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Log C^∞-Functions and Degenerations of Hodge Structures

Kazuya Kato, Toshiharu Matsubara and Chikara Nakayama

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Introduction

In [27], J.H.M. Steenbrink studied degenerations of Hodge structures. For $f\colon X \longrightarrow \Delta = \{z \in \mathbb{C} \; ; \; |z| < 1\}$ projective and of semi-stable degeneration, he showed that a "limit Hodge structure" appears as the limit of the Hodge structures $H^m(X_t, \mathbb{Z})$ $(m \in \mathbb{Z}, t \in \Delta - \{0\})$. In log Hodge theory, as in [23], his theory is interpreted in the form "the higher direct images on Δ of $\mathbb{Z}_X$ carry the natural variations of polarized log Hodge structure."

In this paper, we will generalize the theory of Steenbrink in this form to the theory with coefficients (that is, we will start with general variations of polarized log Hodge structure $\mathcal{H}_\mathbb{Z}$ on X instead of $\mathbb{Z}_X$).

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Our method is different from Steenbrink's. We use "log C^∞-functions" and "log harmonic forms", in the way as we use C^∞-functions and harmonic forms in the case without degeneration in the classical Hodge theory. Our main result is the following. (See Appendix for special terminology of log geometry, if the reader is not familiar with log structures of Fontaine-Illusie. For example, see Appendix 2 for "log smooth fs log analytic space", see Appendix 4 for "log smooth morphism" and for "vertical morphism", and see Appendix 5 for "ket sense". In particular, the word "vertical" in the statement below shows that we assume the degeneration of f and the degeneration of $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ occur only in the "vertical direction" with respect to f.)

Theorem. *Let X, Y be log smooth fs log analytic spaces, and let $f\colon X \longrightarrow Y$ be a projective log smooth vertical morphism. Let $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ be a variation of polarized log Hodge structure on X of weight w in the ket sense. Then:*
(1) The Hodge to de Rham spectral sequence

$$E_1^{p,q} = R^{p+q} f_* \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \Rightarrow E_\infty^m = R^m f_*(\omega^\bullet_{X/Y}(\mathcal{M}))$$

degenerates from E_1 and each $R^m f_* \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M}))$ is a locally free $\mathcal{O}_Y$-module on Y_{ket}. Here $\omega^\bullet_{X/Y}(\mathcal{M})$ denotes the de Rham complex with log poles and with coefficients in $\mathcal{M}$.
(2) For each $m \in \mathbb{Z}$, $\left(R^m f_*^{\log} \mathcal{H}_{\mathbb{Z}},\ R^m f_* \omega^\bullet_{X/Y}(\mathcal{M}),\ (\ ,\ )\right)$ with the Hodge filtration on $R^m f_* \omega^\bullet_{X/Y}(\mathcal{M})$ is a variation of polarized log Hodge structure on Y of weight $w+m$ in the ket sense. Here $(\ ,\ )$ is the intersection form defined by fixing an invertible $\mathcal{O}_X$-module which is relatively very ample with respect to Y.

In the case where $\mathcal{H}_{\mathbb{Z}} = \mathbb{Z}$ and X is semi-stable over $Y = \Delta$, this gives a new proof of a result of Steenbrink [27] as explained in the above. See 8.13 for the details. The theorem also gives new proofs to results of T. Fujisawa [6], L. Illusie [13] and M. Cailotto [1]. See also Remark 8.12.

In [14], the functoriality of log Riemann-Hilbert correspondences was established, which is a generalization of results of the second author [23], [24], [25], F. Kato [16], and S. Usui [29], [30]. This implies that $R^m f_*(\omega^\bullet_{X/Y}(\mathcal{M}))$ is a locally free $\mathcal{O}_Y$-module in the ket sense, and corresponds to $R^m f_*^{\log} \mathcal{H}_{\mathbb{Z}}$ via the log Riemann-Hilbert correspondence on Y. The above Theorem shows that we can add Hodge filtrations in this functoriality.

See Y. Kawamata-Y. Namikawa [21] for another approach by log method to the degenerations of Hodge structures.

Log C^∞-functions are functions which have, together with their "log derivatives", logarithmic growth at the boundary. After we completed our paper, we learned from Prof. S. Zucker that this notion has been already considered by several authors (for example, [10], [9], [11], [12]) and that contents of the sections 1 and 3 are known.

We are very much thankful to Prof. L. Illusie, Prof. S. Usui, Prof. T. Saito and Prof. S. Zucker for their advice.

§1. Log C^∞-functions

1.1. Let X be an fs log analytic space which is log smooth over $\mathbb{C}$. Let $X_{\text{triv}} = \{x \in X; M_{X,x} = \mathcal{O}_{X,x}^\times\}$, which is an open dense subset of X. (See Appendix.) We define the ring of log C^∞-functions on X as a subring of the ring of C^∞-functions on X_{triv}. When X is a complex manifold M whose log structure is given by a divisor D with normal crossings, the sheaf of log C^∞-functions is the same as $\mathcal{A}^0_{sia}(M, D)$ in [12] (2.2). See also [10], [9], and [11] 3.8.

For a function $f \colon X_{\text{triv}} \longrightarrow \mathbb{C}$, we say f is of log growth on X if there exists an open covering $(U_\lambda)_\lambda$ of X with an element $t_\lambda \in \Gamma(U_\lambda, M_X^{\text{gp}})$ and an integer $m(\lambda) \geq 0$ for each λ, for which we have

$$|f(x)| \leq |\log |t_\lambda(x)||^{m(\lambda)}$$

for any λ and any $x \in X_{\text{triv}} \cap U_\lambda$.

By a log C^∞-function on X, we mean a C^∞-function $f \colon X_{\text{triv}} \longrightarrow \mathbb{C}$ having the following property: If U is an open set of X and $(t_j)_{1\leq j\leq n}$ is a family of elements of $\Gamma(U, M_X^{\text{gp}})$ such that $(d\log(t_j))_{1\leq j\leq n}$ is an $\mathcal{O}_U$-basis of ω_U^1 (= the sheaf of analytic differential forms on U with log poles; see [18] (3.5)), then the following condition (C) is satisfied.
(C) For any $a(j),\ b(j) \in \mathbb{N}$ $(1 \leq j \leq n)$,

$$\left(\prod_j \left(t_j \cdot \frac{\partial}{\partial t_j}\right)^{a(j)} \left(\overline{t_j} \cdot \frac{\partial}{\partial \overline{t_j}}\right)^{b(j)}\right)(f)$$

is of logarithmic growth on U.

Note that locally on X, a family $(t_j)_{1\leq j\leq n}$ as above exists and the condition (C) is independent of the choice of such $(t_j)_{1\leq j\leq n}$.

Example 1.2. (1) When X is a complex manifold whose log structure is given by a divisor with normal crossings, a C^∞-function on X is a log C^∞-function.

(2) A meromorphic function on X is a log C^∞-function on X if and only if it is holomorphic.
(3) For any section t of M_X^{gp}, $\log|t|$ and $(\frac{t}{|t|})^n$ ($n \in \mathbb{Z}$) are log C^∞-functions on X. If t is in M_X, $|t|^c$ ($c \in \mathbb{C}$, $\mathrm{Re}(c) \geq 0$) is a log C^∞-function on X.
(4) For any section t of M_X, $|\log|t||^c$ ($c \in \mathbb{C}$) is a log C^∞-function on X outside the points $x \in X$ at which $t_x \in \mathcal{O}_{X,x}^\times$ and $|t(x)| = 1$.

1.3. We show that log C^∞-functions on X form a ring. For this, it is sufficient to show that functions on X_{triv} of log growth on X form a ring. It is sufficient to show that for $x \in X$ and t_1, $t_2 \in \Gamma(X, M_X^{\mathrm{gp}})$, there exist an open neighborhood U of x and $t \in \Gamma(U, M_X)$ such that

$$|\log(|t|)| \geq \max(|\log(|t_1|)|, |\log(|t_2|)|) \text{ on } X_{\mathrm{triv}} \cap U.$$

We may assume that $x \notin X_{\mathrm{triv}}$. Take an element s of $M_{X,x}$ whose image in the fs monoid $M_{X,x}/\mathcal{O}_{X,x}^\times$ belongs to the interior of $M_{X,x}/\mathcal{O}_{X,x}^\times$. Then for some $n \geq 1$, $s^n t_1$, $s^n t_1^{-1}$, $s^n t_2$, $s^n t_2^{-1}$ belong to the interior of $M_{X,x}/\mathcal{O}_{X,x}^\times$. Hence $|s^n t_1| < 1$, $|s^n t_1^{-1}| < 1$, $|s^n t_2| < 1$, $|s^n t_2^{-1}| < 1$ on $X_{\mathrm{triv}} \cap U$ for some open neighborhood U of x. This shows

$$|\log(|s^n|)| > \max(|\log(|t_1|)|, |\log(|t_2|)|)$$

on $X_{\mathrm{triv}} \cap U$.

1.4. Let $\mathcal{A}_X$ be the sheaf $U \mapsto \{\log\ C^\infty\text{-functions on } U\}$ of X. Let $V = X_{\mathrm{triv}}$, let C_V^∞ be the sheaf of C^∞-functions on V, and let $j\colon V \longrightarrow X$ be the canonical morphism. Then $\mathcal{A}_X$ is regarded as a subsheaf of $j_* C_V^\infty$. Let $C_V^{\infty,q}$ ($q \in \mathbb{Z}$) be the sheaf of C^∞ q-forms on V. For p, $q \in \mathbb{Z}$, define the sheaf $\mathcal{A}_X^{p,q}$ of log C^∞ (p,q)-forms on X to be the image of

$$\mathcal{A}_X \otimes \omega_X^p \otimes \omega_X^q \longrightarrow j_*(C_V^{\infty,p+q}); f \otimes \omega \otimes \eta \mapsto f\omega \wedge \overline{\eta},$$

and for $m \in \mathbb{Z}$, let

$$\mathcal{A}_X^m = \bigoplus_{p+q=m} \mathcal{A}_X^{p,q} \subset j_* C_V^{\infty,m}.$$

Proposition 1.5. *Assume that the underlying analytic space $\overset{\circ}{X}$ of X is Hausdorff. For any p, $q \in \mathbb{Z}$, $\mathcal{A}_X^{p,q}$ is a soft sheaf on X.*

Proof. If A is a soft ring, an A-module M is also soft. Hence we can reduce 1.5 to proving that $\mathcal{A}_X$ is soft. Moreover we can assume $X =$

$(\operatorname{Spec}\mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ for some fs monoid $\mathcal{S}$. Then we can find a surjective map $\mathbb{N}^s \longrightarrow \mathcal{S}$ and, hence, a closed immersion $X \overset{i}{\hookrightarrow} Z := (\operatorname{Spec}\mathbb{C}[\mathbb{N}^s])^{\mathrm{an}}$. Then we have a map $i^{-1}\mathcal{A}_Z \longrightarrow \mathcal{A}_X$. Since $C^\infty_{\overset{\circ}{Z}}$ is soft, $C^\infty_{\overset{\circ}{Z}}$-module $\mathcal{A}_Z$ is soft. This implies $\mathcal{A}_X$ is soft. Q.E.D.

Proposition 1.6. *$U \mapsto \mathcal{A}_X(U)$ is a sheaf on X_{ket}. (See [14] or Appendix for the definition of X_{ket}.)*

Proof. It is enough to show that, for a surjective, Kummer log étale morphism $g\colon V \longrightarrow W$ of log smooth fs log analytic spaces, a C^∞-function $f\colon W_{\mathrm{triv}} \longrightarrow \mathbb{C}$ is log C^∞ if and only if $f \circ g$ is log C^∞. This is easily checked with the fact that g is an open map. Q.E.D.

Proposition 1.7. *Let M be a sheaf of $\mathbb{Q}$-vector spaces on X_{ket}. Then $R^q\varepsilon_* M = 0$ for any $q > 0$, where ε is the projection of topoi from X_{ket} to X.*

Proof. See [14]. Q.E.D.

From now, everything is in the ket sense unless the contrary is explicitly stated.

1.8. We define a sheaf $\mathcal{A}_X^{\log}$ on $X^{\log}$ by

$$\mathcal{A}_X^{\log} = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\mathcal{A}_X),$$

where τ is the canonical map $X^{\log} \longrightarrow X_{\mathrm{ket}}$. ($\mathcal{O}_X$, $\mathcal{A}_X$, and $\mathcal{O}_X^{\log}$ here are the ket versions.) Note that $\mathcal{A}_X^{\log} \longrightarrow j_*^{\log} C_V^\infty$ is not necessarily injective since $\tau^{-1}\mathcal{A}_X \longrightarrow j_*^{\log} C_V^\infty$ is not. We define

$$\mathcal{A}_X^{p,q,\log} = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\mathcal{A}_X^{p,q}) \quad (p,\ q \in \mathbb{Z})$$
$$\mathcal{A}_X^{m,\log} = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\mathcal{A}_X^{m}) \quad (m \in \mathbb{Z}).$$

We have a complex conjugate $\mathcal{A}_X^{\log} \longrightarrow \mathcal{A}_X^{\log}$ by extending the complex conjugate of $\mathcal{A}_X$ by $\mathcal{O}_X^{\log} \longrightarrow \mathcal{A}_X^{\log}; \log(f) \mapsto 2\cdot(\log|f|) - \log(f) \otimes 1$.

Proposition 1.9. *We have*

$$R\tau_*(\mathcal{A}_X^{p,q,\log}) = \mathcal{A}_X^{p,q} \text{ for } p,\ q \in \mathbb{Z}.$$

Proof. It is checked stalkwise that $R\tau_*(\mathcal{O}_X^{\log} \otimes_{\tau^{-1}\mathcal{O}_X} \tau^{-1}M) = M$ for any $\mathcal{O}_X$-module M (cf. [14]). The proposition is a special case of this fact. Q.E.D.

§2. Log C^∞ Hodge decompositions

2.1. In this section, we relate log C^∞-functions to degenerations of polarized Hodge structures. In Theorem 2.6 below, we show that a "variation of polarized log Hodge structure" (VPLH) has a "log C^∞ Hodge decomposition". Here VPLH is a notion which is something like "degenerating variation of polarized Hodge structure" and which matches well the theory of Schmid on nilpotent orbits ([26]). The proof of Theorem 2.6 bases on the theory of Cattani-Kaplan-Schmid on $SL(2)$-orbits ([26], [3]).

In the classical theory, if X is a complex manifold and $\mathcal{H}_{\mathbb{Z}}$ is a variation of polarized Hodge structure (VPH) on X of weight w, $\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z},x}$ for each $x \in X$ has Hodge decomposition

$$\mathcal{H}_{\mathbb{C},x} = \bigoplus_{p+q=w} \mathcal{H}_{\mathbb{C},x}^{p,q}$$

where $\mathcal{H}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}$ and $\mathcal{H}_{\mathbb{C},x}^{p,q}$ is the intersection of $\mathrm{Fil}^p(\mathcal{H}_{\mathbb{C},x})$ and the complex conjugate of $\mathrm{Fil}^q(\mathcal{H}_{\mathbb{C},x})$. The $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}$ has a filtration by the definition of VPH, but this $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}$ does not necessarily have a Hodge decomposition (this is because $\mathcal{O}_X$ does not have the complex conjugation). However $C_X^\infty \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}$ has a Hodge decomposition

$$C_X^\infty \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}} = \bigoplus_{p+q=w} (p,q)\text{-part}$$

where (p,q)-part means the intersection of $\mathrm{Fil}^p(C_X^\infty \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}})$ and the complex conjugate of $\mathrm{Fil}^q(C_X^\infty \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}})$. Theorem 2.6 states that a similar Hodge decomposition exists also for a VPLH if we replace C^∞-functions by log C^∞-functions.

2.2. Before we discuss VPLH, we review the theory of log Riemann-Hilbert correspondences studied in [18] and [14] (cf. Remark 2.4). Let X be a log smooth fs log analytic space. The log Riemann-Hilbert correspondence relates the following two categories $L_{\mathrm{qunip}}(X)$ and $V_{\mathrm{qnilp}}(X)$. Let $L_{\mathrm{qunip}}(X)$ be the category of locally constant sheaves L of finite dimensional $\mathbb{C}$-vector spaces on $X^{\log}$ such that for any $x \in X$ and $y \in \tau^{-1}(x) \subset X^{\log}$, the action of $\pi_1(\tau^{-1}(x))$ (called the local monodromy at x) on the stalk L_y is quasi-unipotent. On the other hand, let $V_{\mathrm{qnilp}}(X)$ be the category of $\mathcal{O}_X$-modules V on X_{ket} endowed with an integrable connection with log poles

$$\nabla \colon V \longrightarrow \omega_X^1 \otimes_{\mathcal{O}_X} V$$

which satisfies the following condition locally on X_{ket} (*cf.* [14]).

There exists a finite family of $\mathcal{O}_X$-submodules $(V_i)_{0\leq i\leq n}$ of V satisfying $\nabla(V_i)\subset\omega^1_X\otimes_{\mathcal{O}_X}V_i$ such that

$$0=V_0\subset V_1\subset\cdots\subset V_n=V$$

and such that for each $1\leq i\leq n$, V_i/V_{i-1} is locally free and the connection induced on V_i/V_{i-1} does not have a pole.

Then we have an equivalence of categories

$$L_{\text{qunip}}(X)\xrightarrow{\sim}V_{\text{qnilp}}(X);\ L\mapsto\tau_*(\mathcal{O}^{\log}_X\otimes_{\mathbb{C}}L)$$

whose converse is given by

$$V\mapsto\operatorname{Ker}(\nabla\colon\mathcal{O}^{\log}_X\otimes_{\mathcal{O}_X}V\longrightarrow\omega^{1,\log}_X\otimes_{\mathcal{O}_X}V),$$

where $-\otimes_{\mathcal{O}_X}V=-\otimes_{\tau^{-1}(\mathcal{O}_X)}\tau^{-1}(V)$.

Furthermore, if $L\in L_{\text{qunip}}(X)$ and $V=\tau_*(\mathcal{O}^{\log}_X\otimes_{\mathbb{C}}L)\in V_{\text{qnilp}}(X)$, we have

$$\mathcal{O}^{\log}_X\otimes_{\mathbb{C}}L=\mathcal{O}^{\log}_X\otimes_{\mathcal{O}_X}V.$$

2.3. Now we introduce VPLH. See [19], [20] for generality of log Hodge structures and polarized log Hodge structures (cf. Remark 2.4).

First, we review the definition of VPH. For a complex manifold X and for $w\in\mathbb{Z}$, a VPH on X of weight w is a triple $(\mathcal{H}_{\mathbb{Z}},F,(\ ,\ ))$ where
∘ $\mathcal{H}_{\mathbb{Z}}$ is a locally constant sheaf of finitely generated $\mathbb{Z}$-modules on X,
∘ F is a descending filtration $(F^p)_{p\in\mathbb{Z}}$ on $\mathcal{O}_X\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z}}$ by $\mathcal{O}_X$-submodules such that

$$F^p=\mathcal{O}_X\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z}}\text{ for }p\ll0,\ \ F^p=0\text{ for }p\gg0,$$

and each F^p is locally a direct summand of $\mathcal{O}_X\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z}}$,
∘ $(\ ,\ )$ is a $\mathbb{Q}$-bilinear form $\mathcal{H}_{\mathbb{Q}}\times\mathcal{H}_{\mathbb{Q}}\longrightarrow\mathbb{Q}$,
satisfying the following conditions (1) and (2).
(1) For any $x\in X$, the triple $(\mathcal{H}_{\mathbb{Z},x},\ (\ ,\ )_x,F(x))$ is a polarized Hodge structure of weight w. Here $F(x)$ means the filtration $(\mathbb{C}\otimes_{\mathcal{O}_{X,x}}F^p_x)_{p\in\mathbb{Z}}$ on $\mathbb{C}\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z},x}$ ($\mathcal{O}_{X,x}\longrightarrow\mathbb{C}$ is given by $f\mapsto f(x)$).
(2) (Griffiths transversality) The connection

$$\nabla=d\otimes1\colon\mathcal{O}_X\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z}}\longrightarrow\Omega^1_X\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z}}$$

sends F^p into $\Omega^1_X\otimes_{\mathcal{O}_X}F^{p-1}$ for any $p\in\mathbb{Z}$.

Now let X be a log smooth fs log analytic space and let $w\in\mathbb{Z}$. A VPLH on X of weight w is a triple $(\mathcal{H}_{\mathbb{Z}},\ \mathcal{M},\ (\ ,\ ))$ where
∘ $\mathcal{H}_{\mathbb{Z}}$ is a locally constant sheaf on $X^{\log}$ of finitely generated $\mathbb{Z}$-modules with quasi-unipotent local monodromies.

$\circ$ $\mathcal{M}$ is the object of $V_{\mathrm{qnilp}}(X)$ corresponding to the object $\mathcal{H}_{\mathbb{C}} = \mathbb{C}\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z}}$ of $L_{\mathrm{qunip}}(X)$, endowed with a descending filtration $(\mathcal{M}^p)_{p\in\mathbb{Z}}$ by $\mathcal{O}_X$-submodules such that

$$\mathcal{M}^p = \mathcal{M} \text{ for } p \ll 0, \ \mathcal{M}^p = 0 \text{ for } p \gg 0,$$

and each $\mathcal{M}^p$ is locally a direct summand of $\mathcal{M}$,
$\circ$ $(\ ,\ )$ is a $\mathbb{Q}$-bilinear form $\mathcal{H}_{\mathbb{Q}} \times \mathcal{H}_{\mathbb{Q}} \longrightarrow \mathbb{Q}$
satisfying the following conditions (1) and (2).
(1) Let $x \in X$, let y be a point of $X^{\log}$ lying over x, and let $\mathrm{sp}(y)$ be the set of all ring homomorphisms $\mathcal{O}^{\log}_{X,y} \longrightarrow \mathbb{C}$ whose restrictions to the subring $\mathcal{O}_{X,x}$ of $\mathcal{O}^{\log}_{X,y}$ coincide with the map $\mathcal{O}_{X,x} \longrightarrow \mathbb{C};\ f \mapsto f(x)$. Then if $s \in \mathrm{sp}(y)$ and if the map $M_{X,x} \longrightarrow \mathbb{C}^\times;\ f \mapsto \exp(s(\log(f)))$ is sufficiently near to the canonical composition

$$M_{X,x} \xrightarrow{\alpha} \mathcal{O}_{X,x} \longrightarrow \mathbb{C};\ f \mapsto \alpha(f)(x),$$

then $(\mathcal{H}_{\mathbb{Z},y},\ \mathcal{M}(s),\ (\ ,\ )_y)$ is a polarized Hodge structure of weight w in the classical sense. Here $\log(f)$ is defined in $\mathcal{O}^{\log}_{X,y}/2\pi i\mathbb{Z}$ and $\exp(s(\log(f)))$ is well defined since $\exp(s(2\pi i\mathbb{Z})) = \exp(2\pi i\mathbb{Z}) = 1$, "sufficiently near " is with respect to the topology of simple convergence of the set $\mathrm{Map}(M_{X,x},\ \mathbb{C})$, and $\mathcal{M}(s) = \mathbb{C}\otimes_{\mathcal{O}_{X,y}} \mathcal{M}_y$ endowed with the induced filtration. ($\mathcal{O}_{X,y}$ (resp. $\mathcal{M}_y$) is the stalk at y of the inverse image of $\mathcal{O}_X$ (resp. $\mathcal{M}$) on $X^{\log}$ by $X^{\log} \longrightarrow X_{\mathrm{ket}}$, $\mathcal{O}_{X,y} \longrightarrow \mathbb{C}$ is $f \mapsto f(y)$, and we identify $\mathcal{M}(s)$ with $\mathbb{C}\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z},y}$ by

$$\mathcal{M}(s) = \mathbb{C}\otimes_{\mathcal{O}^{\log}_{X,y}} (\mathcal{O}^{\log}_{X,y}\otimes_{\mathcal{O}_{X,y}} \mathcal{M}_y) = \mathbb{C}\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z},y}$$

where $\mathcal{O}^{\log}_{X,y} \longrightarrow \mathbb{C}$ is s.)
(2) (Griffiths transversality)

$$\nabla(\mathcal{M}^p) \subset \omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}^{p-1} \text{ for any } p \in \mathbb{Z}.$$

Sometimes we denote by $(\mathcal{H}_{\mathbb{Z}}, (\mathcal{M}^p)_{p\in\mathbb{Z}}, (\ ,\ ))$ for $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$.

