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A Proof of the Absolute Purity Conjecture (after Gabber)

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§0. Introduction

This article is an edited version of O. Gabber's talk on his proof of the absolute purity conjecture of A. Grothendieck given at the p-adic conference held in Toulouse in 1994. The details of the proofs given here are supplied by the author following marvelous ideas due to Gabber. The author takes the full responsibility for inaccuracies that may appear in this article.

The absolute purity conjecture is the following.

Conjecture (Grothendieck, [G]). *Let $Y \overset{i}{\hookrightarrow} X$ be a closed immersion of noetherian regular schemes of pure codimension c. Let n be an integer which is invertible on X, and let $\Lambda = \mathbf{Z}/n$. Then*

$$\mathcal{H}^q_Y(\Lambda) \simeq \begin{cases} 0 & \text{for } q \neq 2c, \\ \Lambda_Y(-c) & \text{for } q = 2c. \end{cases}$$

The conjecture has been proved in the following cases:

a) X is smooth over a field k, and Y is also smooth over k ([AGV], exposé XVI, 3.7).
b) X is of equal characteristic ([AGV], exposé XIX for special cases and conditional results, the general case can be deduced from Popescu's general Néron desingularization ([P]) as in §6) .
c) $\dim X \leq 2$ (Gabber (1976), see [Sa], §5, Remark 5.6 for a published proof).

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d) X is of finite type over $\mathbf{Z}$, and any prime divisor l of n satisfies $l \geq \dim X + 2$ ([Thom2], the result in [Thom2] is in fact more general, but in [Thom2] it is necessary to assume that all prime divisors of n are "sufficiently" large).

We explain the outline of the proof by Gabber of the absolute purity conjecture.

The first key step is the construction of the global cycle class in §1

$$\mathrm{cl}(Y) \in H_Y^{2c}(X, \Lambda(c))$$

for X, a noetherian scheme, and $Y \subset X$ a closed local complete intersection subscheme. In the case that Y is an effective Cartier divisor, it is given by the localized first Chern class. In the general case, one uses the blowing up of X along Y, and the cycle class is constructed from the localized Chern classes of the exceptional divisor. The obtained cycle class refines the cycle class defined in [De1], 2.2.

Thanks to the existence of the global cycle class, absolute purity is now reduced to punctual purity (cf. Definition 2.2.1 and Proposition 2.2.4): For a regular strict local ring $\mathcal{O}$ of dimension d with the closed point $i_x : x \to \operatorname{Spec} \mathcal{O}$, the cycle class gives an isomorphism

$$\mathrm{cl}(x) : \Lambda_x \simeq i_x^! \Lambda(d)[2d].$$

By using induction on dimension, this is reduced to showing the following vanishing (cf. Proposition 3.1.2)

$$H^p(\mathrm{Spec}\ \mathcal{O}[f^{-1}], \Lambda) = 0 \quad \text{for } p \neq 0, 1,$$

where $f \in m \backslash m^2$ with $m \subset \mathcal{O}$, the maximal ideal. By a reduction step in §6, we may also assume that $\mathcal{O}$ is arithmetic.

The second key step is the affine Lefschetz theorem (§5, Theorem B) on vanishing of cohomology of affine schemes of arithmetic type. It is a generalization of a theorem of M. Artin ([AGV]) in case of algebraic varieties over a field. Thanks to the theorem, we can show the following vanishing (cf. Proposition 5.2.1) by using the induction hypothesis and the duality result in §4

$$H^p(\mathrm{Spec}\ \mathcal{O}[f^{-1}], \Lambda) = 0 \quad \text{for } p \neq 0, 1, d-1, d,$$

where $\mathcal{O}$ and $f \in m$ are the same as before. In order to deal with the remaining vanishing, we invoke in the final key step the Atiyah-Hirzebruch type spectral sequence for the étale K-theory constructed by Thomason (cf. §7)

$$E_2^{p,q} = \begin{cases} H^p(\mathrm{Spec}\ \mathcal{O}[f^{-1}], \mathbf{Z}/\ell^\nu(i)) & (q=-2i), \\ 0 & (q \text{ is odd}) \end{cases}$$
$$\Longrightarrow (K/\ell^\nu)_{-p-q}(\mathcal{O}[f^{-1}])[\beta^{-1}].$$

One should note that the idea that the étale K-theory of schemes offers a very strong tool in the proof of the absolute purity conjecture is due to R. Thomason ([Thom2]) who has shown the conjecture with ambiguity of bounded torsion. Thanks to Suslin's computation of K-theory of separably closed field ([Sus]) and Gabber's rigidity theorem for algebraic K-theory ([Ga]), $(K/\ell^\nu)_*(\mathcal{O}[f^{-1}])[\beta^{-1}]$ is computed. This implies the degeneracy of the spectral sequence from which the remaining vanishing of étale cohomology follows.

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Notation. For a scheme X, X_{et} denotes the étale topos of X. For an integer $n \geq 1$ and $\Lambda = \mathbf{Z}/n$, $D^+(X_{\mathrm{et}}, \Lambda)$ denotes the derived category of complexes of Λ-sheaves bounded below. The constant sheaf Λ on X is denoted by Λ_X. All cohomology groups are étale cohomology groups.

§1. Cycle class

1.1. Refined cycle class

Let X be a noetherian scheme, $Y \subset X$ be a closed, local complete intersection subscheme of pure codimension c. For an integer n which is invertible on X, let $\Lambda = \mathbf{Z}/n$. Under this condition, Gabber constructs a cycle class $\mathrm{cl}(Y) \in H_Y^{2c}(X, \Lambda(c))$ without any purity assumption.

In the case of effective Cartier divisors, the class

$$\mathrm{cl}(Y) = c_1(\mathcal{O}(Y)) \in H_Y^2(X, \Lambda(1))$$

is given by the localized first Chern class as in [Del], 2.1. For a morphism $f : X' \to X$ such that $Y' = f^*(Y)$ is also an effective Cartier divisor,

$$f^* \mathrm{cl}(Y) = \mathrm{cl}(Y')$$

holds. In the general case, consider the blowing up

$$\pi : \tilde{X} \to X$$

of X along Y. For the defining ideal $\mathcal{I}_Y$ of Y, $\tilde{X} = \operatorname{Proj} \bigoplus_{m \geq 0} \mathcal{I}_Y^m$. Let E be the exceptional divisor defined by $\mathcal{I}_E = \pi^{-1}(\mathcal{I}_Y)\mathcal{O}_{\tilde{X}}$, and let $\mathcal{N}_{Y|X} = \mathcal{I}_Y/\mathcal{I}_Y^2$ be the conormal sheaf of Y in X.

$$E = \operatorname{Proj} \bigoplus_{m \geq 0} \mathcal{I}_Y^m/\mathcal{I}_Y^{m+1} = \mathbf{P}(\mathcal{N}_{Y|X})$$

is the projective bundle of $\mathcal{N}_{Y|X}$ in the notation of EGA II. $\mathcal{O}(1) = \mathcal{I}_E = \mathcal{O}(-E)$ holds on $\tilde{X}$. Since $\mathcal{O}(1)$ is canonically trivialized on $\tilde{X} \setminus E$, one has the localized first Chern class $c_1(\mathcal{O}(1))$ in $H^2_E(\tilde{X}, \Lambda(1))$.

