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Infinitesimal Logarithmic Torelli Problem for Degenerating Hypersurfaces in $\mathbb{P}^n$

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Contents

§0. Introduction

Recently Kato and Usui ([KU]) have constructed partial compactifications of period domains by using logarithmic Hodge structures. It motivates us to study extended period maps and their Torelli problems. The purpose of this article is to give some results on the infinitesimal logarithmic Torelli problem of extended period maps for degenerating hypersurfaces.

We first set up the following problem in the logarithmic algebraic geometry. Let k be a field of characteristic zero and let $\underline{S}_0 = (\mathrm{Spec}(k), N_0)$ be the standard log point where the log structure N_0 is defined by $\mathbb{N} \to k; 1 \to 0$. Let $f_0 : (Z, M_Z) \to \underline{S}_0$ be a log smooth morphism of semistable type whose underlying morphism is proper and flat of relative dimension m. Let ω_{Z/S_0} be the sheaf of logarithmic differentials

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of $(Z, M_Z)/\underline{S}_0$ introduced by [K]. It is a locally free $\mathcal{O}_Z$-module and we put

$$\omega^q_{Z/S_0} = \overset{q}{\wedge}\omega_{Z/S_0} \quad \text{and} \quad \theta_{Z/S_0} = \mathcal{H}om_{\mathcal{O}_Z}(\omega_{Z/S_0}, \mathcal{O}_Z).$$

The main object of study is the map

$$d\rho_Z \ : \ H^1(Z, \theta_{Z/S_0}) \to \bigoplus_{0 \le p \le m-1} \mathrm{Hom}\big(H^p(Z, \omega^{m-p}_{Z/S_0}), H^{p+1}(Z, \omega^{m-1-p}_{Z/S_0})\big)$$

that is given rise to by the pairing $H^1(Z, \theta_{Z/S_0}) \otimes H^p(Z, \omega^{m-p}_{Z/S_0}) \to H^{p+1}(Z, \omega^{m-1-p}_{Z/S_0})$ induced by the contraction $\theta_{Z/S_0} \otimes \omega^q_{Z/S_0} \to \omega^{q-1}_{Z/S_0}$ and cup product. We say that the infinitesimal logarithmic Torelli holds for $(Z, M_Z)/\underline{S}_0$ when $d\rho_Z$ is injective.

Now we consider the case $k = \mathbb{C}$. In [KU] $d\rho_Z$ is interpreted as the logarithmic differential of an extended period map and the injectivity implies that the period map is an embedding on the universal deformation space of f_0. In order to explain the implication more precisely, we consider the following concrete situation. Let $B = \{t \in \mathbb{C} \mid |t| < 1\}$ be the unit disc and let $\mathbb{P}^{m+1}$ be the complex projective space with homogeneous coordinate $X_0, \dots, X_{m+1}$. We will construct a certain moduli space over B that parametrizes families of hypersurfaces $X \subset \mathbb{P}^{m+1} \times B$ over B such that $X \to B$ is smooth outside the point $\{t = X_1 = \cdots X_{m+1} = 0\}$ with some prescribed type of singularity at the point. Every member X/B of the moduli space has a standard desingularization $\widetilde{X} \to X$ such that $\widetilde{X}$ has strictly semistable reduction over B. It gives rise to a log smooth morphism $(Z, M_Z) \to \underline{S}_0$ where Z is the central fiber of $\widetilde{X} \to B$ and M_Z is associated to the embedding $Z \hookrightarrow \widetilde{X}$. Our main result Theorem(2-1) asserts under suitable assumptions that the infinitesimal logarithmic Torelli holds for $(Z, M_Z)/\underline{S}_0$. Letting $\overline{B}^*$ be the universal covering of $B - \{0\}$, it implies that the limiting Hodge structure on $H^m(\widetilde{X} \times_B \overline{B}^*, \mathbb{Q})$ defined by Steenbrink [St] determines (Z, M_Z) locally on the moduli space up to isomorphisms of log schemes over $\underline{S}_0$ (see Theorem(2-2) for the more precise statement).

The proof of the main theorem follows the line of thoughts developed by Griffiths [Grif]. The point is to express $H^1(Z, \theta_{Z/S_0})$ and $H^p(Z, \omega^q_{Z/S_0})$ by Jacobian rings and observe that $d\rho_Z$ is induced by multiplication of polynomials. In order to develop the generalized Jacobian rings in our logarithmic context, we apply the Green's technique of Koszul complexes ([G, Lecture 4]) to a certain toric variety.

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§1. A moduli space of degenerating hypersurfaces in $\mathbb{P}^n$

In the whole paper we fix the base field k of characteristic zero, a variable t over k and a discrete valuation ring Λ over $k[t]$ such that t is a prime element of Λ and that $\Lambda/(t) \simeq k$. Let $\mathbb{P}_\Lambda = \mathrm{Proj}(\Lambda[X_0, \ldots, X_n])$ be the projective space of dimension $n \geq 3$ over $\mathrm{Spec}(\Lambda)$ and let $\mathbb{A}_\Lambda = \mathrm{Spec}(\Lambda[x_1, \ldots, x_n])$ with $x_i = X_i/X_0$ be the affine subspace. We are going to study hypersurfaces in $\mathbb{P}_\Lambda$ which are smooth over $\mathrm{Spec}(\Lambda)[1/t]$ and whose fibers over $\mathrm{Spec}(\Lambda/(t))$ has isolated singularity of some prescribed type at the origin of $\mathbb{A}_\Lambda$. We need introduce some notations.

Definition(1-1). We fix an integer $s \geq 1$.

(1) Let $q \geq 0$ be an integer. Let $P^q_\Lambda \subset \Lambda[X_0, \ldots, X_n]$ be the Λ-module of homogeneous polynomials of degree q. Let $A_\Lambda = \Lambda[x_1, \ldots, x_n]$ and $A^{\leq q}_\Lambda$ be the subspace of polynomials of degree $\leq q$. We constantly use the identification $P^q_\Lambda \xrightarrow{\sim} A^{\leq q}_\Lambda$; $G(X_0, X_1, \ldots, X_n) \to G(1, x_1, \ldots, x_n)$.

(2) For an integer $\nu \geq 0$ let $\mathfrak{m}_\Lambda(\nu) \subset A_\Lambda$ be the ideal generated by the elements

$$t^\beta x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \qquad \text{with } \beta + \alpha_n + s \sum_{1 \leq i \leq n-1} \alpha_i \geq \nu$$

and define

$$P^q_\Lambda(\nu) = \mathrm{Ker}(P^q_\Lambda \to A_\Lambda/\mathfrak{m}_\Lambda(\nu)) = A^{\leq q}_\Lambda \cap \mathfrak{m}_\Lambda(\nu).$$

Now we fix an integer $d > 0$ for the degree of our hypersurfaces and $\delta > 0$ for the multiplicity of designated singularity. Our object to study is hypersurfaces defined by an equation $F \in P^d_\Lambda(s\delta)$ (s is already fixed in Definition(1-1)). By definition $F \in P^d_\Lambda$ lies in $P^d_\Lambda(s\delta)$ if $f = F(1, x_1, \ldots, x_n) \in A^{\leq d}_\Lambda$ is written in the form

$$f = \sum_{\alpha_1 + \cdots + \alpha_n \leq d} a_{\alpha_1, \ldots, \alpha_n} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad (a_{\alpha_1, \ldots, \alpha_n} \in \Lambda),$$

$$|a_{\alpha_1, \ldots, \alpha_n}| + \sum_{1 \leq i \leq n-1} s\alpha_i + \alpha_n \geq s\delta$$

where $|\lambda|$ denotes the normalized additive valuation of $\lambda \in \Lambda$.

Remark(1-1). A typical example of such f is

$$f = x_1^\delta + \cdots + x_{n-1}^\delta + x_n^{s\delta} + t^{s\delta} + h \quad \text{with } h \in A_\Lambda^{\leq d} \cap \mathfrak{m}_\Lambda(s\delta + 1).$$

where we assume $s\delta < d$. The reason why I choose such special type of degenerating hypersurfaces for our stydy of the logarithmic Torelli problem is not so much more than the convenience of computation. By using toric geometry we will construct a birational map $\widetilde{\mathbb{P}}_\Lambda = \widetilde{\mathbb{P}}_\Lambda(\Delta) \to \mathbb{P}_\Lambda$ associated to a certaion cone decomposition Δ in such a way that the inverse images of the generic members of degenerating hypersurfaces defined by equations in $P_\Lambda^d(s\delta)$ have semistable reduction over $\mathrm{Spec}(\Lambda)$. A change of Δ will bring about that of the type of equations, and vice versa. I believe that with more effort, one should be able to carry out the similar computation for degenerating hypersurfaces defined by equations of more general type. For example one may fix appropriate rational numbers $\delta_0, \delta_1, \ldots, \delta_n$ to consider equations of the form

$$f = \sum_{\alpha_1 + \cdots + \alpha_n \leq d} a_{\alpha_1, \ldots, \alpha_n} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad (a_{\alpha_1, \ldots, \alpha_n} \in \Lambda),$$

$$\delta_0 |a_{\alpha_1, \ldots, \alpha_n}| + \sum_{1 \leq i \leq n} \delta_i \alpha_i \geq 1.$$

Now we construct a moduli space over $\mathrm{Spec}(\Lambda)$ parametrizing hypersurfaces in $\mathbb{P}_\Lambda$ of the above type which are smooth in the logarithmic sense (cf. Theorem(1-1)). We start with the following.

Definition(1-2). We fix integers d, δ, r satisfying $r = s\delta$ and $\delta \geq 2$ and $d \geq r$. Let $M = \mathbb{P}(P_\Lambda^d(r)^*)$ be the projective space bundle over $\mathrm{Spec}(\Lambda)$ associated to the Λ-dual of $P_\Lambda^d(r)$. For a morphism $p : \mathrm{Spec}(\Lambda) \to M$ we let $X_p \subset \mathbb{P}_\Lambda$ be the corresponding hypersurface. Note that X_p passes through $0 = (t, x_1, \ldots, x_n) \in \mathbb{A}_\Lambda$ where X_p is singular.

The next task is to construct an appropriate simultaneous desingularization of X_p. For this we use the terminology of toric geometry that we refer to [F]. Let N be the lattice of rank $n+1$ with the standard basis

$$e_0 = (1, 0, \ldots, 0),\ e_1 = (0, 1, \ldots, 0), \cdots, e_n = (0, 0, \ldots, 1)$$

where the vectors are $n+1$-dimensional. Let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ with dual pairing $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$. For a cone σ in $N_{\mathbb{R}}$ write

$$U_\sigma = \mathrm{Spec}(k[M_\sigma]) \quad \text{with } M_\sigma = \{u \in M \mid \langle u, v \rangle \geq 0\ (\forall v \in \sigma)\}.$$

For a fan Δ in N let

$$X(\Delta) = \bigcup_{\sigma\in\Delta} U_\sigma$$

be the corresponding toric variety. In particular we have

$$\mathbb{A} := X(\Delta_0) = \mathrm{Spec}(k[t, x_1, \dots, x_n])$$

for $\Delta_0 = \{\text{faces of } \sigma_0\}$ with $\sigma_0 = \mathbb{R}_{\geq 0}e_0 + \cdots + \mathbb{R}_{\geq 0}e_n$ where $t, x_1, \dots, x_n$ are the elements in $k[M_{\sigma_0}]$ corresponding to the dual basis of $\{e_0, \dots, e_n\}$. For a fan Δ which is a subdivision of Δ_0 we have the natural morphism $\pi : X(\Delta) \to \mathbb{A}$ arising from the natural map $(N, \Delta) \to (N, \Delta_0)$.

Definition(1-3). Let $s \geq 1$ be as in Definition(1-1).

(1) Let Δ_s be the fan consisting of the following cones and their faces

$$\sigma_s^+,\ \sigma_s^-,\ \sigma_{0,\hat{i}}, \sigma_{k,\hat{i}}^+,\ \sigma_{k,\hat{i}}^-\ (1 \leq k \leq s-1, 1 \leq i \leq n-1),$$

where, denoting $v_k = (1, k, \dots, k, 1)$,

$$\begin{aligned}
\sigma_{k,\hat{i}}^+ =& \mathbb{R}_{\geq 0}e_0 + \mathbb{R}_{\geq 0}v_k + \mathbb{R}_{\geq 0}v_{k+1} + \mathbb{R}_{\geq 0}e_1 \\
&+ \cdots + \widehat{\mathbb{R}_{\geq 0}e_i} + \cdots + \mathbb{R}_{\geq 0}e_{n-1}, \\
\sigma_{k,\hat{i}}^- =& \mathbb{R}_{\geq 0}e_n + \mathbb{R}_{\geq 0}v_k + \mathbb{R}_{\geq 0}v_{k+1} + \mathbb{R}_{\geq 0}e_1 \\
&+ \cdots + \widehat{\mathbb{R}_{\geq 0}e_i} + \cdots + \mathbb{R}_{\geq 0}e_{n-1}, \\
\sigma_{0,\hat{i}} =& \mathbb{R}_{\geq 0}e_0 + \mathbb{R}_{\geq 0}e_n + \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}e_1 \\
&+ \cdots + \widehat{\mathbb{R}_{\geq 0}e_i} + \cdots + \mathbb{R}_{\geq 0}e_{n-1}, \\
\sigma_s^+ =& \mathbb{R}_{\geq 0}e_0 + \mathbb{R}_{\geq 0}v_s + \mathbb{R}_{\geq 0}e_1 + \cdots + \mathbb{R}_{\geq 0}e_{n-1}, \\
\sigma_s^- =& \mathbb{R}_{\geq 0}e_n + \mathbb{R}_{\geq 0}v_s + \mathbb{R}_{\geq 0}e_1 + \cdots + \mathbb{R}_{\geq 0}e_{n-1}.
\end{aligned}$$

Let $\widetilde{\mathbb{A}} := X(\Delta_s)$ with $\pi : \widetilde{\mathbb{A}} \to \mathbb{A}$ be the corresponding toric variety. Note that if $s = 1$, π is the blow up of $\mathbb{A}$ with center at $0 = (t, x_1, \dots, x_n)$.