Remark 2.4. In the above 2.2 and 2.3, we work on the ket site. Working on the usual site (of open sets of X) instead, we have the non-ket analogues of 2.2 and 2.3: First, replacing X_{ket} with X (the usual site) in 2.2, we have the definition of the non-ket analogue $V_{\mathrm{nilp}}(X)$ of $V_{\mathrm{qnilp}}(X)$. Then we have the non-ket version of the log Riemann-Hilbert correspondence $L_{\mathrm{unip}}(X) \xrightarrow{\sim} V_{\mathrm{nilp}}(X)$, where $L_{\mathrm{unip}}(X) := \{L \in L_{\mathrm{qunip}}(X)$; the local monodromies of L are unipotent$\}$. See [18] for the details. Next, replacing X_{ket} with X in 2.3, we have the definition of the non-ket version of VPLH, which is called VPLH in [20].

These non-ket versions relate to ours as follows: First, the functor ι from the category of locally free $\mathcal{O}_X$-modules of finite rank on X ($\mathcal{O}_X$ here is in the non-ket sense) to that of locally free $\mathcal{O}_X$-modules of finite rank on X_{ket} is fully faithful and it induces the categorical equivalence between $V_{\mathrm{nilp}}(X)$ and the full subcategory of $V_{\mathrm{qnilp}}(X)$ consisting of the objects whose "V" belong to the essential image of ι. Further ι induces the equivalence between the category of VPLH in the non-ket sense and the full subcategory of that of VPLH in our sense consisting of the objects whose "$\mathcal{H}_{\mathbb{C}}$" belong to $L_{\mathrm{unip}}(X)$.

The following Proposition 2.5 is a reformulation of the nilpotent orbit theorem of Schmid ([26]).

Proposition 2.5. *Let X be a log smooth fs log analytic space and let $w \in \mathbb{Z}$. Then the restriction to X_{triv} induces an equivalence of categories*

$$\{\mathit{VPLH\ on\ } X \mathit{\ of\ weight\ } w\} \xrightarrow{\sim} \{\mathit{VPH\ on\ } X_{\mathrm{triv}} \mathit{\ of\ weight\ } w\}.$$

We show in 2.7–2.9 how Proposition 2.5 is deduced from the nilpotent orbit theorem of Schmid.

See [20] for more details about the relation between nilpotent orbits and polarized log Hodge structures on more general fs log analytic spaces X.

The aim of this section is to prove

Theorem 2.6. *Let X be a log smooth fs log analytic space, let $w \in \mathbb{Z}$, and let $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ be a VPLH on X of weight w. Then we have*

$$\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M} = \bigoplus_{p+q=w} \mathcal{M}_{\mathcal{A}}^{p,q}$$

where $\mathcal{M}_{\mathcal{A}}^{p,q}$ is the intersection of $\mathcal{M}_{\mathcal{A}}^{p} = \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M}^p$ and the complex conjugate of $\mathcal{M}_{\mathcal{A}}^{q}$.

Here the complex conjugation on $\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is the one induced by

$$(\text{complex conjugation}) \otimes 1 \text{ on } \mathcal{A}_X^{\log} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}},$$

via the identification

$$\mathcal{A}_X^{\log} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}} = \mathcal{A}_X^{\log} \otimes_{\mathcal{O}_X} \mathcal{M}.$$

2.7. We prove Proposition 2.5 in 2.7–2.9. In there we fix $w \in \mathbb{Z}$ and VPH (resp. VPLH) means VPH (resp. VPLH) of weight w. The fully faithfulness of the restriction functor

$$\{\text{VPLH on } X\} \longrightarrow \{\text{VPH on } X_{\mathrm{triv}}\}$$

is easily seen. Hence it is sufficient to show that a VPH $(\mathcal{H}_{\mathbb{Z}},\ F,\ (\ ,\ ))$ on X_{triv} extends to a VPLH on X. Since $X^{\log}$ is a topological manifold with the boundary $X^{\log} - X_{\mathrm{triv}}$, $\mathcal{H}_{\mathbb{Z}}$ extends uniquely to $X^{\log}$ as a locally constant sheaf, and $(\ ,\ )$ extends also on $X^{\log}$. Denote this extension of $(\mathcal{H}_{\mathbb{Z}},\ (\ ,\ ))$ on $X^{\log}$ also by $(\mathcal{H}_{\mathbb{Z}},\ (\ ,\ ))$. Since the local monodromy of $\mathcal{H}_{\mathbb{Z}}$ at each point of X is quasi-unipotent by a theorem of Borel [26, 4.5], $\mathcal{H}_{\mathbb{C}}$ is an object of $L_{\mathrm{qunip}}(X)$. Let $\mathcal{M}$ be the object of $V_{\mathrm{qnilp}}(X)$ corresponding to $\mathcal{H}_{\mathbb{C}}$. It remains to show that F extends to a filtration on $\mathcal{M}$ and $(\mathcal{H}_{\mathbb{Z}},\ \mathcal{M},\ (\ ,\ ))$ satisfies the conditions (1) (2) of VPLH. We prove this in 2.8 in the case where X is a complex manifold and the log structure of X is given by a divisor with normal crossings, and in 2.9 in general.

2.8. Assume that X is a complex manifold whose log structure is given by a divisor with normal crossings. We may assume $X = \Delta^{n+m}$ with the log structure given by the divisor which is the complement of $(\Delta^*)^n \times \Delta^m$. Assume that we are given a VPH $(\mathcal{H}_{\mathbb{Z}},\ F,\ (\ ,\ ))$ on $X_{\mathrm{triv}} = (\Delta^*)^n \times \Delta^m$. We show that it extends to a VPLH on X. As is explained in 2.7, $(\mathcal{H}_{\mathbb{Z}}, (\ ,\ ))$ is extended to $X^{\log}$ and we have an $\mathcal{O}_X$-module $\mathcal{M}$ on X_{ket}. We may assume that the local monodromies of $(\mathcal{H}_{\mathbb{Z}},\ F,\ (\ ,\ ))$ are unipotent. Let

$$\begin{aligned} U &= \text{ the upper half plane } = \{x + yi\ ;\ x,\ y \in \mathbb{R},\ y > 0\}, \\ \overline{U} &= \{x + yi\ ;\ x \in \mathbb{R},\ 0 < y \leq \infty\}. \end{aligned}$$

Then we have a commutative diagram

$$\begin{tikzcd} & & \overline{U}^n \times \Delta^m \arrow[d] & U^n \times \Delta^m \arrow[l, hook'] \arrow[d] \\ X^{\log} \arrow[d, "\tau"'] & \cong & (\mathbb{Z}\backslash\overline{U})^n \times \Delta^m \arrow[d, "(1)"'] & (\mathbb{Z}\backslash U)^n \times \Delta^m \arrow[l, hook'] \arrow[d, "(2)"', "\cong"] \\ X & = & \Delta^{n+m} & (\Delta^*)^n \times \Delta^m \arrow[l, hook'] \end{tikzcd}$$

where $\mathbb{Z}\backslash *$ $(* = U,\ \overline{U})$ means the quotient by the action $z \mapsto z + n$ $(n \in \mathbb{Z})$ of the group $\mathbb{Z}$, (2) is the isomorphism induced by $U \longrightarrow \Delta^*;\ z \mapsto \exp(2\pi i z)$, and (1) is the unique continuous extension of (2). The group $\pi_1(X^{\log}) \cong \pi_1(X_{\mathrm{triv}}) \cong \mathbb{Z}^n$ acts on the stalks of $\mathcal{H}_{\mathbb{Z}}$, and since $\pi_1(X^{\log})$ is commutative, we have a unique action of $\pi_1(X^{\log})$ on $\mathcal{H}_{\mathbb{Z}}$ which induces the original action of $\pi_1(X^{\log})$ on each stalk of $\mathcal{H}_{\mathbb{Z}}$. Let $\gamma_j \in \pi_1((\Delta^*)^n \times \Delta^m)$ $(1 \leq j \leq n)$ be the loop in the j-th Δ^*

around 0 in the clockwise direction, and let N_j be the logarithm of the action of γ_j on $\mathcal{H}_\mathbb{Z}$ which is unipotent. It can be shown easily that the inverse image of $\mathcal{M}$ on $\overline{U}^n \times \Delta^m$ is equal to $\mathcal{O}_X \otimes_\mathbb{Z} \exp(\sum_{j=1}^n z_j N_j)\mathcal{H}_\mathbb{Z}$ in the inverse image of $\mathcal{O}_X^{\log} \otimes_\mathbb{Z} \mathcal{H}_\mathbb{Z}$ on $\overline{U}^n \times \Delta^m$, where z_j denotes the coordinate function of the j-th U, and regarding z_j as $(2\pi i)^{-1}$ times a logarithm of the coordinate function of the j-th Δ, we regard z_j as a global section of the inverse image of $\mathcal{O}_X^{\log}$ on $\overline{U}^n \times \Delta^m$.

Since $\overline{U}^n \times \Delta^m$ is contractible, the inverse image of $\mathcal{H}_\mathbb{Z}$ on $\overline{U}^n \times \Delta^m$ is a constant sheaf. By regarding the inverse image of $\mathcal{H}_\mathbb{C}$ on $\overline{U}^n \times \Delta^m$ as a constant $\mathbb{C}$-vector space, let $\overline{D}$ be the set of all descending filtrations $(f^p)_{p\in\mathbb{Z}}$ on this $\mathbb{C}$-vector space and let D be the subset of $\overline{D}$ consisting of $(f^p)_{p\in\mathbb{Z}}$ for which $(\mathcal{H}_\mathbb{Z},\ f,\ (\ ,\ ))$ is a PH of weight w (D is a classifying space of polarized Hodge structures of Griffiths). Let

$$\tilde{\phi}\colon U^n \times \Delta^m \longrightarrow D$$

be the map defined by the filtration F. Then by Schmid [26, Section 4], the map

$$U^n \times \Delta^m \longrightarrow \overline{D};\ (z,w) \mapsto \exp(-\sum_{j=1}^n z_j N_j)\tilde{\phi}(z,w)$$

descends to a holomorphic map $\psi\colon (\Delta^*)^n \times \Delta^m \longrightarrow \overline{D}$ and furthermore ψ extends to a holomorphic map $\Delta^{n+m} \longrightarrow \overline{D}$. This implies that the filtration $\exp(-\sum_{j=1}^n z_j N_j)F$ on the inverse image of $\mathcal{O}_{X_{\mathrm{triv}}} \otimes_\mathbb{Z} \mathcal{H}_\mathbb{Z}$ on $U^n \times \Delta^m$ extends to a filtration F' of $\mathcal{O}_X \otimes_\mathbb{Z} \mathcal{H}_\mathbb{Z}$ on $\overline{U}^n \times \Delta^m$ by $\mathcal{O}_X$-submodules which are locally direct summands of $\mathcal{O}_X \otimes_\mathbb{Z} \mathcal{H}_\mathbb{Z}$, and that there is a filtration $(\mathcal{M}^p)_{p\in\mathbb{Z}}$ of $\mathcal{M}$ by $\mathcal{O}_X$-submodules such that the inverse image of $\mathcal{M}^p$ on $\overline{U}^n \times \Delta^m$ is equal to $\exp(\sum_{j=1}^n z_j N_j)F'$. These $\mathcal{M}^p$ are locally direct summands of $\mathcal{M}$. We show that $(\mathcal{H}_\mathbb{Z},\ \mathcal{M},\ (\ ,\ ))$ satisfies the condition (1) (2) of VPLH. The Griffiths transversality (2) is checked on X_{triv}. For (1), it is enough to check this at $0 \in \Delta^{n+m}$. Let $a \in X^{\log}$ lie over $0 \in \Delta^{n+m}$, let $s \in \mathrm{sp}(a)$, let b be a lifting of a to $\overline{U}^n \times \Delta^m$, and let $s(z_j) \in \mathbb{C}$ be the image of $z_j \in \mathcal{O}_{X,b}^{\log} = \mathcal{O}_{X,a}^{\log}$ by s. Then the filtration of $\mathcal{M}(s)$ is identified with $\exp(\sum_{j=1}^n s(z_j)N_j)\psi(0)$. Since

$$\mathbb{C}^n \longrightarrow \overline{D};\ (z_j)_{1\le j\le n} \mapsto \exp(\sum_{j=1}^n z_j N_j)\psi(0)$$

is a nilpotent orbit ([26, 4.12], [3, 1.15]), the condition (1) of VPLH is satisfied.

2.9. We prove Proposition 2.5 in general. We may assume that X is an open subspace of the toric variety $(\operatorname{Spec}\mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ where $\mathcal{S}$ is a torsion free fs monoid and the log structure of X is given by the divisor which is the complement of $X\cap(\operatorname{Spec}\mathbb{C}[\mathcal{S}^{\mathrm{gp}}])^{\mathrm{an}}$. Here $\mathcal{S}^{\mathrm{gp}}$ is the group $\{ts^{-1}\ ;\ t,\ s\in\mathcal{S}\}$ associated to $\mathcal{S}$.

We recall some facts about toric geometry ([22]). Let $\mathbb{Q}_{\geq 0}=\{a\in\mathbb{Q}\ ;\ a\geq 0\}$ regarded as an additive monoid. For a finitely generated $\mathbb{Q}$-cone σ in $\operatorname{Hom}(\mathcal{S},\mathbb{Q}_{\geq 0})$ (i.e., a subset of $\operatorname{Hom}(\mathcal{S},\mathbb{Q}_{\geq 0})$ having the form $\{a_1h_1+\cdots+a_rh_r\ ;\ a_j\in\mathbb{Q}_{\geq 0}\}$ for some elements $h_1,\dots,h_r$ of $\operatorname{Hom}(\mathcal{S},\mathbb{Q}_{\geq 0})$), we have a log smooth fs log analytic space $X_\sigma = X\times_{(\operatorname{Spec}\mathbb{C}[\mathcal{S}])^{\mathrm{an}}}(\operatorname{Spec}\mathbb{C}[\mathcal{S}_\sigma])^{\mathrm{an}}$ where $\mathcal{S}_\sigma=\{t\in\mathcal{S}^{\mathrm{gp}}\ ;\ h(t)\geq 0 \text{ for all } h\in\sigma\}$. The canonical morphism $f_\sigma\colon X_\sigma\longrightarrow X$ induces an isomorphism $X_\sigma\times_X X_{\mathrm{triv}}\xrightarrow{\sim}X_{\mathrm{triv}}$. If λ is a finite polyhedral cone decomposition of $\operatorname{Hom}(\mathcal{S},\mathbb{Q}_{\geq 0})$, we have a log smooth fs log analytic space $X_\lambda=\cup_{\sigma\in\lambda}X_\sigma$ (open covering) with a proper surjective map $f_\lambda\colon X_\lambda\longrightarrow X$ which induces $X_\lambda\times_X X_{\mathrm{triv}}\xrightarrow{\sim}X_{\mathrm{triv}}$. If λ' is a subdivision of λ, we have a unique morphism $X_{\lambda'}\longrightarrow X_\lambda$ over X.

We endow X_σ and X_λ with the log structures corresponding to the divisors which are the complements of X_{triv}.

Assume that we are given a VPH $(\mathcal{H}_{\mathbb{Z}},\ F,\ (\ ,\ ))$ on X_{triv}.

Claim 2.9.1. *If σ is a simplicial $\mathbb{Q}$-cone (that is, σ is a $\mathbb{Q}$-cone generated by $\dim(\sigma)$ elements), $(\mathcal{H}_{\mathbb{Z}},\ F,\ (\ ,\ ))$ extends to a VPLH on X_σ.*

In fact, there is a finite Galois Kummer log étale covering X'_σ of X_σ such that X'_σ is smooth and such that the reduced part of the complement of the inverse image of X_{triv} in X'_σ is a normal crossing divisor. By 2.8, $(\mathcal{H}_{\mathbb{Z}},\ F,\ (\ ,\ ))$ extends to a VPLH on X'_σ, and by Galois descent, we see that $(\mathcal{H}_{\mathbb{Z}},\ F,(\ ,\ ))$ extends to a VPLH on X_σ.

As in 2.7, we can extend $(\mathcal{H}_{\mathbb{Z}},\ (\ ,\ ))$ to $X^{\log}$ and we have the $\mathcal{O}_X$-module $\mathcal{M}$ on X_{ket}. Let $\overline{D}(\mathcal{M})\longrightarrow X$ be the space classifying descending filtrations $(\mathcal{F}^p)_{p\in\mathbb{Z}}$ on $\mathcal{M}$ such that all $\mathcal{F}^p$ are locally direct summands of $\mathcal{M}$. ($\overline{D}(\mathcal{M})$ is a (finite disjoint union of) flag manifold bundle(s) over X.) By 2.9.1, for a simplicial $\mathbb{Q}$-cone in $\operatorname{Hom}(\mathcal{S},\mathbb{Q}_{\geq 0})$, the Hodge filtration of the extension of $(\mathcal{H}_{\mathbb{Z}},\ F,\ (\ ,\ ))$ to X_σ defines a morphism $\mu_\sigma\colon X_\sigma\longrightarrow\overline{D}(\mathcal{M})$ over X. Take $\lambda\in\Lambda$ such that for any $\sigma\in\lambda$, $\mathcal{S}_\sigma\cong\mathbb{N}^n\times\mathbb{Z}^m$ for some $m,\ n\geq 0$ (such λ exists by [22] I, Theorem 11). Let $\mu_\lambda\colon X_\lambda\longrightarrow\overline{D}(\mathcal{M})$ be the union of μ_σ $(\sigma\in\lambda)$.

Claim 2.9.2. *μ_λ descends to a section $X\longrightarrow\overline{D}(\mathcal{M})$ of $\overline{D}(\mathcal{M})$.*

If we prove 2.9.2, we have a filtration $(\mathcal{M}^p)_{p\in\mathbb{Z}}$ on $\mathcal{M}$ extending F on X_{triv} such that $\mathcal{M}^p$ are locally direct summands of $\mathcal{M}$. We can then prove that with this filtration of $\mathcal{M}$, $(\mathcal{H}_{\mathbb{Z}},\ \mathcal{M},\ (\ ,\ ))$ is a VPLH on X. In fact, Griffiths transversality is checked on X_{triv}, and the condition (1) of VPLH follows from the nilpotent orbit theorem of Schmid [26, 4.12] applied to the manifold X_λ.

We prove 2.9.2. It is sufficient to show that $\mu_\lambda(y) = \mu_\lambda(z)$ for any $y,\ z \in X_\lambda$ whose images in X coincide. Fix $x \in X$ and let $X_\lambda(x) = f_\lambda^{-1}(x) \subset X_\lambda$. Let

$$\mathcal{S}_x = \{a \in \mathcal{S}^{\mathrm{gp}}\ ;\ a \in \mathcal{O}_{X,x}\},$$
$$\mathcal{S}_y = \{a \in \mathcal{S}^{\mathrm{gp}}\ ;\ a \in \mathcal{O}_{X_\lambda,y}\} \quad \text{for } y \in X_\lambda(x).$$

Then

$$\mathcal{S} \subset \mathcal{S}_x \subset \mathcal{S}_y \subset \mathcal{S}^{\mathrm{gp}}.$$

For $p = x$ or for $p \in X_\lambda(x)$, let $C(p) = \mathrm{Hom}\,(\mathcal{S}_p, \mathbb{Q}_{\geq 0})$ and regard $C(p)$ as a $\mathbb{Q}$-cone in $\mathrm{Hom}\,(\mathcal{S}^{\mathrm{gp}}, \mathbb{Q})$. Then for $y \in X_\lambda(x)$, $C(y) \subset C(x)$ and the interior $\{h \in C(y)\ ;\ \mathrm{Ker}\,(h\colon \mathcal{S}_y \longrightarrow \mathbb{Q}_{\geq 0}) = (\mathcal{S}_y)^\times\}$ of $C(y)$ is contained in the interior of $C(x)$.

To prove $\mu_\lambda(y_1) = \mu_\lambda(y_2)$ for any $y_1,\ y_2 \in X_\lambda(x)$, it is sufficient to consider the case $C(y_1)$ is a face of $C(y_2)$ (this is because any two points of $X_\lambda(x)$ are connected by a chain of this relation). Let h_1 be an element of the interior of $C(y_1)$. Since h_1 belongs to the topological closure of the interior of $C(y_2)$, by taking a point h_2 of the interior of $C(y_2)$ which is sufficiently near to h_1, we can find a simplicial $\mathbb{Q}$-cone σ in $C(x)$ such that both h_1 and h_2 are contained in the interior of σ and such that $\dim(\sigma) = \dim(C(x))$. Fix such h_1, h_2 and σ.

Take a finite polyhedral cone decomposition λ' of σ such that the corresponding proper birational $X_{\lambda'} \longrightarrow X_\sigma$ has a morphism $X_{\lambda'} \longrightarrow X_\lambda$ over X. The composite maps $X_{\lambda'} \longrightarrow X_\sigma \xrightarrow{\mu_\sigma} \overline{D}(\mathcal{M})$ and $X_{\lambda'} \longrightarrow X_\lambda \xrightarrow{\mu_\lambda} \overline{D}(\mathcal{M})$ coincide because they coincide on X_{triv}. Hence it is sufficient to show that there are elements y'_j of $X_{\lambda'}$ for $j = 1,\ 2$ such that the image of y'_j in X_λ is y_j for $j = 1,\ 2$ and such that the images of y'_j in X_σ coincide.

