi_k : X_k \to X$ is the projection map.

Remark that if in the definition above we take $d = \dim X_k$, then this means the same as supposing the tautological line bundle $\mathcal{O}_{X_k}(1)$ to be big. Also, for $d = 1$ and $k = 1$, this is equivalent to the ampleness of the cotangent bundle. Now we have the following fact which is an application of Theorem 2.4 (or more precisely of its proof).

Fact 3.2. *Almost ampleness in all dimensions $\geq d = \dim X$ for a manifold X of general type implies the degeneration of entire curves in X.*

This motivates the following conjecture.

Almost ampleness conjecture 3.3. *Let X be a projective manifold of general type with stable tangent bundle. Then there exists k such that the cotangent bundle is almost ample on k-jets in all dimensions $\geq \dim X$.*

We remark that the additional hypothesis of "stable" is necessary in the above conjecture. In fact, exceptional examples like smooth quotients of the bidisk can not have almost ample cotangent bundle on k-jets in dimensions ≥ 2 for any $k > 0$. An important example supporting this conjecture is the class of surfaces of general type with positive indices, i.e., with $\frac{c_1^2 - 2c_2}{3} > 0$. This is due to the work of Y. Miyaoka

[9] cited above. Another support for this almost ampleness conjecture is the following.

Theorem 3.4. *Let $f : X \to B$ be a surface of general type fibred over a hyperbolic curve. Suppose that f is not isotrivial, then $T_X^\star$ is almost ample on 1-jets in all dimensions ≥ 2.*

Proof . — To see that $\mathcal{O}_{X_1}(1)$ is big, we observe that there is a generically injective rational map $\mathcal{O}_{P(T_X^\star)}(1) \to \mathcal{O}_{P(T_B^\star)}(1)$. We have also that $T^\star_{X|F}$, where F is a generic fibre, is ample by a criterion of Gieseker [6]. Then, applying the additivity of Kodaira dimensions of T. Fujita in [5], we are done. In fact, this is a particular case of Sakai's additivity of λ−dimensions [10].

Now, let Y be a surface in X_1 with $\pi_1(Y) = X$. Let $X^{(1)} \subset X_1$ the surface containing the liftings of all the fibres of f. We have to distinguish two cases:

The first is when $Y \neq X^{(1)}$. Then we have a generically injective morphism $(\pi_1 \circ f)^\star : T_B^\star \to \mathcal{O}_{X_1}(1)_{|Y}$ and we conclude the above (applying the additivity of Kodaira dimensions).

It remains to consider the case $Y = X^{(1)}$. Here, $(\pi_1)_\star(\mathcal{O}_{X_1}(1)_{|Y}) = \Omega_{X/B}$, where $\Omega_{X/B}$ is the sheaf of relative differentials with respect to f. We will prove that this sheaf is big. Remark that it suffices to prove this for a semi-stable fibration using the semi-stability reduction theorem. We assume that X is semi-stable. Then, we blow-up the singularities of each fiber so that we have an exceptional curve through each singular point. We use the same notation X for the surface obtained. It suffices to prove that $\Omega_{X/B}$ is big which is equivalent to proving that $\mathcal{O}_{X_1}(1)_{|Y}$ is big with Y is defined as above. Let $\mathcal{O}_{\overline{X}_1}(1)$ be the tautological line bundle associated with the logarithmic tangent bundle along the exceptional curves in X (which form a finite number). Actually, it suffices to prove that $\mathcal{O}_{\overline{X}_1}(1)_{|\overline{X}^{(1)}}$ is big with $\overline{X}^{(1)}$ the associated surface in $\overline{X}_1$. For simplicity, suppose that we have only one singular point. Let $\overline{X}_k$ the k-th logarithmic jet-bundle and $\mathcal{O}_{\overline{X}_k}(1)$ the tautological line bundle on it (see [4] for the definitions). Then we have $\mathcal{O}_{\overline{X}_k}(1)_{|\overline{X}^{(k)}} = \pi_k^\star(\omega_{X/B}) \otimes \mathcal{O}_{\overline{X}_k}(-E_k)$, where E_k is an exceptional curve of the first kind and $\omega_{X/B} = K_X \otimes K_B^{-1}$ the relative dualizing sheaf. By a result in [1], this latter bundle is big. Now we can verify easily that, for large k, the self intersection of the following bundle $\mathcal{O}_{\overline{X}_k}(1) \otimes \mathcal{O}_{\overline{X}_{k-1}}(1) \otimes \ldots \otimes \mathcal{O}_{\overline{X}_1}(1)_{|\overline{X}^{(k)}}$ is positive. This implies that $\mathcal{O}_{\overline{X}_k}(1)_{|\overline{X}^{(k)}}$ is big and then $\mathcal{O}_{\overline{X}_1}(1)_{|\overline{X}^{(1)}}$ is (actually those last bundles have the same sections on $\overline{X}^{(k)}$). □