Lemma 1.1.1. *Assume $c > 0$. There is a canonical isomorphism*

$$\alpha : \bigoplus_{i=1}^{c-1} H^{2(c-i)}(Y,\ \Lambda(c-i)) \oplus H^{2c}_Y(X,\ \Lambda(c)) \xrightarrow{\sim} H^{2c}_E(\tilde{X},\ \Lambda(c)).$$

given by $\alpha((\gamma_i)_{1 \leq i \leq c-1}, \gamma) = \sum_{i=1}^{c-1} \gamma_i \cdot c_1(\mathcal{O}(1))^i + \pi^*\gamma$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{\tilde{i}} & \tilde{X} & \xleftarrow{\tilde{j}} & \tilde{X} \setminus E \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi} & & \| \\ Y & \xrightarrow{i} & X & \xleftarrow{j} & X \setminus Y. \end{array}$$

We write down the localization triangles on $\tilde{X}$ and X.

$$\begin{gathered} \to \tilde{i}^!\Lambda_{\tilde{X}} \to \Lambda_E \to \tilde{i}^*\tilde{j}_*\Lambda_{\tilde{X}} \to \\ \to i^!\Lambda_X \to \Lambda_Y \to i^*j_*\Lambda_X \to . \end{gathered}$$

The first sequence exists on E, and the second on Y. Applying π_{Y*} to the first, we have

$$\operatorname{Cone}(i^!\Lambda_X \to \pi_{Y*}(\tilde{i}^!\Lambda_{\tilde{X}})) = \operatorname{Cone}(\Lambda_Y \to \pi_{Y*}\Lambda_E),$$

since

$$\pi_{Y*}(\tilde{i}^*\tilde{j}_*\Lambda_{\tilde{X}}) = i^*j_*\Lambda_X$$

by the proper base change theorem. Let c' be the image of $c_1(\mathcal{O}(1))$ under $H^2_E(\tilde{X}, \Lambda(1)) \to H^2(\tilde{X}, \Lambda(1)) \to H^2(E, \Lambda(1))$. c' is the first Chern class of $\mathcal{O}(1)|_E$. Since E is a projective bundle over Y of relative dimension $c-1$, the multiplication by c' induces a canonical decomposition

$$\bigoplus_{i=0}^{c-1} \Lambda_Y(-i)[-2i] \xrightarrow{\sim} \pi_{Y*}\Lambda_E$$

in $D^+(Y_{\text{et}}, \Lambda)$, and hence $\text{Cone}(\Lambda_Y \to \pi_{Y*}\Lambda_E)$ is identified with $\bigoplus_{i=1}^{c-1} \Lambda_Y(-i)[-2i]$. By taking the global section $R\Gamma$, the map α defined in 1.1.1 gives an isomorphism. Q.E.D.

By Lemma 1.1.1, we have an equation

$$c_1(\mathcal{O}(1))^c + \sum_{i=1}^{c} c_i \cdot c_1(\mathcal{O}(1))^{c-i} = 0$$

in $H_E^{2c}(\tilde{X}, \Lambda(c))$. For $0 \le i \le c-1$, $c_i \in H^{2i}(Y, \Lambda(i))$ is the i-th Chern class of $\mathcal{N}_{Y|X}^{\vee}$ ([Jou], §3). The constant term $c_c \in H_Y^{2c}(X, \Lambda(c))$ refines the c-th Chern class $c_c(\mathcal{N}_{Y|X}^{\vee})$ in $H^{2c}(Y, \Lambda(c))$.

Definition 1.1.2 (Gabber). $\text{cl}(Y) = c_c \in H_Y^{2c}(X, \Lambda(c))$.

Proposition 1.1.3 (functoriality). *Let X be a noetherian scheme, $Y \subset X$ be a closed, local complete intersection subscheme of pure codimension c. For a morphism $f : X' \to X$ such that $Y' = f^*(Y)$ is also a local complete intersection subscheme of codimension c,*

$$f^* \text{cl}(Y) = \text{cl}(Y').$$

Especially, the formation of cl *commutes with any flat pullback.*

Proposition 1.1.4. *Let X be a noetherian scheme. Let D_i $(1 \le i \le c)$ be effective Cartier divisors on X crossing normally (i.e., their local defining equations form a regular sequence). If $Y = \bigcap_{1 \le i \le c} D_i$, then*

$$\text{cl}(Y) = \bigcup_{i=1}^{c} \text{cl}(D_i).$$

Proof. Let $\tilde{D}_i$ be the strict transform of D_i. $\pi^* D_i = \tilde{D}_i + E$, and hence there is an equality

$$c_1(\mathcal{O}(\pi^* D_i)) + c_1(\mathcal{O}(1)) = c_1(\mathcal{O}(\tilde{D}_i))$$

of localized Chern classes in $H^2_{\pi^* D_i}(\tilde{X}, \Lambda(1))$. By taking the cup product of these classes,

$$\bigcup_{1\le i\le c} (c_1(\mathcal{O}(\pi^* D_i)) + c_1(\mathcal{O}(1))) = \bigcup_{1\le i\le c} c_1(\mathcal{O}(\tilde{D}_i))$$

in $H^{2c}_{\bigcap_{1\le i\le c}\pi^* D_i}(\tilde{X}, \Lambda(c)) = H^{2c}_E(\tilde{X}, \Lambda(c))$. Since $\bigcap_{1\le i\le c} \tilde{D}_i = \emptyset$, $\bigcup_{1\le i\le c} c_1(\mathcal{O}(\tilde{D}_i))$ is zero, and hence we have the following equation

$$\sum_{i=0}^{c} \sigma_i(c_1(\mathcal{O}(\pi^* D_1)), \dots, c_1(\mathcal{O}(\pi^* D_c))) \cdot c_1(\mathcal{O}(1))^{c-i} = 0$$

in $H^{2c}_E(\tilde{X}, \Lambda(c))$. Here $\sigma_i(\alpha_1, \dots, \alpha_c)$ is the i-th fundamental symmetric polynomial in $\alpha_1, \dots, \alpha_c$, and $\sigma_0 = 1$. By the definition of the cycle class,

$$\mathrm{cl}(Y) = \sigma_c(c_1(\mathcal{O}(\pi^* D_1)), \dots, c_1(\mathcal{O}(\pi^* D_c))) = \bigcup_{i=1}^{c} c_1(\mathcal{O}(\pi^* D_i)).$$

On the other hand, by the functoriality of the localized Chern classes of effective divisors,

$$\bigcup_{i=1}^{c} c_1(\mathcal{O}(\pi^* D_i)) = \pi^* \bigcup_{i=1}^{c} c_1(\mathcal{O}(D_i))$$

via $\pi^* : H^{2c}_Y(X, \Lambda(c)) \to H^{2c}_E(\tilde{X}, \Lambda(c))$. This proves the claim. Q.E.D.

Corollary 1.1.5. *The image of* $\mathrm{cl}(Y)$ *in* $H^0(Y, R^{2c} i^! \Lambda(c))$ *coincides with the class defined in* [De1], 2.2.

1.2. Gysin map

Assume that we are given a closed embedding $Y \subset X$, which is a local complete intersection of pure codimention c. For $K \in D_{tf}(X_{\mathrm{et}}, \Lambda)$ (the index tf means "finite Tor dimension") and $L \in D^+(X_{\mathrm{et}}, \Lambda)$, we have the canonical product

$$i^* K \otimes^{\mathbf{L}} i^! L \to i^!(K \otimes^{\mathbf{L}} L).$$

$\mathrm{cl}(Y)$ defines a morphism $\Lambda \overset{\mathrm{cl}(Y)}{\to} i^! \Lambda(c)[2c]$ in $D^+(Y_{\mathrm{et}}, \Lambda)$, and hence we have the canonical map which we call the *Gysin map* for (i, K)

$$Gys_{(i,K)} : i^* K \to i^* K \otimes^{\mathbf{L}} i^! \Lambda(c)[2c] \to i^! K(c)[2c].$$

Proposition 1.2.1 (Compatibility). *Assume that we are given two embeddings $Y \overset{i_1}{\hookrightarrow} Z \overset{i_2}{\hookrightarrow} X$ which are complete intersections of pure codimension c_1 and c_2, respectively. For the map*

$$i_1^! i_2^* K \overset{i_1^! Gys_{(i_2,K)}}{\longrightarrow} i_1^! i_2^! K(c_2)[2c_2]$$

induced by the Gysin map

$$i_2^* K \overset{Gys_{(i_2,K)}}{\longrightarrow} i_2^! K(c_2)[2c_2],$$

the composition of

$$(i_2 \cdot i_1)^* K \overset{Gys_{(i_1,K)}}{\longrightarrow} i_1^! i_2^* K(c_1)[2c_1]$$
$$\overset{i_1^! Gys_{(i_2,K)}(c_1)[2c_1]}{\longrightarrow} (i_2 \cdot i_1)^! K(c_1 + c_2)[2(c_1 + c_2)].$$

is the Gysin map for $(i_2 \cdot i_1, K)$.

