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The ideal structure of graph algebras

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§1. Introduction

For an $n \times n$ $\{0,1\}$-matrix $A = [A(i,j)]$ without zero rows or columns the corresponding Cuntz-Krieger algebra $\mathcal{O}_A$ is defined in [4] as a C^*-algebra generated by partial isometries $\{s_i \mid i = 1, \dots, n\}$ on a Hilbert space satisfying $s_i^* s_i = \sum_{j=1}^{n} A(i,j) s_j s_j^*$. Almost from the start it was observed [25] that instead of a matrix we can use a directed graph to encode this data. It took a little bit longer though before it was realized that graphical approach may be equally successfully applied to infinite graphs. This extension (cf. [16, 15, 6, 2, 19] and references there) allows us to study by similar tools and within the same framework objects as diverse as classical Cuntz-Krieger algebras $\mathcal{O}_n$, $\mathcal{O}_\infty$, AF-algebras, and many other C^*-algebras.

A variety of methods have been employed in the investigations of graph algebras. The arguments in [16] and several subsequent papers (eg see [17]) rely heavily on the machinery of groupoids. A different approach is based on the realization of graph algebras as Cuntz-Pimsner algebras (cf. [18, 13, 7]) corresponding to suitable Hilbert bimodules over discrete abelian C^*-algebras. However it may well be that the direct approach yields the sharpest results (cf. [2, 19]).

The structure of graph algebras is fairly well-known by now. Indeed, after several earlier partial results a criterion for their simplicity has been found [21] (see also [17]). Their K-theory is readily computable [19, 23]. Their stable rank can be determined from the graph [5]. A number of other questions, like injectivity of their homomorphisms (cf. [24]) or direct sum decomposability (cf. [8]) can now be easily answered. Modelling with graph algebras has been employed in the studies of semiprojectivity (cf. [22, 20]) and pure-infiniteness (cf. [10]).

We begin this article with a brief overview of basic facts about graph algebras, illustrated with a number of examples. Then we move to our

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main point of interest, the discussion of the structure of their ideals. First fundamental results about ideals of Cuntz-Krieger algebras were obtained in [3]. A complete discussion of the primitive spectrum of a Cuntz-Krieger algebra corresponding to a finite $\{0, 1\}$-matrix (and hence a finite graph) was later given in [12]. However the ideal structure of graph algebras corresponding to infinite graphs is much more complicated. Most previously obtained results in this direction dealt with the case of ideals of row-finite graphs (ie such that each vertex emits only finitely many edges) which are invariant under the canonical gauge action of the circle group [16, 15, 2].

Similar results for row-finite graph algebras were obtained in [13] by viewing graph algebras as Cuntz-Pimsner algebras of suitable Hilbert bimodules. Very recently a complete description of ideals of all graph algebras has been obtained [1, 9]. That is, all primitive ideals together with the hull-kernel topology on the primitive spectrum are known. These results cover the most general countable directed graphs, with no restrictive assumptions whatever. In this article we present without proofs the description of gauge invariant ideals and then briefly indicate how other ideals arise. For the complete results, see [1, 9].

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§2. Cuntz-Krieger algebras of directed graphs

2.1. Definitions and examples

A directed graph E is a quadruple (E^0, E^1, r, s) with E^0 the set of vertices, E^1 the set of edges, and $r, s : E^1 \to E^0$ the range and source function, respectively. In what follows we always assume that both E^0 and E^1 are at most countable.

If $n \geq 1$ then a path α of length n in E is a sequence $\alpha = (e_1, \ldots, e_n)$ with $e_i \in E^1$ and $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n-1$. Then $s(\alpha) = s(e_1)$, $r(\alpha) = r(e_n)$, and we say that α is a path from $s(\alpha)$ to $r(\alpha)$. A path α (of length at least 1) is a loop if $r(\alpha) = s(\alpha)$. It is a vertex simple loop if the vertices $s(e_i)$ are distinct. The loop has no exits if $s^{-1}(s(e_i)) = \{e_i\}$ for $i = 1, \ldots, n$. A vertex v is called sink if $s^{-1}(v) = \emptyset$.