Theorem 3.5. *Let $f : X \to B$ be a hyperbolic surface fibred over a hyperbolic base. Suppose that f is not isotrivial, then there exists a positive integer k_0 such that $\mathcal{O}_{X_k}(-1)_{|Y}$ has a (singular) metric of negative k-jets curvature for all $Y \subset X_k$ not contained in X_k^{sing} for every $k \geq k_0$.*

Proof . — Let k_0 be the stage where all the liftings of the fibres of f become smooth (this is possible by Proposition 5.11 in [2]). By Theorem 3.4, $T_X^\star$ is almost ample on 1-jets in all dimensions ≥ 2. This implies (see Lemma 7.6 in [2]) that it is also almost ample on k-jets in all dimensions ≥ 2. Then, for $k \geq 2$, we have that $\mathcal{O}_{X_k}(1)_{|Y}$ is big for all $Y \subset X_k$ with $\pi_k(Y) = X$. In addition, using an easy Riemann-Roch calculation, this is also true if $\pi_k(Y)$ is a curve. This implies, using sections, that we have a metric of negative curvature on $\mathcal{O}_{X_k}(-1)_{|Y}$ for all such Y.

Now take $k \geq k_0$. It remains to consider the case when Y is a curve (the case when Y projects to a point to X is treated similarly). If a curve $Y \subset X_k$ is not tangent to V_k except at a finite set of points, we obtain a metric of negative k-jet curvature on $\mathcal{O}_{X_1}(-1)_{|Y}$ just by taking a smooth metric which is equal to the Poincaré metric in the neighbourhood of each of those points. If the curve is a lifting of some fibre, then, as the fibres are hyperbolic and the tangent bundle of such lifting (being smooth) is negative and isomorphic to $\mathcal{O}_{X_1}(-1)_{|Y}$, we are done. The final case is when Y is a lifting of a curve in X which is not a fibre of f. In this case the existence of a nontrivial sheaf morphism shows that the negativity of T_B implies the negativity of $\mathcal{O}_{X_1}(-1)_{|Y}$. □

§4. Application to Demailly's conjecture

Let L be a line bundle on a compact complex manifold and h_0 a fixed smooth metric on it. Consider a singular metric h on L. We write $h = h_0 \exp(-\Phi)$, where Φ is a smooth function outside the singularities of h. Then we obtain the following relation between curvatures

$$\Theta_h(L) = \Theta_{h_0}(L) + i\partial\overline{\partial}\Phi.$$

This relation permits us to reduce the problem of piecing together metrics to piecing together quasi-psh functions (a terminology of J.-P. Demailly which means functions locally a sum of plurisubharmonic functions and smooth functions). Now, we have the following two lemmas needed for piecing together quasi-psh functions which can be easily proved using techniques from [3].