Proof. In case that Z and Y are obtained by intersections of Cartier divisors D_i:

$$Y = \bigcap_{i=1}^{c_1+c_2} D_i, \quad Z = \bigcap_{i=1}^{c_2} D_i,$$

1.2.1 follows from Proposition 1.1.4. In what follows in this paper we use Proposition 1.2.1 only in this special case. The general case of Proposition 1.2.1 is shown by using the method in [F], 9.2, but the details are omitted. Q.E.D.

§2. Statement of purity theorem

2.1. Absolute purity theorem

In this section all schemes are over $\operatorname{Spec} \mathbf{Z}[1/n]$, and let $\Lambda = \mathbf{Z}/n$.

Theorem 2.1.1 (Gabber). *Let $Y \overset{i}{\hookrightarrow} X$ be a closed immersion of noetherian regular schemes of pure codimension c. Then the cycle class defined in* 1.1 *gives an isomorphism*

$$\Lambda_Y \overset{\mathrm{cl}(Y)}{\underset{\sim}{\to}} i^! \Lambda(c)[2c],$$

in $D^+(Y_{\mathrm{et}}, \Lambda)$, i.e.,

$$\mathcal{H}_Y^q(\Lambda) \simeq \begin{cases} 0 & \text{for } q \neq 2c, \\ \Lambda_Y(-c) & \text{for } q = 2c. \end{cases}$$

2.2. Punctual purity

Definition 2.2.1.

a) Let $Y \overset{i}{\hookrightarrow} X$ be a closed immersion of noetherian regular schemes of pure codimension c. We say that the purity holds for (X, Y) if and only if

$$\Lambda_Y \overset{\mathrm{cl}(Y)}{\xrightarrow{\sim}} i^! \Lambda(c)[2c]$$

is an isomorphism in $D^+(Y_{\mathrm{et}}, \Lambda)$.

b) Let X be a regular scheme, and x be a point of X. We say that the punctual purity holds at x, or X is punctually pure at x if and only if

$$\Lambda_x \overset{\mathrm{cl}(x)}{\xrightarrow{\sim}} i_x^! \Lambda(d_x)[2d_x]$$

is an isomorphism in $D^+(x_{\mathrm{et}}, \Lambda)$. Here $d_x = \dim \mathcal{O}_{X,x}$ is the local codimension at x, and $i_x^! = i'^{*}_x \cdot i^!_{\overline{\{x\}}}$ for $\{x\} \overset{i'_x}{\hookrightarrow} \overline{\{x\}} \overset{i_{\overline{\{x\}}}}{\hookrightarrow} X$.

c) Let $\mathcal{O}$ be a regular strict local ring with the maximal ideal m. We say that the punctual purity holds for $(\mathcal{O}, m)$, or $(\mathcal{O}, m)$ is punctually pure, if and only if $\operatorname{Spec} \mathcal{O}$ is punctually pure at the closed point $V(m)$.

Here are very useful propositions to check punctual purity in some special cases.

Proposition 2.2.2. *Let $\mathcal{O}$ be a regular strict local ring with the maximal ideal m, and $\hat{\mathcal{O}}$ be the m-adic completion. Let $X = \operatorname{Spec} \mathcal{O}$, $x = V(m)$, $\hat{X} = \operatorname{Spec} \hat{\mathcal{O}}$, and $\hat{x} = V(m\hat{\mathcal{O}})$. Then X is punctually pure at x if and only if $\hat{X}$ is punctually pure at $\hat{x}$.*

Proof. By the formal base change theorem of Gabber ([Fu], Corollary 6.6.4),

$$H^q(X \setminus \{x\}, \Lambda) \xrightarrow{\sim} H^q(\hat{X} \setminus \{\hat{x}\}, \Lambda),$$

and hence

$$H^q_x(X, \Lambda) \xrightarrow{\sim} H^q_{\hat{x}}(\hat{X}, \Lambda)$$

for any $q \in \mathbf{Z}$. Since $\hat{X} \to X$ is faithfully flat, by 1.1.3 $\mathrm{cl}(x)$ is mapped to $\mathrm{cl}(\hat{x})$ under this isomorphism. The claim follows. Q.E.D.

Corollary 2.2.3. *Let X be a noetherian regular scheme of equal characteristic. Then X is punctually pure at every point.*

Proof. It suffices to prove that $(\mathcal{O}, m)$ is punctually pure for any regular strict local ring $\mathcal{O}$ of equal characteristic with the maximal ideal m. By 2.2.2, we may assume that $\mathcal{O}$ is m-adically complete. By the structure theorem of complete regular local rings, there is a local isomorphism

$$\mathcal{O} \simeq \hat{\mathcal{O}}',$$

where $\mathcal{O}'$ is the strict henselization of a polynomial algebra $k[X_1, \dots, X_N]$ over a separably closed field k at $(X_1, \dots, X_N)$. By the relative purity theorem ([AGV], exposé XVI, 3.7), $\operatorname{Spec} k[X_1, \dots, X_N]$ is punctually pure at any k-rational point. Using 2.2.2 again, we finish the proof. Q.E.D.

Proposition 2.2.4. *Let $Y \overset{i}{\hookrightarrow} X$ be a closed immersion of regular schemes of pure codimension c. Any two of the following conditions imply the third.*

a) *The purity holds for (X, Y).*
b) *Y is punctually pure at every point.*
c) *X is punctually pure at every point of Y.*

The proof of "b), c) $\Rightarrow$ a)" will be given in 3.1.3 later.
Proof of a), b) $\Rightarrow$ c). Take any point y of Y. Put $d_y = \dim \mathcal{O}_{X,y}$, $\{y\} \overset{i_y}{\hookrightarrow} Y \overset{i}{\hookrightarrow} X$. From a), we have an isomorphism

$$\Lambda_Y \overset{\mathrm{cl}(Y)}{\overset{\sim}{\to}} i^! \Lambda(c)[2c],$$

and hence

$$\Lambda \overset{\mathrm{cl}(y)}{\overset{\sim}{\to}} i_y^! \Lambda_Y(d_y - c)[2d_y - 2c] \overset{i_y^! \mathrm{cl}(Y)(d_y - c)[2(d_y - c)]}{\overset{\sim}{\to}} (i \cdot i_y)^! \Lambda(d_y)[2d_y].$$

By the compatibility 1.2.1, the composite of the isomorphisms is obtained by the cycle class.
Proof of a), c) $\Rightarrow$ b). It is proved in a similar way as above, so the proof is omitted. Q.E.D.

Proposition 2.2.5. *Let S be a noetherian regular scheme of dimension at most one, and P be a smooth S-scheme. Then the punctual purity holds at every point of P.*

Proof. By 2.2.3, we may assume that S is a trait. Let s be the closed point of S. By 2.2.3 again, it suffices to prove that P is punctually pure at any point over s. We apply a), b) $\Rightarrow$ c) part of Proposition 2.2.4 (we have already proved this part) with $X = P$, $Y = P_s$. a) is a consequence of the smooth base change theorem ([AGV], XVI, Corollaire 1.2). b) follows from 2.2.3. Q.E.D.

Corollary 2.2.6. *Let V be a strict complete discrete valuation ring, $\mathcal{O} = V[[X_1, \dots, X_n]]$ be a power series ring over V. Then* $\operatorname{Spec} \mathcal{O}$ *is punctually pure at the closed point.*

This is a consequence of Propositions 2.2.2 and 2.2.5.

2.3. Injectivity

Lemma 2.3.1. *Let $\mathcal{O}$ be a regular strict local ring, and $X =$* $\operatorname{Spec} \mathcal{O}$. *For a regular closed subscheme $Y \subset X$ of codimension c, $\Lambda \overset{\mathrm{cl}(Y)}{\to} H^{2c}_Y(X, \Lambda(c))$ is injective.*

Proof. By considering the composition

$$\Lambda \to H^{2c}_Y(X, \Lambda(c) \to H^{2c}_{\bar{y}}(X_{\bar{y}}, \Lambda(c))$$

where $\bar{y}$ is a geometric generic point of Y, we are reduceed to the case $Y = \{x\}$ where x is the closed point of X and c is the dimension of $\mathcal{O}$. By Proposition 2.2.2, we may assume that $\mathcal{O}$ is a complete local ring. By the structure theorem of complete local rings, there are a coefficient ring C of $\mathcal{O}$ and a surjective local homomorphism $f : C[[X_1, \dots, X_n]] \twoheadrightarrow \mathcal{O}$. C is a field, or a Cohen ring, namely a complete discrete valuation ring for which a rational prime p is a prime element. Let $P = \operatorname{Spec} C[[X_1, \dots, X_n]]$, $N = \dim P$. Consider the closed embedding $X \overset{i}{\hookrightarrow} P$ induced by f. P is punctually pure at x by Corollary 2.2.6, and we have the Gysin map

$$H^{2c}_x(X,\ \Lambda(c)) \to H^{2N}_x(P,\ \Lambda(N)) \xleftarrow{\simeq} \Lambda.$$

The last isomorphism is given by the cycle class. By the compatibility of cycle class 1.2.1, the composition of

$$\Lambda \overset{\mathrm{cl}(x)}{\to} H^{2c}_x(X,\ \Lambda(c)) \to \Lambda$$

is the identity, and hence the map is injective. Q.E.D.