(2) Let $\mathbb{P} = \mathrm{Proj}(k[t][X_0, \dots, X_n])$ be the projective n-space over $\mathrm{Spec}(k[t])$ and fix the embedding $\mathbb{A} \hookrightarrow \mathbb{P}$ over $\mathrm{Spec}(k[t])$ via $x_i = X_i/X_0$. Putting

$$\widetilde{\mathbb{P}} = \widetilde{\mathbb{A}} \cup \bigcup_{1\leq i\leq n} \mathbb{A}_i \quad \text{with } \mathbb{A}_i = \{X_i \neq 0\} \subset \mathbb{P}$$

patched together in an evident manner, we get $\pi : \widetilde{\mathbb{P}} \to \mathbb{P}$ that fits into the following cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathbb{A}} & \hookrightarrow & \widetilde{\mathbb{P}} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{A} & \hookrightarrow & \mathbb{P}. \end{array}$$

Proposition(1-1). *$\widetilde{\mathbb{A}}$ is smooth over k, π is proper and $\pi : \widetilde{\mathbb{A}} - \pi^{-1}(0) \xrightarrow{\sim} \mathbb{A} - \{0\}$ where $0 = (t, x_1, \ldots, x_n) \in \mathbb{A}$. The reduced part $\mathbb{E}$ of $\pi^{-1}(0)$ is a simple normal crossing divisor on $\widetilde{\mathbb{A}}$. We can write $\mathbb{E} = \bigcup_{1\leq k\leq s} \mathbb{E}_k$ as a divisor on $\widetilde{\mathbb{A}}$, where $\mathbb{E}_k \subset \widetilde{\mathbb{A}}$ is an irreducible smooth divisor characterized by the following property:*

$$\mathbb{E} \cap U_\sigma \subset \begin{cases} \mathbb{E}_k \cup \mathbb{E}_{k+1} & \textit{if } \sigma = \sigma^{\pm}_{k,\hat{i}} \textit{ with } 1 \leq k \leq s-1 \\ \mathbb{E}_1 & \textit{if } \sigma = \sigma_{0,\hat{i}} \\ \mathbb{E}_s & \textit{if } \sigma = \sigma^{\pm}_{s}. \end{cases}$$

Proof. Easy and left to the readers. Q.E.D.

Definition(1-4). Let

$$\begin{array}{ccc} \widetilde{\mathbb{A}}_\Lambda & \hookrightarrow & \widetilde{\mathbb{P}}_\Lambda \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{A}_\Lambda & \hookrightarrow & \mathbb{P}_\Lambda \end{array}$$

be the base change of the diagram in Definition(1-3)(2) via $\mathrm{Spec}(\Lambda) \to \mathrm{Spec}(k[t])$. For $X_p \subset \mathbb{P}_\Lambda$ as in Definition(1-2), we denote by $\widetilde{X}_p \subset \widetilde{\mathbb{P}}_\Lambda$ the proper transform of X_p by $\pi : \widetilde{\mathbb{P}}_\Lambda \to \mathbb{P}_\Lambda$.

Theorem(1-1). *There exists a Zariski open subset $M_{ls} \subset M$ (the locus of "log-smooth points") characterized by the property that a morphism $p :$ $\mathrm{Spec}(\Lambda) \to M$ factors through M_{ls} if and only if the following condition $(*)$ holds:*
$(*)$ $\widetilde{X}_p$ is regular and $X_p \to \mathrm{Spec}(\Lambda)$ is smooth outside 0 and $\widetilde{X}_p$ is a strict semi-stable reduction over $\mathrm{Spec}(\Lambda)$.

Moreover $M_{ls} \subset M$ is strictly dense, where a dense open immersion $U \hookrightarrow V$ of Λ-schemes is strictly dense if $U \otimes_\Lambda k \hookrightarrow V \otimes_\Lambda k$ is dense.

Fix a morphism $p : \mathrm{Spec}(\Lambda) \to M$ and let $F \in P^d_\Lambda(r)$ be a corresponding non-zero polynomial. Let $f = F(1, x_1, \ldots, x_n) \in A_\Lambda$. By

Definition(1-1) we can write

$$(1\text{-}1) \quad f = \sum_{\alpha_1+\cdots+\alpha_n \le d} a_{\alpha_1,\ldots,\alpha_n} t^{\beta} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad (a_{\alpha_1,\ldots,\alpha_n} \in \Lambda),$$
$$\beta = \beta(\alpha_1, \ldots, \alpha_n) = \max\{0, r - s(\alpha_1 + \cdots + \alpha_{n-1}) - \alpha_n\}.$$

Note $M = \mathrm{Proj}(\Lambda[a_{\alpha_1,\ldots,\alpha_n}])$ by taking the coefficients of f as a homogeneous coordinate. Put

$$(1\text{-}2) \quad \begin{cases} f^{hom} = \displaystyle\sum_{s(\alpha_1+\cdots+\alpha_{n-1})+\alpha_0+\alpha_n=r} a_{\alpha_1,\ldots,\alpha_n} x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\ \qquad\qquad \in \Lambda[x_0, x_1, \ldots, x_n], \\ f_0^{hom} = (f^{hom} \mod t) \in k[x_0, x_1, \ldots, x_n]. \end{cases}$$

Grading $k[x_0, \ldots, x_n]$ by $\deg x_i = s$ for $1 \le i \le n-1$ and $\deg x_0 = \deg x_n = 1$, f_0^{hom} is homogeneous of degree r. Let $\mathbb{P}(Q) := \mathrm{Proj}(k[x_0, x_1, \ldots, x_n])$ be the weighted projective space of type $Q = (1, s, \ldots, s, 1)$. Now Theorem(1-1) is an easy consequence of Theorem(1-2) below.

Theorem(1-2). *X_p satisfies the condition $(*)$ of Theorem(1-1) if and only if the following conditions hold.*

(1) *$\partial F/\partial X_0, \ldots, \partial F/\partial X_n$ have no common zero in $\mathbb{P}_\Lambda - \{0\}$.*

(2) *$x_0 \partial f_0^{hom}/\partial x_0, \partial f_0^{hom}/\partial x_1, \ldots, \partial f_0^{hom}/\partial x_n$ have no common zero in $\mathbb{P}(Q)$.*

Remark(1-2). For f as Rem.(1-1), we have $f_0^{hom} = x_0^r + x_1^\delta + \cdots + x_{n-1}^\delta + x_n^r$ and we easily see that it satisfies Theorem(1-2)(2). Hence it corresponds to a morphism $\mathrm{Spec}(\Lambda) \to M_{ls}$ if we take h sufficiently general so that f satisfies the condition of Theorem(1-2)(1).

Proposition(1-3). *Let $V_1 \subset \mathbb{P}(Q)$ be the hypersurface defined by $f_0^{hom} = 0$ and let $V_2 = V_1 \cap \{x_0 = 0\}$ and $V_3 = V_1 \cap \{x_0 = x_n = 0\}$. The condition of Theorem(1-2)(2) is equivalent to the following condition: If $s = 1$, V_1 and V_2 are regular. If $s \ge 2$, V_3 and $V_1 \cap (D^+(x_0) \cup D^+(x_n))$ and $V_2 \cap D^+(x_n)$ are regular where $D^+(x_i) = \mathbb{P}(Q) - \{x_i = 0\}$.*

Proof. The proof is standard and left to the readers. Q.E.D.

Now we prove Theorem(1-2). Let $H_t \subset \mathbb{P}_\Lambda$ be defined by $\{t = 0\}$ and let $\widetilde{H}_t \subset \widetilde{\mathbb{P}}_\Lambda$ be its proper transform. In view of Proposition(1-1) the conditions of Theorem(1-1) are equivalent to the conditions that $X_p \to \mathrm{Spec}(\Lambda)$ is smooth outside 0 and that:

(*): $\widetilde{X}_p \cap \mathbb{E}_k$, $\widetilde{X}_p \cap \mathbb{E}_k \cap \mathbb{E}_{k+1}$ $(1 \leq k \leq s-1)$, $\widetilde{X}_p \cap \widetilde{H}_t$, $\widetilde{X}_p \cap \widetilde{H}_t \cap \mathbb{E}_k$ and $\widetilde{X}_p \cap \widetilde{H}_t \cap \mathbb{E}_k \cap \mathbb{E}_{k+1}$ intersect transversally.

It is standard that the first condition is equivalent to Theorem(1-2)(1). We show the equivalence of the condition (*) and the condition of Proposition(1-3). We assume $s \geq 2$ and leave the (easier) case $s = 1$ to the readers. Recall that we have

$$\widetilde{\mathbb{A}}_\Lambda = \bigcup_{\sigma \in \Delta_s} U_\sigma \to \mathbb{A}_\Lambda = \mathrm{Spec}(\Lambda[x_1, x_2, \ldots, x_n]),$$

where we write U_σ for $U_\sigma \times_{\mathrm{Spec}(k[t])} \mathrm{Spec}(\Lambda)$. We describe the condition (*) on each open subset U_σ for $\sigma = \sigma_{0,\hat{i}}, \sigma^{\pm}_{k,\hat{i}}, \sigma^{\pm}_{s}$. For simplicity we assume $n = 3$ while the argument in general case is the same.

Case $\sigma = \sigma_{0,\hat{i}}$ *or* $\sigma^{\pm}_{k,\hat{i}}$. We take for example $\sigma = \sigma^{+}_{k,\hat{2}} = \mathbb{R}_{\geq 0} v_k + \mathbb{R}_{\geq 0} v_{k+1} + \mathbb{R}_{\geq 0} e_0 + \mathbb{R}_{\geq 0} e_1$ with $1 \leq k \leq s-1$. We have $U_\sigma = \mathrm{Spec}(R_\sigma)$ where

$$R_\sigma = \Lambda[y_1, y_2, y_3, y_4]/(t - y_1 y_2 y_3)) \quad \text{with} \quad \begin{cases} x_1 = y_1^k y_2^{k+1} y_4 \\ x_2 = y_1^k y_2^{k+1} \\ x_n = y_1 y_2. \end{cases}$$

Note that $\mathbb{E} \cap U_\sigma = (\mathbb{E}_k \cup \mathbb{E}_{k+1}) \cap U_\sigma$ and $\mathbb{E}_k$, $\mathbb{E}_{k+1}$, $\widetilde{H}_t$ are defined by $y_1 = 0$, $y_2 = 0$, $y_3 = 0$ respectively on U_σ. First assume $k \leq s-2$. An easy calculation shows that $f = (y_1^k y_2^{k+1})^\delta (g + y_1 y_2 h)$ with $h \in R_\sigma$ and

$$g = g(y_4) = \sum_{\alpha_1 + \alpha_2 = \delta, \alpha_n = 0} a_{\alpha_1, \alpha_2, 0} \cdot y_4^{\alpha_1}.$$

We note that

$$(*1) \qquad x_2^\delta g\left(\frac{x_1}{x_2}\right) = f^{hom}(0, x_1, x_2, 0)$$

Now the condition (*) on U_σ is equivalent to the regularity of $R_\sigma/(y_1, y_2, y_3, g)$. By (*1) it is equivalent to the regularity of $V_3 \cap D^+(x_2)$ where V_3 is as in Proposition(1-3). The similar computations in cases $\sigma = \sigma_{0,\hat{i}}$ or $\sigma^{\pm}_{k,\hat{i}}$ show that the condition (*) on U_σ for these σ is equivalent to the regularity of V_3. In case $k = s-1$ we see that $f = (y_1^{s-1} y_2^s)^\delta (g +$

$y_1y_2h)$ with $h \in R_\sigma$ and

$$\begin{aligned} g &= g(y_1, y_3, y_4) \\ &= \sum_{s(\alpha_1+\alpha_2)+\alpha_n \le r} a_{\alpha_1,\alpha_2,\alpha_n} \cdot y_1^{\delta-(\alpha_1+\alpha_2)} y_3^{r-s(\alpha_1+\alpha_2)-\alpha_n} y_4^{\alpha_1}. \end{aligned}$$

We note that

$$(*2) \qquad \begin{aligned} x_2^\delta g\Big(\frac{x_n^s}{x_2}, \frac{x_0}{x_n}, \frac{x_1}{x_2}\Big) &= f^{hom}(x_0, x_1, x_2, x_n), \\ g(0, y_3, y_4) &= \sum_{\alpha_1+\alpha_2=\delta, \alpha_n=0} a_{\alpha_1,\alpha_2,0} \cdot y_4^{\alpha_1}. \end{aligned}$$

Now the condition $(*)$ on U_σ is equivalent to the regularity of $R_\sigma/(y_i, g)$ and $R_\sigma/(y_3, y_i, g)$ with $i = 1, 2$ and that of $R_\sigma/(y_1, y_2, g)$ and $R_\sigma/(y_1, y_2, y_3, g)$. By $(*1)$ and $(*2)$ it is equivalent to the regularity of $V_3 \cap D^+(x_2)$ and $V_1 \cap (D^+(x_2) \cap D^+(x_n))$ and $V_2 \cap (D^+(x_2) \cap D^+(x_n))$. The similar computations in cases $\sigma = \sigma^{\pm}_{k,\hat{i}}$ with $k = s - 1$ show that the condition $(*)$ on U_σ for these σ is equivalent to the regularity of V_3 and $V_1 \cap (D^+(x_0) \cup D^+(x_n)) \cap (D^+(x_1) \cup D^+(x_2))$ and $V_2 \cap D^+(x_n) \cap (D^+(x_1) \cup D^+(x_2))$.

Case $\sigma = \sigma_s^{\pm}$. We take for example $\sigma = \sigma_s^+ = \mathbb{R}_{\ge 0} v_s + \mathbb{R}_{\ge 0} e_1 + \mathbb{R}_{\ge 0} e_2 + \mathbb{R}_{\ge 0} e_0$. We have $U_\sigma = \operatorname{Spec}(R_\sigma)$ where

$$R_\sigma = \Lambda[u_0, u_1, u_2, u_3]/(t - u_0u_3) \quad \text{with} \quad \begin{cases} x_1 = u_3^s u_1 \\ x_2 = u_3^s u_2 \\ x_n = u_3. \end{cases}$$

Note that $\mathbb{E} \cap U_\sigma = \mathbb{E}_s \cap U_\sigma$ and $\mathbb{E}_s$ and $\widetilde{H}_t$ are defined by $u_3 = 0$ and $u_0 = 0$ on U_σ respectively. An easy calculation shows that $f = u_3^r(g + u_3h)$ with $h \in R_\sigma$ and

$$g = g(u_0, u_1, u_2) = \sum_{s(\alpha_1+\alpha_2)+\alpha_n \le r} a_{\alpha_1,\alpha_2,\alpha_n} u_0^{r-s(\alpha_1+\alpha_2)-\alpha_n} u_1^{\alpha_1} u_2^{\alpha_2}.$$

We note

$$(*3) \qquad x_n^r g\Big(\frac{x_0}{x_n}, \frac{x_1}{x_n^s}, \frac{x_2}{x_n^s}\Big) = f^{hom}(x_0, x_1, x_2, x_n).$$

Now the condition $(*)$ on U_σ is equivalent to the regularity of $R_\sigma/(u_3, g)$ and $R_\sigma/(u_0, u_3, g)$. By $(*3)$ it is equivalent to the regularity of $V_1 \cap D^+(x_n)$ and $V_2 \cap D^+(x_n)$. The similar computation in case $\sigma = \sigma_s^-$

shows that the condition $(*)$ on U_σ for these σ is equivalent to the regularity of $V_1 \cap (D^+(x_0) \cup D^+(x_n))$ and $V_2 \cap D^+(x_n)$. This completes the proof of Theorem(1-2). Q.E.D.