The following concept of a Cuntz-Krieger E-family for a given directed graph E was introduced in [15].

Definition 1. *Let E be a directed graph and let B be a C^*-algebra. A Cuntz-Krieger E-family $\{S_e, P_v\}$ inside B consists of a collection of partial isometries $\{S_e \in B \mid e \in E^1\}$ and a collection of projections $\{P_v \in B \mid v \in E^0\}$ such that the following conditions are satisfied.*

(G1) $P_v P_w = 0$ *if* $v \neq w$.
(G2) $S_e^* S_f = 0$ *if* $e \neq f$.
(G3) $S_e^* S_e = P_{r(e)}$.
(G4) $S_e S_e^* \leq P_{s(e)}$.
(G5) $\sum_{e \in E^1, s(e)=v} S_e S_e^* = P_v$, *if v emits finitely many (and at least one) edges.*

The following definition of graph algebras was given in [6].

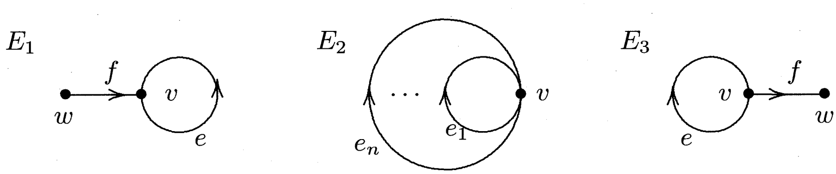
Definition 2. *The C^*-algebra $C^*(E)$ of a directed graph E is a C^*-algebra generated by partial isometries $\{s_e \mid e \in E^1\}$ and projections $\{p_v \mid v \in E^0\}$, which is universal for Cuntz-Krieger E-families. That is, for any Cuntz-Krieger E-family $\{S_e, P_v\}$ inside a C^*-algebra B there exists a unique C^*-algebra homomorphism $\pi_{S,P} : C^*(E) \to B$ such that $\pi_{S,P}(s_e) = S_e$ for all $e \in E^1$ and $\pi_{S,P}(p_v) = P_v$ for all $v \in E^0$.*

Throughout this article we use symbols $\{s_e, p_v\}$ with small s, p for the generators of the C^*-algebra $C^*(E)$. Universality of graph algebras implies that there exists a canonical action γ of the circle group $\mathbf{T}$ on $C^*(E)$, called the gauge action,

$$\gamma : \mathbf{T} \to \mathrm{Aut}(C^*(E))$$

such that $\gamma_t(p_v) = p_v$ and $\gamma_t(s_e) = ts_e$ for all $v \in E^0$, $e \in E^1$, $t \in \mathbf{T}$.

Example 3. Let E_i, i=1,2,3, be the following directed graphs.



We have $C^*(E_1) \cong M_2 \otimes C(\mathbf{T})$ and $C^*(E_3)$ is isomorphic to the Toeplitz algebra generated by a unilateral shift. Also $C^*(E_2) \cong C(\mathbf{T})$ for $n = 1$ and $C^*(E_2) \cong \mathcal{O}_n$ for $n \geq 2$, including $n = \infty$.

2.2. Basic properties of graph algebras

One of the great advantages of working with graph algebras is the ease with which we can read all basic properties of these complicated objects from the underlying graphs. For example, $C^*(E)$ is unital if and only if E^0 is finite. Below we show how to recognize from directed graphs such properties of the corresponding algebras as Cuntz-Krieger uniqueness, simplicity, being AF, pure infiniteness, and K-theory.