§3. Impure cohomology

Definition 3.1.1. Let $\mathcal{O}$ be a regular strict local ring, and $X =$ $\operatorname{Spec} \mathcal{O}$. For a regular closed subscheme $Y \subset X$ of codimension c and

$q \in \mathbf{Z}$, the impure cohomology group $H^q_Y(X, \Lambda)_{\text{impure}}$ is defined as

$$H^q_Y(X,\ \Lambda)_{\text{impure}} = \begin{cases} H^q_Y(X,\ \Lambda) & (q \neq 2c) \\ \operatorname{Coker}(\Lambda(-c) \to H^q_Y(X,\ \Lambda)) & (q = 2c). \end{cases}$$

Here $\Lambda(-c) \overset{\text{cl}(Y)(-c)}{\to} H^{2c}_Y(X, \Lambda)$ is defined by the cycle class. The impure cohomology group $H^q(X \setminus Y,\ \Lambda)_{\text{impure}}$ is defined as $H^{q+1}_Y(X, \Lambda)_{\text{impure}}$ for $q \in \mathbf{Z}$.

By 2.3.1, X is punctually pure at the closed point x if and only if $H^q(X \setminus \{x\}, \Lambda)_{\text{impure}} = \{0\}$ for any $q \in \mathbf{Z}$.

Proposition 3.1.2 (Invariance of impure cohomology groups). *Let $\mathcal{O}$ be a regular strict local ring, $X = \operatorname{Spec} \mathcal{O}$, and x be the closed point of X. Let Y be a non-empty regular closed subscheme different from X. Let n be an integer which is invertible on X, and let $\Lambda = \mathbf{Z}/n$. We assume the following two conditions.*

a) *Y is punctually pure at every point.*
b) *The purity holds for $(X \setminus \{x\}, Y \setminus \{x\})$.*

Then there is a canonical isomorphism

$$H^q(X \setminus \{x\},\ \Lambda)_{\text{impure}} \xrightarrow{\simeq} H^q(X \setminus Y,\ \Lambda)_{\text{impure}}$$

for any $q \in \mathbf{Z}$.

Proof. Let $d = \dim \mathcal{O}$, $c = \operatorname{codim}(Y, X)$. $\tilde{X} = X \setminus \{x\}$, $\tilde{Y} = Y \setminus \{x\}$, $\tilde{i} : \tilde{Y} \hookrightarrow \tilde{X}$, $j_x : \tilde{Y} \hookrightarrow Y$, $i_x : \{x\} \hookrightarrow X$. By our assumption b),

$$\Lambda_{\tilde{Y}} \xrightarrow{\simeq} \tilde{i}^! \Lambda(c)[2c]$$

by the cycle class. Since the claim is obvious in the case of $c = d$, we may assume that $c < d$. Note that there is a following morphism of localization triangles

$$\begin{tikzcd} i_x^! \Lambda_Y(-c)[-2c] \arrow[r] \arrow[d, "\alpha"] & i_x^* \Lambda_Y(-c)[-2c] \arrow[r] \arrow[d, "\beta"] & i_x^* j_{x*} \Lambda_{\tilde{Y}}(-c)[-2c] \arrow[d, "\gamma"] \\ i_x^! \Lambda \arrow[r] & i_x^* i^! \Lambda \arrow[r] & i_x^* j_{x*} \tilde{i}^! \Lambda \end{tikzcd}$$

which is induced by the cycle class $\Lambda_Y(-c)[-2c] \overset{\text{cl}(Y)(-c)[-2c]}{\to} i^! \Lambda$. First we look at the second line. By assumption a), $H^q(i_x^* j_{x*} \tilde{i}^! \Lambda) = 0$ unless

$q = 2c,\ 2d-1$. The long exact sequence associated to the second line reduces to

$$\text{(1)} \qquad H^q_x(X,\Lambda) \xrightarrow{\sim} H^q_Y(X,\Lambda) \quad \text{for } q \neq 2c, 2c+1, 2d-1, 2d,$$

$$\text{(2)} \qquad \begin{aligned} 0 \to H^{2c}_x(X,\Lambda) \to H^{2c}_Y(X,\Lambda) &\to H^{2c}(\tilde{Y}, \tilde{i}^!\Lambda) \\ &\to H^{2c+1}_x(X,\Lambda) \to H^{2c+1}_Y(X,\Lambda), \end{aligned}$$

$$\text{(3)} \qquad \begin{aligned} H^{2d-1}_x(X,\Lambda) \to H^{2d-1}_Y(X,\Lambda) &\to H^{2d-1}(\tilde{Y}, \tilde{i}^!\Lambda) \\ &\to H^{2d}_x(X,\Lambda) \to H^{2d}_Y(X,\Lambda) \to 0, \end{aligned}$$

where we can add $\to 0$ to the right hand side of (2) and $0 \to$ to the left hand side of (3) in the case $c \neq d-1$. Concerning (2), the composition of

$$\Lambda(-c) \overset{H^{2c}(\beta)}{\to} H^{2c}_Y(X,\Lambda) \to H^{2c}(\tilde{Y}, \tilde{i}^!\Lambda) \overset{H^{2c}(\gamma)^{-1}}{\simeq} \Lambda(-c)$$

is the identity. So the surjectivity of $H^{2c}_Y(X,\Lambda) \to H^{2c}_{\mathrm{et}}(\tilde{Y}, \tilde{i}^!\Lambda)$ follows, showing that $H^{2c+1}_x(X,\Lambda) \to H^{2c+1}_Y(X,\Lambda)$ is bijective (resp. injective) in the case $c \neq d-1$ (resp. $c = d-1$), and the canonical decomposition

$$H^{2c}_Y(X,\Lambda) \simeq \Lambda(-c) \oplus \mathrm{Ker}(H^{2c}_Y(X,\Lambda) \to \Lambda(-c))$$

induces

$$H^{2c}_x(X,\Lambda) \xrightarrow{\sim} \mathrm{Ker}(H^{2c}_Y(X,\Lambda) \to \Lambda(-c)) \simeq H^{2c}_Y(X,\Lambda)_{\mathrm{impure}}.$$

Concerning (3), we show that

(4) $H^{2d-1}(\tilde{Y}, \tilde{i}^!\Lambda) \to H^{2d}_x(X,\Lambda)$ is injective,

(5) the image of this map is spanned by the cycle class twisted by $\Lambda(-d)$.

Then we finish the proof: it follows that $H^{2d-1}_x(X,\Lambda) \xrightarrow{\sim} H^{2d-1}_Y(X,\Lambda)$ and $H^{2d}_x(X,\Lambda)_{\mathrm{impure}} \xrightarrow{\sim} H^{2d}_Y(X,\Lambda)$.

To prove (4) and (5), consider the following commutative diagram.

$$\begin{tikzcd} H^{2d-2c-1}(\tilde{Y},\Lambda)(-c) \arrow[r, "\delta"] \arrow[d, "H^{2d-1}(\gamma)"'] & H^{2d-2c}_x(Y,\Lambda)(-c) \arrow[d, "H^{2d}(\alpha)"'] \\ H^{2d-1}(\tilde{Y}, \tilde{i}^!\Lambda) \arrow[r] & H^{2d}_x(X,\Lambda). \end{tikzcd}$$

$H^{2d-1}(\gamma)$ is an isomorphism since $\Lambda_{\tilde{Y}} \xrightarrow{\simeq} \tilde{i}^!\Lambda(c)[2c]$. δ is an isomorphism by the localization sequence and the fact that the higher cohomology of Y vanishes. $H^{2d-2c}_x(Y,\Lambda)(d-c)$ is spanned by the cycle class ϵ of x in Y, and by the compatibility 1.2.1, $H^{2d}(\alpha)(d)(\epsilon)$ is the cycle class of x in X. (5) is shown. (4) follows from Lemma 2.3.1. This completes the proof of 3.1.2 Q.E.D.