Lemma 4.1. *Let Y and Z be two subvarieties of a compact complex manifold X. Let V be a subbundle of T_X and let ω be a smooth metric on X. Suppose there exist a smooth $(1,1)$-form α on X and a smooth function Φ_Y (resp. Φ_Z) on Y (resp. on Z) such that*

$$\alpha + i\partial\overline{\partial}\Phi_Y \geq \epsilon\omega \text{ on } V \cap T_{Y_{\text{reg}}},$$

and

$$\alpha + i\partial\overline{\partial}\Phi_Z \geq \epsilon\omega \text{ on } V \cap T_{Z_{\text{reg}}}.$$

Then, there exists a smooth function $\Phi_{Y\cup Z}$ on a neighbourhood U of $Y \cup Z$ such that

$$\alpha + i\partial\overline{\partial}\Phi_{Y\cup Z} \geq \frac{\epsilon}{4}\omega \text{ on } V_{|U}.$$

Lemma 4.2. *Let $Y \subset Z$ be two subvarieties of a compact complex manifold X. Let V be a subbundle of T_X and ω a smooth metric on X. Suppose there exist a smooth $(1,1)$-form α on X and a smooth function Φ_Y (resp. $\Phi_{Z\setminus Y}$) on Y (resp. on $Z \setminus Y$ locally bounded from above on Y) such that*

$$\alpha + i\partial\overline{\partial}\Phi_Y \geq \epsilon\omega \text{ on } V \cap T_{Y_{\text{reg}}},$$

and

$$\alpha + i\partial\overline{\partial}\Phi_{Z\setminus Y} \geq \epsilon\omega \text{ on } V \cap T_{Z_{\text{reg}}}.$$

Then, there exists a smooth function Φ_Z on Z such that

$$\alpha + i\partial\overline{\partial}\Phi_Z \geq \frac{\epsilon}{2}\omega \text{ on } V_{|U}.$$

By Theorem 3.5 we obtain a collection of metrics of negative curvatures on X_k for $k \geq k_0$: We start from a metric on X_k of negative k-jet curvature and we consider its base locus which is a finite union of irreducible proper subvarieties. By the same theorem, the restriction of $\mathcal{O}_{X_k}(-1)$ to each of those components (not contained in X_k^{sing}) has a metric with negative k-jet curvature with smaller base locus and so on. In order to obtain a global metric with non degenerate k-jet curvature we should piece together these metrics. For a stable fibration (where singularities of fibres are nodal), we can take $k_0 = 1$ in Theorem 3.5. As X_1^{sing} is empty, we can thus do this piecing together easily using lemmas 4.1 and 4.2 above. We obtain:

Theorem 4.3. *Let $X \to B$ be a stable fibration as in Theorem 3.5. Then X has nondegenerate Demailly's 1-jet negative curvature.*

For $k \geq 2$ the piecing together procedure is complicated because, in this case, $\mathcal{O}_{X_k}(1)$ is not relatively nef with respect to π_2 and X_k^{sing} is nonempty. Nevertheless, using a weaker condition on singularities than stability, we can achieve the construction for the 2-jet stage. In fact, for this stage, we have a good alternative tautological bundle $L_2 := \mathcal{O}_{X_2}(1) \otimes \pi_{2,1}^{\star}(\mathcal{O}_{X_1}(2))$ which is relatively nef with respect to the projection to X by Proposition 6.16 in [2]. We have the following:

Theorem 4.4. *Let $f : X \to B$ be a fibration as in Theorem 3.5. Suppose that L_2 has positive degree on every lifting to X_2 of the singular fibres of f. Then X has nondegenerate Demailly's 2-jet negative curvature.*

Proof . — By theorem 3.4, the restriction $\mathcal{O}_{X_1}(1)_{|Y}$ is big for all $Y \subset X_1$ which projects surjectively onto X. As $X_2^{\text{sing}} = D_2$ is equal to $\mathcal{O}_{X_2}(1) \otimes \mathcal{O}_{X_1}(-1)$, we have $L_2 = \pi_{2,1}^{\star}\mathcal{O}_{X_1}(3) \otimes \mathcal{O}(D_2)$. Consequently, $L_{2|Y}$ is big for all $Y \subset X_2$ not contained in D_2 and which projects surjectively onto X. In particular L_2 is big on X_2. This implies that the line bundle $L_2^{\epsilon} := \mathcal{O}_{X_2}(1) \otimes \pi_{2,1}^{\star}(\mathcal{O}_{X_1}(2+\epsilon))$, which is relatively ample with respect to $X_2 \to X$, is also big for small ϵ.