As an immediate consequence of the above proof we get the following.

Proposition(1-4). *For* $p : \mathrm{Spec}(\Lambda) \to M_{ls}$, $\pi^{-1}(X_p) = \widetilde{X}_p + \delta \cdot \sum_{1\le k\le s} k\mathbb{E}_k$ *as a divisor on* $\widetilde{\mathbb{P}}_\Lambda$.

§2. Injectivity of infinitesimal period map

Fix a morphism $p : \mathrm{Spec}(\Lambda) \to M_{ls}$ and let $X \subset \mathbb{P}_\Lambda$ and $\widetilde{X} \subset \widetilde{\mathbb{P}}_\Lambda$ be the corresponding hypersurface and its proper transform with the projection $\pi : \widetilde{X} \to X$. Let $Z \subset \widetilde{X}$ be the closed fiber of $\widetilde{X} \to \mathrm{Spec}(\Lambda)$.

Definition(2-1). Let $\omega_{\widetilde{X}/S} = \Omega_{\widetilde{X}/S}(\log Z)$ be the sheaf of logarithmic differentials of the semistable family $\widetilde{X}$ over $S := \mathrm{Spec}(\Lambda)$ in the sense of [St]. By [St] $\omega_{\widetilde{X}/S}$ is a locally free $\mathcal{O}_{\widetilde{X}}$-module. We put

$$\omega^q_{\widetilde{X}/S} = \overset{q}{\wedge}\omega_{\widetilde{X}/S} \quad \text{and} \quad \theta_{\widetilde{X}/S} = \mathcal{H}om_{\mathcal{O}_{\widetilde{X}}}(\omega_{\widetilde{X}/S}, \mathcal{O}_{\widetilde{X}}).$$

We also define a locally free $\mathcal{O}_Z$-modules

$$\omega_{Z/S_0} = \omega_{\widetilde{X}/S} \otimes_\Lambda k, \quad \omega^q_{Z/S_0} = \overset{q}{\wedge}\omega_{Z/S_0}, \quad \theta_{Z/S_0} = \mathcal{H}om_{\mathcal{O}_Z}(\omega_{Z/S_0}, \mathcal{O}_Z).$$

Remark(2-1). In the language of log geometry (cf. [K]), $\omega_{\widetilde{X}/S}$ (resp. ω_{Z/S_0}) is the sheaf of logarithmic differentials of the log smooth morphisms $(\widetilde{X}, M_{\widetilde{X}}) \to (\mathrm{Spec}(\Lambda), N_\Lambda)$ (resp. $(Z, M_Z) \to (\mathrm{Spec}(k), N_0)$). Here N_0 and N_Λ are the log structure defined by $\mathbb{N} \to \Lambda \to k;\ 1 \to t \to 0$ and $M_{\widetilde{X}}$ is associated to the embedding $Z \subset \widetilde{X}$ and M_Z is its inverse image on Z.

Let $m = n - 1$ be the relative dimension of $\widetilde{X}/S$ and let

$$H^1(Z, \theta_{Z/S_0}) \otimes H^p(Z, \omega^{m-p}_{Z/S_0}) \to H^{p+1}(Z, \omega^{m-1-p}_{Z/S_0})$$

be the map induced by the contraction $\theta_{Z/S_0} \otimes \omega^q_{Z/S_0} \to \omega^{q-1}_{Z/S_0}$ and the cup product. It induces

$$H^1(Z, \theta_{Z/S_0}) \overset{d\rho_Z}{\to} \bigoplus_{0\le p\le m-1} \mathrm{Hom}(H^p(Z, \omega^{m-p}_{Z/S_0}), H^{p+1}(Z, \omega^{m-1-p}_{Z/S_0})).$$

Theorem(2-1). *Let the assumption be as in Definition(1-1) and Definition(1-2). Assume the following conditions:*

(i) $d \geq r+1$ *and either* $n \geq 4$ *or* $n = 3$ *and* $d \neq 4$.
(ii) $\delta < n$.
(iii) $H^0(Z, \theta_{Z/S_0}) = 0$.

The map $d\rho_Z$ *is injective if* n *is even or* $\delta \geq 3$. *If* n *is odd and* $\delta = 2$ *then* $\dim_k(\mathrm{Ker}(d\rho_Z)) = r-1$.

Remark(2-2). We will see (cf. Remark(4-1)) that the singularity $(X, 0)$ is canonical if and only if $\delta < n$ or $\delta = n$, $s = 1$. Thus Theorem(2-1) suggests that the canonicality of the singularity plays an important role in the infinitesimal logarithmic Torelli problem.

Concerning the second assumption of Theorem (2-1), we have the following.

Proposition(2-1). *The condition* $H^0(Z, \theta_{Z/S_0}) = 0$ *defines strictly dense open subset* M_{ls}^{st} *of* M_{ls}.

Proposition(2-1) follows from Proposition(2-2)(2) below that will be proven in §4. Indeed, by the semicontinuity, the vanishing of $H^0(Z, \theta_{Z/S_0})$ is an open condition on the moduli space. Thus it suffices to show that there exists a morphism $p : \mathrm{Spec}(\Lambda) \to M_{ls}$ satisfying the condition. Indeed we may take for example

$$F = X_0^{d-\delta}(X_1^\delta + \cdots X_{n-1}^\delta) + X_0^{d-r}X_n^r + \Phi_d + t^r G \in P_\Lambda^d(r)$$

where Φ_d (resp. G) is a sufficiently general homogeneous polynomials of degree d in $X_1, \ldots, X_n$ (resp. in $X_0, X_1, \ldots, X_n$) (cf. Theorem(1-2)).

Proposition(2-2). *Assume* $d \geq r$ *and* $\delta < n$.

(1) $H^j(Z, \theta_{Z/S_0}) = 0$ *for* $j \geq 2$ *if either* $j \leq n-3$ *or* $j = n-2$ *and* $d \neq n+1$ *or* $j = n-1$ *and* $d \leq n+1$.
(2) *Let* $F \in P_\Lambda^d(r)$ *be a polynomial defining* $X \subset \mathbb{P}_\Lambda$ *and put* $F_0 = (F \bmod t) \in k[X_0, \ldots, X_n]$. *Then* $H^0(Z, \theta_{Z/S_0}) = 0$ *if either* $s \geq 2$ *and*

$$X_j \partial F_0/\partial X_i \ (1 \leq i, j \leq n-1), \ X_j \partial F_0/\partial X_i \ (1 \leq j \leq n, \ i = 0, n)$$

are linearly independent over k, *or if* $s = 1$ *and*

$$X_j \partial F_0/\partial X_i \ (1 \leq i, j \leq n), \ X_j \partial F_0/\partial X_0 \ (0 \leq j \leq n)$$

are linearly independent over k.

In the rest of this section we give a geometric implication of Theorem(2-1). We suppose that the readers are familiar with basic notions of log geometry.

Let M^{st}_{ls} be as Proposition(2-1) and put $M^{st}_{ls,0} = M^{st}_{ls} \otimes_\Lambda k$. Note that M^{st}_{ls} is smooth over $\mathrm{Spec}(\Lambda)$. The construction of §1 gives us the following cartesian diagram of log schemes

$$\begin{array}{lccc} & (Z^{univ}, M_{Z^{univ}}) & \hookrightarrow & (\widetilde{X}^{univ}, M_{\widetilde{X}^{univ}}) \\ (2\text{-}1) & \downarrow f_0^{univ} & & \downarrow f^{univ} \\ & (M^{st}_{ls,0}, N_0) & \hookrightarrow & (M^{st}_{ls}, N_\Lambda) \end{array}$$

with the following properties: Let

$$\underline{S}_0 = (\mathrm{Spec}(k), N_0) \hookrightarrow \underline{S} = (\mathrm{Spec}(\Lambda), N_\Lambda)$$

denote the exact closed immersion of log schemes, where N_0 and N_Λ are defined in Remark(2-1).

(a): N_Λ and N_0 in (2-1) are defined in the same way as Remark(2-1).

(b): $Z^{univ} \subset \widetilde{X}^{univ}$ is a simple normal crossing divisor defined by $t = 0$.

(c): $M_{\widetilde{X}^{univ}}$ is associated to the embedding $Z^{univ} \subset \widetilde{X}^{univ}$ and $M_{Z^{univ}}$ is its inverse image.

(d): f^{univ} and f_0^{univ} are log smooth of semistable type.

(e): Let $p : \underline{S} \to (M^{st}_{ls}, N_\Lambda)$ be an exact closed immersion and let $p_0 : \underline{S}_0 \to (M^{st}_{ls,0}, N_0)$ be the induced exact closed immersion. By pulling back the above diagram via p and p_0, we get the following cartesian diagram of log smooth morphisms

$$\begin{array}{ccc} (Z, M_Z) & \hookrightarrow & (\widetilde{X}, M_{\widetilde{X}}) \\ \downarrow & & \downarrow \\ \underline{S}_0 & \hookrightarrow & \underline{S} \end{array}$$

where the underlying morphisms are those associated to p as in §1 and $M_{\widetilde{X}}$ and M_Z are defined as Remark(2-1).

By (a) (M^{st}_{ls}, N_Λ) is a log scheme over $\underline{S}$. A morphism $p_0 : \mathrm{Spec}(k) \to M^{st}_{ls,0}$ extends in the unique way to an exact closed immersion $p_0 : \underline{S}_0 \to (M^{st}_{ls,0}, N_0)$ over $\underline{S}_0$. By pulling back f_0^{univ} in (2-1) via p_0 we get the log smooth morphism

$$f_{p_0} \ : \ (Z_{p_0}, M_{Z_{p_0}}) \to \underline{S}_0.$$

Now assume $k = \mathbb{C}$ and let $\Lambda \subset \mathbb{C}[[t]]$ be the ring of convergent formal power series. By [Mat] and [FK3] f_{p_0} gives rise to a log Hodge structure $H(p_0)$ over $\underline{S}_0$ that is underlain by

$$(R^m f^{log}_{p_0 *}\mathbb{Q}, R^m f_{p_0 *}\omega^{\cdot}_{Z_{p_0}/S_0}),$$

a pair of a local system on $\underline{S}^{log}_0 (\simeq \{t \in \mathbb{C} \mid \ |t| = 1\})$ and a $\mathbb{C}$-vector space with the descending filtration given by subspaces $R^m f_{p_0 *}\omega^{\geq q}_{Z_{p_0}/S_0} \subset R^m f_{p_0 *}\omega^{\cdot}_{Z_{p_0}/S_0}$. We note that $H(p_0)$ is determined by the limiting Hodge structure defined by Steenbrink [St] on the space

$$H^m(\widetilde{X}^{an} \times_{B_\epsilon} \overline{B}^*_\epsilon, \mathbb{Q}).$$

Here $\widetilde{X}^{an} \to B_\epsilon := \{t \in \mathbb{C} \mid \ |t| < \epsilon\}$ with sufficiently small $\epsilon > 0$ is the morphism of complex analytic space that arises from $\widetilde{X} \to \mathrm{Spec}(\Lambda)$ corresponding to a lift $p : \mathrm{Spec}(\Lambda) \to M^{st}_{ls}$ of p_0 over $\mathrm{Spec}(\Lambda)$ and $\overline{B}^*_\epsilon$ is the universal covering of $B_\epsilon - \{0\}$.

Theorem(2-2). *Assume the following conditions:*

(i) $d \geq r+1$ *and either* n *is even or* $\delta \geq 3$.

(ii) $\delta < n$

(iii) $n \geq 4$.

Locally on $M^{st}_{ls,0}$, $H(p_0)$ *determines* $(Z_{p_0}, M_{Z_{p_0}})$ *up to isomorphisms of log schemes over* $\underline{S}_0$.

Proof Fix an exact closed immersion $p_0 : \underline{S}_0 \to (M^{st}_{ls,0}, N_0)$ and denote simply by $f_0 : (Z_0, M_{Z_0}) \to \underline{S}_0$ the log smooth morphism obtained by pulling back f^{univ}_0 via p_0. We recall the logarithmic deformation theory of f_0 (cf. [KN], [FK1] and [FK2]). Let $\mathcal{C}_k$ (resp. $\mathcal{C}^\wedge_k$) be the category of artinian (resp. Noetherian complete) local k-algebra with residue field k. Let $L\mathcal{C}_k$ (resp. $L\mathcal{C}^\wedge_k$) be the category of pairs $(\underline{T}, 0_{\underline{T}})$ where $\underline{T} = (\mathrm{Spec}(A), M)$ with $A \in \mathcal{C}_k$ is a log scheme (resp. $\underline{T} = (\mathrm{Spf}(A), M)$ with $A \in \mathcal{C}^\wedge_k$ is a formal log scheme) whose log structure M is isomorphic to the inverse image of the log structure on $\underline{S}_0$ via $\mathrm{Spec}(A) \to \mathrm{Spec}(k)$ and $0_{\underline{T}} : \underline{S}_0 \hookrightarrow \underline{T}$ is an exact closed immersion whose underlying morphisms come from $\mathcal{C}_k$ (resp. $\mathcal{C}^\wedge_k$). One defines the functor $D_{Z_0/S_0} : L\mathcal{C}_k \to \mathit{Sets}$ by setting $D_{Z_0/S_0}(\underline{T})$ to be the set of isomorphism classes of log smooth liftings of f_0 to $\underline{T}$ (cf. [FK2, Definition 4.1]). We remark that by definition the log structure $\alpha : M \to A$ for an object $\underline{T} = (\mathrm{Spec}(A), M) \in L\mathcal{C}_k$ is isomorphic to $\mathbb{N} \oplus A^* \to A$; $(n, u) \to 0^n u$. Hence D_{Z_0/S_0} deals only with locally trivial deformations of f_0. By [FK1, §8], [FK2, §4] and [FK4] we have the following facts:

(a): D_{Z_0/S_0} is pro-represented by an object

$$\underline{\mathcal{T}} = (\mathcal{T} = \mathrm{Spf}(R), N_{\mathcal{T}})$$

in $LC_k^\wedge$ with the universal family $\varphi : (\mathcal{X}, M_{\mathcal{X}}) \to (\mathcal{T}, N_{\mathcal{T}})$. Moreover R is formally smooth over k. By definition the fiber of φ over $0_{\mathcal{T}} : \underline{S}_0 \to \underline{\mathcal{T}}$ is f_0.