3.1.3. Here we give the proof of the part b), c) $\Rightarrow$ a) of Proposition 2.2.4. Take a point y in Y, and let $X_y = \operatorname{Spec} \mathcal{O}^h_{X,y}$, $Y_y = \operatorname{Spec} \mathcal{O}^h_{Y,y}$. Here $(-)^h$ denotes the henselization. It suffices to prove that the purity holds for (X_y, Y_y) for any $y \in Y$. We proceed by induction on $\delta = \dim \mathcal{O}_{Y,y}$. If δ is zero, the statement is obvious.

Let $\tilde{X}_y = X_y \setminus \{y\}$, $\tilde{Y}_y = Y_y \setminus \{y\}$. By our induction hypothesis,

$$\Lambda_{\tilde{Y}_y} \overset{\mathrm{cl}(\tilde{Y}_y)}{\xrightarrow{\simeq}} \tilde{i}^!\Lambda(c)[2c]$$

by the cycle class. By Proposition 3.1.2, the punctual purity of X_y at y implies the purity for (X_y, Y_y). Q.E.D.

Corollary 3.1.4. *In* 3.1.2, *condition* b) *can be replaced by*

b') X *is punctually pure except possibly at the closed point* x.

Proof. By the part b), c) $\Rightarrow$ a) of Proposition 2.2.4 applied to $(X \setminus \{x\}, Y \setminus \{x\})$, the assumption a) of 3.1.2 and b') of 3.1.4 implies condition b) of 3.1.2. Q.E.D.

§4. Duality formalism for arithmetic schemes

In this section, we fix a noetherian regular scheme S of dimension at most one. By an arithmetic S-scheme, we mean a separated scheme X of finite type over S. Let n be an integer which is invertible on S, and let $\Lambda = \mathbf{Z}/n$.

For a morphism of arithmetic S-schemes $f : X \to Y$, we make use of the six operations ([De2])

$$\begin{aligned}
f_*, f_! &: D^b_c(X_{\mathrm{et}}, \Lambda) \to D^b_c(Y_{\mathrm{et}}, \Lambda),\\
f^*, f^! &: D^b_c(Y_{\mathrm{et}}, \Lambda) \to D^b_c(X_{\mathrm{et}}, \Lambda),\\
\mathit{Hom} &: D^-_c(X_{\mathrm{et}}, \Lambda) \times D^+_c(X_{\mathrm{et}}, \Lambda) \to D^+_c(X_{\mathrm{et}}, \Lambda),\\
\otimes^{\mathbf{L}} &: D^-_c(X_{\mathrm{et}}, \Lambda) \times D^-_c(X_{\mathrm{et}}, \Lambda) \to D^-_c(X_{\mathrm{et}}, \Lambda).
\end{aligned}$$

Here $D^*_c(X_{\mathrm{et}}, \Lambda)$, $D^*_c(Y_{\mathrm{et}}, \Lambda)$ denote the derived categories of the complexes of Λ-sheaves with constructible cohomology sheaves and a suitable boundedness condition.

4.1. Normalization of dualizing complexes

For the structural morphism $f_X : X \to S$ of an arithmetic S-scheme X, let $\delta_{f_X} : X \to \mathbf{Z}$ be the dimension function defined by

$$\delta_{f_X}(x) = \dim \overline{\{f_X(x)\}} + \mathrm{tr.deg}\, k(x)/k(f_X(x)) \quad (x \in X).$$

We put

$$\delta_X = \sup_{x\in X} \delta_{f_X}(x),$$

and call it the *total dimension* of X. For an arithmetic S-scheme X, we say that X is *δ-equidimensional* if and only if δ_Y, where Y is any irreducible component of X, are equal. X is called locally δ-equidimensional if and only if every point of X admits an open neighbourhood which is δ-equidimensional. When X is locally δ-equidimensional, we also denote by δ_X the locally constant function that maps a point x to δ_U of a δ-equidimensional open neighborhood U of x.

For an arithmetic S-scheme X, we put

$$K_X = f_X^! \Lambda(\delta_S)[2\delta_S].$$

Here δ_S is the total dimension of S, which we view as a locally constant function on S. K_X has a finite injective dimension. For any $K \in D_c^b(X, \Lambda)$, define $D_X K$ by

$$D_X K = \mathit{Hom}(K, K_X).$$

By a theorem of Deligne ([De2], 4.3 and 4.7), K_X satisfies the local biduality:

a) $D_X K$ belongs to $D_c^b(X, \Lambda)$.
b) $K \xrightarrow{\simeq} D_X D_X K$ for any $K \in D_c^b(X, \Lambda)$.

We call an object K_X of $D_c^b(X, \Lambda)$ having a finite quasi-injective dimension ([G], Définition 1.7) with these two properties a), b), a *dualizing complex* of X. For a connected scheme, dualizing complexes are unique up to a shift, and the twist by a smooth locally free Λ-sheaf of rank one ([G], Théorème 2.1).

We make use of the dualizing complex K_X chosen as above. We will construct the *fundamental class* of X in some special cases.

Let us start with the smooth case. If $f : X \to S$ is smooth of relative dimension d, there is a canonical trace map ([AGV], exposé XVIII, 2.9)

$$Tr_f : R^{2d} f_! f^* K_S(d) \to K_S$$

which induces

$$\Lambda(\delta_X)[2\delta_X] \overset{Tr_f}{\xrightarrow{\simeq}} K_X.$$

Here we regard d and δ_X as a locally constant function on X.

Let X be a δ-equidimensional local complete intersection scheme admitting a closed embedding $X \overset{i}{\hookrightarrow} P$ into a smooth S-scheme P. Then we have the cycle class

$$\Lambda \overset{\mathrm{cl}(X)}{\longrightarrow} i^!\Lambda(\delta_P - \delta_X)[2(\delta_P - \delta_X)] \overset{i^!\, Tr_{f_P}(-\delta_X)[-2\delta_X]}{\xrightarrow{\simeq}} i^! f_P^! K_S(-\delta_X)[-2\delta_X],$$

and hence

$$\Lambda(\delta_X)[2\delta_X] \overset{[X]}{\longrightarrow} K_X$$

as the composition.

Lemma 4.1.1. *Let X be a δ-equidimensional local complete intersection scheme admitting a closed embedding $X \overset{i}{\hookrightarrow} P$ into a smooth S-scheme P. Then $[X]$ is independent of any choice of i. We call $[X]$ the fundamental class of X.*

Proof. The proof is standard. Assume that we have two embeddings $i_s : X \to P_s$ $(s = 1, 2)$. The composition

$$X \overset{(i_1,i_2)}{\longrightarrow} P_1 \times_S P_2 \overset{\mathrm{pr}_s}{\longrightarrow} P_s$$

is i_s. By replacing $P_1 \times_S P_2$ by P_1, we may assume that P_1 is a P_2-scheme, $F : P_1 \to P_2$ is a smooth morphism of relative dimension d, and $i_2 = F \cdot i_1$. By the compatibility of the trace map, the composition of

$$\Lambda(\delta_{P_1})[2\delta_{P_1}] \overset{F^*\, Tr_{f_{P_2}}(d)[2d]}{\longrightarrow} F^* f_{P_2}^! \Lambda(d)[2d] \overset{Tr_F}{\longrightarrow} F^! f_{P_2}^! \Lambda$$

is $Tr_{f_{P_1}}$. It suffices to prove

Sublemma 4.1.2. *Let X be a noetherian scheme. Assume that $F : P \to X$ is a smooth morphism of relative dimension d, $Y \overset{i_1}{\hookrightarrow} X$ is a local complete intersection of pure codimension c, and $i_2 : Y \to P$ is a closed immersion such that $F \cdot i_2 = i_1$. Then the composition*

$$\Lambda \overset{\mathrm{cl}(i_2)}{\longrightarrow} i_2^!\Lambda(d+c)[2(d+c)] \overset{Tr_F(c)[2c]}{\xrightarrow{\simeq}} i_2^! F^!(c)\Lambda[2c]$$

is $\mathrm{cl}(i_1)$. *Here we denote the cycle map $\Lambda \to i^!\Lambda(c)[2c]$ defined for a local complete intersection subscheme $W \overset{i}{\to} Z$ of codimension c as* $\mathrm{cl}(i)$.