Since graph algebras are defined via a universal property, it is not too difficult to construct homomorphisms from these algebras to other C^*-algebras. However, it is usually much more difficult to verify whether such a homomorphism is injective or not. To this end we often use the following gauge invariant uniqueness theorem, which essentially says that the universality in the definition of $C^*(E)$ is equivalent to the existence of the gauge action. This result was proved in [2] for row-finite graphs, and in full generality in [19].

Theorem 4. *Let E be a directed graph, $\{S_e, P_v\}$ be a Cuntz-Krieger E-family, and $\pi_{S,P} : C^*(E) \to C^*(\{S_e, P_v\})$ be a C^*-algebra homomorphism such that $\pi_{S,P}(s_e) = S_e$ and $\pi_{S,P}(p_v) = P_v$ for all $e \in E^1$ and all $v \in E^0$. Suppose that each P_v is non-zero, and that there is a strongly continuous action β of $\mathbf{T}$ on $C^*(\{S_e, P_v\})$ such that $\beta_t \circ \pi_{S,P} = \pi_{S,P} \circ \gamma_t$ for all $t \in \mathbf{T}$. Then $\pi_{S,P}$ is injective.*

The classical Cuntz-Krieger uniqueness has also been generalized to the context of graph algebras. The following result was proved in [16] for row-finite graphs, and in full generality in [6].

Theorem 5. *Let E be a directed graph in which every loop has an exit. Then for all Cuntz-Krieger E-family $\{S_e, P_v\}$ such that each P_v is different from 0, the corresponding C^*-algebra homomorphism $\pi_{S,P} : C^*(E) \to C^*(\{S_e, P_v\})$ (with $\pi_{S,P}(s_e) = S_e$ and $\pi_{S,P}(p_v) = P_v$ for all $e \in E^1$, $v \in E^0$) is an isomorphism.*

An easy consequence of Theorem 5 is the uniqueness of the C^*-algebra, generated by a proper isometry (ie the classical result due to Coburn), corresponding to the graph E_3. A common generalization of Theorems 4 and 5 is given in [24].

A convenient criterion of simplicity of graph algebras is known [21]. It is formulated in terms of hereditary and saturated sets of vertices, which also play a crucial role in our description of ideals in the next section. A subset $H \subseteq E^0$ is called;

(i) *saturated* if any $v \in E^0$, emitting finitely many (and at least one) edges and such that $r(e) \in H$ for all $e \in E^1$ with $s(e) = v$, belongs to H,

(ii) *hereditary* if $r(e) \in H$ for any $e \in E^1$ such that $s(e) \in H$.

Theorem 6. *Let E be a directed graph. Then $C^*(E)$ is simple if and only if the following two conditions are satisfied.*

1. *All loops in E have exits.*
2. *The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.*

The graphs E_2 $(n \geq 2)$ satisfy the conditions of Theorem 6, thus Cuntz algebras $\mathcal{O}_n$ $(n \geq 2)$ and $\mathcal{O}_\infty$ are simple. But the Toeplitz algebra is not, since the graph E_3 has a nontrivial proper hereditary and saturated subset $\{v\}$. Earlier partial results in this direction may be found in [4, 16, 7, 6, 2]. A different (based on the groupoid approach) of an equivalent simplicity criterion has been recently found in [17].

It turns out that all simple graph algebras are either AF or purely infinite (cf. [15, 2, 19]). In fact, $C^*(E)$ is AF if and only if E has no loops, and $C^*(E)$ is purely infinite in the sense of [15] (but not necessarily simple) if and only if all loops in E have exits and every vertex connects to a loop by a directed path (cf. [15, 2, 10]).

The K-theory of graph algebras is readily computable by the Cuntz method. Namely, the crossed product of $C^*(E)$ by the gauge action of the circle group $\mathbf{T}$ is known as an AF-algebra ([14, 19]). Thus $C^*(E)$ is stably isomorphic to a crossed product of an AF-algebra by an action of the integers $\mathbf{Z}$ (dual to the gauge action), which allows us to apply the Pimsner-Voiculescu exact sequence.