(b): Let $k[\epsilon]$ be the ring of dual numbers and let N_ϵ be the inverse image of the log structure N_0 on S_0 via the map $\mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k)$ induced by the canonical map $k \to k[\epsilon]$. By definition $\underline{S}_\epsilon := (\mathrm{Spec}(k[\epsilon]), N_\epsilon)$ with the canonical exact closed immersion $\underline{S}_0 \hookrightarrow \underline{S}_\epsilon$ induced by the residue map $k[\epsilon] \to k$ is an object in LC_k. For an object $\underline{T}$ in LC_k or $LC_k^\wedge$ we call $T_0(\underline{T}) = \mathrm{Hom}_{LC_k^\wedge}(\underline{S}_\epsilon, \underline{T})$ the logarithmic tangent space of $\underline{T}$ at $0_{\mathcal{T}}$, where the space on the right hand side denotes the set of morphisms $\underline{S}_\epsilon \to \underline{\mathcal{T}}$ in LC_k. The logarithmic Kodaira-Spencer map induces the isomorphism

$$H^1(Z_0, \theta_{Z_0/S_0}) \xrightarrow{\sim} T_0(\underline{\mathcal{T}}) = D_{Z_0/S_0}(\underline{S}_\epsilon).$$

(c): For a morphism

$$\phi : \underline{\mathcal{T}} = (\mathrm{Spf}(R), M) \to \underline{\mathcal{T}}' = (\mathrm{Spf}(R'), M')$$

in $LC_k^\wedge$, we define its logarithmic differential of ϕ to be the map

$$d\phi : T_0(\underline{\mathcal{T}}) \to T_0(\underline{\mathcal{T}}').$$

If R and R' are formally smooth over k and if $d\phi$ is injective, then the underlying morphism $\mathrm{Spf}(R) \to \mathrm{Spf}(R')$ is an embedding.

The pro-representability follows from the assumption $H^0(Z_0, \theta_{Z_0/S_0}) = 0$ and the rigidity of D_{Z_0/S_0} (cf. [FK2, §3 and §4]). The formally smoothness of R over k is a consequence of the fact that D_{Z_0/S_0} has no obstruction that follows from Proposition(2-2)(1) (with $j = 2$) and the assumption (ii) (In cases $n = 4,\ d = 5$, $H^2(Z_0, \theta_{Z_0/S_0}) \neq 0$ so that one needs an extra argument that we omit).

Now we assume $k = \mathbb{C}$ and let $\varphi^{an} : (\mathcal{X}^{an}, M_{\mathcal{X}^{an}}) \to \underline{\mathcal{T}}^{an} = (\mathcal{T}^{an}, N_{\mathcal{T}^{an}})$ be the corresponding morphism of log analytic spaces over $\mathbb{C}$. By the universality there is an open neighborhood $V \subset M_{ls,0}^{st}$ of p_0 and a strict morphism of log analytic spaces $g : (V, N_{0|V}) \to \underline{\mathcal{T}}^{an}$ mapping p_0 to 0 such that the restriction to V of f_0^{univ} in the diagram (2-1) is isomorphic to the pullback of φ^{an} via g. By the theory of logarithmic Hodge structures and their moduli space (cf. [KU], [Mat], [FK3] and [Us]) φ^{an} gives rise to the extended period map

$$\rho : \underline{\mathcal{T}}^{an} \to \Gamma \backslash D_\Sigma$$

where the space on the right hand side is the classifying space of logarithmic Hodge structures of suitable type equipped with its canonical log structure. We have $\rho(g(p_0')) = [H(p_0')]$ for $p_0' : \mathrm{Spec}(\mathbb{C}) \to V \subset M_{ls}^{st}$. Theorem(2-1) implies that under the assumption of Theorem(2-2) the logarithmic differential $d\rho$ at $0_{\mathcal{I}}$ of ρ is injective, which implies ρ is an embedding. This proves the assertion of Theorem(2-2). Q.E.D.

§3. Jacobian rings of degenerating hypersurfaces

Let the assumption be as in §2. In this section we express the cup product

$$(3\text{-}1) \qquad H^1(\widetilde{X}, \theta_{\widetilde{X}/S}) \otimes_\Lambda H^p(\widetilde{X}, \omega^{m-p}_{\widetilde{X}/S}) \to H^{p+1}(\widetilde{X}, \omega^{m-p-1}_{\widetilde{X}/S})$$

$$(3\text{-}2) \quad H^p(\widetilde{X}, \omega^{m-p}_{\widetilde{X}/S}) \otimes_\Lambda H^{m-p-1}(\widetilde{X}, \omega^{p+1}_{\widetilde{X}/S}) \to H^{m-1}(\widetilde{X}, \omega^1_{\widetilde{X}/S} \otimes \omega^m_{\widetilde{X}/S})$$

in terms of Jacobian rings and we prove Theorem(2-1).

Definition(3-1). Let $F \in P_\Lambda^d(r)$ be an equation defining $X \subset \mathbb{P}_\Lambda$.

(1) For an integer $q \geq 0$ we write $B_F^q = P_\Lambda^q / J_F^q$ where

$$J_F^q = \Big\{ \sum_{0 \leq i \leq n} H_i \partial F / \partial X_i \;\Big|\; H_i \in P_\Lambda^{q-d+1} \Big\} \subset P_\Lambda^q.$$

(2) For integers $q, \nu \geq 0$ we write $B_F^q(\nu) = P_\Lambda^q(\nu)/J_F^q(\nu)$ where

$$J_F^q(\nu) = \Big\{ \sum_{0 \leq i \leq n} H_i \partial F / \partial X_i \;\Big|\; H_i \in P_\Lambda^{q-d+1}(\nu - r + \mu_i) \Big\} \subset J_F^q,$$

where $\mu_i = s$ if $1 \leq i \leq n-1$ and $\mu_n = 1$ and $\mu_0 = 0$. Note that $J_F^q(\nu) \subset P_\Lambda^q(\nu)$ since $\partial F / \partial X_i \in P_\Lambda^{d-1}(r - \mu_i)$.

(3) Write $f = F(1, x_1, x_2, \ldots, x_n) \in A_\Lambda$. For integers $q, \nu \geq 0$ we define

$$R_F^q(\nu) = \mathrm{Ker}(B_F^q \to A_\Lambda / (I_f + \mathfrak{m}_\Lambda(\nu))),$$

where $I_f = \big\{ g_0 f + \sum_{1 \leq i \leq n} g_i \partial f / \partial x_i \;\big|\; g_i \in A_\Lambda \ (0 \leq i \leq n) \big\}$.

Lemma(3-1). *The natural map $\iota : B_F^q(\nu) \to R_F^q(\nu)$ is injective if $B_F^q(\nu)$ is torsion free as a Λ-module. It is surjective if $q - \nu \geq d - r + s - 2$.*

Proof. It is easy to see $\mathrm{Ker}(\iota)$ is torsion, which implies the first assertion immediately. To show the second assertion it suffices to show the surjectivity of $P_\Lambda^q(\nu) \to R_F^q(\nu)$. We have the following commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & J_F^q & \xrightarrow{\alpha} & I_f + \mathfrak{m}_\Lambda(\nu)/\mathfrak{m}_\Lambda(\nu) \\
 & & & & \downarrow & & \downarrow \\
0 \to & & P_\Lambda^q(\nu) & \to & P_\Lambda^q & \to & A_\Lambda/\mathfrak{m}_\Lambda(\nu) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & R_F^q(\nu) & \to & B_F^q & \to & A_\Lambda/I_f + \mathfrak{m}_\Lambda(\nu) \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
\end{array}$$

By the diagram the desired assertion follows from the surjectivity of α. Note that under $P_\Lambda^q \xrightarrow{\sim} A_\Lambda^{\leq q}$

$$J_F^q \xrightarrow{\sim} \Big\{ h_0\big(d\cdot f - \sum_{1\leq i\leq n} x_i \partial f/\partial x_i\big) + \sum_{1\leq i\leq n} h_i \partial f/\partial x_i \;\Big|\; h_i \in A_\Lambda^{\leq q-(d-1)} \Big\}.$$

Take $\phi = g_0 f + \sum_{1\leq i\leq n} g_i \partial f/\partial x_i \in I_f$ with $g_i \in A_\Lambda$. We have

$$\phi = \frac{1}{d} g_0\Big(d\cdot f - \sum_{1\leq i\leq n} x_i\partial f/\partial x_i\Big) + \sum_{1\leq i\leq n}\Big(g_i + \frac{1}{d} g_0 x_i\Big)\partial f/\partial x_i.$$

We may write

$$\frac{1}{d} g_0 = h_0 + g_0', \;\; g_i + \frac{1}{d} g_0 x_i = h_i + g_i' \;\; (1 \leq i \leq n)$$

with $h_i \in A_\Lambda^{\leq q-(d-1)}$ and $g_i' \in (x_1, \ldots, x_n)^{q-d+2} \subset \mathfrak{m}_\Lambda(q-d+2)$ for $0 \leq i \leq n$. The assumption of Lemma(3-1) implies $\nu \leq q-d+2+r-\mu_i$ (cf. Definition(3-1)(2)) and we have

$$g_i' \partial f/\partial x_i \in \mathfrak{m}_\Lambda(q-d+2+r-\mu_i) \subset \mathfrak{m}_\Lambda(\nu).$$

Since $h_0\big(d\cdot f - \sum_{1\leq i\leq n} x_i\partial f/\partial x_i\big)$, $h_i \partial f/\partial x_i \in J_F^q$, this completes the proof. Q.E.D.

Theorem(3-1). *Assume $n \geq 3$ and $d \geq r+1$.*

(1) *For $0 \leq p \leq m := n-1$ there is the natural isomorphism of free Λ-modules*

$$\phi_F^p \;:\; B_F^{\kappa(p)}(\nu(p)) \xrightarrow{\sim} R_F^{\kappa(p)}(\nu(p)) \xrightarrow{\sim} H^p(\widetilde{X}, \omega^{m-p}_{\widetilde{X}/S})_{prim}$$

where $\kappa(p) = d(p+1) - n - 1$ *and* $\nu(p) = s(\delta(p+1) - n + 1) - 1$. *Here the primitive part* $H^p(\widetilde{X}, \omega^q_{\widetilde{X}/S})_{prim}$ *is defined to be*

$$\mathrm{Ker}\big(H^p(\widetilde{X}, \omega^q_{\widetilde{X}/S}) \to H^p(X_\eta, \Omega^q_{X_\eta/\eta})/H^p(X_\eta, \Omega^q_{X_\eta/\eta})_{prim}\big),$$

where X_η/η *is the generic fiber of* X/S.

(2) *Assume* $\delta < n$ *and either* $n \geq 4$ *or* $n = 3$ *and* $d \neq 4$. *There is the natural isomorphism of free* Λ*-modules*

$$\phi_F^{tan*} \;:\; B_F^{d^*}(r^*) \xrightarrow{\sim} R_F^{d^*}(r^*) \xrightarrow{\sim} H^{m-1}(\widetilde{X}, \omega^1_{\widetilde{X}/S} \otimes \omega^m_{\widetilde{X}/S}),$$

where $d^* = dn - 2(n+1)$ *and* $r^* = rn - s(2n-2) - 2 = s(\delta n - 2n + 2) - 2$.

(3) *Assume* $H^0(Z, \theta_{Z/S_0}) = 0$ *and* $\delta < n$. *There is the natural injective homomorphism of free* Λ*-modules*

$$\phi_F^{tan} \;:\; R_F^d(r) \to H^1(\widetilde{X}, \theta_{\widetilde{X}/S}).$$

It is an isomorphism if either $n \geq 4$ *or* $n = 3$ *and* $d \neq 4$. *There is an exact sequence*

$$0 \to B_F^d(r) \to R_F^d(r) \to \bigoplus_{1 \leq i \leq n-1} \mathrm{Coker}(A_\Lambda^{\leq 1} \to A_\Lambda/\mathfrak{m}_\Lambda(s)) \to 0.$$

In particular $B_F^d(r) \xrightarrow{\sim} R_F^d(r)$ *if and only if* $s \leq 2$.

Moreover the cup products (3-1) *and* (3-2) *are compatible with the multiplication of the Jacobian rings.*

The proof of Theorem(3-1) is given in §4.

Remark(3-1). *Under the assumption* $d \geq r+1$ *we can verify* $\kappa(p) - \nu(p)$, $d^* - r^* \geq d - r + s - 2$. *Hence Lemma*(3-1) *implies the first isomorphisms of Theorem*(3-1)(1) *and* (2) *if we already know that the groups on the left hand side are torsion free.*

We have the following auxiliary result that will be proven in §4.

Proposition(3-1). *Assume* $d \geq r$ *and* $j \geq 2$ *and* $\delta < n$. *Then* $H^j(\widetilde{X}, \theta_{\widetilde{X}/S})$ *is a free* Λ*-module and it vanishes if either* $j \leq n-3$ *or* $j = n-2$ *and* $d \neq n+1$ *or* $j = n-1$ *and* $d \leq n+1$.

To deduce Theorem(2-1) from Theorem(3-1) we need the following.

Lemma(3-2). *Assume* $d \geq r+1$ *and* $n \geq 3$. *For an integer* $0 \leq p \leq n-2$ *let*

$$\psi_p \;:\; P_\Lambda^{\kappa(p)}(\nu(p)) \otimes P_\Lambda^{\kappa(n-2-p)}(\nu(n-2-p)) \to P_\Lambda^{d^*}(r^*)$$

be the multiplication. Recall that $P_\Lambda^{d^*}(r^*)$ *is generated over* k *by those polynomials of the form* $\phi_\gamma x_n^\alpha t^\beta$ *where* $s\gamma+\alpha+\beta \geq r^*$ *and* $\alpha+\gamma \leq d^*$ *and* ϕ_γ *is a homogeneous polynomial of degree* γ *in* $x_1, \dots, x_{n-1}$. *Assume that* $\nu(p) \geq 0$ *and* $\nu(n-2-p) \geq 0$.

(1) *If* $s = 1$, ψ_p *is surjective. If* $s \geq 2$, $\mathrm{Im}(\psi_p)$ *contains all the above polynomials except for* $\phi_{\mu-1} x_n^\alpha t^\beta$ *with* $\alpha+\beta = s-2$ *where* $\mu = (\delta-2)n+2$.