Proof. We decompose i_2 as $Y \overset{\tilde{i}_2}{\to} P_Y = P \times_X Y \overset{i_{1P}}{\hookrightarrow} P$.

$$\begin{tikzcd} P \arrow[d, "F"'] & P_Y \arrow[l, "i_{1P}"'] \arrow[d, "F_Y"] & Y \arrow[l, "\tilde{i}_2"'] \\ X & Y \arrow[l, "i_1"'] & \end{tikzcd}$$

The base change morphism([AGV], expose XVIII (3.1.14.2)) is an isomorphism:

$$i_{1P}^* F^! \Lambda_X \overset{(1)}{\underset{\to}{\simeq}} F_Y^! \Lambda_Y,$$

and the isomorphism (1) is compatible with the trace maps: the identification $Tr_F : \Lambda_P(d)[2d] \underset{\to}{\simeq} F^! \Lambda_X$ induces $Tr_{F_Y} : \Lambda_{P_Y}(d)[2d] \underset{\to}{\simeq} F_Y^! \Lambda_Y$ by the functoriality.

On the other hand, since $F^! \Lambda_X$ is a shift of a Λ-smooth sheaf, we have the canonical isomorphism induced by the canonical product

$$i_{1P}^! F^! \Lambda_X \underset{\leftarrow}{\simeq} i_{1P}^! \Lambda_P \otimes^{\mathbf{L}} i_{1P}^* F^! \Lambda_X.$$

So the Gysin map

$$i_{1P}^* F^! \Lambda_X \overset{Gys_{(i_{1P}, F^! \Lambda_X)}}{\longrightarrow} i_{1P}^! F^! \Lambda_X(c)[2c]$$

is identified with

$$i_{1P}^* F^! \Lambda_X \overset{\mathrm{cl}(i_{1P}) \otimes^{\mathbf{L}} \mathrm{id}_{i_{1P}^* F^! \Lambda_X}}{\longrightarrow} i_{1P}^! \Lambda_P(c)[2c] \otimes^{\mathbf{L}} i_{1P}^* F^! \Lambda_X,$$

and also with

$$F_Y^! \Lambda_Y \overset{\mathrm{cl}(i_{1P}) \otimes^{\mathbf{L}} \mathrm{id}_{F_Y^! \Lambda_Y}}{\longrightarrow} i_{1P}^! \Lambda_P(c)[2c] \otimes^{\mathbf{L}} F_Y^! \Lambda_Y$$

by (1). Using this identification and (1),

$$(2) \qquad \begin{aligned} \tilde{i}_2^* i_{1P}^* F^! \Lambda_X(-d)[-2d] &\overset{Gys_{(\tilde{i}_2, i_{1P}^* F^! \Lambda(-d)[-2d])}}{\longrightarrow} \tilde{i}_2^! i_{1P}^* F^! \Lambda_X \\ &\overset{\tilde{i}_2^! Gys_{(i_{1P}, F^! \Lambda_X)}}{\longrightarrow} \tilde{i}_2^! i_{1P}^! F^! \Lambda_X(c)[2c] \end{aligned}$$

is identified with

$$(3) \qquad \begin{aligned} \tilde{i}_2^* F_Y^! \Lambda_Y(-d)[-2d] &\overset{Gys_{(\tilde{i}_2, F_Y^! \Lambda_Y(-d)[-2d])}}{\longrightarrow} \tilde{i}_2^! F_Y^! \Lambda_Y \\ &\overset{\tilde{i}_2^! (\mathrm{cl}(i_{1P}) \otimes^{\mathbf{L}} \mathrm{id}_{F_Y^! \Lambda})}{\longrightarrow} \tilde{i}_2^! (i_{1P}^! \Lambda_P(c)[2c] \otimes^{\mathbf{L}} F_Y^! \Lambda_Y). \end{aligned}$$

The composition of (2) is

$$Gys_{(i_2, F^! \Lambda_X(-d)[-2d])} = \mathrm{cl}(i_2) \otimes \mathrm{id}_{\tilde{i}_2^* F_Y^! \Lambda(-d)[-2d]}$$

by the compatibility 1.2.1. By the functoriality 1.1.3, $\Lambda_{P_Y} \overset{\mathrm{cl}(i_{1P})}{\longrightarrow} i_{1P}^! \Lambda_P(c)[2c]$ is canonically isomorphic to $\Lambda_{P_Y} \overset{F_Y^* \mathrm{cl}(i_1)}{\longrightarrow} F_Y^* i_1^! \Lambda_X(c)[2c]$, and hence the second arrow of (3) is $\tilde{i}_2^! F_Y^! \Lambda_Y \overset{\tilde{i}_2^! F_Y^! \mathrm{cl}(i_1)}{\longrightarrow} \tilde{i}_2^! F_Y^! i_1^! \Lambda_X(c)[2c]$ which is equal to $\Lambda_Y \overset{\mathrm{cl}(i_1)}{\longrightarrow} i_1^! \Lambda_X(c)[2c]$.

Since (1) is compatible with the trace map, it suffices to see that the composition of

$$\Lambda_Y \overset{\mathrm{cl}(\tilde{i}_2)}{\longrightarrow} \tilde{i}_2^! \Lambda_{P_Y}(d)[2d] \overset{\tilde{i}_2^! \, Tr_{F_Y}}{\overset{\simeq}{\to}} \tilde{i}_2^! F_Y^! \Lambda_Y = \Lambda_Y$$

is the identity of Λ_Y as a constant sheaf on Y.

So we are reduced to the case when $Y = X$, and $i = i_2$ is a section of F. Since the problem is étale local, we may assume that $P = \mathbf{A}_X^N$, and i is the zero section. Using the compatibility 1.2.1 and the compatibility of the trace map with compositions, we may assume that $P = \mathbf{A}_X^1$. It suffices to check it at the maximal points of X, so that we may assume that X is zero dimensional. By considering the geometric closed fibers over X, we reduce to the case when $X = \operatorname{Spec} k$, where k is a separably closed field. Since our cycle class coincides with the classical one for divisors, the claim is obvious by the definition of the trace map. Q.E.D.

Proposition 4.1.3. *Let X be a δ-equidimensional regular scheme admitting a closed embedding into a smooth S-scheme. For any $x \in X$, the following two conditions are equivalent:*

a) *The fundamental class $[X]$ gives an isomorphism*

$$i_y^* \Lambda(\delta_X)[2\delta_X] \overset{i_y^*[X]}{\overset{\simeq}{\to}} i_y^* K_X$$

at any generization y of x.

b) *X is punctually pure at any generization y of x.*

Proof. We choose an embedding $X \overset{i}{\hookrightarrow} P$ into a smooth S-scheme P. By 2.2.5, P is punctually pure at every point. Then the claim follows from the implications b), c) $\Rightarrow$ a) and a), c) $\Rightarrow$ b) of Proposition 2.2.4. Q.E.D.

As a corollary, the fundamental class gives an isomorphism if X is regular of equal characteristic by 2.2.3. Since for each point x over $s \in S$, the closure $\overline{\{x\}}$ of x in X_s has a dense affine open regular subscheme of equal characteristic, we have a canonical isomorphism

$$i_x^! K_X \xleftarrow{\simeq} \Lambda(\delta_{f_X}(x))[2\delta_{f_X}(x)].$$

Remark. For any δ-equidimensional local complete intersection arithmetic S-scheme X, the fundamental class $[X]$ is defined as follows. Take an affine dense open subscheme U of X. Then it is easy to see that

$$H^{2\delta_X}(X, K_X(-\delta_X)) \xrightarrow{\simeq} H^{2\delta_X}(U, K_U(-\delta_X)).$$

Note that $\delta_X = \delta_U$ by the definition. For U, we have $[U] \in H^{2\delta_U}(U, K_U(-\delta_U))$ by taking an embedding into an affine space over S. Then we put $[X] = [U]$. This is independent of any choice of U, and for X which admits an embedding into a smooth S-scheme we have the same fundamental class as before.