For $j = 1,\ 2$, let $K_j = \mathrm{Ker}\,(h_j : \mathcal{S}^{\mathrm{gp}} \longrightarrow \mathbb{Q})$. Then $K_j \supset (\mathcal{S}_{y_j})^\times$. Extend the homomorphism $(\mathcal{S}_{y_j})^\times \longrightarrow \mathbb{C}^\times;\ f \mapsto f(y_j)$ to a homomorphism $s_j\colon K_j \longrightarrow \mathbb{C}^\times$. For $j = 1,\ 2$, take $\sigma_j \in \lambda'$ such that $h_j \in \sigma_j$ and let y'_j be the point of $X_{\sigma_j} \subset (\mathrm{Spec}\,\mathbb{C}[\mathcal{S}_{\sigma_j}])^{\mathrm{an}}$ characterized by the following property. For $t \in \mathcal{S}_{\sigma_j}$, $t(y'_j) = s_j(t)$ if $t \in K_j$ and $t(y'_j) = 0$ otherwise. Then the image of y'_j in X_λ coincides with y_j. By $\dim(\sigma) = \dim(C(x))$, we have $(\mathcal{S}_\sigma)^\times = (\mathcal{S}_x)^\times$, and we have $K_j \cap \mathcal{S}_\sigma = (\mathcal{S}_\sigma)^\times$ since h_j is in the interior of σ. Hence for $j = 1,\ 2$ and for $t \in \mathcal{S}_\sigma$, $t(y'_j) = t(x)$ if

$t \in (\mathcal{S}_\sigma)^\times$ and $t(y'_j) = 0$ otherwise. Hence the images of y'_1 and y'_2 in X_σ coincide. This completes the proof of Proposition 2.5.

The following proposition is useful in the proof of Theorem 2.6 and also in other places in this paper.

Proposition 2.10 (Cf. [11], 3.8.2). *Let X be a log smooth fs log analytic space, and let $f\colon Z \longrightarrow X$ be a blowing up along log structure. Then*

$$f_*(\mathcal{A}_Z) = \mathcal{A}_X.$$

Proof. It is easy to see that the equality $X_{\mathrm{triv}} = Z_{\mathrm{triv}}$ induces the bijection between the set of functions of log growth on X and that for Z. On the other hand, $\omega^1_Z = \mathcal{O}_Z \otimes_{\mathcal{O}_X} \omega^1_X$ since f is log étale. These imply the desired equality. Q.E.D.

2.11. By 2.10, we can reduce the proof of Theorem 2.6 to the case where X is a manifold and the log structure of X is given by a divisor with normal crossings.

Lemma 2.12. *Let X be a log smooth fs log analytic space and let $f \in \Gamma(X, \mathcal{A}_X)$. Assume that f does not have zero on X_{triv} and that the function $\frac{1}{f}$ on X_{triv} is of log growth on X. Then $\frac{1}{f} \in \Gamma(X, \mathcal{A}_X)$.*

Proof. We may assume that X is an open subspace of $(\operatorname{Spec} \mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ for a torsion free fs monoid $\mathcal{S}$ and the log structure is given by the divisor which is the complement of $X \cap (\operatorname{Spec} \mathbb{C}[\mathcal{S}^{\mathrm{gp}}])^{\mathrm{an}}$. Let $(t_j)_{j\in J}$ be a $\mathbb{Z}$-basis of $\mathcal{S}^{\mathrm{gp}}$ and let

$$\Theta = \{t_j \frac{\partial}{\partial t_j},\ \overline{t_j} \frac{\partial}{\partial \overline{t_j}}\ ;\ j \in J\}.$$

Then 2.12 is reduced to

Claim 2.12.1. *For any $\partial_1, \dots, \partial_k \in \Theta$, $\partial_1 \cdots \partial_k(\frac{1}{f})$ is contained in the ring generated over $\mathbb{Z}$ by $\{\frac{1}{f},\ \delta_1 \cdots \delta_l(f)\ ;\ l \geq 0,\ \delta_1, \dots, \delta_l \in \Theta\}$.*

This 2.12.1 is deduced from $\partial(\frac{1}{f}) = f^{-2}\partial(f)$ $(\partial \in \Theta)$ by induction on k.

2.13. We prove Theorem 2.6.

Assume that X is a complex manifold and the log structure of X is given by a divisor with normal crossings. Let $(\mathcal{H}_\mathbb{Z}, \mathcal{M}, (\ ,\ ))$ be a VPLH on X. We may assume that the local monodromies of $\mathcal{H}_\mathbb{Q}$ are unipotent.

It is sufficient to show that the map

$$\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M}^p \oplus \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M}^{w+1-p} \longrightarrow \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M};\ (f,g) \mapsto f + \overline{g}$$

is an isomorphism for any $p \in \mathbb{Z}$. Locally on X, take an $\mathcal{O}_X$-basis $(e_j)_j$ of $\mathcal{M}^p$, an $\mathcal{O}_X$-basis $(e'_k)_k$ of $\mathcal{M}^{w+1-p}$, and an $\mathcal{O}_X$-basis $(e''_l)_l$ of $\mathcal{M}$, and let φ be the matrix which expresses the pair $((e_j)_j,\ (\overline{e'_k})_k)$ by $(e''_l)_l$. Then $\det(\varphi) \in \mathcal{A}_X$ and $\det(\varphi)$ does not have zero on X_{triv}. It is sufficient to prove $\det(\varphi)^{-1} \in \mathcal{A}_X$. By 2.12, it is enough to show that $\det(\varphi)^{-1}$ is of log growth. Hence it is enough to prove

Claim 2.13.1. *For $p,\ q \in \mathbb{Z}$ such that $p+q=w$, the projector*

$$C^\infty_{X_{\mathrm{triv}}} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}} \longrightarrow (p,q)\text{-}part$$

of the Hodge decomposition on X_{triv} is of log growth on X, that is, in $j_*C^\infty_{X_{\mathrm{triv}}} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M})$ (j denotes the inclusion $j\colon X_{\mathrm{triv}} \longrightarrow X$), the projector belongs to $A' \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M})$ where A' is the subsheaf of $j_*C^\infty_{X_{\mathrm{triv}}}$ consisting of functions of log growth.

In the following proof of 2.13.1, we use the arguments in section 5 of [3] which were used for the estimate of the Hodge metric of a degenerating VPH ([3, Theorem 5.21], [15]).

We may assume $X = \Delta^{n+m}$ and the log structure of X is given by the divisor which is the complement of $(\Delta^*)^n \times \Delta^m$. As in [3], for a subset I of $\{1,\dots,n\}$ containing n, and for $K > 1$, let

$$(\mathbb{R}^n_{>0})^I_K \subset \mathbb{R}^n_{>0}, \qquad ((\Delta^*)^n)^I_K \subset (\Delta^*)^n$$

($\mathbb{R}_{>0} = \{r \in \mathbb{R}\ ;\ r > 0\}$) be as follows. Write $I = \{i_\alpha\ ;\ 1 \le \alpha \le r\}$, $i_\alpha < i_\beta$ if $\alpha < \beta$. Let

$$\begin{aligned}(\mathbb{R}^n_{>0})^I_K = \{y = (y_j)_j \in \mathbb{R}^n_{>0}\ ;\ & y_{i_\alpha}/y_{i_{\alpha+1}} > K\ (1 \le \alpha \le r, y_{i_{r+1}} \text{ means } \\ & 1),\ K^{-1} \le y_j/y_{i_\alpha} \le K \text{ for any } \alpha\ (1 \le \alpha \le r) \text{ and} \\ & j \text{ such that } i_{\alpha-1} < j < i_\alpha\ (i_0 \text{ means } 0)\},\end{aligned}$$
$$((\Delta^*)^n)^I_K = \{(t_j)_j \in (\Delta^*)^n\ ;\ (-(2\pi)^{-1}\log|t_j|)_{1\le j\le n} \in (\mathbb{R}^n_{>0})^I_K\}.$$

Then, when σ ranges over all permutations on the set $\{1,\dots,n\}$ and I ranges over all subsets of $\{1,\dots,n\}$ containing n, the union $\cup_{\sigma,I}\sigma((\Delta^*)^n)^I_K$ contains a set of the form $V \cap (\Delta^*)^n$ for some neighborhood V of 0 in Δ^n ([3, 5.7]). Hence Claim 2.13.1 is reduced to

Claim 2.13.2. *Fix a subset I of $\{1,\dots,n\}$ containing n. Then if $K > 1$ is sufficiently large, the projectors of the Hodge decomposition* *on* X_{triv} *are of log growth at* $0 \in \Delta^{n+m}$ *when they are restricted to* $((\Delta^*)^n)^I_K \times \Delta^m$.

We prove a more precise

Claim 2.13.3. *Fix a subset* I *of* $\{1,\dots,n\}$ *containing* n. *For* $K > 1$, *let* B^I_K *be the subring of* $(j_*C^\infty_{X_{\mathrm{triv}}})_0$ *consisting of elements which are bounded on* $V \cap (((\Delta^*)^n)^I_K \times \Delta^m)$ *for some open neighborhood* V *of* 0 *in* Δ^{n+m}. *Then, if* $K > 1$ *is sufficiently large, the projectors of the Hodge decomposition on* X_{triv} *are contained in the subring*

$$B^I_K[y_1,\dots,y_n] \otimes_{\mathcal{O}_{X,0}} \mathrm{End}_{\mathcal{O}_{X,0}}(\mathcal{M}_0)$$

of

$$(j_*C^\infty_{X_{\mathrm{triv}}})_0 \otimes_{\mathcal{O}_{X,0}} \mathrm{End}_{\mathcal{O}_{X,0}}(\mathcal{M}_0)$$

where y_j *are defined by* $z_j = x_j + iy_j$ *with* x_j, y_j *real.*

(If t_j denotes the coordinate function of the j-th Δ, $y_j = -(2\pi)^{-1}\log(|t_j|)$ and it is of log growth.)

Let D, $\overline{D}$, $(N_j)_{1\le j\le n}$, $\tilde{\phi}\colon U^n \times \Delta^m \longrightarrow D$ and $\psi\colon \Delta^{n+m} \longrightarrow \overline{D}$ be as in 2.8. By regarding the inverse image of $\mathcal{H}_{\mathbb{R}}$ on $U^n \times \Delta^m$ as a constant finite dimensional $\mathbb{R}$-vector space, let $G_{\mathbb{R}}$ be the group of all automorphisms of this $\mathbb{R}$-vector space preserving $(\ ,\ )$. Let $I' := \{1,\dots,n\} - I$. Let

$$S = \{(u,w)\ ;\ u = (u_j)_{j\in I'},\ u_j \in \mathbb{R}_{>0},\ w \in \Delta^m\}.$$

For $(u,w) \in S$, the pair

$$\left(\Big(\big(\sum_{i_{\alpha-1}<j<i_\alpha} u_jN_j\big) + N_{i_\alpha}\Big)_{1\le\alpha\le r},\ \psi(0,w)\right)$$

yields a nilpotent orbit, and hence by the theory of $SL(2)$-orbits ([3]), this pair defines a homomorphism

$$\rho_{u,w}\ :\ SL(2,\mathbb{R})^r \longrightarrow G_{\mathbb{R}}$$

of algebraic groups over $\mathbb{R}$. This homomorphism $\rho_{u,w}$ depends real analytically on $(u,w) \in S$. For $a_1,\dots,a_r \in \mathbb{R}_{>0}$, let

$$t(a_1,\dots,a_r) = \left(\begin{pmatrix} a_1 & 0 \\ 0 & \frac{1}{a_1}\end{pmatrix},\dots,\begin{pmatrix} a_r & 0 \\ 0 & \frac{1}{a_r}\end{pmatrix}\right) \in SL(2,\mathbb{R})^r.$$

By [3, Proposition 5.10], there exists $K_0 > 1$ such that for any $K > K_0$, we can find a compact set C of D and a neighborhood V of 0 in Δ^{n+m} such that

$$\rho_{u(y),w}(t(y_{i_1}^{-1/2}, \ldots, y_{i_r}^{-1/2})) \exp(-\sum_{j=1}^{n} x_j N_j)\tilde{\phi}(z,w) \in C$$

for all $(z,w) \in U^n \times \Delta^m$ such that $(\exp(2\pi i z), w) \in V \cap (((\Delta^*)^n)_K^I \times \Delta^m)$, where $z_j = x_j + iy_j$ with x_j, y_j real and $u(y) = (y_j/y_{i_\alpha})_{j\in I'}$, $i_{\alpha-1} < j < i_\alpha$. This proves 2.13.3.

Example 2.14. We describe an example of the log C^∞ Hodge decomposition, for the VPLH arising from a family $f\colon E \longrightarrow \Delta$ of elliptic curves on Δ^* degenerating at $0 \in \Delta$. Let

$$E = \{(u,v) \in \mathbb{C}^2 \ ; \ |uv| < 1\}/\sim$$

where $\sim$ is the equivalence relation defined as follows: $(u,v) \sim (u',v')$ if and only if either one of the following (1) (2) is satisfied.
(1) $uv = u'v' \neq 0$ and if we denote uv $(= u'v')$ by t, $u' = ut^n$ and $v' = vt^{-n}$ for some $n \in \mathbb{Z}$.
(2) $(u,v) = (c,0)$ and $(u',v') = (0,1/c)$ for some $c \in \mathbb{C}^\times$, or $(u,v) = (0,1/c)$ and $(u',v') = (c,0)$ for some $c \in \mathbb{C}^\times$, or $(u,v) = (u',v')$.

Then E is a complex manifold. Let $f\colon E \longrightarrow \Delta$ be the holomorphic map $(u,v) \mapsto uv$. Then for $t \in \Delta^*$, $f^{-1}(t)$ is identified with the elliptic curve $\mathbb{C}^\times/t^{\mathbb{Z}}$ where we identify the coordinate u on E with the coordinate of $\mathbb{C}^\times$, and $f^{-1}(0)$ is identified with the singular space obtained from $\mathbb{P}^1(\mathbb{C})$ by identifying 0 and ∞. We endow Δ with the log structure corresponding to the divisor $\{0\}$, and E with the log structure corresponding to the divisor $f^{-1}(0)$ with normal crossings.

The family of H^1 of the elliptic curves $\mathbb{C}^\times/t^{\mathbb{Z}}$ forms a VPH on Δ^* and this VPH is extended to a VPLH $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ on Δ, where

$$\mathcal{H}_{\mathbb{Z}} = R^1 f_*^{\log}\mathbb{Z}, \ \mathcal{M} = R^1 f_*(\omega^\bullet_{E/\Delta}), \ \mathcal{M}^p = R^1 f_*(\omega^{\bullet \geq p}_{E/\Delta})$$

and $(\ ,\ )$ is explained later. The sheaf $\mathcal{H}_{\mathbb{Z}}$ is a locally constant sheaf which is described as follows. Let $\overline{U} \longrightarrow \Delta^{\log} \longrightarrow \Delta$ be as in 2.8. The pull back of the family $E - f^{-1}(0) \longrightarrow \Delta^*$ to U is identified with the family $\{\mathbb{C}/(\mathbb{Z}z + \mathbb{Z})\}_{z\in U}$ of elliptic curves (we identify $\mathbb{C}/(\mathbb{Z}z+\mathbb{Z})$ with $\mathbb{C}^\times/t^{\mathbb{Z}}$, where $t = \exp(2\pi i z)$, by $\exp(2\pi i -)$) and $\mathrm{H}_1(\mathbb{C}/(\mathbb{Z}z+\mathbb{Z}), \mathbb{Z})$ is identified with $\mathbb{Z}z + \mathbb{Z}$. Hence $\mathcal{H}_{\mathbb{Z}} = R^1 f_*^{\log}\mathbb{Z}$ is identified with the local system $\mathcal{H}om_{\mathbb{Z}}(\mathbb{Z}z + \mathbb{Z}, \mathbb{Z})$ where z is regarded as a local section $(2\pi i)^{-1}\log(t)$ of $\mathcal{O}_\Delta^{\log}$ (t denotes here the coordinate function of Δ) and the inverse image of $\mathcal{H}_{\mathbb{Z}}$ on $\overline{U}$ is identified with the constant sheaf $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}z+\mathbb{Z},\mathbb{Z})$ where z is regarded here as a global section of the inverse image of $\mathcal{O}_{\Delta}^{\log}$ on $\overline{U}$. Let $(e_j)_{j=1,2}$ be the $\mathbb{Z}$-basis of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}z+\mathbb{Z},\mathbb{Z})$ where e_1 sends z to 1 and 1 to 0, and e_2 sends z to 0 and 1 to 1. Then

$$\Gamma(\Delta,\ R^1f_*\mathbb{Z}) = \Gamma(\Delta^{\log},\ R^1f_*^{\log}\mathbb{Z}) = \mathbb{Z}e_1,\ (R^1f_*\mathbb{Z})_0 = \mathbb{Z}e_1.$$

The $\mathbb{Q}$-bilinear form $(\ ,\ ) \colon \mathcal{H}_{\mathbb{Q}} \times \mathcal{H}_{\mathbb{Q}} \longrightarrow \mathbb{Q}$ is the unique anti-symmetric form satisfying $(e_2, e_1) = 1$.

Next, $\mathcal{M}$ is a free $\mathcal{O}_{\Delta}$-module of rank 2 with basis (e_1, ω) where

$$\omega = d\log(u) = -d\log(v) \in \Gamma(\Delta,\ f_*\omega^1_{E/\Delta}),$$

and the filtration of $\mathcal{M}$ is described as

$$\mathcal{M}^p = \mathcal{M} \text{ for } p \le 0, \quad \mathcal{M}^p = 0 \text{ for } p \ge 2,$$
$$\mathcal{M}^1 = f_*\omega^1_{E/\Delta} = \mathcal{O}_{\Delta}\cdot\omega.$$

On $\overline{U}$, we have

2.14.1. $\qquad \omega = 2\pi i z e_1 + 2\pi i e_2.$

In fact, the pull back of ω to each elliptic curve $\mathbb{C}/(\mathbb{Z}z+\mathbb{Z})$ for $z \in U$ is $2\pi i ds$ where s is the coordinate of $\mathbb{C}$, and 2.14.1 follows from $\int_0^z 2\pi i ds = 2\pi i z$ and $\int_0^1 2\pi i ds = 2\pi i$.

Now the log C^∞ Hodge decomposition

$$\mathcal{A}_{\Delta} \otimes_{\mathcal{O}_{\Delta}} \mathcal{M} = \mathcal{M}^{1,0}_{\mathcal{A}} \oplus \mathcal{M}^{0,1}_{\mathcal{A}}$$

is described as follows: $\mathcal{M}^{1,0}_{\mathcal{A}}$ is a free $\mathcal{A}_{\Delta}$-module of rank 1 with basis ω, $\mathcal{M}^{0,1}_{\mathcal{A}}$ is a free $\mathcal{A}_{\Delta}$-module of rank 1 with basis

2.14.2. $\qquad \overline{\omega} = -2\pi i \overline{z} e_1 - 2\pi i e_2.$

The relation with the basis (e_1, ω) of $\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is given by

$$\overline{\omega} = -\omega + 2\log(|t|)e_1,$$
$$e_1 = \tfrac{1}{2}\log(|t|)^{-1}\omega + \tfrac{1}{2}\log(|t|)^{-1}\overline{\omega},$$

as is seen from 2.14.1 and 2.14.2. Note that $\log(|t|)$ and $\log(|t|)^{-1}$ are log C^∞-functions on Δ, but not C^∞-functions on Δ. This tells that C^∞-functions are not enough to obtain the Hodge decomposition in the situation of degeneration.

§3. Log $\overline{\partial}$-Poincaré lemma

The purpose of this section is to prove

Theorem 3.1. *Let X be an fs log analytic space which is log smooth over $\mathbb{C}$. Then we have an exact sequence on X_{ket}*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A}_X \xrightarrow{\overline{\partial}} \mathcal{A}_X^{0,1} \xrightarrow{\overline{\partial}} \mathcal{A}_X^{0,2} \longrightarrow \cdots,$$

where $\overline{\partial}\colon \mathcal{A}_X^{0,q} \longrightarrow \mathcal{A}_X^{0,q+1}$ is the map induced by $d\colon \mathcal{A}_X^q \longrightarrow \mathcal{A}_X^{q+1}$.
(The non-ket version of this is also true.)

When X is a complex manifold whose log structure is given by a divisor with normal crossings, the non-ket version of this theorem is a case of Proposition (2.2.4) in [12]. The part after 3.3 of this section is essentially included in [12] section 2. See also [10].

3.2. To prove 3.1, first we show that we may assume $X = \Delta^{n+m}$ with the log structure given by the complement of $(\Delta^*)^n \times \Delta^m$. In fact, locally on X, take a blowing up $f\colon Z \longrightarrow X$ along log structure such that Z is a complex manifold and the complement of X_{triv} in Z is a divisor with normal crossings. If Theorem 3.1 is true for Z, then by $Rf_*\mathcal{O}_Z = \mathcal{O}_X$ ([22] I, Corollary 1 c) to Theorem 12, GAGA ([8] XII Théorème 4.2), and 1.7), $Rf_*\mathcal{A}_Z = \mathcal{A}_X$ (1.5, 1.7, and 2.10), and $\mathcal{A}_Z^{p,q} = \mathcal{A}_Z \otimes_{\mathcal{A}_X} \mathcal{A}_X^{p,q}$ (log étaleness of f), Theorem 3.1 is true for X. Hence we may assume that $X = \Delta^{n+m}$ with the log structure as above.

3.3. We fix notation concerning Δ^{n+m}.

For $1 \le j \le n+m$, let t_j be the j-th coordinate function of Δ^{n+m}. For $1 \le j \le n$, let

$$r_j = |t_j|, \quad u_j = t_j/r_j$$

(u_j is defined on $(\Delta^*)^n \times \Delta^m$). Let

$$\partial_j = t_j \cdot \frac{\partial}{\partial t_j}, \qquad \overline{\partial}_j = \overline{t_j} \cdot \frac{\partial}{\partial \overline{t_j}} \qquad \text{for } 1 \le j \le n,$$
$$\partial_j = \frac{\partial}{\partial t_j}, \qquad \overline{\partial}_j = \frac{\partial}{\partial \overline{t_j}} \qquad \text{for } n+1 \le j \le n+m.$$

Then for $1 \le j \le n$,

3.3.1. $\partial_j = \frac{1}{2}(r_j \frac{\partial}{\partial r_j} + u_j \frac{\partial}{\partial u_j}), \quad \overline{\partial}_j = \frac{1}{2}(r_j \frac{\partial}{\partial r_j} - u_j \frac{\partial}{\partial u_j}).$

Let $|\Delta| = \{s \in \mathbb{R}\ ;\ 0 \le s < 1\}$, $|\Delta^*| = \{s \in \mathbb{R}\ ;\ 0 < s < 1\}$.