(2) *The composite of* ψ_p *with the projection* $P_\Lambda^{d^*}(r^*) \to B_F^{d^*}(r^*)$ *is surjective.*

Proof. First we deduce Lemma(3-2)(2) from (1). We may assume $s \geq 2$ and fix $\phi_{\mu-1} x_n^\alpha t^\beta$ with $\alpha+\beta = s-2$ as in (1). By Definition(3-1)(2) $J_F^{d^*}(r^*)$ contains the polynomials

$$x_n^\alpha t^\beta \sum_{1 \leq i \leq n-1} h_i \partial f / \partial x_i$$

where h_i is a homogeneous polynomial of degree $\mu-\delta$ in $x_1, \dots, x_{n-1}$ and f is as in Definition(3-1)(3). In fact we need note that the assumptions $d \geq r+1$ and $\delta \geq 2$ imply $\mu - \delta = (\delta-2)(n-1) \geq 0$ and $d^* - ((s-2) + (\mu-\delta) + (d-1)) = (n-1)(d-\delta) - s - 1 \geq 0$. We can write

$$f = \sum_{a,b,c \geq 0} \Phi_{a,b,c} x_n^a t^b$$

where $\Phi_{a,b,c}$ is a homogeneous polynomial of degree c in $x_1, \dots, x_{n-1}$ and $a, b, c \geq 0$ are integers satisfying $sc + a + b \geq r$ and $c + a \leq d$. By Theorem(1-2) and Proposition(1-3) $\Phi_{0,0,\delta}$ is non-degenerate. By Macaulay's theorem ([D, §2]) any homogeneous polynomial of degree $> (\delta-2)(n-1) = \mu - \delta$ in $x_1, \dots, x_{n-1}$ is in the homogeneous ideal generated by $\partial \Phi_{0,0,\delta} / \partial x_i$ with $1 \leq i \leq n-1$. Noting $\mu - 1 > \mu - \delta \geq 0$ if $\delta \geq 2$, we can thus find h_i homogeneous of degree $\mu - \delta$ in $x_1, \dots, x_{n-1}$ such that

$$\phi_{\mu-1} x_n^\alpha t^\beta \equiv \sum_{1 \leq i \leq n-1} \sum_{(a,b,c) \neq (0,0,\delta)} x_n^{\alpha+a} t^{\beta+b} h_i \partial \Phi_{a,b,c} / \partial x_i \mod J_F^{d^*}(r^*).$$

Thus the desired assertion follows from Lemma(3-2)(1).

Next we prove Lemma(3-2)(1). We prove only the statement in case $s \geq 2$ and leave the (easier) case $s = 1$ to the readers. Put

$$\begin{cases} \kappa_1 = \kappa(p), \ \kappa_2 = \kappa(n-2-p), \ \nu_1 = \nu(p), \ \nu_2 = \nu(n-2-p) \\ \lambda_1 = \delta(p+1) - (n-1), \ \lambda_2 = \delta(n-p-1) - (n-1) \end{cases}$$

We note

$$\begin{cases} d^* = \kappa_1 + \kappa_2, \ \mu = \lambda_1 + \lambda_2, \ \lambda_1, \lambda_2 \geq 1 \\ r^* = s\mu - 2, \ \nu_1 = s\lambda_1 - 1, \ \nu_2 = s\lambda_2 - 1. \end{cases}$$

We have

$$P_\Lambda^{d^*}(r^*) = \sum_{\epsilon \leq \mu} P_\Lambda^{d^*}(r^*)_\epsilon \quad \text{and} \quad P_\Lambda^{\kappa_i}(\nu_i) = \sum_{\tau_i \leq \lambda_i} P_\Lambda^{\kappa_i}(\nu_i)_{\tau_i} \ (i = 1, 2).$$

Here $P_\Lambda^{d^*}(r^*)_\epsilon \subset P_\Lambda^{d^*}(r^*)$ and $P_\Lambda^{\kappa_i}(\nu_i)_{\tau_i} \subset P_\Lambda^{\kappa_i}(\nu_i)$ are the submodule generated by those polynomials of the form

$$(*) \qquad \phi_{\mu-\epsilon} x_n^\alpha t^\beta \quad \text{with } \alpha + \beta \geq s\epsilon - 2 \text{ and } \alpha \leq d^* - \mu + \epsilon,$$

$$\phi_{\lambda_i - \tau_i} x_n^{\alpha_i} t^{\beta_i} \quad \text{with } \alpha_i + \beta_i \geq s\tau_i - 1 \text{ and } \alpha_i \leq \kappa_i - \lambda_i + \tau_i,$$

respectively where ϕ_γ is as in Lemma(3-2). When $\tau_1 + \tau_2 = \epsilon$, the multiplication induces

$$\psi_{\epsilon,\tau_1,\tau_2} \ : \ P_\Lambda^{\kappa_1}(\nu_1)_{\tau_1} \otimes_\Lambda P_\Lambda^{\kappa_1}(\nu_1)_{\tau_1} \to P_\Lambda^{d^*}(r^*)_\epsilon.$$

We note that the assumption $d \geq r + 1$, $n \geq 3$ and $s \geq 2$ implies that $d^* > r^*$ and $\kappa_i > \nu_i$ $(i = 1, 2)$, so that $d^* - \mu + \epsilon > s\epsilon - 2$ and $\kappa_i - \lambda_i + \tau_i > s\tau_i - 1$. From this we see that $\psi_{\epsilon,\tau_1,\tau_2}$ is surjective if either $s\epsilon - 2 \geq 0$, $s\tau_i - 1 \geq 0$ $(i = 1, 2)$ or $s\epsilon - 2 \leq 0$, $s\tau_i - 1 \leq 0$ $(i = 1, 2)$. Unless $\epsilon = 1$, for a given $\epsilon \leq \mu$ we can find $\tau_i \leq \lambda_i$ $(i = 1, 2)$ such that $\tau_1 + \tau_2 = \epsilon$ and that the above condition is satisfied. Thus we get $P_\Lambda^{d^*}(r^*)_\epsilon \subset \mathrm{Im}(\psi_p)$ if $\epsilon \neq 1$. When $\epsilon = 1$, it is easy to see that $\mathrm{Im}(\psi_{1,1,0})$ contains all the polynomials of the form $(*)$ except $\alpha + \beta = s - 2$. This completes the proof. Q.E.D.

Now we deduce Theorem(2-1) from Theorem(3-1) and Lemma(3-2). We use the following duality theorem.

Theorem(3-2). *Let $f : \widetilde{X} \to S = \mathrm{Spec}(\Lambda)$ be the natural morphism. For a locally free $\mathcal{O}_{\widetilde{X}}$-module $\mathcal{F}$, we have the isomorphism in the derived category of bounded complexes of $\mathcal{O}_S$-modules.*

$$\mathbb{R}f_*\mathbb{R}\mathcal{H}om_{\mathcal{O}_{\widetilde{X}}}(\mathcal{F}, \omega_{\widetilde{X}/S}^m[m]) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{O}_S}(\mathbb{R}f_*\mathcal{F}, \mathcal{O}_S).$$

Writing $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_{\widetilde{X}}}(\mathcal{F}, \mathcal{O}_{\widetilde{X}})$, it gives rise to the exact sequence

$$0 \to \mathrm{Ext}_\Lambda^1(H^{m+1-p}(\widetilde{X}, \mathcal{F}), \Lambda) \to H^p(\widetilde{X}, \mathcal{F}^\vee \otimes \omega_{\widetilde{X}/S}^m) \to \mathrm{Hom}(H^{m-p}(\widetilde{X}, \mathcal{F}), \Lambda) \to 0.$$

Proof. This is a consequence of [H]. The key fact is $f^!\mathcal{O}_S = \omega^m_{\widetilde{X}/S}[m]$.
Q.E.D.

By [St] $H^p(\widetilde{X}, \omega^q_{\widetilde{X}/S})$ is a free Λ-module. By Proposition(3-1) the first two assumptions of Theorem(2-1) imply that $H^2(\widetilde{X}, \theta_{\widetilde{X}/S})$ is a free Λ-module. By the long exact sequence induced by

$$0 \longrightarrow \theta_{\widetilde{X}/S} \xrightarrow{t} \theta_{\widetilde{X}/S} \longrightarrow \theta_{Z/S_0} \longrightarrow 0,$$

the last assumption of Theorem(2-1) and the freeness of $H^2(\widetilde{X}, \theta_{\widetilde{X}/S})$ imply that $H^1(\widetilde{X}, \theta_{\widetilde{X}/S})$ is a free Λ-module and

$$H^1(\widetilde{X}, \theta_{\widetilde{X}/S}) \otimes_\Lambda k \xrightarrow{\sim} H^1(Z, \theta_{Z/S_0}).$$

Hence Theorem(3-1) and Theorem(3-2) imply that the dual of $d\rho_Z$ of Theorem(2-1) is equal to the multiplication

$$\Psi : \bigoplus_{0 \le p \le n-2} B_F^{\kappa(p)}(\nu(p)) \otimes_\Lambda B_F^{\kappa(n-2-p)}(\nu(n-2-p)) \otimes_\Lambda k \longrightarrow B_F^{d^*}(r^*) \otimes_\Lambda k.$$

We easily see that $\nu(p) \ge 0$ and $\nu(n-2-p) \ge 0$ if and only if $\frac{n}{\delta} - 1 \le p \le n - 1 - \frac{n}{\delta}$. If n is even or $\delta \ge 3$, there exists $0 \le p \le n-2$ satisfying the condition. Hence Theorem(2-1) in this case follows from Lemma(3-2). Now assume that $\delta = 2$ and n is odd. We note $r^* = r - 2 = 2s - 2$. For $0 \le \forall p \le n-2$, we have either $\nu(p) \ge 2s-1$ or $\nu(n-2-p) \ge 2s-1$ so that the image of multiplication

$$\psi_p : P_\Lambda^{\kappa(p)}(\nu(p)) \otimes_\Lambda P_\Lambda^{\kappa(n-2-p)}(\nu(n-2-p)) \longrightarrow P_\Lambda^{d^*}(r^*) = P_\Lambda^{d^*}(2s-2)$$

is contained in $P_\Lambda^{d^*}(2s-1)$. Taking $q = \frac{n-3}{2}$, we have $0 \le q \le n-2$, $\nu(q) = -1$ and $\nu(n-2-q) = 2s-1$. By the same argument as the proof of Lemma(3-2) we can prove $\mathrm{Im}(\psi_q) = P_\Lambda^{d^*}(2s-1)$. This shows that

$$(*) \qquad \mathrm{Coker}(\Psi) \xrightarrow{\sim} P_\Lambda^{d^*}(2s-2)/P_\Lambda^{d^*}(2s-1) + J_F^{d^*}(2s-2).$$

If $s = 1$, we easily see that the right hand side is of dimension 1 over k. Assume $s \ge 2$. A direct computation shows that $P_\Lambda^{d^*}(2s-2)/P_\Lambda^{d^*}(2s-1)$ is a k-linear space with a basis

$t^\mu x_n^\nu$ with $\mu + \nu = 2s-2$, and $x_i t^a x_n^b$ with $1 \le i \le n-1$, $a + b = s-2$.

By Definition(3-1)(2) the image of $J_F^{d^*}(2s-2)$ in $P_\Lambda^{d^*}(2s-2)/P_\Lambda^{d^*}(2s-1)$ is generated by the classes of $t^a x_n^b \partial f/\partial x_j$ with $1 \le j \le n-1$ and $a+b=s-2$. We can write

$$f = \sum_{\substack{0\le\gamma\le 2\\ \nu+\mu=s(2-\gamma)}} \phi_{\gamma,\nu,\mu} t^\nu x_n^\mu + g \qquad \text{with } g \in P_\Lambda^d(r+1) = P_\Lambda^d(2s+1)$$

where $\phi_{\gamma,\nu,\mu}$ is homogeneous of degree γ in $x_1,\dots,x_{n-1}$. By Theorem(1-2) and Proposition(1-3), $\phi_2 = \phi_{2,0,0}$ is non-degenerate and hence x_i for $1 \le \forall i \le n-1$ is a linear combination of $\partial\phi_2/\partial x_1,\dots,\partial\phi_2/\partial x_{n-1}$. Noting $t^a x_n^b \partial g/\partial x_j \in P_\Lambda^{d^*}(2s-1)$ for $1 \le j \le n-1$ and $a+b=s-2$, this shows that the right hand side of $(*)$ is generated by $t^\mu x_n^\nu$ with $\mu+\nu = 2s-2$ and they are linearly independent over k. This completes the proof of Theorem(2-1). Q.E.D.

§4. Proof of main results

In this section we prove Theorem(3-1), Proposition(2-2) and Proposition(3-1). The key is Proposition(4-1) and Proposition(4-2) below. We maintain the assumption in §2.

Definition(4-1). Let the notation be as in Definition(1-3). Let $H_i \subset \mathbb{P}$ for $1 \le i \le n$ be the hyperplane $X_i = 0$ and let $H_t \subset \mathbb{P}$ be defined by $t=0$. As a divisor on $\widetilde{\mathbb{P}}$ write $\pi^{-1}H_i = \widetilde{H}_i + \mathbb{E}(i)$ and $\pi^{-1}H_t = \widetilde{H}_t + \mathbb{E}(t)$ where $\widetilde{H}_i$ and $\widetilde{H}_t$ are the proper transforms of H_i and H_t respectively. From the computation in the proof of Theorem(1-2) we see (cf. Proposition(1-1))

$$\mathbb{E}(i) = \sum_{1\le k\le s} k\mathbb{E}_k \text{ if } 1 \le i \le n-1 \qquad \text{and} \qquad \mathbb{E}(n) = \mathbb{E}(t) = \sum_{1\le k\le s} \mathbb{E}_k.$$

The first key result concerns the cohomology of the sheaf

$$\mathcal{F} := \pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(\ell) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-a\mathbb{E}(t) - b\mathbb{E}(1)) \quad (\ell, a, b \in \mathbb{Z}).$$

Recall the notation in Definition(1-1). Let $\mathfrak{m}(a,b) \subset k[t,x_1,\dots,x_n]$ be the ideal generated by

$$t^\beta x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

$$\text{with } \beta + \alpha_n + k\Big(\sum_{1\le i\le n-1} \alpha_i\Big) \ge a + kb \quad \text{for } 1 \le \forall k \le s.$$

Let $\mathfrak{m}_\Lambda(a,b) = \mathfrak{m}(a,b) \otimes_{k[t]} \Lambda \subset A_\Lambda$ and write $P_\Lambda^\ell(a,b) = \mathrm{Ker}(P_\Lambda^\ell \to A_\Lambda/\mathfrak{m}_\Lambda(a,b))$. We note $P_\Lambda^\ell(a,b) = P_\Lambda^\ell$ if $a,b \le 0$ and that $\mathfrak{m}_\Lambda(a,b) = \mathfrak{m}_\Lambda(a+sb)$ if $a \le 0$.