For general X, one may take the following formula as the definition of the fundamental class:

$$[X] = \sum_{x \in I_X} \operatorname{length} \mathcal{O}_{X,x} \cdot [U_x] \ \in H^{2\delta_X}(X, K_X(-\delta_X)).$$

Here I_X is the set of the maximal points of X such that $\delta_{f_X}(x) = \delta_X$, and U_x is an open dense subscheme of the closure $\overline{\{x\}}$ which is a local complete intersection. When X is a δ-equidimensional local complete intersection, the coincidence of the two definitions is shown as follows.

By localization, one can assume that X is irreducible and embedded into an affine space over S. If X dominates an irreducible component of S, one uses the compatibility of cycle classes with proper intersection of cycles ([De1], over a field). In general one may use the formalism of local Chern classes of perfect complexes (as in [Iv]).

4.2. Local duality

We describe the local duality theory for regular local rings of arithmetic type by using another normalization of dualizing complexes.

Definition 4.2.1. We say that a strict local ring $\mathcal{O}$ is of arithmetic type over S if

$$\mathcal{O} \simeq \mathcal{O}_{X,\bar{x}}^{sh}$$

for an arithmetic S-scheme X and a point x of X. Here $\bar{x}$ is a geometric point of X localized at x, and $(-)^{sh}$ denotes the strict henselization.

Let $\mathcal{O}$ be a regular strict local ring of arithmetic type over S. We choose an isomorphism

$$\mathcal{O} \simeq \mathcal{O}^{sh}_{X,\bar{x}}$$

for an affine arithmetic regular S-scheme X and a geometric point $\bar{x}$ above x. Let $X_{\bar{x}} = \operatorname{Spec} \mathcal{O}^{sh}_{X,\bar{x}}$, $\tilde{X}_{\bar{x}} = X_{\bar{x}} \setminus \{x_s\}$, $i_{\bar{x}} : \{x_s\} \hookrightarrow X_{\bar{x}}$, and $j_{\bar{x}} : \tilde{X}_{\bar{x}} \hookrightarrow X_{\bar{x}}$. Here x_s is the closed point of $X_{\bar{x}}$.

Put

$$K^{\mathrm{ren}}_{X_{\bar{x}}} = K_X|_{X_{\bar{x}}}(-\delta_X)[-2\delta_X].$$

Then $K^{\mathrm{ren}}_{X_{\bar{x}}}$ is a dualizing complex of $X_{\bar{x}}$: $K^{\mathrm{ren}}_{X_{\bar{x}}}$ has a finite injective dimension, and the two properties in 4.1 characterizing a dualizing complex are satisfied for this $K_{X_{\bar{x}}}$.

By our renormalization,

$$i^!_{\bar{x}} K^{\mathrm{ren}}_{X_{\bar{x}}} \xleftarrow{\sim} \Lambda(-d_x)[-2d_x], \quad d_x = \dim \mathcal{O},$$

and there is a canonical map

$$\Lambda \overset{\mathrm{can}}{\to} K^{\mathrm{ren}}_{X_{\bar{x}}}$$

obtained by the fundamental class. From Proposition 4.1.3, we deduce the main result of this section:

Proposition 4.2.2. *For any $y \in X_{\bar{x}}$, the following two conditions are equivalent*:

a) *The canonical map*

$$\Lambda \overset{\mathrm{can}}{\to} K^{\mathrm{ren}}_{X_{\bar{x}}}$$

defined above gives an isomorphism

$$i^*_{y'}\Lambda \overset{i^*_{y'}\,\mathrm{can}}{\xrightarrow{\sim}} i^*_{y'} K^{\mathrm{ren}}_{X_{\bar{x}}}$$

at any generization y' of y.

b) *$X_{\bar{x}}$ is punctually pure at any generization y' of y.*

4.2.3. Finally we deduce the *local duality theorem* from the biduality of $K^{\mathrm{ren}}_{X_{\bar{x}}}$ ([G], 4.7). For $L \in D^b_c(X_{\bar{x}}, \Lambda)$, we put

$$D_{X_{\bar{x}}} L = \mathcal{H}om(L, K^{\mathrm{ren}}_{X_{\bar{x}}}).$$

For $K \in D^b_c(\tilde{X}_{\bar{x}}, \Lambda)$, apply the biduality to $j_{\bar{x}!}K$, and we get

$$i^!_{\bar{x}} j_{\bar{x}!} K \xrightarrow{\sim} i^!_{\bar{x}} D_{X_{\bar{x}}} D_{X_{\bar{x}}} j_{\bar{x}!} K \xrightarrow{\sim} \mathcal{H}om(i^*_{\bar{x}} D_{X_{\bar{x}}} j_{\bar{x}!} K, i^!_{\bar{x}} K^{\mathrm{ren}}_{X_{\bar{x}}})$$

$$\simeq Hom(i_{\bar{x}}^* j_{\bar{x}*} D_{\tilde{X}_{\bar{x}}} K, \Lambda(-d_x)[-2d_x]).$$

Here

$$D_{\tilde{X}_{\bar{x}}} K = Hom(K, K_{\tilde{X}_{\bar{x}}}^{\mathrm{ren}}), \ K_{\tilde{X}_{\bar{x}}}^{\mathrm{ren}} = j_{\bar{x}}^* K_{X_{\bar{x}}}^{\mathrm{ren}}.$$

Using that $i_{\bar{x}}^! j_{\bar{x}!} K \xrightarrow{\simeq} i_{\bar{x}}^* j_{\bar{x}*} K[-1]$, we have a perfect pairing of Λ-modules

$$H^q(\tilde{X}_{\bar{x}}, \ K) \times H^{2d_x-1-q}(\tilde{X}_{\bar{x}}, \ D_{\tilde{X}_{\bar{x}}} K) \to \Lambda(-d_x)$$

for any $q \in \mathbf{Z}$.

§5. Vanishing theorems

5.1. Affine Lefschetz theorems

Let S be a noetherian regular scheme of dimension at most one. For an integer n which is invertible on S, let $\Lambda = \mathbf{Z}/n$. Gabber proved the following affine Lefschetz theorems. We will use Theorem C in this paper.

Theorem A. *Let $f : X \to Y$ be an affine morphism of finite type between arithmetic S-schemes. Let $\mathcal{F}$ be a Λ-sheaf on X. Then*

$$\delta_{\mathrm{Supp}\, R^q f_* \mathcal{F}} \leq \delta_{\mathrm{Supp}\, \mathcal{F}} - q.$$

To state the next theorem, we define a dimension function in a general setting. Let Y be the spectrum of a universally catenary local ring, $f : X \to Y$ be a morphism of finite type. For $x \in X$, we put

$$\delta_f(x) = \dim \overline{\{f(x)\}} + \mathrm{tr.deg}\, k(x)/k(f(x)),$$

and $\delta_{X,f} = \sup\{\delta_f(x); x \in X\}$.

Theorem B. *Let Y be the spectrum of a strict local ring of arithmetic type over S, $f : X \to Y$ be an affine morphism of finite type. If $\mathcal{F}$ is a Λ-sheaf on X,*

$$H^q(X, \ \mathcal{F}) = 0$$

for $q > \delta_{\mathrm{Supp}\, \mathcal{F}, f}$.

Especially, Theorem B implies

Theorem C. *Let $\mathcal{O}$ be a strict local ring of arithmetic type over S. For a non-zerodivisor f of $\mathcal{O}$ and a Λ-sheaf $\mathcal{F}$ on* $\mathrm{Spec}\, \mathcal{O}[f^{-1}]$,

$$H^q(\mathrm{Spec}\, \mathcal{O}[f^{-1}], \mathcal{F}) = 0 \quad \textit{for } q > \dim \mathcal{O}$$

holds.

Remark. For algebraic varieties over a field, these theorems A–C are due to M. Artin ([AGV], exposé XIV, Théorème 3.1, Corollaire 3.4). Amazingly, the original proof by Artin basically works well in the arithmetic setting, except one important step. The missing step is Theorem C for two dimensional regular local rings of arithmetic type, which is treated by the local duality formalism. See [Il2] for the details.