We use the theory of Fourier expansions as in [31, Proposition 6.4].

Lemma 3.4. (1) *Via the Fourier expansion*

$$f = \sum_{l \in \mathbb{Z}^n} f_l \cdot \prod_{j=1}^{n} u_j^{l(j)},$$

a C^∞-function f on $(\Delta^*)^n \times \Delta^m$ corresponds bijectively to a family $(f_l)_{l \in \mathbb{Z}^n}$ of C^∞-functions on $|\Delta^*|^n \times \Delta^m$ satisfying the following condition 3.4.1.

3.4.1. *For each $v \in |\Delta^*|^n \times \Delta^m$ and each $a \in \mathbb{N}^n$, there exists $C > 0$ such that*

$$\left(\prod_{j=1}^{n} |l(j)|^{a(j)}\right) \cdot |f_l(v)| \leq C$$

for any $l \in \mathbb{Z}^n$.

(2) *Endow Δ^{n+m} with the log structure as in* 3.2. *Then, in the correspondence in* (1), *f is a log C^∞-function on Δ^{n+m} if and only if the family $(f_l)_{l \in \mathbb{Z}^n}$ satisfies the following condition* 3.4.2.

3.4.2. *For each a, $b \in \mathbb{N}^n$, each c, $d \in \mathbb{N}^m$ and each compact subset K of $|\Delta|^n \times \Delta^m$, there exists $C > 0$ and $h \in \mathbb{N}^n$ such that*

$$\begin{aligned}&\left(\prod_{j=1}^{n} |l(j)|^{a(j)}\right) \cdot \left| \left(\prod_{j=1}^{n} \left(r_j \frac{\partial}{\partial r_j}\right)^{b(j)}\right) \left(\prod_{j=1}^{m} \partial_{n+j}^{c(j)} \overline{\partial}_{n+j}^{d(j)}\right)(f_l)(r, z) \right| \\ &\leq C \cdot \prod_{j=1}^{n} |\log(r_j)|^{h(j)}\end{aligned}$$

for any $l \in \mathbb{Z}^n$ and any $(r, z) \in K \cap (|\Delta^*|^n \times \Delta^m)$.

Proof. As is well known in the theory of Fourier expansions, a C^∞-function on $(\mathbb{S}^1)^n$ corresponds bijectively to a rapidly decreasing function on $\mathbb{Z}^n$. (1) follows from this. (2) is deduced from the relation 3.3.1 of ∂_j, $\overline{\partial}_j$ and $r_j \frac{\partial}{\partial r_j}$, $u_j \frac{\partial}{\partial u_j}$ $(1 \leq j \leq n)$. Q.E.D.

3.5. We prove that if f is a log C^∞-function on Δ^{n+m} with the log structure as in 3.2 and if $\overline{\partial}(f) = 0$, then f is a holomorphic function on Δ^{n+m}. Let $f = \sum_l f_l \cdot \prod_{j=1}^n u_j^{l(j)}$ be the Fourier expansion of f. Then by $\overline{\partial}_j(f) = 0$ for $1 \leq j \leq n+m$ and by 3.3.1, we have for each $l \in \mathbb{Z}^n$

$$r_j \frac{\partial}{\partial r_j}(f_l) - l(j) f_l = 0 \qquad \text{for } 1 \leq j \leq n,$$

$$\overline{\partial}_j(f_l) = 0 \qquad \text{for } n+1 \leq j \leq n+m.$$

This shows

$$f_l(r,z) = (\prod_{j=1}^{n} r_j^{l(j)}) \cdot h_l(z)$$

where $h_l(z)$ $(z = (t_j)_{n+1 \le j \le n+m})$ is a holomorphic function in $z \in \Delta^m$. The log growth of f shows the log growth of f_l for each $l \in \mathbb{Z}^n$, and this shows $h_l = 0$ unless $l(j) \ge 0$ for all $1 \le j \le n$. Hence

$$f = \sum_{l \in \mathbb{N}^n} (\prod_{j=1}^{n} t_j^{l(j)}) \cdot h_l(z).$$

This and 3.4 (1) show that f is a holomorphic function on Δ^{n+m}.

3.6. By 3.5, for the proof of Theorem 3.1 for $X = \Delta^{n+m}$ with the log structure as in 3.2, it remains to prove $\mathcal{H}^q(\mathcal{A}_X^{0,\bullet}) = 0$ for $q \ge 1$. As in the argument of the proof of the classical $\overline{\partial}$-Poincaré lemma (*cf.* [7, p. 25]), this is reduced to proving the following 3.6.1.

3.6.1. *Let $1 \le k \le n+m$ and let S be a subset of $\{1, \ldots, n+m\}$ which does not contain k. Let f be a log C^∞-function on Δ^{n+m} and assume $\overline{\partial}_j(f) = 0$ for $j \in S$. Then locally on Δ^{n+m}, there exists a log C^∞-function g satisfying*

$$\overline{\partial}_j(g) = 0 \ \textit{for}\ j \in S, \quad \textit{and}\ \overline{\partial}_k(g) = f.$$

We prove 3.6.1 in the case $1 \le k \le n$ (resp. $n+1 \le k \le n+m$) in 3.7 (resp. 3.9).

3.7. First assume $1 \le k \le n$. Let $f = \sum_l f_l \cdot \prod_{j=1}^n u_j^{l(j)}$ be the Fourier expansion of f. For each $l \in \mathbb{Z}^n$, define a C^∞-function $g_{f,l}$ on $|\Delta^*|^n \times \Delta^m$ as follows. Fix a positive number $A < 1$. Let $e = l(k)$, and define

$$g_{f,l}(r,z) = 2r_k^e \int_B^{r_k} s^{-e} f_l(r_1, \ldots, r_{k-1}, s, r_{k+1}, \ldots, r_n, z) \frac{ds}{s}$$

where $B = A$ in the case $e \ge 0$ and $B = 0$ in the case $e < 0$. We estimate $g_{f,l}$. Let $a \in \mathbb{N}^n$, let K be a compact subset of $|\Delta|^n \times \Delta^m$, and by putting $b = 0 \in \mathbb{N}^n$ and $c = d = 0 \in \mathbb{N}^m$ in 3.4 (2), let $C > 0$ and $h \in \mathbb{N}^n$ be as in 3.4 (2) for the family $(f_l)_l$. Then by lemma 3.8 below, for any $l \in \mathbb{Z}^n$ and any $(r,z) \in K \cap (|\Delta^*|^n \times \Delta^m)$,

$$(\prod_{j=1}^{n} |l(j)|^{a(j)}) |g_{f,l}(r,z)| \le 2C \cdot \theta(r_k) \cdot \prod_{\substack{j=1 \\ j \ne k}}^{n} |\log(r_j)|^{h(j)}$$

where

$$\theta(r_k) = \begin{cases} h(k)! \cdot \sum_{i=0}^{h(k)} (|\log(r_k)|^i + |\log(A)|^i A^{-e}) & \text{if } e > 0, \\ |\log(r_k)|^{h(k)+1} + |\log(A)|^{h(k)+1} & \text{if } e = 0, \\ h(k)! \cdot \sum_{i=0}^{h(k)} |\log(r_k)|^i & \text{if } e < 0. \end{cases}$$

Hence

$$g_f = \sum_{l \in \mathbb{Z}^n} g_{f,l} \cdot \prod_{j=1}^{n} u_j^{l(j)}$$

is a C^∞-function on $(\Delta^*)^n \times \Delta^m$ and is of log growth on Δ^{n+m}. We can check easily

3.7.1. $\overline{\partial}_k(g_f) = f$,

3.7.2. $D(g_f) = g_{D(f)}$ for $D = \prod_{j=1}^{n+m} \partial_j^{a(j)} \overline{\partial}_j^{b(j)}$

for any $a,\ b \in \mathbb{N}^{n+m}$.

By 3.7.2, we have $\overline{\partial}_j(g_f) = 0$ for $j \in S$. Furthermore, by 3.7.2, what we have proved concerning the log growth of g_f shows that g_f is a log C^∞-function.

Lemma 3.8. *Let $e,\ h \in \mathbb{Z},\ h \geq 0$ and let $B,\ x \in \mathbb{R},\ 0 < x < 1$. Assume $0 < B < 1$ in the case $e \geq 0$, and $B = 0$ in the case $e < 0$. Then*

$$x^e \int_B^x t^{-e} \log(t)^h \cdot \frac{dt}{t}$$

is equal to

$$\begin{cases} -\sum_{i=0}^{h} \frac{h!}{i!} \cdot e^{i-h-1}(\log(x)^i - \log(B)^i (\frac{x}{B})^e) & \textit{if } e > 0, \\ \frac{1}{h+1} \cdot (\log(x)^{h+1} - \log(B)^{h+1}) & \textit{if } e = 0, \\ -\sum_{i=0}^{h} \frac{h!}{i!} \cdot e^{i-h-1} \log(x)^i & \textit{if } e < 0. \end{cases}$$

3.9. We prove the case $n+1 \leq k \leq n+m$ of 3.6.1 by the method in [7]. Fix $v = (v_j)_j \in \Delta^{n+m}$. Take a positive number ϵ such that $|v_j| + \epsilon < 1$

for $1 \le j \le n+m$ and $|v_k| + 3\epsilon < 1$. Let

$$\begin{aligned}
U &= \{w \in \Delta^{n+m} \ ; \ |w_j - v_j| < \epsilon \quad \text{for } 1 \le j \le n+m\}, \\
K &= \{w \in \Delta^{n+m} \ ; \ |w_j - v_j| \le \epsilon \quad \text{if } 1 \le j \le n+m \text{ and } j \ne k, \\
&\qquad\qquad\qquad |w_k - v_k| \le 3\epsilon\}, \\
M &= \{z \in \Delta \ ; \ |z - v_k| \le \epsilon\}, \\
N &= \{(r,\ u) \ ; \ r \in \mathbb{R},\ u \in \mathbb{C},\ 0 \le r \le 2\epsilon,\ |u| = 1\}.
\end{aligned}$$

We define a C^∞-function g_f on $U \cap ((\Delta^*)^n \times \Delta^m)$ by

$$g_f(w) = \frac{1}{2\pi i}\int_M (z - w_k)^{-1} f(w_1, \ldots, w_{k-1}, z, w_{k+1}, \ldots, w_{n+m}) dz \wedge d\bar{z}.$$

Since

$$(z - w_k)^{-1} dz \wedge d\bar{z} = -2u^{-2} dr \wedge du,$$

where $r = |z - w_k|$, $u = (z - w_k)/r$, and since

$$M \subset \{w_k + ru \ ; \ (r,\ u) \in N\},$$

we see that the integral defining g_f converges, g_f is a C^∞-function on $U \cap ((\Delta^*)^n \times \Delta^m)$, and
3.9.1.

$$|g_f(w)| \le \frac{1}{\pi}\int_N |f(w_1, \ldots, w_{k-1}, w_k + ru, w_{k+1}, \ldots, w_{n+m})| \cdot |dr \wedge du|.$$

It is checked easily that $f \mapsto g_f$ satisfies 3.7.1 and 3.7.2. By 3.7.2, $\bar{\partial}_j(g_f) = 0$ for $j \in S$. By 3.7.2, to show that g_f is a log C^∞-function on U, it is sufficient to prove that g_f is of log growth on U. Since K is compact and f is of log growth, there are $C > 0$ and $h \in \mathbb{N}^n$ such that

$$|f(w)| \le C \cdot \prod_{j=1}^{n} |\log(|w_j|)|^{h(j)}$$

for any $w \in K \cap ((\Delta^*)^n \times \Delta^m)$. By

$$\{(w_1, \ldots, w_{k-1}, w_k + ru, w_{k+1}, \ldots, w_{n+m}) \ ; \ w \in U,\ (r,\ u) \in N\} \subset K,$$

and by 3.9.1,

$$|g_f(w)| \le 4\epsilon C \cdot \prod_{j=1}^{n} |\log(|w_j|)|^{h(j)}$$

for any $w \in U \cap ((\Delta^*)^n \times \Delta^m)$. This shows that g_f is of log growth on U.

§4. Relative log Poincaré lemma

Here everything is in the ket sense except in the latter part of 4.4.

Let X, Y be fs log analytic spaces which are log smooth over $\mathbb{C}$, and let $f\colon X \longrightarrow Y$ be a log smooth morphism. Let $\mathcal{A}^1_{X/Y}$ be the cokernel of $\mathcal{A}_X \otimes_{\mathcal{A}_Y} \mathcal{A}^1_Y \longrightarrow \mathcal{A}^1_X$, $\mathcal{A}^p_{X/Y} := \bigwedge^p_{\mathcal{A}_X} \mathcal{A}^1_{X/Y}$, and $\mathcal{A}^{p,\log}_{X/Y} := \mathcal{O}^{\log}_X \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\mathcal{A}^p_{X/Y})$ for each $p \geq 0$.

Theorem 4.1. *Let $f\colon X \longrightarrow Y$ be a log smooth morphism of log smooth fs log analytic spaces. Let x be a point of $X^{\log}$. Assume that f is exact at $\tau(x)$. Then the stalk at x of*

$$0 \longrightarrow (f^{\log})^{-1}(\mathcal{A}^{\log}_Y) \longrightarrow \mathcal{A}^{\log}_X \longrightarrow \mathcal{A}^{1,\log}_{X/Y} \longrightarrow \mathcal{A}^{2,\log}_{X/Y} \longrightarrow \cdots$$

is exact.

For the proof we use;

Proposition 4.2. *Under the same assumption as in Theorem* 4.1, *let $y = f^{\log}(x) \in Y^{\log}$. Assume that the cokernel of $M^{\mathrm{gp}}_{Y,\tau(y)}/\mathcal{O}^\times_{Y,\tau(y)} \longrightarrow M^{\mathrm{gp}}_{X,\tau(x)}/\mathcal{O}^\times_{X,\tau(x)}$ is torsion free. Then the followings hold.*
(1) *There exists an open neighborhood U_0 of $\tau(y)$ having the following property: For any open neighborhood W of x, there is a continuous map $s\colon U := U_0^{\log} \longrightarrow X^{\log}$ satisfying the following* 4.2.1–4.2.4.

4.2.1. $f^{\log} \circ s = \mathrm{id}_U$.

4.2.2. $s(y) \in W$.

4.2.3. $s(U_{\mathrm{triv}}) \subset X_{\mathrm{triv}}$.

4.2.4. *For any open set V of X, U_1 of U_0 such that $s(U_1^{\log}) \subset V^{\log}$, and for any $g \in \Gamma(V, \mathcal{A}_X)$, $g \circ s$ belongs to $\Gamma(U_1, \mathcal{A}_Y)$.*

(2) *If f is vertical, s can be chosen to satisfy $s(y) = x$.*

4.3. We prove Proposition 4.2. We may assume the following: $X = \mathrm{Spec}\,(\mathbb{C}[\mathcal{T}])^{\mathrm{an}}$, $Y = \mathrm{Spec}\,(\mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ for fs monoids $\mathcal{S}$, $\mathcal{T}$ such that $\mathcal{S} \subset \mathcal{T}$, $\mathcal{S}^{\mathrm{gp}} \cap \mathcal{T} = \mathcal{S}$, $\mathcal{S}^\times = \{1\}$, $\mathcal{T}^\times = \{1\}$, $\mathcal{T}^{\mathrm{gp}}/\mathcal{S}^{\mathrm{gp}}$ is torsion free, $x \in X^{\log}$ lies over the origin of X, and $f\colon X \longrightarrow Y$ is the natural projection so that $y \in Y^{\log}$ lies over the origin of Y. It is enough to prove the following claim on monoids.

In the rest of this subsection, for $\mathcal{U} = \mathcal{S}$ or $\mathcal{T}$, we denote by $|\mathcal{U}^\vee|$ the topological space $\mathrm{Hom}\,(\mathcal{U}, \mathbb{R}^{\mathrm{mult}}_{\geq 0})$. By a log C^∞-function on an open U of $|\mathcal{U}^\vee|$, we mean a C^∞-function $g\colon U \cap \mathrm{Hom}\,(\mathcal{U}, \mathbb{R}_{>0}) \longrightarrow \mathbb{C}$ having the following property: If $(t_j)_{1\le j\le n}$ is a basis of $\mathcal{U}^{\mathrm{gp}}$, for any $a_j \in \mathbb{N}$ $(1 \le j \le n)$, $\prod_j \left(t_j \cdot \frac{\partial}{\partial t_j}\right)^{a_j}(g)$ is of log growth on U (cf. 1.1).

Claim. For $\mathcal{S} \subset \mathcal{T}$ as above, let x (resp. y) be the origin of $|\mathcal{T}^\vee|$ (resp. $|\mathcal{S}^\vee|$). Then for any open neighborhood W of x, there is a continuous map $s\colon |\mathcal{S}^\vee| \longrightarrow |\mathcal{T}^\vee|$ satisfying the following 4.2.1′–4.2.4′.

4.2.1′ $f\circ s = \mathrm{id}$, where f is the canonical map $|\mathcal{T}^\vee| \longrightarrow |\mathcal{S}^\vee|$.

4.2.2′ $s(y) \in W$.

4.2.3′ $s(\mathrm{Hom}\,(\mathcal{S}, \mathbb{R}_{>0})) \subset \mathrm{Hom}\,(\mathcal{T}, \mathbb{R}_{>0})$.

4.2.4′ For any open set V of $|\mathcal{T}^\vee|$ and for any log C^∞-function g on V, $g\circ s$ is a log C^∞-function on $s^{-1}(V)$.

Further, if $\mathcal{S} \longrightarrow \mathcal{T}$ is dominating (i.e., any $t \in \mathcal{T}$ divides an element of $\mathcal{S}$), 4.2.2′ can be replaced by $s(y) = x$.

In the rest of this subsection, we prove this claim. By induction on $\mathrm{rank}(\mathcal{T}^{\mathrm{gp}}) - \mathrm{rank}(\mathcal{S}^{\mathrm{gp}})$, we may assume that $\mathrm{rank}(\mathcal{T}^{\mathrm{gp}}) = \mathrm{rank}(\mathcal{S}^{\mathrm{gp}})+1$. Fix an embedding $\mathcal{T} \subset \mathcal{S}^{\mathrm{gp}}_{\mathbb{Q}} \oplus \mathbb{Q}$ which sends each $s \in \mathcal{S} \subset \mathcal{T}$ to $(s,\ 0)$. Take a finite family $((a_\lambda,\ e(\lambda)))_{\lambda\in\Lambda}$ $(a_\lambda \in \mathcal{S}^{\mathrm{gp}}_{\mathbb{Q}},\ e(\lambda) \in \{\pm 1\})$ of elements of $\mathcal{T}_{\mathbb{Q}_{\ge 0}} := \mathcal{T} \otimes_{\mathbb{N}} \mathbb{Q}_{\ge 0} \subset \mathcal{S}^{\mathrm{gp}}_{\mathbb{Q}} \oplus \mathbb{Q}$ which together with $\mathcal{S}$ generates $\mathcal{T}_{\mathbb{Q}_{\ge 0}}$. Let

$$\Lambda_+ = \{\lambda \in \Lambda,\ e(\lambda) = 1\}, \quad \Lambda_- = \{\lambda \in \Lambda,\ e(\lambda) = -1\}.$$

Then the exactness $\mathcal{S} = \mathcal{T} \cap \mathcal{S}^{\mathrm{gp}}$ implies the following 4.3.1.

4.3.1. If $(a, 1)$ and $(a', -1)$ belong to $\mathcal{T}_{\mathbb{Q}_{\ge 0}}$, then $aa' \in \mathcal{S}_{\mathbb{Q}_{\ge 0}}$.

The condition "dominating" implies

4.3.2. $\Lambda_+ \ne \emptyset$ and $\Lambda_- \ne \emptyset$.

In the followings, for $\mathcal{U} = \mathcal{S}$ or $\mathcal{T}$, we identify each element of $|\mathcal{U}^\vee|$ with its natural extension in $\mathrm{Hom}\,(\mathcal{U} \otimes_{\mathbb{N}} \mathbb{R}^{\mathrm{add}}_{\ge 0}, \mathbb{R}^{\mathrm{mult}}_{\ge 0})$. We will define $s(h)$ for each $h \in |\mathcal{S}^\vee|$.

In the non-dominating case, define $s(h)$ as follows. We may assume that $\mathcal{S} \ne \{1\}$. We have $\Lambda = \Lambda_+$ or $\Lambda = \Lambda_-$. So assume $\Lambda = \Lambda_+$. Take an element b of $\mathcal{S}$ such that $a_\lambda b$ belongs to the interior of $\mathcal{S}_{\mathbb{Q}_{\ge 0}}$ for any $\lambda \in \Lambda$, and define a homomorphism $\theta\colon \mathcal{T} \longrightarrow \mathcal{S}_{\mathbb{Q}_{\ge 0}}$ by sending $(a_\lambda, 1)$ to $a_\lambda b$. Define $s(h) = h\circ\theta$.

Now we assume that $\mathcal{S} \longrightarrow \mathcal{T}$ is dominating, that is, Λ_+ and Λ_- are non-empty sets. Let $I := \Lambda_+ \times \Lambda_-$. By 4.3.1, we have $s_i := a_\lambda a_\mu \in \mathcal{S}_{\mathbb{Q}_{\ge 0}}$ for any $i = (\lambda, \mu) \in I$.

In the case $h(s_i) = 0$ for any $i \in I$, define $s(h)$ to be the unique homomorphism $\mathcal{T} \longrightarrow \mathbb{R}_{\ge 0}$ which coincides on $\mathcal{S}$ with h and which sends $(a_\lambda,\ e(\lambda))$ to 0 for all $\lambda \in \Lambda$. (It is easy to see that such homomorphism exists.)