Proposition(4-1). *Let $\mathbb{E} \subset \widetilde{\mathbb{P}}_\Lambda$ be the exceptional divisor of $\pi : \widetilde{\mathbb{P}}_\Lambda \to \mathbb{P}_\Lambda$.*

(1) *Letting $i : 0 \to \mathbb{P}_\Lambda$ be the inclusion, we have the exact sequences*

$$0 \to \pi_* \mathcal{F} \to \mathcal{O}_{\mathbb{P}_\Lambda}(\ell) \to i_*(A_\Lambda/\mathfrak{m}_\Lambda(a,b)) \to 0.$$

(2) $H^0(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F}) = P^\ell_\Lambda(a,b)$.

(3) *Assume $a \geq -2$, $b \geq 1-n$ and $a+b \geq -n$. Then we have $R^\nu \pi_* \mathcal{F} = 0$ if $\nu \geq 1$.*

(4) *Under the same assumption as (3) we have:*

$H^\nu(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F}) = 0$ *if* $2 \leq \nu \neq n$ *or* $\nu = n,\ \ell \geq -n$,

$H^1(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F}) = \mathrm{Coker}(P^\ell_\Lambda \to A_\Lambda/\mathfrak{m}_\Lambda(a,b))$,

$H^1(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F}) = 0$ *if* $\ell \geq \max\{a+sb, a+b\} - 1$.

(5) *Under the same assumption as (3) we have*

$$H^i_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F}) \xrightarrow{\sim} \begin{cases} 0 & \text{if } 0 \leq i \neq 1, n+1 \\ A_\Lambda/\mathfrak{m}_\Lambda(a,b) & \text{if } i = 1. \end{cases}$$

The proof of Proposition(4-1) is given in §5.

Definition(4-2). Let $\omega_{\widetilde{\mathbb{P}}_\Lambda/S}$ be the sheaf of logarithmic differentials of the semistable family $\widetilde{\mathbb{P}}_\Lambda$ over $\mathrm{Spec}(\Lambda)$ in the sense of [St]. Let $\omega_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X})$ be the sheaf of logarithmic differentials with additional logarithmic pole along $\widetilde{X}$. By [St] these are locally free $\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}$-modules. Let $\theta_{\widetilde{\mathbb{P}}_\Lambda/S}$ (resp. $\theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log \widetilde{X})$) be the $\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}$-dual of $\omega_{\widetilde{\mathbb{P}}_\Lambda/S}$ (resp. $\omega_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X})$). We write

$$\omega^p_{\widetilde{\mathbb{P}}_\Lambda/S} = \overset{p}{\wedge} \omega_{\widetilde{\mathbb{P}}_\Lambda/S} \quad \text{and} \quad \omega^p_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X}) = \overset{p}{\wedge} \omega_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X}).$$

Recall the Euler exact sequence

$$0 \to \Omega^1_{\mathbb{P}_\Lambda/S} \to \bigoplus_{0 \leq i \leq n} \mathcal{O}_{\mathbb{P}_\Lambda}(-1) \cdot dX_i \to \mathcal{O}_{\mathbb{P}_\Lambda} \to 0 \tag{4-1}$$

and its $\mathcal{O}_{\mathbb{P}_\Lambda}$-dual

$$0 \to \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda} \to \Sigma \to T_{\mathbb{P}_\Lambda/S} \to 0,$$

where $\Sigma = \bigoplus_{0 \leq i \leq n} \mathcal{O}_{\mathbb{P}_\Lambda}(1)$. The second key result gives us the similar exact sequences for $\omega_{\widetilde{\mathbb{P}}_\Lambda/S}$. Write $\mathcal{L} = \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(\widetilde{X})$. By Proposition(1-4) we have the canonical isomorphism

$$\mathcal{L} \simeq \pi^* \mathcal{O}_{\mathbb{P}_\Lambda}(d) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-\delta \cdot \mathbb{E}(1)). \tag{4-2}$$

Proposition(4-2). *We have the exact sequences*

$$0 \to \omega_{\widetilde{\mathbb{P}}_\Lambda/S} \to \pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(-1)\cdot dX_0 \oplus \bigoplus_{1\le i\le n} \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(\mathbb{E}(i))\otimes \pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(-1)\cdot dX_i \to \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda} \to 0,$$

and its $\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}$*-dual*

$$0 \to \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda} \xrightarrow{\iota} \pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(1)\cdot\partial_0 \oplus \bigoplus_{1\le i\le n} \pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(1)\otimes\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-\mathbb{E}(i))\cdot\partial_i \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S} \to 0$$

where $\iota(1) = \sum_{0\le i\le n} X_i\partial_i$. *Denoting by* $\widetilde{\Sigma}$ *the sheaf at the middle of the second exact sequence, we have the following exact sequence*

$$0 \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X}) \to \widetilde{\Sigma} \xrightarrow{j_F} \mathcal{L} \to 0,$$

where $j_F(\partial_i) = \partial F/\partial X_i \in P_\Lambda^{d-1}(r-\mu_i)$ *for* $0\le i\le n$ *(cf. Definition*(3-1)(2)) *that induces by* (4-2) *and Proposition*(4-1)(2) *the map*

$$\pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(1) \to \mathcal{L}\ (i=0)$$

$$\textit{and}\quad \pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(1)\otimes\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-\mathbb{E}(i)) \to \mathcal{L}\ (1\le i\le n).$$

The proof will be given in the next section. We also need the following auxiliary results.

Proposition(4-3). *We have the exact sequences*

$$0 \to \omega^p_{\widetilde{\mathbb{P}}_\Lambda/S} \to \omega^p_{\widetilde{\mathbb{P}}_\Lambda/S}(\log\widetilde{X}) \to \omega^{p-1}_{\widetilde{X}/S} \to 0,$$

$$0 \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S}\otimes\mathcal{L}^{-1} \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X}) \to \theta_{\widetilde{X}/S} \to 0.$$

Proof. The proof is standard and left to the readers. Q.E.D.

Proposition(4-4). *We have the natural isomorphism*

$$\omega^m_{\widetilde{X}/S} \xrightarrow{\sim} \pi^*(\mathcal{O}_{\mathbb{P}_\Lambda}(d-n-1))\otimes\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(\mathbb{E}(t)+(n-1-\delta)\mathbb{E}(1))\otimes\mathcal{O}_{\widetilde{X}}.$$

Proof. By the first sequence of Proposition(4-2)

$$\omega^n_{\widetilde{\mathbb{P}}_\Lambda/S}(\log\widetilde{X}) = \mathcal{L}\otimes\mathcal{E} \tag{4-3}$$

with $\mathcal{E} = \pi^*(\mathcal{O}_{\mathbb{P}_\Lambda}(-n-1))\otimes\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(\mathbb{E}(t)+(n-1)\mathbb{E}(1))$.

Thus Proposition(4-4) follows from Proposition(4-3). Q.E.D.

Remark(4-1). Proposition(4-4) immediately implies the following fact: The singularity $(X,0)$ is canonical if and only if either $\delta \leq n-1$ or $\delta = n,\ s=1$. If $\delta \geq n+2$, $(X,0)$ is not log-canonical.

Now we prove Theorem(3-1) and Proposition(3-1). First we show Theorem(3-1)(1). By [St] we know that $H^p(\widetilde{X}, \omega^q_{\widetilde{X}/S})$ is a free Λ-module. By Remark(3-1) it suffices to show $B_F^{\kappa(p)}(\nu(p)) \xrightarrow{\sim} H^p(\widetilde{X}, \omega^q_{\widetilde{X}/S})_{prim}$. We note

$$(4\text{-}4) \qquad H^q(\mathbb{P}_\Lambda, \Omega^p_{\mathbb{P}_\Lambda/S}) \xrightarrow{\sim} H^q(\widetilde{\mathbb{P}}_\Lambda, \omega^p_{\widetilde{\mathbb{P}}_\Lambda/S}) \quad \text{for } \forall p, q \geq 0.$$

To see this it suffices to show $\pi_*(\omega^p_{\widetilde{\mathbb{P}}_\Lambda/S}) = \Omega^p_{\mathbb{P}_\Lambda/S}$ and $R^\nu \pi_*(\omega^p_{\widetilde{\mathbb{P}}_\Lambda/S}) = 0$ for $\forall \nu \geq 1$. The first sequence of Proposition(4-2) induces the following exact sequence

$$0 \to \omega^p_{\widetilde{\mathbb{P}}_\Lambda/S} \to \overset{p}{\wedge}\widetilde{\Sigma}^* \to \overset{p-1}{\wedge}\widetilde{\Sigma}^* \to \cdots \to \widetilde{\Sigma}^* \to \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda} \to 0,$$

where $\widetilde{\Sigma}^*$ is the $\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}$-dual of $\widetilde{\Sigma}$. By Proposition(4-1)(3) and (1), $R^\nu\pi_*(\overset{\mu}{\wedge}\widetilde{\Sigma}^*) = 0$ for $\forall \nu \geq 1$, $\forall \mu \geq 0$ and $\pi_*(\overset{\mu}{\wedge}\widetilde{\Sigma}^*) = \overset{\mu}{\wedge}\Sigma^*$ with $\Sigma^* = \bigoplus_{0\leq i\leq n} \mathcal{O}_{\mathbb{P}_\Lambda}(-1)$. This shows the second assertion and that we have the exact sequence

$$0 \to \pi_*\omega^p_{\widetilde{\mathbb{P}}_\Lambda/S} \to \overset{p}{\wedge}\Sigma^* \to \overset{p-1}{\wedge}\Sigma^* \to \cdots \Sigma^* \to \mathcal{O}_{\mathbb{P}_\Lambda} \to 0.$$

Compared with the similar exact sequence induced by (4-1), this proves $\pi_*\omega^p_{\widetilde{\mathbb{P}}_\Lambda/S} = \Omega^p_{\mathbb{P}_\Lambda/S}$. By (4-4) and Proposition(4-3) and the Bott vanishing of $H^q(\mathbb{P}_\Lambda, \Omega^q_{\mathbb{P}_\Lambda/S})$, Theorem(3-1)(1) follows from the isomorphism

$$B_F^{\kappa(p)}(\nu(p)) \xrightarrow{\sim} H^p(\widetilde{\mathbb{P}}_\Lambda, \omega^{n-p}_{\widetilde{\mathbb{P}}_\Lambda/S}(\log\widetilde{X})).$$

By (4-3) we have $\overset{p}{\wedge}\theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X}) \otimes \mathcal{L} \otimes \mathcal{E} = \omega^{n-p}_{\widetilde{\mathbb{P}}_\Lambda/S}(\log\widetilde{X})$. Therefore the last sequence of Proposition(4-2) gives rise to the following exact sequence

$$0 \to \omega^{n-p}_{\widetilde{\mathbb{P}}_\Lambda/S}(\log\widetilde{X}) \to \mathcal{C}^0_p \to \mathcal{C}^1_p \to \cdots \to \mathcal{C}^p_p \to 0,$$

where $\mathcal{C}^a_p = \overset{p-a}{\wedge}\widetilde{\Sigma} \otimes \mathcal{L}^{a+1} \otimes \mathcal{E}$ for $0 \leq a \leq p$. Thus we obtain the spectral sequence

$$E_1^{a,b} = H^b(\widetilde{\mathbb{P}}_\Lambda, \mathcal{C}^a_p) \Rightarrow H^{a+b}(\widetilde{\mathbb{P}}_\Lambda, \omega^{n-p}_{\widetilde{\mathbb{P}}_\Lambda/S}(\log\widetilde{X})).$$

By definition $\mathcal{C}_p^a$ is the direct sum of $\pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(\ell)\otimes\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-\alpha\mathbb{E}(t)-\beta\mathbb{E}(1))$ with

$$\begin{cases}\ell = d(a+1)-n-1+p-a,\\ \beta=\delta(a+1)+p-a-(n-1)-\epsilon \text{ with } \epsilon=0 \text{ or } 1 \text{ or } 2,\\ \alpha=-1 \text{ or } 0 \text{ and } (\alpha,\epsilon)\neq(0,0),\end{cases}$$

We have $\ell\geq -n$ and $\beta\geq 1-n$ by the assumption that $d\geq 3$, $\delta\geq 2$ and $p\geq a\geq 0$. By Proposition(4-1)(4), $E_1^{a,b}=0$ for $\forall b\geq 1$ if

$$d(a+1)-n-1+p-a\geq c(\delta(a+1)+p-a-(n-1))-1 \quad \text{for } c=1,s.$$

It is easy to see that this holds under the assumption $d\geq r+1$, $s\geq 1$ and $p\leq n-1$. Thus the spectral sequence degenerates at E_2 and we get the isomorphism

$$H^p(\widetilde{\mathbb{P}}_\Lambda,\omega^{n-p}_{\widetilde{\mathbb{P}}_\Lambda/S}(\log\widetilde{X}))\xrightarrow{\sim}\mathrm{Coker}(H^0(\widetilde{\mathbb{P}}_\Lambda,\widetilde{\Sigma}\otimes\mathcal{L}^p\otimes\mathcal{E})\xrightarrow{j_F}H^0(\widetilde{\mathbb{P}}_\Lambda,\mathcal{L}^{p+1}\otimes\mathcal{E})).$$

This shows the desired isomorphism by Proposition(4-1)(2) and Definition(3-1)(2). Q.E.D.