The following thoerem was obtained in [19] for row-finite graphs, and then extended to the directed graphs with finitely many edges in [22]. These and several other results about the K-theory of graph or Cuntz-Krieger algebras are all based on the original calculation in [3].

Theorem 7. *Let E be a directed graph and let V denote the collection of all those vertices which emit at least one but at most finitely many edges. Let $\mathbf{Z}V$ and $\mathbf{Z}E^0$ be free abelian groups on free generators V and E^0, respectively. Then*

$$K_0(C^*(E)) \cong \operatorname{coker}(\Delta_E) \quad \textit{and} \quad K_1(C^*(E)) \cong \ker(\Delta_E),$$

where $\Delta_E : \mathbf{Z}V \to \mathbf{Z}E^0$ is the map defined as

$$\Delta_E(w) = \sum_{e \in E^1,\, s(e)=w} r(e) - w.$$

It follows from Theorem 7 that K_1 groups of graph algebras must be free abelian. It turns out that this is the only restriction. Namely, for any pair of countable abelian groups A_0, A_1 with A_1 free abelian, there exists a stable, purely infinite and simple graph algebra $C^*(E)$ such that $K_i(C^*(E)) \cong A_i$ for $i = 0, 1$ [23]. An easy way to check criterion for stability of graph algebras is given in [8].

Note that all graph algebras are separable according to our definition, since we only deal with countable graphs. All of them are also nuclear and satisfy the Universal Coefficient Theorem. Therefore purely infinite and simple algebras $C^*(E)$ serve as convenient models of a large subclass of the classifiable algebras. This fact has been recently exploited in [22] and [20] to show that all Kirchberg algebras with K_0 finitely generated and K_1 finitely generated free abelian are semiprojective.

§3. Ideals of graph algebras

In this section we present the ideal structure of graph algebras. We focus primarily on gauge invariant ideals J of $C^*(E)$ such that $\gamma_t(J) = J$ for all $t \in \mathbf{T}$. We begin by recalling the classification of gauge invariant ideals for algebras of row-finite graphs. Then we discuss the general case of arbitrary graphs, and conclude with a brief indication of how other ideals, ie non gauge invariant ideals, arise.

3.1. Gauge invariant ideals

3.1.1. *Row-finite directed graphs* It turns out that that gauge invariant ideals of algebras of row-finite graphs are in a one-to-one correspondence with hereditary and saturated sets of vertices. For a directed graph E we denote by Σ_E the collection of all hereditary and saturated subsets of E^0. If $X \subseteq E^0$ then We denote by $\Sigma(X)$ the smallest hereditary and saturated subset of E^0 containing X. If J is a closed two-sided ideal of $C^*(E)$ then we define $V_J := \{v \in E^0 \mid p_v \in J\}$. It is easy to see that V_J is hereditary and saturated. For a hereditary and saturated set $K \subseteq E^0$ we define J_K to be the closed two-sided ideal of $C^*(E)$ generated by $\{p_v \mid v \in K\}$.

The following theorem is given in [2]. Its earlier versions, with some additional restrictions on the underlying directed graphs, are in [4, 13, 16].

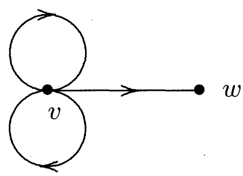
Theorem 8. *If E is a row-finite directed graph then there is a one-to-one correspondence between the collection of closed, two-sided, gauge invariant ideals of $C^*(E)$ and Σ_E, via $J \to V_J$ and $J_K \leftarrow K$.*

The key fact used in the proof of Theorem 8 is that for a gauge invariant ideal J of $C^*(E)$ the quotient $C^*(E)/J$ is again a graph algebra, corresponding to the graph obtained by restriction of E to $E^0 \setminus V_J$.