Let h_0 be a metric of negative curvature on $(L_2^{\epsilon})^{\star}$, and Σ_{h_0} its singular set. Then Σ_{h_0} is a finite union of subvarieties of X_2 of dimensions at most 3. Now, for components Y of Σ_{h_0} not contained in $X_2^{\text{sing}} = D_2$ and which projects onto X, the same argument as above shows that $(L_2^{\epsilon})^{\star}{}_{|Y}$ has also a metric of negative curvature (though perhaps for a smaller ϵ). This is also true when $Y = D_2$ by a Riemann-Roch calculation. This gives a collection of metrics $h_j, j = 1,..,s$ (for some integer s) with a finite union $\cup_{j=1,..,s}\Sigma_j$ of subvarieties of dimensions at most 2 as singular loci.

It remains to study the restrictions of $(L_2^{\epsilon})^{\star}$ to components Y of $\cup_{j=0,..,s}\Sigma_j$ which projects to a curve in X (the case $Y \subset D_2$ projects on X surjectively is treated similarly). If Y is a curve, the hypothesis and a similar argument as in the proof of Theorem 3.5 show that $(L_2^{\epsilon})^{\star}{}_{|Y}$ has a metric of negative 2-jet curvature. Suppose now that Y is not a curve and $\pi_2(Y)$ is a curve C in X. Then the intersection of the tangent sheaf to Y and V_2 consists of the tangent sheaf of curves contained in the fibres and of the lifting of C to two jets. As L_2^{ϵ} is relatively ample and $(L_2^{\epsilon})^{\star}$ has negative 2-jet curvature on C, we obtain, using lemma 4.2, that $(L_2^{\epsilon})^{\star}{}_{|Y}$ has a smooth metric of negative 2-jet curvature.

Finally, using lemmas 4.1 and 4.2, we glue together all the metrics we have now to obtain a smooth metric of negative 2-jet curvature on $(L_2^{\epsilon})^{\star}$. This gives a non degenerate metric of negative 2-jet curvature on $\mathcal{O}_{X_2}(-1)$ by the identity $L_2^{\epsilon} = \pi_{2,1}^{\star}(\mathcal{O}_{X_1}(3+\epsilon)) \otimes \mathcal{O}(D_2)$. □

Remark 4.5. *Suppose we have a sequence $(L_k)_{k\in\mathbf{N}}$ of relatively nef line bundles on X_k such that, for k sufficiently large, L_k has positive degree on the lifting to X_k of each singular fibres of f. Then, using the same proof as for Theorem 4.4, we can prove Demailly's conjecture without additional hypothesis for our fibration.*

References

[1] S.J. Arakelov, Families of algebraic curves with fixed degeneracies, *Eng. Trans. USSR Izv* , 5, 1971, 1277-1302.

[2] J.-P. Demailly, Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials, *Proceedings of Symposia in Pure Maths*, vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp.285–360.

[3] J.-P. Demailly, Cohomology of q-convex spaces in top degrees, *Math. Z.*, 204, 1990, 283–295.

[4] G. Dethloff & S. Lu, Logarithmic jet bundles and applications, *Osaka J. Math.*, 2000.

[5] T. Fujita, Some remarks on Kodaira dimensions of fiber spaces, *Proc. Japan Acad.*, 53Ser. A, 1977, 28–30.

[6] D. Gieseker, P-ample bundles and their Chern classes, *Nagoya Math. J.*, 43, 1971, 91–116.

[7] S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Marcel Dekker, New York, 1970.

[8] S. Lu, On meromorphic maps into varieties of log-general type, *Proceedings of Symposia in Pure Maths*, Amer. Math. Soc, 52(2), 1991, 305–333.

[9] Y. Miyaoka, Algebraic surfaces with positive indices, Katata Sympo. Proc. 1982, *Progress in Mathematics*, 39, 1983, 281–301.

[10] F. Sakai, Symmetric powers of the cotangent bundle and classification of algebraic varieties, *Proc. Copenhagen Meeting in alg. geom.*, 1978.

Laboratoire E. Picard
UFR MIG, Université Paul Sabatier
118, route de Narbonne
31062 Toulouse
France