Before defining $s(h)$ for $h \in V := \{h \in |\mathcal{S}^\vee| \ ; \ h(s_i) \neq 0$ for some $i \in I\}$, we choose a partition of unity on V, which is subordinate to the covering $(U_i)_{i \in I}$ as follows. Here $U_i := \{h \ ; \ 3h(s_i) > h(s_j)$ for any $j \in I\}$. Take a C^∞-function χ on $\mathbb{R}_{\geq 0}$ such that $\chi(t) = 0$ for $t \geq 2$ and $\chi(t) + \chi(t^{-1}) = 1$ for all $t > 0$. For any $i, j \in I$, let $\varphi_{ij} := \chi(s_i^{-1}s_j)$, which is defined on $\{h \ ; \ h(s_i) \neq 0\}$, and $\varphi_i := \prod_{j \neq i} \varphi_{ij}$. Then $\{\varphi_i\}_{i \in I}$ is the desired partition of unity. Note that each φ_i is log C^∞ in the sense explained before the above claim. Now let $c := \prod_{i \in I} s_i^{\varphi_i}$, $a_+ := \prod_{i=(\lambda,\mu) \in I} a_\lambda^{\varphi_i}$, and $a_- := \prod_{i=(\lambda,\mu) \in I} a_\mu^{\varphi_i}$. Then $c(h) = h(a_+ a_-) \neq 0$ for any $h \in V$. Let $h \in V$. Define $s(h)$ to be the unique homomorphism $\mathcal{T} \longrightarrow \mathbb{R}_{\geq 0}$ which coincides on $\mathcal{S}$ with h and which sends $(a_\lambda, 1)$ $(\lambda \in \Lambda_+)$ to $h(a_\lambda a_-) \cdot c^{-1/2}$, and $(a_\mu, -1)$ $(\mu \in \Lambda_-)$ to $h(a_+ a_\mu) \cdot c^{-1/2}$. (It is easy to see that such homomorphism exists.)

Thus we have defined a map s. It is easy to see that s is continuous and has the desired properties.

4.4. We prove Theorem 4.1. The proof is essentially the same as the proof of the classical Poincaré lemma. We may assume that the following: $X = \operatorname{Spec}(\mathbb{C}[\mathcal{T}])^{\mathrm{an}} \times \mathbb{C}^T$, $Y = \operatorname{Spec}(\mathbb{C}[\mathcal{S}])^{\mathrm{an}} \times \mathbb{C}^S$ for fs monoids $\mathcal{S}$, $\mathcal{T}$ such that $\mathcal{S} \subset \mathcal{T}$, $\mathcal{S}^{\mathrm{gp}} \cap \mathcal{T} = \mathcal{S}$, $\mathcal{S}^\times = \{1\}$, $\mathcal{T}^\times = \{1\}$ and finite sets $S \subset T$, $x \in X^{\log}$ lies over the origin of X and $f \colon X \longrightarrow Y$ is the natural projection so that $y = f^{\log}(x) \in Y^{\log}$ lies over the origin of Y. We will prove the exactness at $\mathcal{A}^{p,\log}_{X/Y,x}$, $p \geq 0$.

First we reduce to the case where the relative dimension $d := \dim X - \dim Y$ is one by the standard induction argument (cf. [7] p.25) as follows: Supposing that the statement is valid for the case of the relative dimension $< d$, we will prove the case where it is d. We will assume that $\mathcal{S} \neq \mathcal{T}$; the other case is similar. Let $\omega \in \mathcal{A}^{p,\log}_{X/Y,x}$ such that $d\omega = 0$. We will prove that ω comes from $\mathcal{A}^{p-1,\log}_{X/Y,x}$ (resp. $\mathcal{A}^{\log}_{Y,y}$) for $p \geq 1$ (resp. $p = 0$). Take an fs monoid $\mathcal{S}' \subset \mathcal{T}$ such that $\mathcal{S} \subset \mathcal{S}' = \mathcal{S}'^{\mathrm{gp}} \cap \mathcal{T}$ and such that $\operatorname{rank}(\mathcal{S}')^{\mathrm{gp}} = \operatorname{rank}(\mathcal{S})^{\mathrm{gp}} + 1$ and an element $t \in \mathcal{S}'$ such that $t \notin \mathcal{S}^{\mathrm{gp}} \otimes \mathbb{Q}$. Denote $\operatorname{Spec}(\mathbb{C}[\mathcal{S}'])^{\mathrm{an}} \times \mathbb{C}^S$ by Y' and the image of x in Y' by y'. Since the image of ω in $\mathcal{A}^{p,\log}_{X/Y',x}$ is closed, the induction hypothesis implies that we may assume that the image is zero if $p \geq 1$; ω lies in $\mathcal{A}^{\log}_{Y',y'}$ if $p = 0$. Hence the case $p = 0$ follows. In the case where $p \geq 1$, we can write ω as $\omega_0 d\log t + \omega_1 d\log \bar{t}$ $(\omega_i \in \mathcal{A}^{p-1,\log}_{X/Y,x},\ i = 0,\ 1)$. Similarly we may assume that the image of ω_i in $\mathcal{A}^{p-1,\log}_{X/Y',x}$ is zero $(i = 0,\ 1)$ if $p \geq 2$; ω_i lies in $\mathcal{A}^{\log}_{Y',y'}$ $(i = 0,\ 1)$ if $p = 1$. Thus the case $p = 1$ follows. If $p \geq 2$, we can write $\omega = \omega_2 d\log t \wedge d\log \bar{t}$ $(\omega_2 \in \mathcal{A}^{p-2,\log}_{X/Y,x})$.

Then the similar argument shows that ω is exact ($p \geq 3$) or lies in $\mathcal{A}^{2,\log}_{Y'/Y,x}$ ($p = 2$). Thus we may assume that $d = 1$ and $0 \leq p \leq 2$.

Further we may assume that the cokernel of $\mathcal{S}^{\mathrm{gp}} \longrightarrow \mathcal{T}^{\mathrm{gp}}$ is torsion free and it is enough to prove the non-ket version of the statement. In the following we will assume that $\mathcal{S} \neq \mathcal{T}$; the proof for the other case where $S \neq T$ is similar and simpler. Let $\omega \in \mathcal{A}^{p,\log}_{X/Y,x}$ with $d\omega = 0$ and we will prove that ω is exact ($p = 1$, 2) or comes from Y ($p = 0$). In the following, fix an element $t \in \mathcal{T} - \mathcal{S}$ and consider it as a relative coordinate function.

Assume that $p = 0$. Take a base $s_1, \dots, s_r$ of $\mathcal{S}^{\mathrm{gp}}$. Then ω is regarded as a polynomial in $\mathcal{A}_{X,\tau(x)}[l_1, \dots, l_r, l]$, where $l_i = \log s_i$ $(1 \leq i \leq r)$ and $l = \log t$. Write $\omega = \sum \omega_{i_1 \cdots i_r} l_1^{i_1} \cdots l_r^{i_r}$, $\omega_{i_1 \cdots i_r} \in \mathcal{A}_{X,\tau(x)}[l]$. Then $d\omega = 0$ implies $d\omega_{i_1 \cdots i_r} = 0$ for the highest degree $(i_1, \dots, i_r)$ in the sense of the lexicographic order. Hence the induction works when $\omega_{i_1 \cdots i_r}$ comes from $\mathcal{A}_{Y,\tau(y)}$. Thus the problem is reduced to show that $d\omega = 0$ for $\omega \in \mathcal{A}_{X,\tau(x)}[l]$ implies $\omega \in \mathcal{A}_{Y,\tau(y)}$. We will show this. By induction of the degree of l with the fact that $df_0 + f_1 d\log t = 0$ implies $f_1 = 0$ for any $f_0 \in \mathcal{A}_{X,\tau(x)}$ and $f_1 \in \mathcal{A}_{Y,\tau(y)}$, we may assume that $\omega \in \mathcal{A}_{X,\tau(x)}$. (We have that, by seeing each fiber near y, the above fact is reduced to another simple fact that $\alpha d\log t$ ($\alpha \in \mathbb{C}$) is not exact on an annulus $\{re^{i\theta}\ ;\ 0 \leq \theta \leq 2\pi,\ R_1 < r < R_2\}$, $R_2 > R_1 \geq 0$ in the complex t-plane unless $\alpha = 0$.) Fix a set of generators $\{t_0 = t, t_1, \dots, t_s\}$ of $\mathcal{T}$. Then there is a positive real number ε such that ω is defined and $d\omega = 0$ on the neighborhood $X' = \{x \in X\ ;\ |t_i(x)| < \varepsilon$ for any $i = 0, \dots, s\}$ of $\tau(x)$. By Proposition 4.2, we may assume that there exist open neighborhood Y' of $\tau(y)$, a continuous map $s\colon U := Y'^{\log} \longrightarrow X'^{\log} \subset X^{\log}$ satisfying 4.2.1, 4.2.3, and 4.2.4 (U_0 there being replaced with Y'). Then we see that ω comes from $\mathcal{A}_{Y,\tau(y)}$ by 4.2.4.

Next let $p = 1$ or 2. Let $l_1, \dots, l_r$, l as above. Write $\omega = \sum \omega_{i_1 \cdots i_r i}$ $l_1^{i_1} \cdots l_r^{i_r} l^i$, $\omega_{i_1 \cdots i_r i} \in \mathcal{A}^p_{X/Y,\tau(x)}$. Then $d\omega = 0$ implies $d\omega_{i_1 \cdots i_r i} = 0$ for the highest degree (in the same sense as above). Hence the induction works and the problem is reduced to show that $\omega \in \mathcal{A}^p_{X/Y,\tau(x)}$ with $d\omega = 0$ comes from $\mathcal{A}^{p-1}_{X/Y,\tau(x)}$. We take the same X', Y' and s as above. In the following, we regard each fiber of $X'_{\mathrm{triv}} \longrightarrow Y'_{\mathrm{triv}}$ as an annulus in the t-plane. Assume that $p = 1$. For $y' \in Y'_{\mathrm{triv}}$, define

$$c(y') = \int_\gamma \omega,$$

where γ is any loop $\{Re^{i\theta}\ ;\ 0 \leq \theta \leq 2\pi\}$, $R > 0$, in the fiber of $X'_{\mathrm{triv}} \longrightarrow Y'_{\mathrm{triv}}$ at y' in the t-plane. Then c is a log C^∞-function on Y'

and $d(c\log(t)) = cd\log(t)$. Replacing ω with $\omega - \frac{c}{2\pi i}d\log(t)$, we may assume that $c = 0$. For $x' \in X'_{\mathrm{triv}}$ which maps into Y', define

$$\phi(x') = \int_\gamma \omega,$$

where γ is any route from $s(f(x'))$ to x' in the fiber $f^{-1}(f(x')) \cap X'_{\mathrm{triv}}$. Then we have $\omega = d\phi$. To show that ϕ is log C^∞, we have to estimate the growth. It is achieved by using special routes; for example, jointed ones of the routes on which either u or v is constant, where $u = \arg(t)$ and $v = \log|t|$.

Assume that $p = 2$ and $\omega = h(u,v)dudv$. Here we take the above u, v as coordinates of the fiber. For $x' \in X'_{\mathrm{triv}}$ which maps into Y', define

$$H(x') = \int_{v(s(f(x')))}^{v(x')} h(u,v)dv.$$

Then H is log C^∞ and $d(-Hdu) = h(u,v)dudv$.

§5. Consequences of the relative log Poincaré lemma

Everything is ket here.

Let $f\colon X \longrightarrow Y$ be as in the beginning of section 4. For an object V of $V_{\mathrm{qnilp}}(X)$, let $\omega^\bullet_{X/Y}(V)$ (resp. $\mathcal{A}^\bullet_{X/Y}(V)$) be the complex $i \mapsto \omega^i_{X/Y} \otimes_{\mathcal{O}_X} V$ (resp. $\mathcal{A}^i_{X/Y} \otimes_{\mathcal{O}_X} V$) ($i \in \mathbb{Z}$) with the differentials induced by those of $\omega^\bullet_{X/Y}$ (resp. $\mathcal{A}^\bullet_{X/Y}$) and the connection $V \longrightarrow \omega^1_X(V) \longrightarrow \omega^1_{X/Y}(V)$.

The aim of this section is to prove the following proposition.

Proposition 5.1. *Let $f\colon X \longrightarrow Y$ be a proper separated log smooth morphism between log smooth fs log analytic spaces. Let V be an object of $V_{\mathrm{qnilp}}(X)$. Then for any $m \in \mathbb{Z}$, the canonical map*

$$\mathcal{A}_Y \otimes_{\mathcal{O}_Y} R^m f_*(\omega^\bullet_{X/Y}(V)) \longrightarrow R^m f_*(\mathcal{A}^\bullet_{X/Y}(V)) = \mathcal{H}^m(f_*(\mathcal{A}^\bullet_{X/Y}(V)))$$

is an isomorphism.

Here the identity of the right hand side is by f_*-acyclicity of $\mathcal{A}^p_{X/Y}$ (V) $(p \in \mathbb{Z})$ which is deduced from Propositions 1.5 and 1.7.

We use the following result on the functoriality of the log Riemann-Hilbert correspondences in [14] (generalization of results of the second author, F. Kato, and S. Usui).

Theorem 5.2. *Let $f\colon X \longrightarrow Y$, V be as in the hypothesis of Proposition* 5.1. *Let L be the corresponding object to V of $L_{\mathrm{qunip}}(X)$ with respect to the log Riemann-Hilbert correspondence. Then for any $m \in \mathbb{Z}$, we have:*
(1) *$R^m f_*^{\log}(L)$ is an object of $L_{\mathrm{qunip}}(Y)$.*
(2) *$R^m f_*(\omega^\bullet_{X/Y}(V))$, endowed with the Gauss-Manin connection, is an object of $V_{\mathrm{qnilp}}(Y)$.*
(3) *$R^m f_*^{\log}(L)$ and $R^m f_*(\omega^\bullet_{X/Y}(V))$ are in the log Riemann-Hilbert correspondence. In particular,*

$$\mathcal{O}_Y^{\log} \otimes_{\mathbb{C}} R^m f_*^{\log}(L) \cong \mathcal{O}_Y^{\log} \otimes_{\mathcal{O}_Y} R^m f_*(\omega^\bullet_{X/Y}(V)) \quad \text{on } Y^{\log}.$$

(4) $\mathcal{O}_Y^{\log} \otimes_{\mathbb{C}} R f_*^{\log}(L) \cong \mathcal{O}_Y^{\log} \otimes_{\mathcal{O}_Y} R f_*(\omega^\bullet_{X/Y}(V)) \quad \text{on } Y^{\log}.$

To prove Proposition 5.1, since the problem is local on Y, we may assume that we have a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "a"'] \arrow[d, "f'"] \\ Y & Y', \arrow[l, "b"] \end{tikzcd}$$

where a, b are blowing ups along log structures such that f' is exact ([14]). Further we may assume that $Rf_*(\omega^\bullet_{X/Y}(V))$ is bounded above.

Lemma 5.3. *On $(Y')^{\log}$, we have*

$$\mathcal{A}_{Y'}^{\log} \otimes^{\mathbb{L}}_{\mathcal{O}_Y} Rf_*(\omega^\bullet_{X/Y}(V)) \cong R(f')_*^{\log}(\mathcal{A}^{\bullet,\log}_{X'/Y'}(V)).$$

Proof. This is obtained by the sequence of isomorphisms

$$\begin{aligned} &\mathcal{A}_{Y'}^{\log} \otimes^{\mathbb{L}}_{\mathcal{O}_Y} Rf_*(\omega^\bullet_{X/Y}(V)) \\ &\cong \mathcal{A}_{Y'}^{\log} \otimes_{\mathbb{C}} Rf_*^{\log}(L) \\ &\cong \mathcal{A}_{Y'}^{\log} \otimes_{\mathbb{C}} R(f')_*^{\log}(L) \\ &\cong R(f')_*^{\log}((f'^{\log})^{-1}\mathcal{A}_{Y'}^{\log} \otimes_{\mathbb{C}} L) \\ &\cong R(f')_*^{\log}(\mathcal{A}^{\bullet,\log}_{X'/Y'} \otimes_{\mathbb{C}} L) \\ &\cong R(f')_*^{\log}(\mathcal{A}^{\bullet,\log}_{X'/Y'}(V)). \end{aligned}$$

Here the first isomorphism is by Theorem 5.2 (4), the second one is by the following Lemma 5.4, the third is by the projection formula, the fourth is by log Poincaré lemma Theorem 4.1 for $X' \longrightarrow Y'$, and the last isomorphism is the evident one. Q.E.D.

Lemma 5.4. $(b^{\log})^{-1} R^m f_*^{\log}(L) \cong R^m (f')_*^{\log}(L)$ *on* $(Y')^{\log}$. *(The right hand side means* $R^m (f')_*^{\log}((a^{\log})^{-1}(L))$.*)*

Proof. The both sides are locally constant sheaves (5.2 (1)), and the restrictions of them to Y'_{triv} coincide. Q.E.D.

Lemma 5.5. *On* Y', *we have*

$$\mathcal{A}_{Y'} \otimes^{\mathbb{L}}_{\mathcal{O}_Y} Rf_*(\omega^\bullet_{X/Y}(V)) \cong R(f')_*(\mathcal{A}^\bullet_{X'/Y'}(V)).$$

Proof. We apply $R\tau_{Y'*}$ to Lemma 5.3. Proposition 1.9 implies that

$$R\tau_{Y'*}(\text{l.h.s. of 5.3}) \cong \mathcal{A}_{Y'} \otimes^{\mathbb{L}}_{\mathcal{O}_Y} Rf_*(\omega^\bullet_{X/Y}(V)).$$

On the other hand

$$\begin{aligned} R\tau_{Y'*}(\text{r.h.s. of 5.3}) &\cong R(\tau_{Y'} \circ (f')^{\log})_*(\mathcal{A}^{\bullet,\log}_{X'/Y'}(V)) \\ &= R(f' \circ \tau_{X'})_*(\mathcal{A}^{\bullet,\log}_{X'/Y'}(V)) \\ &\cong R(f')_*(\mathcal{A}^\bullet_{X'/Y'}(V)). \end{aligned}$$

Q.E.D.

5.6. Now we prove Proposition 5.1 by applying Rb_* to Lemma 5.5. Propositions 1.5, 1.7, and 2.10 imply that

$$Rb_*(\text{l.h.s. of 5.5}) \cong \mathcal{A}_Y \otimes^{\mathbb{L}}_{\mathcal{O}_Y} Rf_*(\omega^\bullet_{X/Y}(V)).$$

On the other hand

$$\begin{aligned} Rb_*(\text{r.h.s. of 5.5}) &\cong R(b \circ f')_*(\mathcal{A}^\bullet_{X'/Y'}(V)) \\ &= R(f \circ a)_*(\mathcal{A}^\bullet_{X'/Y'}(V)) \\ &\cong Rf_*(\mathcal{A}^\bullet_{X/Y}(V)). \end{aligned}$$

Since $R^m f_*(\omega^\bullet_{X/Y}(V))$ is locally free (5.2 (2)), we obtain 5.1 by taking $\mathcal{H}^m$.

Remark: The authors do not know whether $\mathcal{A}_Y$ is flat over $\mathcal{O}_Y$ or not.

§6. Log Kähler metrics

Everything is in the ket sense here except in a part of the proof of Proposition 6.4.

6.1. Let X and Y be log smooth fs log analytic spaces and let $X \longrightarrow Y$ be a log smooth morphism.

For p, q, $m \in \mathbb{Z}$, let $\mathcal{A}^{p,q}_{X/Y}$ be the image of $\mathcal{A}^{p,q}_X \longrightarrow \mathcal{A}^{p+q}_{X/Y}$,

$$\mathcal{A}_{X/Y,p,q} = \mathcal{H}om_{\mathcal{A}_X}(\mathcal{A}^{p,q}_{X/Y}, \mathcal{A}_X), \qquad \mathcal{A}_{X/Y,m} = \mathcal{H}om_{\mathcal{A}_X}(\mathcal{A}^{m}_{X/Y}, \mathcal{A}_X).$$

We call $\mathcal{A}_{X/Y,1}$ the sheaf of log vector fields on X over Y.

6.2. As is easily seen, we have a bijection between the set of Hermitian forms

$$\langle\ ,\ \rangle\colon \mathcal{A}_{X/Y,1,0} \times \mathcal{A}_{X/Y,1,0} \longrightarrow \mathcal{A}_X$$

and the set $\{\omega \in \Gamma(X, \mathcal{A}^{1,1}_{X/Y})\ ;\ \bar{\omega} = -\omega\}$ given by

$$\langle f, g\rangle = (f \wedge \bar{g}, \omega)$$

where $(\ ,\ )$ means the natural pairing between $\mathcal{A}_{X/Y,1,1}$ and $\mathcal{A}^{1,1}_{X/Y}$.

6.3. By a log Hermitian metric on X over Y, we mean a Hermitian form

$$\langle\ ,\ \rangle\colon \mathcal{A}_{X/Y,1,0} \times \mathcal{A}_{X/Y,1,0} \longrightarrow \mathcal{A}_X$$

which is "positive definite" in the following sense: The map

$$\mathcal{A}_{X/Y,1,0} \longrightarrow \mathcal{H}om_{\mathcal{A}_X}(\mathcal{A}_{X/Y,1,0}, \mathcal{A}_X)\ ;\ g \mapsto (f \mapsto \langle f, g\rangle)$$

is an isomorphism and the restriction of $\langle\ ,\ \rangle$ to X_{triv} is positive definite. By a log Kähler metric on X over Y, we mean a log Hermitian metric on X over Y such that the corresponding global section ω of $\mathcal{A}^{1,1}_{X/Y}$ (6.2) satisfies $d\omega = 0$.

Proposition 6.4. *Let $f\colon X \longrightarrow Y$ be a log smooth projective morphism between log smooth fs log analytic spaces, and fix an invertible $\mathcal{O}_X$-module $\mathcal{L}$ that is relatively very ample with respect to Y. Assume that X is a complex manifold and the log structure of X is given by a divisor on X with simple normal crossings having only finite number of irreducible components.*

Then, locally on Y, there exists a log Kähler metric on X over Y such that the class of the corresponding global section of $\mathcal{A}^{1,1}_{X/Y}$ in $\mathcal{H}^2(f_*\mathcal{A}^\bullet_{X/Y})$ coincides with the image of the Chern class of $\mathcal{L}$ under

$$R^2 f_*\mathbb{Z}(1) \longrightarrow R^2 f_*\mathcal{A}^\bullet_{X/Y} \cong \mathcal{H}^2(f_*\mathcal{A}^\bullet_{X/Y}).$$

Proof. The log Kähler metric which we construct below is essentially the same as the metric which appeared in [4] and [31].