It is also possible to identify those directed graphs E such that all ideals of $C^*(E)$ are gauge invariant. For Cuntz-Krieger algebras of finite

$\{0,1\}$-matrices, this situation is captured by condition II of Cuntz. Its analogue for row-finite graphs was introduced in [16] by the so-called condition K (an analogue of Cuntz's condition II). Condition K requires that any vertex in E^0 lies on either none or at least two distinct vertex simple loops. If a row-finite graph E satisfies condition K then all ideals of $C^*(E)$ are automatically gauge invariant, and consequently Theorem 8 describes all ideals of $C^*(E)$ in this case.

Example 9. *The graph E below satisfies condition K. Thus Σ_E gives all ideals of $C^*(E)$. Since $\Sigma_E = \{\emptyset, \{w\}, E^0\}$, we have $J_{\{w\}} \cong \mathcal{K}$ (compact operators on a separable Hilbert space) is the only nontrivial ideal of $C^*(E)$. Note that the quotient $C^*(E)/J_{\{w\}}$ is isomorphic to $C^*(E_2)$ (with $n = 2$), which is Cuntz algebra $\mathcal{O}_2$.*



Unfortunately, even for graphs as simple as E_1 or E_3 Theorem 8 does not give the full description of ideals. The reason is that non gauge invariant ideals are present.

3.1.2. *The general case* We now give a brief outline of the results obtained recently by the author in collaboration with the group from the University of Newcastle. Proofs of these results will be published in [1].

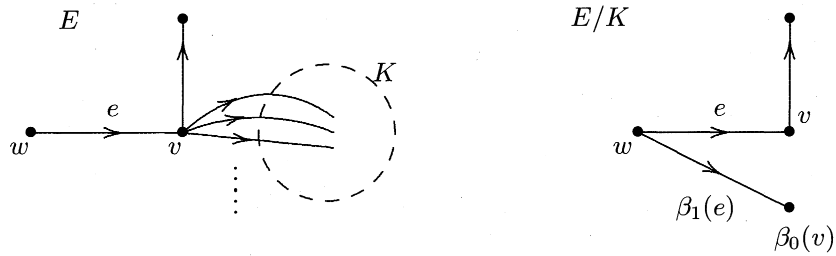
Unlike in the previously discussed much simpler case of row-finite graphs, the collection of hereditary and saturated subsets is not sufficient to describe all gauge inariant ideals in general. In order to do this we must first understand quotients of graph algebras by gauge invariant ideals. We first introduce the notion of the quotient graph.

Let E be an arbitrary directed graph and let $K \subseteq E^0$ be a hereditary and saturated subset. We denote by K_∞^{fin} the collection of all those vertices $v \in E^0 \setminus K$ such that $s^{-1}(v) \cap r^{-1}(K)$ is infinite and $s^{-1}(v) \cap r^{-1}(E^0 \setminus K)$ is finite and non-empty. We then define the quotient graph E/K as follows.

$$\begin{aligned}
(E/K)^0 &= (E^0 \setminus K) \cup \{\beta(v) \mid v \in K_\infty^{\text{fin}}\}, \\
(E/K)^1 &= r^{-1}(E^0 \setminus K) \cup \{\beta(e) \mid e \in E^1,\ r(e) \in K_\infty^{\text{fin}}\},
\end{aligned}$$

with $s(\beta(e)) = s(e)$ and $r(\beta(e)) = \beta(r(e))$. The β is just a symbol helping to distinguish a vertex $v \in E^0$ and an edge $e \in E^1$ from the extra vertex $\beta(v)$ and the extra edge $\beta(e)$ in E/K, respectively. Note that E/K is a subgraph of E if $K_\infty^{\text{fin}} = \emptyset$, and this is always the case when E is row-finite.

Example 10. *Let E be the graph below. It is assumed here that $K \in \Sigma_E$ and v emits infinitely many edges into K. We have $K_\infty^{\text{fin}} = \{v\}$ and the quotient graph E/K looks as follows.*



The following lemma provides a key tool for analyzing gauge invariant ideals of arbitrary graph algebras.