Next we show Proposition(3-1). First we show

$$\text{(4-5)} \qquad H^\nu(\widetilde{\mathbb{P}}_\Lambda,\theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X}))=0 \quad \text{for } \nu\geq 2$$

Indeed this follows from the second sequence of Proposition(4-2) and the fact $H^\nu(\widetilde{\mathbb{P}}_\Lambda,\mathcal{L})=H^{\nu+1}(\widetilde{\mathbb{P}}_\Lambda,\widetilde{\Sigma})=0$ for $\forall\nu\geq 1$ by Proposition(4-1)(4). We note that the assumption $d\geq r$ is used for the vanishing of $H^1(\widetilde{\mathbb{P}}_\Lambda,\mathcal{L})$. By (4-5) Proposition(4-3) implies

$$H^j(\widetilde{X},\theta_{\widetilde{X}/S})\xrightarrow{\sim}H^{j+1}(\widetilde{\mathbb{P}}_\Lambda,\theta_{\widetilde{\mathbb{P}}_\Lambda/S}\otimes\mathcal{L}^{-1}) \quad \text{for } j\geq 2.$$

By using the exact sequence

$$0\to\mathcal{L}^{-1}\to\widetilde{\Sigma}\otimes\mathcal{L}^{-1}\to\theta_{\widetilde{\mathbb{P}}_\Lambda/S}\otimes\mathcal{L}^{-1}\to 0$$

coming from Proposition(4-2), the same argument as the proof of (4-4) shows

$$\text{(4-6)} \qquad \begin{aligned} H^i(\widetilde{\mathbb{P}}_\Lambda,\theta_{\widetilde{\mathbb{P}}_\Lambda/S}\otimes\mathcal{L}^{-1}) &\xrightarrow{\sim}H^i(\mathbb{P}_\Lambda,T_{\mathbb{P}_\Lambda/S}(-d))\\ &\xrightarrow{\sim}H^i(\mathbb{P}_\Lambda,\Omega^{n-1}_{\mathbb{P}_\Lambda/S}(n+1-d)) \quad \text{for } \forall i\geq 0.\end{aligned}$$

We note that the assumption $\delta<n$ is used to get

$$R^\nu\pi_*\mathcal{L}^{-1}=R^\nu\pi_*(\widetilde{\Sigma}\otimes\mathcal{L}^{-1})=0 \quad \text{for } \forall\nu\geq 1$$

by applying Proposition(4-1)(3). Noting that the last group in (4-6) is torsion free and that it vanishes if either $i \leq n-2$ or $i = n-1$ and $d \neq n+1$ or $i = n$ and $d \leq n+1$, it implies Proposition(3-1). Q.E.D.

Next we show Theorem(3-1)(2). Proposition(3-1) implies that $H^2(\widetilde{X}, \theta_{\widetilde{X}/S})$ is free, which implies by Theorem(3-2) that $H^{m-1}(\widetilde{X}, \omega^1_{\widetilde{X}/S} \otimes \omega^m_{\widetilde{X}/S})$ is free. By Proposition(4-3) and Proposition(4-4) we have the exact sequence

$$0 \to \omega^2_{\widetilde{\mathbb{P}}_\Lambda/S} \otimes \mathcal{L} \otimes \mathcal{E} \to \omega^2_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X}) \otimes \mathcal{L} \otimes \mathcal{E} \to \omega^1_{\widetilde{X}/S} \otimes \omega^m_{\widetilde{X}/S} \to 0.$$

By the same argument as the proof of Theorem(3-1)(1) we can show the isomorphsim

$$B_F^{d^*}(r^*) \xrightarrow{\sim} H^{n-2}(\widetilde{\mathbb{P}}_\Lambda, \omega^2_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X}) \otimes \mathcal{L} \otimes \mathcal{E}).$$

By the same argument as the proof of (4-4) we can show

$$H^q(\widetilde{\mathbb{P}}_\Lambda, \omega^2_{\widetilde{\mathbb{P}}_\Lambda/S} \otimes \mathcal{L} \otimes \mathcal{E}) \xrightarrow{\sim} H^q(\mathbb{P}_\Lambda, \Omega^2_{\mathbb{P}_\Lambda/S}(d-n-1))$$

and it vanishes for $1 \leq \forall q \neq n$ unless $q = 2$ and $d = n+1$. It implies

$$H^{n-2}(\widetilde{\mathbb{P}}_\Lambda, \omega^2_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X}) \otimes \mathcal{L} \otimes \mathcal{E}) \xrightarrow{\sim} H^{m-1}(\widetilde{X}, \omega^1_{\widetilde{X}/S} \otimes \omega^m_{\widetilde{X}/S})$$

if $n \geq 4$ or $n = 3, d \neq 4$. Hence Theorem(3-1)(2) is proven. Q.E.D.

Next we prove Theorem(3-1)(3). Let

$$H^1(\widetilde{\mathbb{P}}_\Lambda, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log \widetilde{X})) \to H^1(\widetilde{X}, \theta_{\widetilde{X}/S})$$

be the map induced by the last sequence of Proposition(4-3). By (4-6) it is injective if $n \geq 3$ and an isomorphism if either $n \geq 4$ or $n = 3$ and $d \neq 4$. Therefore it suffices to show the following.

Proposition(4-5). *Assume $d \geq r$. There are the exact sequences*

$$0 \to B_F^d(r) \to H^1(\widetilde{\mathbb{P}}_\Lambda, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log \widetilde{X})) \to \bigoplus_{1 \leq i \leq n-1} A_\Lambda/\mathfrak{m}_\Lambda(s) + A_\Lambda^{\leq 1} \to 0,$$

$$H^1(\widetilde{\mathbb{P}}_\Lambda, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log \widetilde{X})) \xrightarrow{\iota} B_F^d \to A_\Lambda/(\mathfrak{m}_\Lambda(r) + I_f)$$

and $\mathrm{Ker}(\iota)$ *is torsion.*

Proof. The first sequence follows immediately from the last sequence of Proposition(4-2) and Proposition(4-1). To show the second we consider the localization sequence

$$H^1_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \Theta) \to H^1(\widetilde{\mathbb{P}}_\Lambda, \Theta) \to H^1(\mathbb{U}, \Theta_{|\mathbb{U}}) \to H^2_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \Theta),$$

where $\Theta = \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X})$ and $\mathbb{U} = \widetilde{\mathbb{P}}_\Lambda - \mathbb{E} = \mathbb{P}_\Lambda - \{0\}$. We know that $H^i_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \Theta)$ is a torsion Λ-module if $i \geq 1$. Thus it suffices to show

$$H^1(\mathbb{U}, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X})_{|\mathbb{U}}) \xrightarrow{\sim} B^d_F$$

and

$$H^2_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X})) \xrightarrow{\sim} A_\Lambda/\mathfrak{m}_\Lambda(r) + I_f.$$

The first isomorphism follows easily from the exact sequence

$$0 \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X})_{|\mathbb{U}} \to \bigoplus_{0\leq i\leq n} \mathcal{O}_{\mathbb{P}_\Lambda}(1)_{|\mathbb{U}} \to \mathcal{O}_{\mathbb{P}_\Lambda}(d)_{|\mathbb{U}} \to 0$$

induced by the last sequence of Proposition(4-2). It also induces the exact sequence

$$H^1_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \widetilde{\Sigma}) \to H^1_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \mathcal{L}) \to H^2_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X})) \to H^2_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \widetilde{\Sigma}).$$

Hence the second isomorphism follows from Proposition(4-1)(5). Q.E.D.

Finally we show Proposition(2-2). Proposition(2-2)(1) is a direct consequence of Proposition(3-1). To show Proposition(2-2)(2) we start with the following exact sequence induced by Proposition(4-3):

$$0 \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S} \otimes \mathcal{L}^{-1} \otimes_\Lambda k \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X}) \otimes_\Lambda k \to \theta_{Z/S_0} \to 0.$$

By (4-6) it induces $H^0(\widetilde{\mathbb{P}}_{\Lambda,0}, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X}) \otimes_\Lambda k) \xrightarrow{\sim} H^0(Z, \theta_{Z/S_0})$ where $\widetilde{\mathbb{P}}_{\Lambda,0}$ is the special fiber of $\widetilde{\mathbb{P}}_\Lambda$. To compute the left hand side we use the exact sequence induced by the last sequence of Proposition(4-2):

$$(*) \qquad 0 \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X}) \otimes_\Lambda k \to \widetilde{\Sigma} \otimes_\Lambda k \to \mathcal{L} \otimes_\Lambda k \to 0.$$

Writing $\mathcal{F} = \pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(1) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-\mathbb{E}(i)))$, we have the following exact sequence by Proposition(4-1)

$$0 \to \pi_*\mathcal{F} \to \mathcal{O}_{\mathbb{P}_\Lambda}(1) \to i_*A_\Lambda/\mathfrak{m}_\Lambda(\mu_i) \to 0$$

where μ_i is as in Definition(3-1)(2). It induces the exact sequence

$$\begin{aligned}0 \to A_\Lambda/\mathfrak{m}_\Lambda(\mu_i)[t] &\to H^0(\mathbb{P}_{\Lambda,0}, (\pi_*\mathcal{F}) \otimes_\Lambda k)\\ &\to H^0(\mathbb{P}_{\Lambda,0}, \mathcal{O}_{\mathbb{P}_\Lambda}(1) \otimes_\Lambda k) \to (A_\Lambda/\mathfrak{m}_\Lambda(\mu_i)) \otimes_\Lambda k\end{aligned}$$

where $\mathbb{P}_{\Lambda,0}$ is the special fiber of $\mathbb{P}_\Lambda$ and $M[t] = \mathrm{Ker}(M \xrightarrow{t} M)$ for a Λ-module M. Hence we get the exact sequences

$$\begin{aligned}0 \to &A_\Lambda/\mathfrak{m}_\Lambda(\mu_i)[t] \\ &\to H^0(\widetilde{\mathbb{P}}_{\Lambda,0}, (\pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(1) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-\mathbb{E}(i))) \otimes_\Lambda k) \to P_k^1(\mu_i) \to 0,\end{aligned}$$

by noting that $(\pi_*\mathcal{F})\otimes_\Lambda k = \pi_*(\mathcal{F}\otimes_\Lambda k)$ since $R^1\pi_*\mathcal{F} = 0$ by Proposition(4-1)(3). Here, for integers $q, \nu \geq 0$ we write $P_k^q(\nu) = \mathrm{Ker}(P_k^q \to A_k/\mathfrak{m}_k(\nu))$ with $P_k^q = P_\Lambda^q \otimes_\Lambda k$, $A_k = A_\Lambda \otimes_\Lambda k$ and $\mathfrak{m}_k(\nu) \subset A_k$ is the image of $\mathfrak{m}_\Lambda(\nu)$. By the same argument we get the exact sequence

$$0 \to A_\Lambda/\mathfrak{m}_\Lambda(r)[t] \to H^0(\widetilde{\mathbb{P}}_{\Lambda,0}, \mathcal{L} \otimes_\Lambda k) \to P_k^d(r) \to 0.$$

Combining $(*)$ with the last two exact sequences, we get the exact sequence

$$0 \to T_0 \to H^0(\widetilde{\mathbb{P}}_{\Lambda,0}, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X}) \otimes_\Lambda k) \to T_1$$

where T_0 is the kernel of the map

$$\bigoplus_{1\leq i\leq n} A_\Lambda/\mathfrak{m}_\Lambda(\mu_i)[t] \to A_\Lambda/\mathfrak{m}_\Lambda(r);$$
$$(h_i)_{1\leq i\leq n} \mapsto \sum_{1\leq i\leq n} h_i \cdot \partial f/\partial x_i \mod \mathfrak{m}_\Lambda(r)$$

and T_1 is the kernel of the map

$$\bigoplus_{0\leq i\leq n} P_k^1(\mu_i) \to P_k^d(r);\ (H_i)_{0\leq i\leq n} \to \sum_{0\leq i\leq n} H_i \cdot \partial F_0/\partial X_i$$

and hence it vanishes under the assumption of Proposition(2-2)(2). Thus it suffices to show $T_0 = 0$. In what follows we assume $s \geq 2$ and leave the case $s = 1$ to the readers. We see

$$T_0 = \left\{\partial = \sum_{1\leq i\leq n-1} \sum_{\substack{\mu+\nu=s \\ \mu\geq 1, \nu\geq 0}} a_{i,\mu,\nu} t^{\mu-1} x_n^\nu \partial/\partial x_i + b\partial/\partial x_n \;\middle|\; a_{i,\mu,\nu},\ b \in k,\ \partial f \in \mathfrak{m}_\Lambda(r)\right\}$$

It is easy to see that $T_0 = 0$ if and only if there is no non-trivial relation such as

$$(**) \qquad \sum_{1\leq i\leq n-1} \sum_{\substack{\mu+\nu=s \\ \mu\geq 1, \nu\geq 0}} a_{i,\mu,\nu} x_0^{\mu-1} x_n^\nu \partial f_0^{hom}/\partial x_i + b\partial f_0^{hom}/\partial x_n = 0$$

in $k[x_0, x_1, \ldots, x_n]$ where f_0^{hom} is as (1-2) in §1. First assume $b \neq 0$. Let $I \subset k[x_0, \ldots, x_n]$ be the ideal generated by

$$x_0 \partial f_0^{hom}/\partial x_0, \partial f_0^{hom}/\partial x_1, \ldots \partial f_0^{hom}/\partial x_n.$$

The relation implies that I is generated only by n elements so that $\mathbb{P}(Q) \supset \mathrm{Sup}(k[x_0, \ldots, x_n]/I) \neq \emptyset$. This contradicts Theorem(1-2). Next assume $b = 0$. We can write $f_0^{hom} = \sum_{s\alpha+\nu+\mu=r} \Phi_{\alpha,\nu,\mu} x_0^\nu x_n^\mu$ where $\Phi_{\alpha,\nu,\mu}$ is homogeneous of degree α in $x_1, \ldots, x_{n-1}$. Then, putting $\Phi_\delta = \Phi_{\delta,0,0}$, $(**)$ implies

$$0 = \sum_{\substack{\mu+\nu=s \\ \mu\geq 1, \nu\geq 0}} x_0^{\mu-1} x_n^\nu \Big(\sum_{1\leq i\leq n-1} a_{i,\mu,\nu} \partial \Phi_\delta / \partial x_i \Big)$$

$$+ \text{ (terms of degree } \leq \delta - 2 \text{ in } x_1, \ldots, x_{n-1}).$$

Since $\Phi_\delta = f_0^{hom}(0, x_1, \ldots, x_{n-1}, 0)$ is non-degenerated by Theorem(1-2) and Proposition(1-3), it implies $a_{i,\mu,\nu} = 0$ for $\forall i, \nu, \mu$ and the proof is completed. Q.E.D.