Forgetting the log structure of X, take an immersion from X to a projective bundle P over Y such that $\mathcal{L}$ is isomorphic to the pull back of $\mathcal{O}_P(1)$. Let ω_0 be the pull back of the global section of $C^{\infty,1,1}_{P/Y}$ corresponding to the (classical) Kähler metric on P relative to Y. Let D be the divisor on X which gives the log structure of X, and let $(D_j)_j$ be the set of all irreducible components of D. For each j, let f_j be the global section of $M_X/\mathcal{O}_X^\times$ corresponding to D_j, and let $\log(|f_j|)$ be the global section of $\mathcal{A}_X/C^\infty_X$ defined to be the image of f_j under the homomorphism

$$\log(|-|)\ :\ M^{\mathrm{gp}}_X/\mathcal{O}_X^\times \longrightarrow \mathcal{A}_X/C^\infty_X.$$

Here $\mathcal{A}_X$ is in the non-ket sense. By the exact sequence

$$0 \longrightarrow C^\infty_X \longrightarrow \mathcal{A}_X \longrightarrow \mathcal{A}_X/C^\infty_X \longrightarrow 0$$

and by $\mathrm{H}^1(X, C^\infty_X) = 0$, there exists a global section s_j of $\mathcal{A}_X$ such that $s_j \equiv -\log(|f_j|) \bmod C^\infty_X$. By replacing s_j by $\frac{1}{2}(s_j + \overline{s}_j) + t_j$ for a C^∞-function t_j on X with sufficiently large positive values, we find s_j such that $s_j > 0$ on X_{triv}. Take a positive real number C, and let

$$\omega = \omega_0 + C \cdot \sum_j \overline{\partial}\partial(\log(s_j))$$

(∂ (resp. $\overline{\partial}$) denotes the part $\mathcal{A}^{p,q}_{X/Y} \longrightarrow \mathcal{A}^{p+1,q}_{X/Y}$ (resp. $\mathcal{A}^{p,q}_{X/Y} \longrightarrow \mathcal{A}^{p,q+1}_{X/Y}$) of d: $\mathcal{A}^{p,q}_{X/Y} \longrightarrow \mathcal{A}^{p+1,q}_{X/Y} \oplus \mathcal{A}^{p,q+1}_{X/Y}$). Then, locally on Y, if C is sufficiently small, ω corresponds to a relative log Kähler metric on X over Y.

Since $\overline{\partial} \circ \partial = d \circ \partial$, we have class $(\omega) =$ class (ω_0) in $\mathcal{H}^2(f_*\mathcal{A}^\bullet_{X/Y})$. It is known that class (ω_0) coincides with the image of the Chern class of $\mathcal{L}$. Q.E.D.

§7. Log harmonic forms

Let $f\colon X \longrightarrow Y$ be a projective log smooth vertical morphism between log smooth fs log analytic spaces. Let n be the relative dimension of X over Y (that is, the rank of the locally free sheaf $\omega^1_{X/Y}$ which is a locally constant function on X) and we assume that n is constant.

We assume further that we are given a log Kähler metric on X over Y.

Everything in this section is in the ket sense except in a part of the proof of Proposition 7.6.

Assume that we are given a VPLH $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ of weight w on X.

7.1. Following the classical theory of Laplacian, we introduce the star operator

$$*\colon \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^{2n-m}_{X/Y}(\mathcal{M})$$

which is $\mathcal{A}_X$-linear, the δ-operator

$$\delta\colon \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^{m-1}_{X/Y}(\mathcal{M}) \ ; \ \delta = - * d*,$$

where d denotes $\nabla\colon \mathcal{A}^q_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^{q+1}_{X/Y}(\mathcal{M})$, and then the Laplacian

$$\Delta\colon \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^m_{X/Y}(\mathcal{M}) \ ; \ \Delta = d\delta + \delta d.$$

The definition of $*$ is as follows. The Hermitian metric

$$\langle\ ,\ \rangle\colon \mathcal{A}_{X/Y,1,0} \times \mathcal{A}_{X/Y,1,0} \longrightarrow \mathcal{A}_X$$

induces by duality an Hermitian metric

$$\langle\ ,\ \rangle\colon \mathcal{A}^{1,0}_{X/Y} \times \mathcal{A}^{1,0}_{X/Y} \longrightarrow \mathcal{A}_X.$$

Clearly, this Hermitian metric is extended to a unique Hermitian metric

$$\langle\ ,\ \rangle\colon \mathcal{A}^1_{X/Y} \times \mathcal{A}^1_{X/Y} \longrightarrow \mathcal{A}_X$$

having the properties that $\mathcal{A}^{1,0}_{X/Y}$ and $\mathcal{A}^{0,1}_{X/Y}$ are orthogonal under $\langle\ ,\ \rangle$ and $\langle \bar{a}, \bar{b}\rangle$ is the complex conjugate of $\langle a, b\rangle$ for any $a, b \in \mathcal{A}^{1,0}_{X/Y}$. This Hermitian metric on $\mathcal{A}^1_{X/Y}$ is extended naturally to an Hermitian metric

$$\langle\ ,\ \rangle\colon \mathcal{A}^m_{X/Y} \times \mathcal{A}^m_{X/Y} \longrightarrow \mathcal{A}_X$$

for any m.

We have an Hermitian metric

$$\begin{aligned}\langle\ ,\ \rangle\colon \mathcal{A}^m_{X/Y}(\mathcal{M}) \times \mathcal{A}^m_{X/Y}(\mathcal{M}) &\longrightarrow \mathcal{A}_X \\ (a \otimes u, b \otimes v) &\mapsto \langle a, b\rangle \cdot \langle u, v\rangle\ (a, b \in \mathcal{A}^m_{X/Y}, u, v \in \mathcal{M}_{\mathcal{A}}).\end{aligned}$$

Here $\langle u, v\rangle = i^{p-q}(u, \bar{v})$ when $u \in \mathcal{M}^{p,q}_{\mathcal{A}}$ $(p+q=w)$. In this Hermitian metric, the direct summands $\mathcal{A}^{r,s}_{X/Y} \otimes_{\mathcal{A}_X} \mathcal{M}^{p,q}_{\mathcal{A}}$ $(r+s=m,\ p+q=w)$ are orthogonal to each other, and $\langle \bar{u}, \bar{v}\rangle$ coincides with the complex conjugate of $\langle u, v\rangle$ for any $u, v \in \mathcal{A}^m_{X/Y}(\mathcal{M})$.

We define the star operator

$$*\colon \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^{2n-m}_{X/Y}(\mathcal{M})$$

by the formula

$$u \wedge *(\bar{v}) = \langle u, v \rangle (i\omega)^n \qquad \text{for } u, v \in \mathcal{A}^m_{X/Y}(\mathcal{M})$$

where ω is the global section of $\mathcal{A}^{1,1}_{X/Y}$ corresponding to the log Kähler metric of X over Y, and $\wedge$ denotes the pairing

$$\mathcal{A}^m_{X/Y}(\mathcal{M}) \otimes \mathcal{A}^{2n-m}_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^{2n}_{X/Y}$$

induced by the exterior product $\mathcal{A}^m_{X/Y} \times \mathcal{A}^{2n-m}_{X/Y} \longrightarrow \mathcal{A}^{2n}_{X/Y}$ and the $\mathcal{O}_X$-bilinear form $(\ ,\ )\colon \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{O}_X$.

Then we have

7.1.1. $*$ commutes with complex conjugation.

7.1.2. $*(*(u)) = (-1)^p u$ for $u \in \mathcal{A}^p_{X/Y}(\mathcal{M})$.

7.2. We define an $\mathcal{A}_Y$-submodule $\mathrm{har}^m_{X/Y}(\mathcal{M})$ of $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$, called the sheaf of harmonic m-forms with coefficients in $\mathcal{M}$, by

$$\mathrm{har}^m_{X/Y}(\mathcal{M}) = \mathrm{Ker}\,(\Delta\colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow f_*\mathcal{A}^m_{X/Y}(\mathcal{M})).$$

Then $\mathrm{har}^m_{X/Y}(\mathcal{M})$ coincides with the intersection of the kernels of the two operators

$$d\colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$$
$$\delta\colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow f_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M}).$$

In fact, it is clear that $\mathrm{Ker}\,(d) \cap \mathrm{Ker}\,(\delta) \subset \mathrm{Ker}\,(\Delta)$, and the converse inclusion can be checked on Y_{triv}.

The aim of this section is to prove the following log version of the classical direct decomposition theorem.

Theorem 7.3. *For each $m \in \mathbb{Z}$, we have:*
(1) $f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) = \mathrm{har}^m_{X/Y}(\mathcal{M}) \oplus df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M}) \oplus \delta f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$.
(2) $\mathrm{Ker}\,(d\colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \to f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})) = \mathrm{har}^m_{X/Y}(\mathcal{M}) \oplus df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$.
(3) $\mathrm{Ker}\,(\delta\colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \to f_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})) = \mathrm{har}^m_{X/Y}(\mathcal{M}) \oplus \delta f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$.
(4) $\Delta\colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$ *induces an automorphism of the space* $df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M}) \oplus \delta f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$.

We prove 7.3 after preliminaries on the L^2-metric on $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$ (7.4) and on Lie derivatives on $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$ (7.5).

7.4. We define a pairing

$$\langle\langle\ ,\ \rangle\rangle\colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \times f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}_Y$$

(called the L^2-metric on $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$) by

$$(u,v) \mapsto \left(y \mapsto \int_{X_y} u \wedge *\bar{v}\right) \qquad (y \in Y_{\mathrm{triv}}, X_y = f^{-1}(y)).$$

Here, we have to show that the function $y \mapsto \int_{X_y} u \wedge *\bar{v}$ is log C^∞. This is reduced to the following

Claim: For $u \in \Gamma(X, \mathcal{A}^{2n}_{X/Y})$, the function $y \mapsto \int_{X_y} u$ $(y \in Y_{\mathrm{triv}})$ is a log C^∞-function on Y.

This follows from the fact that the above function coincides with the image of u under

$$\begin{aligned} f_*\mathcal{A}^{2n}_{X/Y} \longrightarrow \mathcal{H}^{2n}(f_*\mathcal{A}^\bullet_{X/Y}) \cong \mathcal{A}_Y \otimes_{\mathcal{O}_Y} R^{2n}f_*\omega^\bullet_{X/Y} &\text{ (by Proposition 5.1)} \\ \longrightarrow \mathcal{A}_Y. \end{aligned}$$

(The last homomorphism comes from $R^{2n}f_*\omega^\bullet_{X/Y} \longrightarrow \mathcal{O}_Y$ which follows from the fact that the canonical homomorphism $R^{2n}f^{\log}_*(\mathbb{Z}) \longrightarrow \mathbb{Z}$ on Y_{triv} is canonically extended to $Y^{\log}$.)

The pairing $\langle\langle\ ,\ \rangle\rangle$ satisfies

7.4.1. $\langle\langle u,u\rangle\rangle \geq 0$ for any $u \in f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$.

7.5. Let α be a global section of $\mathcal{A}_{X,1} := \mathcal{A}_{X/\mathbb{C},1}$. Then α is identified with a homomorphism of $\mathcal{A}_X$-modules $\mathcal{A}^1_X \longrightarrow \mathcal{A}_X$. We have a homomorphism of $\mathcal{A}_X$-modules

$$i_\alpha\colon \mathcal{A}^q_X \longrightarrow \mathcal{A}^{q-1}_X$$

characterized by

$$i_\alpha(a_1 \wedge \cdots \wedge a_q) = \sum_{j=1}^{q} (-1)^{j-1} \cdot \alpha(a_j) \cdot a_1 \wedge \cdots \wedge a_{j-1} \wedge a_{j+1} \cdots \wedge a_q.$$

Define

$$\partial_\alpha = d \circ i_\alpha + i_\alpha \circ d\colon \mathcal{A}^q_X \longrightarrow \mathcal{A}^q_X.$$

Then we have:

7.5.1. $\partial_\alpha(u \wedge v) = \partial_\alpha(u) \wedge v + u \wedge \partial_\alpha(v)$ for any $u \in \mathcal{A}_X^p$, $v \in \mathcal{A}_X^q$ $(p, q \in \mathbb{Z})$.
7.5.2. $d \circ \partial_\alpha = \partial_\alpha \circ d\colon \mathcal{A}_X^p \longrightarrow \mathcal{A}_X^{p+1}$ $(p \in \mathbb{Z})$.
Let

$$\partial_\alpha\colon \mathcal{A}_X^q(\mathcal{M}) \longrightarrow \mathcal{A}_X^q(\mathcal{M})$$

be the additive map characterized by

$$\partial_\alpha(a \otimes u) = \partial_\alpha(a) \otimes u + a \otimes \partial_\alpha(u) \quad (a \in \mathcal{A}_X^q, u \in \mathcal{M})$$

where $\partial_\alpha(u)$ means the image of u under

$$\mathcal{M} \xrightarrow{\nabla} \omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\alpha \otimes \mathrm{id}} \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M}.$$

Then:
7.5.3. $\partial_\alpha(au) = \partial_\alpha(a)u + a\partial_\alpha(u)$ for any $a \in \mathcal{A}_X$ and $u \in \mathcal{A}_X^p(\mathcal{M})$ $(p \in \mathbb{Z})$,
7.5.4. $d \circ \partial_\alpha = \partial_\alpha \circ d\colon \mathcal{A}_X^p(\mathcal{M}) \longrightarrow \mathcal{A}_X^{p+1}(\mathcal{M})$ $(p \in \mathbb{Z})$.
If $\alpha\colon \mathcal{A}_X^1 \longrightarrow \mathcal{A}_X$ sends $\mathcal{A}_Y^1$ into $\mathcal{A}_Y \subset \mathcal{A}_X$, then it is seen from 7.5.1 that $\partial_\alpha\colon \mathcal{A}_X^q(\mathcal{M}) \longrightarrow \mathcal{A}_X^q(\mathcal{M})$ induces $\mathcal{A}_{X/Y}^q(\mathcal{M}) \longrightarrow \mathcal{A}_{X/Y}^q(\mathcal{M})$. We call this induced map $\partial_\alpha\colon \mathcal{A}_{X/Y}^q(\mathcal{M}) \longrightarrow \mathcal{A}_{X/Y}^q(\mathcal{M})$ the Lie derivative defined by α.

Proposition 7.6. *Let $u \in \Gamma(X_{\mathrm{triv}}, \mathcal{A}_{X/Y}^m(\mathcal{M}))$. Then the following (1) and (2) are equivalent.*
(1) $u \in \Gamma(X, \mathcal{A}_{X/Y}^m(\mathcal{M}))$.
(2) Locally on Y, for any $k \geq 0$ and any sections $\alpha_1, \dots, \alpha_k$ of $f_*\mathcal{A}_{X,1}$ which send $\mathcal{A}_Y^1$ into $\mathcal{A}_Y$, the section

$$\langle\langle \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_k}(u), \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_k}(u) \rangle\rangle$$

of $\mathcal{A}_Y$ on Y_{triv} is of logarithmic growth on Y.

Proof. It is clear that (1) implies (2). We will reduce the converse to the well-known inequality $\sup|f| \leq (2||f||_{L^2} \cdot ||f'||_{L^2})^{\frac{1}{2}}$ for a compactly supported C^∞-function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, which is a direct consequence of the Schwartz' inequality. We may assume that Y is Hausdorff. We will prove that u is log C^∞ around a point x of X. Let $y = f(x)$. First note that any non-ket germ $\alpha \in \mathcal{A}_{X,1,x}$ which sends $\mathcal{A}_{Y,y}^1$ into $\mathcal{A}_{Y,y}$ is extended to an $\tilde{\alpha} \in (f_*\mathcal{A}_{X,1})_y$ such that $\tilde{\alpha}_{x'}$ sends $\mathcal{A}_{Y,y}^1$ into $\mathcal{A}_{Y,y}$ for any $x' \in f^{-1}(y)$ and $\tilde{\alpha}_x = \alpha$. This is because $\mathcal{A}_Y^1 \otimes_{\mathcal{A}_Y} \mathcal{A}_X$ is a direct summand of $\mathcal{A}_X^1$ and the non-ket version of $\mathcal{A}_{X/Y,1}$ is soft. Then it is enough to show that the section u satisfying (2) is of log growth.

We will prove that u is of log growth around x. Take an open neighborhood U_0 of x, a ket subneighborhood $U \to U_0$, and $t_1, \dots, t_{n+d} \in \Gamma(U_0, M_X)$ such that $(d\log(t_i))_{1\le i\le n}$ is a basis of $\omega^1_{U_0/Y}$; $(d\log(t_i))_{1\le i\le n+d}$ is a basis of $\omega^1_{U_0}$; and such that $\mathcal{M}|_U$ is free. Let $u_i = \arg(t_i)$ and $v_i = \log|t_i|$, $1 \le i \le n+d$. Let S be the subset $\{\frac{\partial}{\partial u_i}, \frac{\partial}{\partial v_i} \ ; \ 1 \le i \le n\}$ of $\{\frac{\partial}{\partial u_i}, \frac{\partial}{\partial v_i} \ ; \ 1 \le i \le n+d\}$, the dual basis of $\{du_i, dv_i\}$. Take a compact subneighborhood $K \subset U$ such that for any $\alpha \in S$, there exists an extension $\tilde\alpha \in f_*\mathcal{A}_{X,1}$ such that $\tilde\alpha$ sends $\mathcal{A}^1_Y$ into $\mathcal{A}_Y$ and such that α and $\tilde\alpha$ coincide on a neighborhood of K in U. Fix such an $\tilde\alpha$ for each $\alpha \in S$. Further, multiplying u by a log C^∞-function φ on X such that $\varphi \equiv 1$ near x and $\varphi \equiv 0$ outside the image K_0 of K in U_0, we may suppose that $u \equiv 0$ outside K_0.

Consider the metric on $\mathcal{A}^1_{U/Y}$ such that $\{du_1, dv_1, \dots, du_n, dv_n\}$ is an orthonormal basis with respect to it. Denote by $\langle\ ,\ \rangle'$ the induced metric on $\mathcal{A}^m_{U/Y}(\mathcal{M})$ and by $\langle\langle\ ,\ \rangle\rangle'$ the induced pairing $f_*\mathcal{A}^m_{U_{\mathrm{triv}}/Y_{\mathrm{triv}}}(\mathcal{M}) \times f_*\mathcal{A}^m_{U_{\mathrm{triv}}/Y_{\mathrm{triv}}}(\mathcal{M}) \longrightarrow \mathcal{A}_{Y_{\mathrm{triv}}} \colon (u,v) \mapsto \left(y \mapsto \int_{K_y} u \wedge *\bar v\right)$ $(y \in Y_{\mathrm{triv}}, K_y = f^{-1}(y) \cap K)$.

We prove that u satisfies the following condition $(2)'$.

$(2)'$: For any $\alpha_1, \dots, \alpha_k \in S$, $\langle\langle \partial_{\alpha_1}\circ \cdots \circ \partial_{\alpha_k}(u), \partial_{\alpha_1}\circ \cdots \circ \partial_{\alpha_k}(u)\rangle\rangle'$ is of log growth.

Taking a function h of log growth on X such that $\langle\langle v,v\rangle\rangle' \le \sup\limits_{f^{-1}(y)} |h| \cdot \langle\langle v,v\rangle\rangle$ for any $v \in f_*\mathcal{A}^m_{X_{\mathrm{triv}}/Y_{\mathrm{triv}}}(\mathcal{M})$ ($\sup\limits_{f^{-1}(y)} |h|$ denotes the function $y \mapsto \sup\limits_{x\in f^{-1}(y)} |h(x)|$ on Y_{triv}), we see that the function in $(2)'$ is pointwise less than $y \mapsto \sup\limits_{f^{-1}(y)} |h| \cdot \langle\langle \partial_{\tilde\alpha_1}\circ \cdots \circ \partial_{\tilde\alpha_k}(u), \partial_{\tilde\alpha_1}\circ \cdots \circ \partial_{\tilde\alpha_k}(u)\rangle\rangle$ $(y \in Y_{\mathrm{triv}})$. Since f is vertical, $\sup|h|$ is of log growth on Y. Thus u satisfies $(2)'$.

The rest is to show that this condition $(2)'$ implies that u is of log growth. Take an orthonormal basis $(e_i)_i$ of $\mathcal{M}_{\mathcal{A}}|_U$. Writing $u = \sum f_i e_i$, we see that $(2)'$ for u implies $(2)'$ for each f_i by induction on k. Hence we may assume that $\mathcal{M} = \mathcal{O}_X$. Then we may assume that $m = 0$. By repeated use of the usual Schwartz' inequality on the real line, we have $\sup\limits_{K_y}|u| \le c_0 \prod_{I\subset S} \langle\langle \partial_I(u), \partial_I(u)\rangle\rangle'^{2^{-2n}}$ for a positive constant c_0. Here $\partial_I(u) = \partial_{\alpha_1}\circ \cdots \circ \partial_{\alpha_k}(u)$ when we denote by $\alpha_1, \dots, \alpha_k$ all the distinct elements of I. Hence u is of log growth. Q.E.D.

7.7. We prove Theorem 7.3 (1). Let $j \colon Y_{\mathrm{triv}} \longrightarrow Y$ be the canonical morphism. Theorem 7.3 (1) is true in the case $Y = Y_{\mathrm{triv}}$ (that implies $X = X_{\mathrm{triv}}$ since f is vertical) by the classical theory (Deligne, [31]).

Hence for $u \in f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$, we have a unique decomposition

$$u = p_h(u) + p_d(u) + p_\delta(u) \qquad \text{in } j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$$

where

$$p_h(u) \in j_*j^*\mathrm{har}^m_{X/Y}(\mathcal{M}),$$
$$p_d(u) \in j_*j^*df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M}), \qquad p_\delta(u) \in j_*j^*\delta f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M}).$$

By Proposition 5.1 and Theorem 5.2, $\mathcal{H}^m(f_*\mathcal{A}^\bullet_{X/Y}(\mathcal{M})) \cong \mathcal{A}_Y \otimes_{\mathcal{O}_Y} R^mf_*\omega^\bullet_{X/Y}(\mathcal{M})$ is a locally free $\mathcal{A}_Y$-module, and hence the map $\mathcal{H}^m(f_*\mathcal{A}^\bullet_{X/Y}(\mathcal{M})) \longrightarrow j_*j^*\mathcal{H}^m(f_*\mathcal{A}^\bullet_{X/Y}(\mathcal{M}))$ is injective. Hence we have

7.7.1. $j_*j^*df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M}) \cap f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) = df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$,

and (applying the star operator $*$ to this) we also have the similar equality concerning δ. These imply that, for the proof of 7.3 (1), it is sufficient to show that $p_h(u)$, $p_d(u)$, and $p_\delta(u)$ belong to $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$. We prove that $p_d(u)$ belongs to $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$. (Then this will show that $p_\delta(u) = (-1)^m * (p_d(*u))$ belongs to $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$, and hence $p_h(u) = u - p_d(u) - p_\delta(u)$ also belongs to $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$.)

Let $\alpha_1, \dots, \alpha_k$ be sections of $f_*\mathcal{A}_{X,1}$ which send $\mathcal{A}^1_Y$ into $\mathcal{A}_Y$. By Proposition 7.6, it is sufficient to show that

$$\langle\langle \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_k} \circ p_d(u), \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_k} \circ p_d(u) \rangle\rangle \in j_*C^\infty_{Y_{\mathrm{triv}}}$$

is of logarithmic growth. Let

$$l_j = (-1)^m * \circ \partial_{\alpha_j} \circ * - \partial_{\alpha_j} : \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^m_{X/Y}(\mathcal{M}).$$

Then l_j is a homomorphism of $\mathcal{A}_X$-modules. We have

7.7.2. $\partial_{\alpha_j} \circ p_d = p_d \circ (\partial_{\alpha_j} + l_j - l_j \circ p_d)$ on $j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$.