Lemma 11. *Let E be a directed graph and let $K \in \Sigma_E$. Then there is a natural isomorphism*

$$C^*(E)/J_K \cong C^*(E/K).$$

Now let $K \in \Sigma_E$ and $X \subseteq K_\infty^{\text{fin}}$. We define $J_{K,X}$ as the closed two-sided ideal of $C^*(E)$ generated by J_K and $\{p_v - q_v \mid v \in X\}$, where $q_v = \sum_{s(e)=v, r(e)\notin K} s_e s_e^*$ is a subprojection of p_v. As a special case we have $J_{K,\emptyset} = J_K$, which always occurs in row-finite graphs. Clearly the ideal $J_{K,X}$ is gauge invariant. We denote by $\wp(K_\infty^{\text{fin}})$ the collection of all subsets of K_∞^{fin}.

Theorem 12. *If E is an arbitrary directed graph then there is a one-to-one correspondence between $\bigcup_{K\in\Sigma_E}\{K\} \times \wp(K_\infty^{\text{fin}})$ and the collection of all closed, two-sided gauge invariant ideals of $C^*(E)$, given by the map*

$$(K, X) \mapsto J_{K,X}.$$

Theorem 8 is then an immediate consequence of Theorem 12.

Lemma 11 says that quotients of graph algebras by gauge invariant ideals are themselves graph algebras. Thus, in order to describe primitive gauge invarinat ideals it suffices to know what graphs E result in primitive algebras $C^*(E)$, since the quotient of a C^*-algebra by a primitive ideal is a primitive C^*-algebra. We have the following.

Proposition 13. *If E is an arbitrary directed graph then $C^*(E)$ is primitive if and only if E satisfies the following two conditions.*

1. *All loops in E have exits.*
2. *$\Sigma(v) \cap \Sigma(w) \neq \emptyset$ for any $v, w \in E^0$.*

We remark that an easy double application of Lemma 11 gives an isomorphism

$$C^*(E)/J_{K,X} \cong C^*((E/K)/\beta(X)),$$

for all $K \in \Sigma_E$ and $X \subseteq K_\infty^{\text{fin}}$. Thus, combining Theorem 12 with Proposition 13 we get a criterion of primitivity of gauge invariant ideals of $C^*(E)$. Namely, a gauge invariant ideal $J_{K,X}$ of $C^*(E)$ is primitive if and only if the quotient graph $(E/K)/\beta(X)$ satisfies the conditions of Proposition 13.

3.2. Other idelas

In many cases, the graph algebra $C^*(E)$ may contain non gauge invariant ideals. For example, since $\Sigma_{E_3} = \{\emptyset, \{w\}, E^0\}$, $J_{\{w\}} \cong \mathcal{K}$ is the only nontrivial gauge invariant ideal of $C^*(E_3)$. However, since the quotient graph $E_3/\{w\}$ is E_2 with $n = 1$, $C^*(E_3)/J_{\{w\}} \cong C^*(E_3/\{w\}) \cong C(\mathbf{T})$ and consequently we see that $C^*(E_3)$ contains a circle of non gauge invariant primitive ideal.

For an arbitrary directed graph E it may be shown that all non gauge invariant primitive ideals arise essentially in the same way as described in the preceding paragraph. Namely, let J be a non gauge invariant primitive ideal of $C^*(E)$. Then there exists a unique maximal gauge invariant ideal J' of $C^*(E)$ contained in J. Furthermore, the quotient $C^*(E)/J'$ is isomorphic to $C^*(F)$ for a suitable graph F. It is that F must contain a unique vertex simple loop with no exits in F and that the ideal J corresponds to a point on that loop.

A complete discussion of all primitive ideals of $C^*(E)$ for an arbitrary directed graph E, including the hull-kernel topology on the primitive spectrum, are presented in articles [1, 9].

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