§5. Proof of key propositions

In this section we prove Proposition(4-1) and Proposition(4-2). First we show Proposition(4-1). By the projection formula

$$R^\nu \pi_* \mathcal{F} = \mathcal{O}_{\mathbb{P}_\Lambda}(\ell) \otimes R^\nu \pi_* \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-a\mathbb{E}(t) - b\mathbb{E}(1)) \quad \text{for } \forall \nu \geq 0.$$

Hence Proposition(4-1)(1) and (3) follow from Proposition(5-1) below. Proposition(4-1)(2) is a direct consequence of (1). Next we show Proposition(4-1)(4). Proposition(4-1)(3) implies $H^\nu(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F}) \xrightarrow{\sim} H^\nu(\mathbb{P}_\Lambda, \pi_*\mathcal{F})$ for $\forall \nu \geq 0$. Therefore Proposition(4-1)(1) implies the following isomorphism and the exact sequence

$$H^\nu(\mathbb{P}_\Lambda, \pi_*\mathcal{F}) \xrightarrow{\sim} H^\nu(\mathbb{P}_\Lambda, \mathcal{O}_{\mathbb{P}_\Lambda}(\ell)) \quad \text{for } \nu \geq 2,$$

$$\begin{aligned} 0 \to H^0(\mathbb{P}_\Lambda, \pi_*\mathcal{F}) &\to H^0(\mathbb{P}_\Lambda, \mathcal{O}_{\mathbb{P}_\Lambda}(\ell)) \\ &\to A_\Lambda/\mathfrak{m}_\Lambda(a,b) \to H^1(\mathbb{P}_\Lambda, \pi_*\mathcal{F}) \to 0. \end{aligned}$$

Thus Proposition(4-1)(4) follow from the standard vanishing of $H^\nu(\mathbb{P}_\Lambda, \mathcal{O}_{\mathbb{P}_\Lambda}(\ell))$ except that the vanishing of $H^1(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F})$ requires Lemma(5-1) below. Finally, to show Proposition(4-1)(5), we use the localization sequence

$$\cdots \to H^i_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F}) \to H^i(\widetilde{\mathbb{A}}_\Lambda, \mathcal{F}) \to H^i(\widetilde{\mathbb{A}}_\Lambda - \mathbb{E}, \mathcal{F}) \to \cdots.$$

By [H, §9 Theorem9.1] we have

$$H^i(\widetilde{\mathbb{A}}_\Lambda - \mathbb{E}, \mathcal{F}) = H^i(\mathbb{A}_\Lambda - \{0\}, \mathcal{O}_{\mathbb{A}_\Lambda}) = \begin{cases} 0 & \text{if } i \neq 0, n, \\ A_\Lambda & \text{if } i = 0. \end{cases}$$

Thus Proposition(4-1)(5) follows from Proposition(5-1). Q.E.D.

Lemma(5-1). *The map $P^\ell_\Lambda \to A_\Lambda/\mathfrak{m}_\Lambda(a,b)$ is surjective if $\ell \geq \max\{a+sb, a+b\} - 1$.*

Proof. This follows from the fact that $(x_1, \ldots, x_n)^\nu \subset \mathfrak{m}_\Lambda(a,b)$ with $\nu = \max\{a+sb, a+b\}$. Q.E.D.

Proposition(5-1). *Let the notation be as in Definition*(1-3). *Let a, b be integers.*

(1) $H^0(\widetilde{\mathbb{A}}, \mathcal{O}_{\widetilde{\mathbb{A}}}(-a\mathbb{E}(t) - b\mathbb{E}(1))) = \mathfrak{m}(a,b) \subset k[t, x_1, \ldots, x_n]$.

(2) $H^i(\widetilde{\mathbb{A}}, \mathcal{O}_{\widetilde{\mathbb{A}}}(-a\mathbb{E}(t) - b\mathbb{E}(1))) = 0$ *for* $\forall i \geq 1$ *if* $a \geq -2$, $b \geq 1-n$ *and* $a + b \geq -n$.

Proof. Proposition(5-1)(1) follows from the standard description of the space of global sections of line bundles on toric varieties (cf. [F,§3]). For the proof of Proposition(5-1)(2) we need a preparation. Let $\mathbb{A}^\ell = \mathrm{Spec}(k[x_1, \ldots, x_\ell])$ be an affine space over k. A refinement of $\mathbb{A}^\ell$ is the proper morphism of toric varieties $X(\Delta) \to X(\Delta_0) = \mathbb{A}^\ell$ where $\Delta_0 = \{\sigma_0\}$ is as in §1 and Δ is a refinement of Δ_0. We use the following standard fact from toric geometry (cf. [F, §3.5, Proposition]).

Proposition(5-2). *Let $X(\Delta) \to \mathbb{A}^\ell$ be a refinement and let $\mathcal{F} = \mathcal{O}_{X(\Delta)}(D)$ with a Cartier divisor D such that $\mathcal{F}$ is generated by global sections. Then $H^i(X(\Delta), \mathcal{F}) = 0$ for $\forall i \geq 1$.*

Now we show Proposition(5-1)(2). First assume $a \geq 0$. If $b \geq 0$, $\mathcal{O}_{\widetilde{\mathbb{A}}}(-a\mathbb{E}(t) - b\mathbb{E}(1))$ is generated by global sections. Hence the vanishing follows from Proposition(5-2). If $1 - n \leq b \leq -1$ we use the exact sequence

$$0 \to \mathcal{O}_{\widetilde{\mathbb{A}}}(-a\mathbb{E}(t) - b\mathbb{E}(1)) \to \mathcal{O}_{\widetilde{\mathbb{A}}}(-a\mathbb{E}(t) - (b+1)\mathbb{E}(1)) \\ \to \mathcal{O}_{\widetilde{H}_{n-1}}(-a\mathbb{E}_{\widetilde{H}_{n-1}}(t) - (b+1)\mathbb{E}_{\widetilde{H}_{n-1}}(1)) \to 0$$

that follows from $\mathcal{O}_{\widetilde{\mathbb{A}}}(\mathbb{E}(1)) \overset{x_{n-1}}{\xrightarrow{\sim}} \mathcal{O}_{\widetilde{\mathbb{A}}}(-\widetilde{H}_{n-1})$, where $\widetilde{H}_i$ is as in Definition(4-1) and $\mathbb{E}_{\widetilde{H}_i}(*)$ is the pull back of $\mathbb{E}(*)$ to $\widetilde{H}_i$. Note that

$\widetilde{H}_{n-1}$ is a refinement of the affine space H_{n-1} and that the map

$$H^0(\mathcal{O}_{\widetilde{\mathbb{A}}}(-a\mathbb{E}(t)-(b+1)\mathbb{E}(1))) \to H^0(\mathcal{O}_{\widetilde{H}_{n-1}}(-a\mathbb{E}_{\widetilde{H}_{n-1}}(t) - (b+1)\mathbb{E}_{\widetilde{H}_{n-1}}(1)))$$

is surjective by Proposition(5-1)(2) and its variant for the toric variety $\widetilde{H}_{n-1}$. The desired vanishing is now reduced to the case $b = 0$ by induction.

In case $a = -1$ we use the exact sequence

$$0 \to \mathcal{O}_{\widetilde{\mathbb{A}}}(\mathbb{E}(t) - b\mathbb{E}(1)) \to \mathcal{O}_{\widetilde{\mathbb{A}}}(-b\mathbb{E}(1)) \to \mathcal{O}_{\widetilde{H}_t}(-b\mathbb{E}_{\widetilde{H}_t}(1)) \to 0$$

that follows from $\mathcal{O}_{\widetilde{\mathbb{A}}}(\mathbb{E}(t)) \overset{t}{\underset{\sim}{\to}} \mathcal{O}_{\widetilde{\mathbb{A}}}(-\widetilde{H}_t)$. Thus we are reduced to the case $a = 0$ by the same argument as before.

In case $a = -2$ we use the exact sequence

$$0 \to \mathcal{O}_{\widetilde{\mathbb{A}}}(2\mathbb{E}(t) - b\mathbb{E}(1)) \to \mathcal{O}_{\widetilde{\mathbb{A}}}(\mathbb{E}(t) - b\mathbb{E}(1)) \to \mathcal{O}_{\widetilde{H}_n}(\mathbb{E}_{\widetilde{H}_n}(t) - b\mathbb{E}_{\widetilde{H}_n}(1)) \to 0$$

that follows from $\mathcal{O}_{\widetilde{\mathbb{A}}}(\mathbb{E}(t)) \overset{x_n}{\underset{\sim}{\to}} \mathcal{O}_{\widetilde{\mathbb{A}}}(-\widetilde{H}_n)$. Thus we are reduced to the case $a = -1$ by the same argument as before. This completes the proof of Proposition(5-1)(2). Q.E.D.

Next we show Proposition(4-2). Recall the notation of Definition(1-3). We start with the exact sequence

$$0 \to \Omega_{\mathbb{P}/k}\Big(\log \sum_{1\le i\le n} H_i + H_t\Big) \to \mathcal{O}_{\mathbb{P}}(-1)\cdot dX_0 \oplus \bigoplus_{1\le i\le n} \mathcal{O}_{\mathbb{P}}\frac{dX_i}{X_i} \oplus \mathcal{O}_{\mathbb{P}}\frac{dt}{t} \to \mathcal{O}_{\widetilde{\mathbb{P}}} \to 0.$$

It is easy to see

$$\pi^*\Omega_{\mathbb{P}/k}\Big(\log \sum_{0\le i\le n} H_i + H_t\Big) \overset{\sim}{\to} \Omega_{\widetilde{\mathbb{P}}/k}\Big(\log \sum_{1\le i\le n} \widetilde{H}_i + \widetilde{H}_t + \mathbb{E}\Big).$$

Hence we get the exact sequence

$$0 \to \Omega_{\widetilde{\mathbb{P}}/k}\Big(\log \sum_{1\le i\le n} \widetilde{H}_i + \widetilde{H}_t + \mathbb{E}\Big) \to \pi^*\mathcal{O}_{\mathbb{P}}(-1)\cdot dX_0 \oplus \bigoplus_{1\le i\le n} \mathcal{O}_{\widetilde{\mathbb{P}}}\frac{dX_i}{X_i} \oplus \mathcal{O}_{\widetilde{\mathbb{P}}}\frac{dt}{t} \to \mathcal{O}_{\widetilde{\mathbb{P}}} \to 0.$$

Noting $\mathcal{O}_{\widetilde{\mathbb{P}}}(\widetilde{H}_i)\otimes\mathcal{O}_{\widetilde{\mathbb{P}}}(\mathbb{E}(i))=\pi^*\mathcal{O}_{\mathbb{P}}(H_i)\overset{X_i}{\underset{\sim}{\to}}\pi^*\mathcal{O}_{\mathbb{P}}(1)$, it induces the exact sequence

$$\begin{aligned}0\to&\Omega_{\widetilde{\mathbb{P}}/k}(\log\widetilde{H}_t+\mathbb{E})\\ \to&\pi^*\mathcal{O}_{\mathbb{P}}(-1)\cdot dX_0\oplus\bigoplus_{1\le i\le n}\mathcal{O}_{\widetilde{\mathbb{P}}}(\mathbb{E}(i))\otimes\pi^*\mathcal{O}_{\mathbb{P}}(-1)\cdot dX_i\oplus\mathcal{O}_{\widetilde{\mathbb{P}}}\frac{dt}{t}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\to\mathcal{O}_{\widetilde{\mathbb{P}}}\to 0.\end{aligned}$$

The first exact sequence of Proposition(4-2) is an immediate consequence of this. To deduce the last exact sequence from the first one, we use the commutative diagram

$$\begin{array}{ccccccccc}0\to&\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}&\to&\widetilde{\Sigma}&\to&\theta_{\widetilde{\mathbb{P}}_\Lambda/S}&\to 0\\ &\|&&\downarrow j_F&&\downarrow i_F&\\ 0\to&\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}&\to&\mathcal{L}&\to&\mathcal{L}\otimes\mathcal{O}_{\widetilde{X}}&\to 0\end{array}$$

where i_F is induced by j_F. It suffices to show that i_F is surjective and that its kernel is $\theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X})$. Restricted on $\widetilde{\mathbb{P}}_\Lambda-\mathbb{E}=\mathbb{P}_\Lambda-\{0\}$, the assertion follows from the condition Theorem(1-2)(1). Thus it suffices to verify it on each U_σ with $\sigma\in\Delta_s$. Restricted on $\widetilde{\mathbb{A}}_\Lambda$, $\mathcal{L}_{|\widetilde{\mathbb{A}}_\Lambda}=\mathcal{O}_{\widetilde{\mathbb{A}}_\Lambda}(-\delta\mathbb{E}(1))$ and we have

$$i_F(\partial)=\text{the class of }\partial f\in\mathcal{O}_{\widetilde{\mathbb{A}}_\Lambda}(-\delta\mathbb{E}(1))\quad\text{for }\partial\in\theta_{\widetilde{\mathbb{P}}_\Lambda/S}.$$

Taking $\sigma=\sigma^+_{k,\hat{i}}$, we have (cf. the proof of Theorem(1-2))

$$U_\sigma=\mathrm{Spec}(R_\sigma)\qquad\text{with }R_\sigma=\Lambda[y_1,\dots,y_{n+1}]/(t-y_1y_2y_3).$$

The exceptional divisors on U_σ are defined by $y_1=0$ and $y_2=0$. We can write $f=(y_1^ky_2^{k+1})^\delta\cdot\tilde{f}$ in R_σ where $\tilde{f}$ is an equation of $\widetilde{X}\cap U_\sigma\subset U_\sigma$ and $\mathcal{O}_{U_\sigma}(-\delta\mathbb{E}(1))=(y_1^ky_2^{k+1})^\delta\mathcal{O}_{U_\sigma}$. We have locally on U_σ

$$\begin{cases}\theta_{\widetilde{\mathbb{P}}_\Lambda/S}=\Big\{\sum_{1\le i\le 3}a_i\cdot y_i\partial/\partial y_i+\sum_{4\le j\le n+1}a_j\cdot\partial/\partial y_j\ \Big|\\ \qquad\qquad\qquad\qquad\qquad a_i,\ a_j\in\mathcal{O}_{U_\sigma},\ a_1+a_2+a_3=0\Big\},\\ \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X})=\{\partial\in\theta_{\widetilde{\mathbb{P}}_\Lambda/S}\mid\partial\tilde{f}\in\tilde{f}\mathcal{O}_{U_\sigma}\}.\end{cases}$$

We compute

$$\begin{cases}(y_1^ky_2^{k+1})^{-\delta}\cdot y_i\partial f/\partial y_i\equiv y_i\partial\tilde{f}/\partial y_i\mod(\tilde{f})\quad(i=1,2),\\ (y_1^ky_2^{k+1})^{-\delta}\cdot\partial f/\partial y_j=\partial\tilde{f}/\partial y_j\quad(3\le j\le n+1),\end{cases}$$

The condition of Theorem(1-1) on U_σ implies the regularity of

$$R_\sigma/(\tilde{f}),\ R_\sigma/(\tilde{f}, y_3),\ R_\sigma/(\tilde{f}, y_i),\ R_\sigma/(\tilde{f}, y_3, y_i)\ (i = 1, 2),$$
$$R_\sigma/(\tilde{f}, y_1, y_2),\ R_\sigma/(\tilde{f}, y_1, y_2, y_3).$$

By a standard argument this implies the desired assertion restricted on U_σ. The same computation shows the assertion on U_σ for other σ and the proof is complete. Q.E.D.

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