We prove 7.7.2. Since ∂_{α_j} commutes with d, ∂_{α_j} preserves $j_*j^*df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$ (resp. $\mathrm{Ker}\,(d\colon\ j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow j_*j^*f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})) = j_*j^*\mathrm{har}^m_{X/Y}(\mathcal{M}) \oplus j_*j^*df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$), and this shows the following 7.7.3 (resp. 7.7.4).

7.7.3. $p_d \circ \partial_{\alpha_j} \circ p_d = \partial_{\alpha_j} \circ p_d$ on $j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$.

7.7.4. $(1 - p_\delta) \circ \partial_{\alpha_j} \circ (1 - p_\delta) = \partial_{\alpha_j} \circ (1 - p_\delta)$ on $j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$.

By taking $(-1)^m * \circ (7.7.4) \circ *$, we obtain

$$(1 - p_d) \circ (\partial_{\alpha_j} + l_j) \circ (1 - p_d) = (\partial_{\alpha_j} + l_j) \circ (1 - p_d),$$

which can be rewritten as

7.7.5. $p_d\circ(\partial_{\alpha_j}+l_j)\circ p_d = p_d\circ(\partial_{\alpha_j}+l_j)$.
7.7.2 is obtained by taking (7.7.5) minus (7.7.3).

Now by 7.7.2 and by the fact that $\partial_{\alpha_i}\circ a - a\circ\partial_{\alpha_i}\colon \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^m_{X/Y}(\mathcal{M})$ is a homomorphism of $\mathcal{A}_X$-modules for any i and any $\mathcal{A}_X$-homomorphism $a\colon \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^m_{X/Y}(\mathcal{M})$, we have:
7.7.6. $\partial_{\alpha_1}\circ\cdots\circ\partial_{\alpha_k}\circ p_d$ is a finite sum of elements of the form

$$s_1\circ\cdots\circ s_q\circ t_1\circ\cdots\circ t_r \qquad (q\geq 0, r\geq 0)$$

where each s_i is an operator $j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$ of the form $p_d\circ a_i$ where a_i is a homomorphism of $\mathcal{A}_X$-modules $\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^m_{X/Y}(\mathcal{M})$, and each t_i is ∂_{α_j} for some j.

Locally on Y, there exists a log C^∞-function b_i on Y such that
7.7.7. $\langle\langle a_iv, a_iv\rangle\rangle \leq |b_i|\cdot\langle\langle v,v\rangle\rangle$ for any $v\in j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$.
($|b_i|$ denotes the function $y\mapsto |b_i(y)|$ on Y_{triv}.) Since $j_*j^*\mathrm{har}^m_{X/Y}(\mathcal{M})$, $j_*j^*df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$ and $j_*j^*\delta f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$ are orthogonal under the pairing

$$\langle\langle\ ,\ \rangle\rangle\colon j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M})\times j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow j_*C^\infty_{Y_{\mathrm{triv}}}$$

(by the classical theory), we have
7.7.8. $\langle\langle p_d(v), p_d(v)\rangle\rangle \leq \langle\langle v,v\rangle\rangle$ for any $v\in j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$.
By 7.7.7 and 7.7.8, locally on Y, there exists a log C^∞-function b on Y such that

$$\begin{aligned}&\langle\langle s_1\circ\cdots\circ s_q\circ t_1\circ\cdots\circ t_r(u), s_1\circ\cdots\circ s_q\circ t_1\circ\cdots\circ t_r(u)\rangle\rangle\\ &\leq |b|\cdot\langle\langle t_1\circ\cdots\circ t_r(u), t_1\circ\cdots\circ t_r(u)\rangle\rangle.\end{aligned}$$

Since $t_1\circ\cdots\circ t_r(u)$ is log C^∞, $\langle\langle t_1\circ\cdots\circ t_r(u), t_1\circ\cdots\circ t_r(u)\rangle\rangle$ is of log growth. Hence by 7.7.6, $\langle\langle\partial_{\alpha_1}\circ\cdots\circ\partial_{\alpha_k}\circ p_d(u), \partial_{\alpha_1}\circ\cdots\circ\partial_{\alpha_k}\circ p_d(u)\rangle\rangle$ is of log growth.

7.8. We prove Theorem 7.3 (2), (3). Since (3) is obtained by applying the star operator $*$ to (2), it is sufficient to prove (2). Let $u\in f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$, and assume $du=0$. Then $u-p_h(u)$ is in $j_*j^*df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$. By 7.7.1, this shows that $u-p_h(u)$ belongs to $df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$.

7.9. We prove Theorem 7.3 (4). Since $\mathrm{har}^m_{X/Y}(\mathcal{M}) = \mathrm{Ker}\,(\Delta$ on $f_*\mathcal{A}^m_{X/Y}(\mathcal{M}))$, the injectivity of Δ on $df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})\oplus\delta f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$ is clear. We prove the surjectivity of Δ on $df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})\oplus\delta f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$. Let

$u \in df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M}) \oplus \delta f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$. Then of course $u = du_1 + \delta u_2$ for some $u_1 \in f_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$ and $u_2 \in f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$. Write $p_\delta(u_1) = \delta(v_1)$ and $p_d(u_2) = d(v_2)$ for v_1, $v_2 \in f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$. Then

$$\begin{aligned}\Delta(p_d(v_1) + p_\delta(v_2)) &= (d\delta + \delta d)p_d(v_1) + (d\delta + \delta d)p_\delta(v_2)\\ &= d\delta p_d(v_1) + \delta d p_\delta(v_2)\\ &= d\delta(v_1) + \delta d(v_2)\\ &= du_1 + \delta u_2.\end{aligned}$$

Example 7.10. Take $f\colon E \longrightarrow \Delta$ in 2.14 as $f\colon X \longrightarrow Y$ here. Then X has a log Kähler metric over Y corresponding to the $(1,1)$-form $d\log(u) \wedge d\log(\overline{u}) \in \Gamma(X, \mathcal{A}^{1,1}_{X/Y})$ (u is as in 2.14). For this log Kähler metric, $*\colon \mathcal{A}^m_{X/Y} \longrightarrow \mathcal{A}^{2-m}_{X/Y}$ $(m \geq 0)$ are $\mathcal{A}_X$-linear maps which operate on the bases of $\mathcal{A}^m_{X/Y}$ as

$$\begin{aligned}&*(1) = i d\log(u) \wedge d\log(\overline{u}),\ *(d\log(u)) = -i d\log(u),\\ &*(d\log(\overline{u})) = i d\log(\overline{u}),\ *(d\log(u) \wedge d\log(\overline{u})) = -i,\end{aligned}$$

and the Laplacian $\Delta\colon \mathcal{A}^m_{X/Y} \longrightarrow \mathcal{A}^m_{X/Y}$ $(m \geq 0)$ are described as

$$\begin{aligned}&\Delta(g) = -2(u\tfrac{\partial}{\partial u})(\overline{u}\tfrac{\partial}{\partial \overline{u}})(g) \text{ for } g \in \mathcal{A}_X,\\ &\Delta(g d\log(u)) = \Delta(g) d\log(u),\ \Delta(g d\log(\overline{u})) = \Delta(g) d\log(\overline{u}),\\ &\Delta(g d\log(u) \wedge d\log(\overline{u})) = \Delta(g) d\log(u) \wedge d\log(\overline{u}) \text{ for } g \in \mathcal{A}_X.\end{aligned}$$

If we take as $(\mathcal{H}_\mathbb{Z}, \mathcal{M}, (\ ,\ ))$ the "unit object" on X ($\mathcal{H}_\mathbb{Z} = \mathbb{Z}$, $\mathcal{M} = \mathcal{O}_X$, and the filtration on $\mathcal{M}$ and $(\ ,\ )$ are the evident ones),

$$\begin{aligned}\mathrm{har}^0_{X/Y}(\mathcal{M}) &= \mathcal{A}_Y,\\ \mathrm{har}^1_{X/Y}(\mathcal{M}) &= \mathcal{A}_Y d\log(u) + \mathcal{A}_Y d\log(\overline{u}),\\ \mathrm{har}^2_{X/Y}(\mathcal{M}) &= \mathcal{A}_Y d\log(u) \wedge d\log(\overline{u}).\end{aligned}$$

Let

$$g = \exp(2\pi i \cdot \log(|u|)/\log(|t|)) \text{ where } t = uv$$

(t is the coordinate function on $Y = \Delta$). Then

$$\Delta(g) = 2\pi^2 \log(|t|)^{-2} g.$$

The inverse of Δ (Green operator) on Image(d) + Image(δ) (Theorem 7.3 (4)) sends g to $(2\pi^2)^{-1}\log(|t|)^2 g$, and we see that Theorem 7.3 (4) is related to the fact that the inverse $(2\pi^2)^{-1}\log(|t|)^2$ of the non-zero eigen value $2\pi^2\log(|t|)^{-2}$ of the Laplacian is of log growth.

§8. Higher direct images of variations of polarized log Hodge structure

Everything here is in the ket sense except 8.11–8.14. In this section we prove:

Theorem 8.1. *Let $f\colon X \longrightarrow Y$ be a projective vertical log smooth morphism between log smooth fs log analytic spaces. Let $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ be a VPLH on X of weight w. For $m \in \mathbb{Z}$, let*

$$\mathcal{L}_m = R^m f_*^{\log}\mathcal{H}_{\mathbb{Z}}, \quad \mathcal{V}_m = R^m f_*(\omega^\bullet_{X/Y}(\mathcal{M})),$$

which are in the log Riemann-Hilbert correspondence (Theorem 5.2). For $p \in \mathbb{Z}$, let $\mathrm{Fil}^p(\omega^\bullet_{X/Y}(\mathcal{M}))$ be the subcomplex $(\omega^q_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}^{p-q})^q$ of $\omega^\bullet_{X/Y}(\mathcal{M})$, and let $\mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) = \mathrm{Fil}^p(\omega^\bullet_{X/Y}(\mathcal{M}))/\mathrm{Fil}^{p+1}(\omega^\bullet_{X/Y}(\mathcal{M}))$.
(1) The Hodge to de Rham spectral sequence

$$E_1^{p,q} = R^{p+q} f_* \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \Rightarrow E_\infty^m = \mathcal{V}_m$$

degenerates from E_1, and each $R^m f_* \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M}))$ $(m,\ p \in \mathbb{Z})$ is a locally free $\mathcal{O}_Y$-module. Consequently, for any $m,\ p \in \mathbb{Z}$, the canonical map $R^m f_* \mathrm{Fil}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \longrightarrow R^m f_*(\omega^\bullet_{X/Y}(\mathcal{M}))$ is injective and the image is locally an $\mathcal{O}_Y$-direct summand of $R^m f_*(\omega^\bullet_{X/Y}(\mathcal{M}))$.
(2) Fix an invertible $\mathcal{O}_X$-module $\mathcal{L}$ which is relatively very ample with respect to Y, and for $m \in \mathbb{Z}$, let

$$(\ ,\ )\colon \mathcal{L}_{m,\mathbb{Q}} \times \mathcal{L}_{m,\mathbb{Q}} \longrightarrow \mathbb{Q}$$

be the pairing below in 8.2 defined by $\mathcal{L}$. Then, with the Hodge filtration $R^m f_* \mathrm{Fil}^p(\omega^\bullet_{X/Y}(\mathcal{M}))$ on $\mathcal{V}_m$,

$$(\mathcal{L}_m,\ \mathcal{V}_m,\ (\ ,\ ))$$

is a VPLH of weight $w+m$ on Y.

8.2. Let the situation be as in the above theorem.
Define a pairing

$$(\ ,\ )\colon \mathcal{L}_{m,\mathbb{Q}} \times \mathcal{L}_{m,\mathbb{Q}} \longrightarrow \mathbb{Q}$$

as follows. We will assume that the relative dimension n of X over Y is constant (the beginning of section 7). The definition is obviously extended to the general case. In the case $m \le n$, define $\mathcal{L}_{m,\mathbb{Q},\mathrm{prim}}$ to be the kernel of

$$c(\mathcal{L})^{n-m+1}\colon \mathcal{L}_{m,\mathbb{Q}} \longrightarrow \mathcal{L}_{2n-m+2,\mathbb{Q}}.$$

Then for any m,

$$\bigoplus_j \mathcal{L}_{j,\mathbb{Q},\mathrm{prim}} \longrightarrow \mathcal{L}_{m,\mathbb{Q}} \ ; \ (a_j)_j \longmapsto \sum_j c(\mathcal{L})^{(m-j)/2} \cdot a_j$$

is an isomorphism where j ranges over all integers such that $j \le n$, $j \le m$, and $j \equiv m \bmod 2$. (This is proved by restricting to Y_{triv}.) Let $(\ ,\ ) \colon \mathcal{L}_{m,\mathbb{Q}} \times \mathcal{L}_{m,\mathbb{Q}} \longrightarrow \mathbb{Q}$ be the unique $\mathbb{Q}$-bilinear form such that the subspaces $c(\mathcal{L})^{(m-j)/2}\mathcal{L}_{j,\mathbb{Q},\mathrm{prim}}$ of $\mathcal{L}_{m,\mathbb{Q}}$ for j as above are orthogonal to each other under $(\ ,\ )$ and

$$(c(\mathcal{L})^{(m-j)/2}u, c(\mathcal{L})^{(m-j)/2}v)$$

for j as above and for $u, v \in \mathcal{L}_{j,\mathbb{Q},\mathrm{prim}}$ is the image of $(-1)^{j(j-1)/2}u \otimes v$ under

$$\begin{aligned}\mathcal{L}_{j,\mathbb{Q}} \otimes \mathcal{L}_{j,\mathbb{Q}} \longrightarrow R^{2j} f_*^{\log}(\mathcal{H}_{\mathbb{Q}} \otimes \mathcal{H}_{\mathbb{Q}}) &\longrightarrow R^{2j} f_*^{\log}\mathbb{Q} \quad (\text{by } (\ ,\ ) \text{ of } \mathcal{H}_{\mathbb{Z}}) \\ &\longrightarrow R^{2n} f_*^{\log}\mathbb{Q} \qquad (\text{by } c(\mathcal{L})^{n-j}) \\ &\longrightarrow \mathbb{Q}.\end{aligned}$$

8.3. Since f is vertical, $X_{\mathrm{triv}} \longrightarrow Y_{\mathrm{triv}}$ is projective. Hence, by Deligne ([5], [31]), the restriction of $(\mathcal{L}_m, (R^m f_* \mathrm{Fil}^p(\omega^\bullet_{X/Y}(\mathcal{M})))_{p\in\mathbb{Z}}, (\ ,\ ))$ to Y_{triv} is a VPH of weight $w+m$. By Proposition 2.5, this VPH extends to a VPLH on Y of weight $w+m$. This extension must have the form $(\mathcal{L}_m, (\mathcal{V}^p_m)_{p\in\mathbb{Z}}, (\ ,\ ))$ for some $\mathcal{O}_Y$-submodules $\mathcal{V}^p_m$ of $\mathcal{V}_m$ such that each $\mathcal{V}^p_m$ is locally free and is locally a direct summand of $\mathcal{V}_m$. We will see below that the canonical map $R^m f_* \mathrm{Fil}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \longrightarrow \mathcal{V}^p_m$ is an isomorphism.

For the proof of 8.1, it is sufficient to prove 8.1 (1). In fact, if we prove 8.1 (1), then we have $R^m f_* \mathrm{Fil}^p(\omega^\bullet_{X/Y}(\mathcal{M})) = \mathcal{V}^p_m$, and hence we obtain 8.1 (2).

Note that 8.1 (1) is a local problem on Y and we may suppose that the relative dimension is constant.

8.4. Locally on Y, take a blowing up $g \colon Z \longrightarrow X$ along the log structure such that Z is a complex manifold and the log structure of Z is given by a divisor with normal crossings whose irreducible components are non-singular and the number of whose irreducible components is finite. Since $Rg_*g^*\mathcal{F} = \mathcal{F}$ for any locally free $\mathcal{O}_X$-module $\mathcal{F}$ of finite rank, the proof of 8.1 (1) is reduced to the case $X = Z$.

In the rest of section 8, we assume that X satisfies the condition on Z in the above and we fix $\mathcal{L}$. Further, shrinking Y, we take and fix a log Kähler metric on X over Y related to $\mathcal{L}$ as in section 7.

8.5. For $p, q \in \mathbb{Z}$ such that $p+q = w+m$, let

$$f_*\mathcal{A}^m_{X/Y}(\mathcal{M})^{p,q} = \bigoplus f_*(\mathcal{A}^{r,s}_{X/Y} \otimes_{\mathcal{A}_X} \mathcal{M}^{j,k}_{\mathcal{A}}) \subset f_*\mathcal{A}^m_{X/Y}(\mathcal{M}),$$

where r, s, j, k range integers satisfying $r+s = m$, $j+k = w$, $r+j = p$, $s+k = q$, and let

$$\mathrm{har}^m_{X/Y}(\mathcal{M})^{p,q} = \mathrm{har}^m_{X/Y}(\mathcal{M}) \cap f_*\mathcal{A}^m_{X/Y}(\mathcal{M})^{p,q} \subset f_*\mathcal{A}^m_{X/Y}(\mathcal{M}).$$

Since Δ: $f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$ preserves $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})^{p,q}$ for any p, q such that $p+q = w+m$ (this is reduced to the classical situation on Y_{triv} described in [31]), we have

8.5.1. $\qquad \mathrm{har}^m_{X/Y}(\mathcal{M}) = \bigoplus_{p+q=w+m} \mathrm{har}^m_{X/Y}(\mathcal{M})^{p,q}.$

8.6. By Griffiths transversality as in [31] pp.420–421, the map d: $\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$ sends $\mathcal{A}^m_{X/Y}(\mathcal{M})^{p,q}$ $(p+q = w+m)$ into $\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})^{p+1,q} \oplus \mathcal{A}^{m+1}_{X/Y}(\mathcal{M})^{p,q+1}$. Hence d can be written as

$$d = d' + d''$$

in the unique way where d' and d'' are additive maps $\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$ such that

$$d'(\mathcal{A}^m_{X/Y}(\mathcal{M})^{p,q}) \subset \mathcal{A}^{m+1}_{X/Y}(\mathcal{M})^{p+1,q}, d''(\mathcal{A}^m_{X/Y}(\mathcal{M})^{p,q}) \subset \mathcal{A}^{m+1}_{X/Y}(\mathcal{M})^{p,q+1}$$

$(p+q = w+m)$. Let

$$\delta'(u) = - * d'' * (u), \quad \delta''(u) = - * d' * (u)$$

for $u \in \mathcal{A}^m_{X/Y}(\mathcal{M})$. Then we have:

8.6.1. $d' \circ d' = 0$, $d'' \circ d'' = 0$.

8.6.2. d', d'', δ', δ'' kill $\mathrm{har}^m_{X/Y}(\mathcal{M})$.

8.6.3. $\Delta = 2(d'\delta' + \delta' d') = 2(d''\delta'' + \delta'' d'')$.

8.6.4. $\Delta d' = d'\Delta$, $\Delta d'' = d''\Delta$, $\Delta\delta' = \delta'\Delta$, $\Delta\delta'' = \delta''\Delta$.

These 8.6.1–8.6.4 are proved by restricting to Y_{triv}.

Proposition 8.7. *The canonical map from* $\mathrm{har}^m_{X/Y}(\mathcal{M})$ *to the m-th cohomology sheaf of the complex* $(f_*\mathcal{A}^\bullet_{X/Y}(\mathcal{M}), d')$ *(resp.* $(f_*\mathcal{A}^\bullet_{X/Y}(\mathcal{M}), d'')$*) is an isomorphism.*

Proof. We consider the case of d' (the proof for the case of d'' is similar.) The injectivity can be checked by restricting to Y_{triv} and reducing to the classical theory (Deligne, [31]). We prove the surjectivity. Let $u \in f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$ and assume $d'u = 0$. By Theorem 7.3 (4), there exists $v \in f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$ such that $u - p_h(u) = \Delta(v)$ (p_h is as in 7.7). It is sufficient to prove that $u - 2d'\delta'v$ belongs to $\mathrm{har}^m_{X/Y}(\mathcal{M})$, that is, $\Delta(u) = 2\Delta d'\delta'v$. We have

$$\begin{aligned} \Delta d'\delta' v &= d'\delta'\Delta v && (8.6.4)\\ &= d'\delta'(u - p_h(u)) \\ &= d'\delta' u && (8.6.2)\\ &= (d'\delta' + \delta' d')u \\ &= \tfrac{1}{2}\Delta u && (8.6.3). \end{aligned}$$

Q.E.D.

By 8.5.1, Proposition 8.7 shows

Corollary 8.8. *For m, p, $q \in \mathbb{Z}$, such that $p+q = w+m$, the canonical map from $\mathrm{har}^m_{X/Y}(\mathcal{M})^{p,q}$ to the m-th cohomology sheaf of the complex*

$$f_*(\mathcal{A}_X(\mathcal{M})^{w-q,q} \xrightarrow{d'} \mathcal{A}^1_{X/Y}(\mathcal{M})^{w+1-q,q} \xrightarrow{d'} \mathcal{A}^2_{X/Y}(\mathcal{M})^{w+2-q,q} \xrightarrow{d'} \cdots)$$

(resp.

$$f_*(\mathcal{A}_X(\mathcal{M})^{p,w-p} \xrightarrow{d''} \mathcal{A}^1_{X/Y}(\mathcal{M})^{p,w+1-p} \xrightarrow{d''} \mathcal{A}^2_{X/Y}(\mathcal{M})^{p,w+2-p} \xrightarrow{d''} \cdots))$$

is an isomorphism.

8.9. By the Hodge decomposition in 2.6 applied to the VPLH $(\mathcal{L}_m, (\mathcal{V}^p_m)_{p\in\mathbb{Z}}, (\ ,\ ))$ on Y (8.2), we have a Hodge decomposition

8.9.1. $\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{V}_m = \bigoplus_{p+q=w+m} \mathcal{V}^{p,q}_{m,\mathcal{A}}$

where $\mathcal{V}^{p,q}_{m,\mathcal{A}}$ denotes the intersection of $\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{V}^p_m$ and the complex conjugate of $\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{V}^q_m$. If we identify $\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{V}_m$ with $\mathrm{har}^m_{X/Y}(\mathcal{M})$ via the canonical isomorphism, the decomposition 8.9.1 coincides with the decomposition 8.5.1. (To see this, it is enough to show that the projectors of the direct decompositions coincide, but the coincidence of the projectors can be checked on Y_{triv}, and hence we are reduced to the classical theory on Y_{triv}.) In particular, we have

8.9.2. $\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{V}^p_m/\mathcal{V}^{p+1}_m \cong \mathcal{V}^{p,w+m-p}_{m,\mathcal{A}} \cong \mathrm{har}^m_{X/Y}(\mathcal{M})^{p,w+m-p}$.

8.10. Now we prove Theorem 8.1 (1).

Fix $p \in \mathbb{Z}$. By the log $\overline{\partial}$-Poincaré lemma on X (3.1), we have an exact sequence of complexes

8.10.1.

$$0 \longrightarrow \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \longrightarrow \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \xrightarrow{\bar\partial} \mathcal{A}_X^{0,1} \otimes_{\mathcal{O}_X} \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \xrightarrow{\bar\partial} \mathcal{A}_X^{0,2} \otimes_{\mathcal{O}_X} \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \longrightarrow \cdots .$$

Let $\mathcal{H}^m$ be the m-th cohomology sheaf of the complex

$$f_*(\mathcal{A}_X(\mathcal{M})^{p,w-p} \xrightarrow{d''} \mathcal{A}^1_{X/Y}(\mathcal{M})^{p,w+1-p} \xrightarrow{d''} \mathcal{A}^2_{X/Y}(\mathcal{M})^{p,w+2-p} \to \cdots).$$

From 8.10.1, since $\mathcal{A}_X^{0,q}$ ($q \in \mathbb{Z}$) has a descending filtration whose r-th graded quotient is $\mathcal{A}^{0,q-r}_{X/Y} \otimes_{\mathcal{A}_Y} \mathcal{A}_Y^{0,r}$ for any $r \in \mathbb{Z}$, and since $\mathcal{A}_X \otimes_{\mathcal{O}_X} \omega^m_{X/Y} = \mathcal{A}^{m,0}_{X/Y}$ for $m \in \mathbb{Z}$, we obtain a spectral sequence

8.10.2. $\quad E_1^{s,t} = \mathcal{A}_Y^{0,s} \otimes_{\mathcal{A}_Y} \mathcal{H}^t \Longrightarrow E_\infty^m = R^m f_* \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M}))$

in which $E_1^{s,t} \longrightarrow E_1^{s+1,t}$ is

$$\bar\partial\colon \mathcal{A}_Y^{0,s} \otimes_{\mathcal{A}_Y} \mathcal{H}^t \longrightarrow \mathcal{A}_Y^{0,s+1} \otimes_{\mathcal{A}_Y} \mathcal{H}^t.$$

By Corollary 8.8 and 8.9.2, we have

8.10.3. $\quad \mathcal{H}^m \cong \mathrm{har}^m_{X/Y}(\mathcal{M})^{p,w+m-p} \cong \mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{V}^p_m/\mathcal{V}^{p+1}_m.$

Hence the complex $E_1^{\bullet,m}$ in 8.10.2 is rewritten as

$$\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{V}^p_m/\mathcal{V}^{p+1}_m \xrightarrow{\bar\partial} \mathcal{A}_Y^{0,1} \otimes_{\mathcal{A}_Y} \mathcal{V}^p_m/\mathcal{V}^{p+1}_m \xrightarrow{\bar\partial} \mathcal{A}_Y^{0,2} \otimes_{\mathcal{A}_Y} \mathcal{V}^p_m/\mathcal{V}^{p+1}_m \to \cdots .$$

Hence by the log $\bar\partial$-Poincaré lemma on Y (3.1), the spectral sequence 8.10.2 satisfies

$$E_2^{s,m} = \begin{cases} \mathcal{V}^p_m/\mathcal{V}^{p+1}_m & \text{if } s = 0, \\ 0 & \text{if } s \neq 0. \end{cases}$$

Hence the spectral sequence 8.10.2 gives a canonical isomorphism

$$R^m f_* \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \cong \mathcal{V}^p_m/\mathcal{V}^{p+1}_m.$$

Hence $R^m f_* \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M}))$ is a locally free $\mathcal{O}_Y$-module. The Hodge to de Rham spectral sequence in 8.1 (1) degenerates on Y_{triv} from E_1. Since each E_1-term is a locally free $\mathcal{O}_Y$-module as we have just seen, the degeneration on Y_{triv} implies the degeneration on Y. This completes the proof of Theorem 8.1.

The non-ket version of Theorem 8.1 is deduced directly from it as follows. This is a generalization of results in [23], [25] (but the proof is different).

Theorem 8.11. *Let $f\colon X \longrightarrow Y$ be as in Theorem* 8.1. *Let* $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ *be a VPLH on X in the non-ket sense* (*cf. Remark* 2.4). *Then the followings hold.*
(1) *The associated Hodge to de Rham spectral sequence* (*in the classical sense*) *degenerates from E_1.*
(2) *Assume that the underlying analytic space $\overset{\circ}{Y}$ of Y is smooth. Then each E_1-term of the spectral sequence in* (1) *is locally free.*
(3) *Assume that for any $x \in X$, the cokernel of $M^{\mathrm{gp}}_{Y,f(x)}/\mathcal{O}^{\times}_{Y,f(x)} \longrightarrow M^{\mathrm{gp}}_{X,x}/\mathcal{O}^{\times}_{X,x}$ is torsion free. Then $(\mathcal{L}_m, \mathcal{V}_m := R^m f_*(\omega^{\bullet}_{X/Y}(\mathcal{M})), (\ ,\ ))$, defined similarly as in Theorem* 8.1, *is a VPLH on Y in the non-ket sense, and each E_1-term of the classical spectral sequence in* (1) *is locally free.*

Proof. This is obtained by applying Theorem 8.1 to $(\mathcal{H}_{\mathbb{Z}}, \varepsilon^*\mathcal{M}, (\ ,\ ))$ as follows. Here and hereafter ε denotes the projection from the ket site to the usual site.

(1) By Proposition 1.7, $\varepsilon_* Rf_* \varepsilon^* M = Rf_* M$ for any locally free $\mathcal{O}_X$-module M of finite rank on X, and the Hodge to de Rham spectral sequence associated to $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ is the direct image by ε of the one associated to $(\mathcal{H}_{\mathbb{Z}}, \varepsilon^*\mathcal{M}, (\ ,\ ))$. Thus the degeneracy follows.

(2) When $\overset{\circ}{Y}$ is smooth, the direct image by ε of a locally free $\mathcal{O}_X$-module of finite rank on X_{ket} is locally free ([14]). This proves (2).

(3) By [14], under the assumption in (3), $R^m f^{\log}_* \mathcal{H}_{\mathbb{C}}$ belongs to $L_{\mathrm{unip}}(Y)$. Hence (3) follows. Note that in this case the spectral sequence in Theorem 8.1 (1) is the pull back to X_{ket} of the classical spectral sequence in the above (1). Q.E.D.

Remark 8.12. In the case where $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ is the unit ojbect $\mathbb{Z}$, Theorem 8.11 (1)(2) gives alternative proofs of results of
(a) J.H.M. Steenbrink [27], [28] and T. Fujisawa [6] without use of CMHC; and
(b) L. Illusie [13] and M. Cailotto [1] without use of algebraic methods.

8.13. Here we explain a relation between our work and the works of J.H.M. Steenbrink [27] and T. Fujisawa [6] on limit Hodge structures.

Let $Y = \Delta^n$ endowed with the log structure given by $\Delta^n - (\Delta^*)^n$. By the works of Cattani-Kaplan and Schmid, if $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ is a VPLH on Y of weight w in the non-ket sense, we have a polarized mixed Hodge structure ([3] Definition (2.26)) as follows. Let y be a point of $Y^{\log}$ lying over the origin $0 \in Y$. By identifying $\mathcal{H}_{\mathbb{C},y}$ with $\mathcal{M}(0) = \mathbb{C} \otimes_{\mathcal{O}_{Y,0}} \mathcal{M}_0$, define a descending filtration F on $\mathcal{H}_{\mathbb{C},y}$ by $F^p = \mathcal{M}^p(0)$. Let $W(N)$ be the weight filtration on $\mathcal{H}_{\mathbb{Q},y}$ associated to the nilpotent operator

$$N = c_1N_1 + \cdots + c_nN_n \ ; \ \mathcal{H}_{\mathbb{Q}} \longrightarrow \mathcal{H}_{\mathbb{Q}} \quad \text{for } c_1, \dots, c_n > 0$$

($N_1, \dots, N_n$ are as in 2.8), and let $W = W(N)[-w]$ be the $-w$ shift of $W(N)$. Then W is independent of the choice of N ([2]), and $(\mathcal{H}_{\mathbb{Z},y}, F, (\ ,\ ), W, N)$ is a polarized mixed Hodge structure for any N as above (cf. [3] 3.4).

Now let $f\colon X \longrightarrow Y = \Delta^n$ be a projective log smooth vertical morphism satisfying the assumption of Theorem 8.11 (3). Assume further that the underlying morphism of f of analytic spaces is flat. Let $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ be a VPLH on X of weight w in the non-ket sense. By Theorem 8.11 (3), we have a VPLH $(\mathcal{L}_m, \mathcal{V}_m, (\ ,\ ))$ on Y of weight $w + m$ in the non-ket sense. Fix a point $y \in Y^{\log}$ lying over $0 \in Y$ and fix a point $t \in (\Delta^*)^n$. By fixing a path connecting t and y, we identify $\mathrm{H}^m(X_t, \mathcal{H}_{\mathbb{Z}})$ with $\mathcal{L}_{m,y}$ via the isomorphisms

$$\mathrm{H}^m(X_t, \mathcal{H}_{\mathbb{Z}}) \cong (R^m f_*^{\log}\mathcal{H}_{\mathbb{Z}})_t \cong (R^m f_*^{\log}\mathcal{H}_{\mathbb{Z}})_y = \mathcal{L}_{m,y},$$

and identify $\mathrm{H}^m(X_t, \mathcal{H}_{\mathbb{C}})$ with $\mathrm{H}^m(X_0, \mathcal{O}_{X_0} \otimes_{\mathcal{O}_X} \omega^\bullet_{X/Y}(\mathcal{M}))$ via the isomorphisms

$$\mathrm{H}^m(X_t, \mathcal{H}_{\mathbb{C}}) \cong \mathcal{L}_{m,\mathbb{C},y} \cong \mathcal{V}_m(0) \cong \mathrm{H}^m(X_0, \mathcal{O}_{X_0} \otimes_{\mathcal{O}_X} \omega^\bullet_{X/Y}(\mathcal{M})).$$

Applying the above result of Cattani-Kaplan and Schmid to the VPLH $(\mathcal{L}_m, \mathcal{V}_m, (\ ,\ ))$ on Y, we obtain the following result.

Proposition 8.14. *Let $Y = \Delta^n$ and let X and $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ be as in* 8.13. *Then the map*

$$F^p := \mathrm{H}^m(X_0, \mathrm{Fil}^p(\mathcal{O}_{X_0} \otimes_{\mathcal{O}_X} \omega^\bullet_{X/Y}(\mathcal{M}))) \to \mathrm{H}^m(X_0, \mathcal{O}_{X_0} \otimes_{\mathcal{O}_X} \omega^\bullet_{X/Y}(\mathcal{M}))$$

is injective for any p, and $(\mathrm{H}^m(X_t, \mathcal{H}_{\mathbb{Z}}), F, (\ ,\ ), W, N)$ is a polarized mixed Hodge structure for any N as in 8.13, *where $W = W(N)[-w-m]$ which is independent of such N.*

In the case where $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ is the unit object $\mathbb{Z}$, this result was obtained by Steenbrink [27] under the assumption that $n = 1$ and X is semistable over $Y = \Delta$. See also Fujisawa's [6] for the case where X is multi-semistable over Y. (For such X, the assumption of 8.11 (3) is satisfied.)

Remark 8.15. The authors hope that Theorem 8.1 would be generalized to the case where the base is not necessarily log smooth over $\mathbb{C}$. When it would be established, it would give a new proof of Lemma 4.1

in [21]. For this, they hope to define the ring of log C^∞-functions $\mathcal{A}_X$ for an fs log analytic space X which need not be log smooth over $\mathbb{C}$ by the following idea: Locally on X, we can take an exact closed immersion $X \longrightarrow Z$ with Z log smooth over $\mathbb{C}$. When we have such an embedding, let I be the ideal of $\mathcal{O}_Z$ which defines X. Then we define $\mathcal{A}_X$ to be the quotient of $\mathcal{A}_Z$ by the ideal generated by I and the complex conjugate of I. The authors do not know that $\mathcal{A}_X$ does not depend on the local choice of $X \longrightarrow Z$. If it is the case, $\mathcal{A}_X$ is defined globally and $\mathcal{A}_X^{\log}$ is also defined by $\mathcal{A}_X^{\log} = \mathcal{O}_X^{\log} \otimes_{\mathcal{O}_X} \mathcal{A}_X$.

Appendix. Terminology in log geometry.

Here we give explanations on special terminologies in log geometry. [18] is a basic reference for what follows.

1. Concerning monoids.

In this paper, a monoid means a commutative monoid with a unit element and a homomorphism of monoids is assumed to respect the unit elements. An fs monoid is a finitely generated monoid $\mathcal{S}$ satisfying the following (i) (ii). (i) $ab = ac$ $(a,\ b,\ c \in \mathcal{S})$ implies $b = c$. (Hence $\mathcal{S}$ is embedded in the associated group $\mathcal{S}^{\mathrm{gp}} := \{ab^{-1}\ ;\ a,\ b \in \mathcal{S}\}$.) (ii) If $a \in \mathcal{S}^{\mathrm{gp}}$ and $a^n \in \mathcal{S}$ for some $n \geq 1$, then $a \in \mathcal{S}$.

2. Concerning log structures.

A log structure on a ringed space $(X,\ \mathcal{O}_X)$ is a sheaf of monoids M endowed with a homomorphism $\alpha\colon M \longrightarrow \mathcal{O}_X$ of sheaves of monoids satisfying $\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\cong} \mathcal{O}_X^\times$ by α. An fs log analytic space is an analytic space over $\mathbb{C}$ endowed with a log structure satisfying a certain "fs condition" (see [18]). In this paper, only "log smooth fs log analytic spaces" appear except in Remark 8.15. A log smooth fs log analytic space is an analytic space with a log structure which is locally isomorphic to an open set of $(\operatorname{Spec}\mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ with $\mathcal{S}$ an fs monoid. Here $Y = (\operatorname{Spec}\mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ is endowed with the log structure

$$\{f \in \mathcal{O}_Y\ ;\ f \text{ is invertible on } (\operatorname{Spec}\mathbb{C}[\mathcal{S}^{\mathrm{gp}}])^{\mathrm{an}}\} = \mathcal{O}_Y^\times \cdot \mathcal{S} \subset \mathcal{O}_Y.$$

For example, if X is a complex manifold and D is a divisor on X with normal crossings, and if X is endowed with the log structure $\{f \in \mathcal{O}_X\ ;\ f$ is invertible outside $D\}$ (called the log structure given by D), then X is a log smooth fs log analytic space.

For a log smooth fs log analytic space X, we denote by M_X its log structure. Let $X_{\mathrm{triv}} := \{x \in X \ ; \ M_{X,x} = \mathcal{O}^\times_{X,x}\}$. If X is an open set of $(\operatorname{Spec} \mathbb{C}[\mathcal{S}])^{\mathrm{an}}$, then $X_{\mathrm{triv}} = X \cap (\operatorname{Spec} \mathbb{C}[\mathcal{S}^{\mathrm{gp}}])^{\mathrm{an}}$.

3. Concerning $(X^{\log}, \mathcal{O}^{\log}_X)$.

For an fs log analytic space X, a topological space $X^{\log}$ endowed with a proper map $\tau \colon X^{\log} \longrightarrow X$ is defined (see [18]). If $X = (\operatorname{Spec} \mathbb{C}[\mathcal{S}])^{\mathrm{an}} = \operatorname{Hom}(\mathcal{S}, \mathbb{C})$, $X^{\log} = \operatorname{Hom}(\mathcal{S}, \mathbb{R}^{\mathrm{mult}}_{\geq 0}) \times \operatorname{Hom}(\mathcal{S}, \mathbb{S}^1)$ where $\mathbb{S}^1 := \{z \in \mathbb{C}^\times \ ; \ |z| = 1\}$ and $\tau \colon X^{\log} \longrightarrow X$ is the map induced by

$$\mathbb{R}_{\geq 0} \times \mathbb{S}^1 \longrightarrow \mathbb{C} \ ; \ (r, \ u) \mapsto ru.$$

We have two important sheaves of rings on $X^{\log}$, the non-ket version of $\mathcal{O}^{\log}_X$ and the ket version of $\mathcal{O}^{\log}_X$ (we use the same notation $\mathcal{O}^{\log}_X$). In this paper, $\mathcal{O}^{\log}_X$ is the ket version unless the contrary is explicitly stated. We explain these two $\mathcal{O}^{\log}_X$ in the case where X is a log smooth fs log analytic space. The inverse image of X_{triv} in $X^{\log}$ is isomorphic to X_{triv} via the canonical projection and hence X_{triv} is identified with an open set of $X^{\log}$. Let $j^{\log} \colon X_{\mathrm{triv}} \longrightarrow X^{\log}$ be the inclusion map. If X is a log smooth fs log analytic space, the non-ket version (resp. ket-version) of $\mathcal{O}^{\log}_X$ is the subring of $j^{\log}_*(\mathcal{O}_{X_{\mathrm{triv}}})$ generated over $\mathcal{O}_X = \tau^{-1}(\mathcal{O}_X)$ locally by $\log(t)$ for $t \in M_X$ (resp. by $\log(t)$, $t^{1/n}$ for $t \in M_X$ and $n \geq 1$). The non-ket version of $\mathcal{O}^{\log}_X$ is used in [18], [19], [20], [23], [24], [25], but the ket version of $\mathcal{O}^{\log}_X$ appear and play an essential role in [14].

4. Concerning morphisms between log smooth fs log analytic spaces.

A morphism between analytic spaces with log structures is defined in the evident way. For log smooth fs log analytic spaces X and Y, a morphism $X \longrightarrow Y$ is the same thing as a morphism of underlying analytic spaces $f \colon X \longrightarrow Y$ satisfying $f(X_{\mathrm{triv}}) \subset Y_{\mathrm{triv}}$. f is said to be vertical if $f^{-1}(Y_{\mathrm{triv}}) = X_{\mathrm{triv}}$. If X and Y are log smooth fs log analytic spaces, a morphism $f \colon X \longrightarrow Y$ is log smooth (resp. log étale) if and only if the following holds locally on X and on Y: There are fs monoids $\mathcal{S}$ and $\mathcal{T}$ and a homomorphism $h \colon \mathcal{S} \longrightarrow \mathcal{T}$ which is injective (resp. which is injective with $\mathcal{T}^{\mathrm{gp}}/h(\mathcal{S}^{\mathrm{gp}})$ finite) such that X is an open set of $(\operatorname{Spec} \mathbb{C}[\mathcal{T}])^{\mathrm{an}}$, Y is an open set of $(\operatorname{Spec} \mathbb{C}[\mathcal{S}])^{\mathrm{an}}$, and f is induced by h. For a morphism $f \colon X \longrightarrow Y$ of log smooth fs log analytic spaces, we say f is a blowing up along log structure if locally on Y, Y is an open set of $(\operatorname{Spec} \mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ for an fs monoid $\mathcal{S}$ and $f \colon X \longrightarrow Y$ is the proper birational morphism associated to a finite polyhedral cone decomposition λ of $\operatorname{Hom}(\mathcal{S}, \mathbb{Q}_{\geq 0})$ such that λ comes from an ideal of $\mathcal{S}$ (cf. 2.9 and [22] I). We say a morphism $f \colon X \longrightarrow Y$ of fs log analytic spaces is exact at $x \in X$ if the induced homomorphism of fs monoids $M_{Y,y}/\mathcal{O}^\times_{Y,y} \longrightarrow M_{X,x}/\mathcal{O}^\times_{X,x}$ $(y = f(x))$ is exact (a homomorphism $\mathcal{S} \longrightarrow \mathcal{T}$ of fs monoids is said to be exact if the inverse image of $\mathcal{T}$ under $\mathcal{S}^{\mathrm{gp}} \longrightarrow \mathcal{T}^{\mathrm{gp}}$ coincides with $\mathcal{S}$). f is said to be exact if it is exact at any point of X.

5. Concerning the ket site.

An exact log étale morphism is called also a Kummer log étale morphism. Roughly speaking, "Kummer log étale over X" is something like "nearly étale over X but possibly ramified outside X_{triv}". For a log smooth fs log analytic space X, the Kummer log étale site X_{ket} is the following site: As a category, it is the category of log smooth fs log analytic spaces over X which are Kummer log étale over X. A covering is a surjection. The structure sheaf of X_{ket}; $U \mapsto \mathcal{O}_U(U)$ is denoted by $\mathcal{O}_X$. The canonical morphism of topoi $X^{\log} \longrightarrow X_{\mathrm{ket}}$ is denoted by τ (the same notation as $\tau\colon X^{\log} \longrightarrow X$).

Notations.

$\mathcal{A}_X$	sheaf of log C^∞-functions on X
$\mathcal{A}_X^{\log}$	sheaf of log C^∞-functions on $X^{\log}$
$\mathcal{A}_X^{p,q}$	sheaf of log C^∞ (p,q)-forms
$\mathcal{A}_{X/Y}^{p,\log}$	sheaf of relative log C^∞ p-forms on $X^{\log}$
$\mathcal{A}_{X/Y}^{p,q}$	sheaf of relative log C^∞ (p,q)-forms
$\mathrm{har}^m_{X/Y}(\mathcal{M})$	sheaf of harmonic m-forms with coefficients in $\mathcal{M}$
VPH	variation of polarized Hodge structure
VPLH	variation of polarized log Hodge structure
ω_X^p	sheaf of analytic p-forms with log poles
$\omega_{X/Y}^p$	sheaf of relative analytic p-forms with log poles
$\omega_{X/Y}^p(\mathcal{M})$	sheaf of relative analytic p-forms with log poles and with coefficients in $\mathcal{M}$

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Kazuya Kato
Graduate School of Sciences
Kyoto University
Sakyo-ku
Kyoto, 606
Japan
kazuya@kusm.kyoto-u.ac.jp

Toshiharu Matsubara
Graduate School of Mathematical Sciences
the University of Tokyo
8-1 Komaba 3-chome, Meguro-ku
Tokyo, 153-8914
Japan
matubara@ms357.ms.u-tokyo.ac.jp

Chikara Nakayama
Department of Mathematics
Tokyo Institute of Technology
12-1 Oh-okayama 2-chome, Meguro-ku
Tokyo, 152-8551
Japan
cnakayam@math.titech.ac.jp