5.2. Application to purity

The following theorem is an important step towards the proof of the absolute purity conjecture.

Theorem 5.2.1 (Vanishing theorem). *Let $\mathcal{O}$ be a regular strict local ring of arithmetic type, and $X = \operatorname{Spec}\mathcal{O}$. Let Y be a regular divisor of X, and x be the closed point of X. We assume the following two conditions.*

a) *Y is punctually pure at every point.*
b) *X is punctually pure except possibly at x.*

Then

$$H^q(X \setminus Y,\ \Lambda) = 0$$

if $q \neq 0,\ 1,\ d-1,\ d$. Here $d = \dim \mathcal{O}$.

Proof. By Theorem C,

$$H^q(X \setminus Y,\ \Lambda) = 0$$

if $q > d$. By the invariance of impure cohomology groups 3.1.2 (cf. Corollary 3.1.4), we have

$$H^q(X \setminus \{x\},\ \Lambda) = H^q(X \setminus Y,\ \Lambda) = 0$$

for $d < q < 2d-1$. By assumption *b*), the renormalized dualizing complex $K^{\mathrm{ren}}_{X\setminus\{x\}}$ is isomorphic to Λ by Proposition 4.2.2. Hence by the local duality theorem 4.2.3, $H^q(X\setminus\{x\},\Lambda)$ is the Λ-dual of $H^{2d-1-q}(X\setminus\{x\},\Lambda)$. Hence we have

$$H^q(X \setminus \{x\},\ \Lambda) = 0$$

for $0 < q < d-1$. By using Proposition 3.1.2 again, we have the desired vanishing. Q.E.D.

§6. Proof of purity: reduction to the arithmetic case

In the following two sections, we prove the absolute purity conjecture. First we make reductions to the arithmetic case over $\mathbf{Z}$ to apply the vanishing theorem and étale K-theories.

6.1. Limit argument

Lemma 6.1.1 (descent lemma). *Let A be a regular local ring with the maximal ideal m and the residue field k, and $I \subset I' \subset A$ be two ideals of A. Assume the following two conditions.*

a) *I is an ideal generated by a part of a regular system of parameters of A of height c, and I'/I is an ideal generated by a part of a regular system of parameters of A/I of height d.*
b) *There is a directed inductive system $\{A_j\}_{j\in J}$ of regular local rings such that each transition map $\varphi_{ij} : A_i \to A_j$ for $i \leq j$ is a local homomorphism, and there is a local isomorphism*

$$A \simeq \varinjlim_{j\in J} A_j.$$

Then there is an index $j_0 \in J$ and two subsystems $\{I_j\}_{j\in J,\ j\geq j_0}$, $\{I'_j\}_{j\in J,\ j\geq j_0}$ of $\{A_j\}_{j\in J,\ j\geq j_0}$ satisfying the following properties:

a') *$I_j \subset I'_j \subset A_j$ are ideals of A_j for $j \geq j_0$. I_j is an ideal generated by a part of a regular system of parameters of A_j of height c, I'_j/I_j is an ideal generated by a part of a regular system of parameters of A_j/I_j of height d for $j \geq j_0$.*
b') *$\varphi_{ij}(I_i)A_j = I_j$, $\varphi_{ij}(I'_i)A_j = I'_j$ for $j \geq i \geq j_0$. $I = I_jA$, $I' = I'_jA$ for any $j \geq j_0$.*

Proof. Let m and m_j $(j \in J)$ be the maximal ideals of A and A_j $(j \in J)$, respectively.

Let f_s $(1 \leq s \leq c)$ be elements of I such that $\{f_s\}_{1\leq s\leq c}$ generate I, and forms a part of a regular system of parameters of A.

Take elements g_t $(1 \leq t \leq d)$ of I' such that $\{g_t \bmod I\}_{1\leq t\leq d}$ generate I'/I, and forms a part of a regular system of parameters of A/I. $\{f_s, g_t\}_{1\leq s\leq c, 1\leq t\leq d}$ forms a part of a regular system of parameters of A.

We identify A with the inductive limit of $\{A_j\}_{j\in J}$. Then $m_jA, j \in J$ generates m, and hence for some $j_0 \in J$ $m_jA = m$ for any $j \geq j_0$.

We take elements $F_s \in m_{j_0}$ $(1 \leq s \leq c)$ and $G_t \in m_{j_0}$ $(1 \leq t \leq d)$ such that F_s and G_t are mapped to f_s and g_t, respectively.

Put $F_s(j) = \varphi_{j_0 j}(F_s)$ and $G_t(j) = \varphi_{j_0 j}(G_t)$ for $j \geq j_0$. We show that these elements $F_s(j)$ $(1 \leq s \leq c)$ and $G_t(j)$ $(1 \leq t \leq d)$ form a part of a regular system of parameters of A_j. For this, it suffices to see that $c + d$-elements $F_s(j) \bmod m_j^2$ $(1 \leq s \leq c)$ and $G_t(j) \bmod m_j^2$ $(1 \leq t \leq d)$

are linearly independent in m_j/m_j^2. The images of these elements in m/m^2 are $f_s \bmod m^2 (1 \le s \le c)$ and $g_t \bmod m^2 (1 \le t \le d)$, and span a $c+d$-dimensional subspace of m/m^2. So the linear independence follows.

We put $I_j = (F(j)_s,\ 1 \le s \le c)$ and $I'_j = (F_s(j), (1 \le s \le c),\ G_t(j), (1 \le t \le d)) \subset A_j$ for $j \ge j_0$. I_j, I'_j define regular subschemes of $\operatorname{Spec} A_j$ of codimension c and $c+d$, respectively. φ_{ij} induces $\varphi_{ij}(I_i) = I_j$, $\varphi_{ij}(I'_i) = I'_j$ for $j \ge i \ge j_0$. $I_j A = I$, $I'_j A = I'$ for $j \ge j_0$ by the construction. The descent lemma follows. Q.E.D.

Proposition 6.1.2. *Assume that the purity is true for any closed immersion $Y \to X$ of arithmetic regular schemes over a Cohen ring. Then the punctual purity is true for any regular strict local ring.*

Proof. By Proposition 2.2.4, it suffices to show the punctual purity for a regular strict complete local ring $\mathcal{O}$. Let m be the maximal ideal of $\mathcal{O}$, and let $d = \dim \mathcal{O}$. By Corollary 2.2.3, we may assume that $\mathcal{O}$ is of mixed characteristic. Let p be the residue characteristic. By the structure theorem of complete regular local rings, there is a presentation

$$\mathcal{O} \simeq C[[X_1, \dots, X_n]]/(f), \quad f \in m' \setminus {m'}^2,\ m' = (p, X_1, \dots, X_n).$$

Here C is a Cohen ring for k.

Sublemma 6.1.3 *There is a directed inductive system $\{S_j\}_{j \in J}$ of regular local rings satisfying the following properties:*

a) *There is a local C-isomorphism $S_j \simeq C\{\{Y_1, \dots, Y_{n_j}\}\}$ for some $n_j \in \mathbf{N}$. Here $C\{\{Y_1, \dots, Y_N\}\}$ is the strict henselization of $C[Y_1, \dots, Y_N]$ at $(p, Y_1, \dots, Y_N)$.*
b) *Each transition map $\varphi_{ij} : S_i \to S_j$ for $i \le j$ is a local C-homomorphism, and*

$$C[[X_1, \dots, X_n]] \simeq \varinjlim_{j \in J} S_j$$

as a local C-algebra.

Proof. By the theorem of Artin-Rotthaus ([AR]), there is a directed inductive system $\{R_j\}_{j \in J}$ of smooth C-algebras, and $C[[X_1, \dots, X_n]]$ is isomorphic to the inductive limit: $C[[X_1, \dots, X_n]] \xrightarrow{\sim} \varinjlim_{j \in J} R_j$. Let $x_j \in \operatorname{Spec} R_j$ be the image of the closed point x of $\operatorname{Spec} C[[X_1, \dots, X_n]]$, and S_j be the henselization of R_j at x_j. $S_j \simeq C\{Y_1, \dots, Y_{n_j}\}$ for some $n_j \ge 0$ since the residue field $k(x)$ at x is k. The claim follows. Q.E.D.

We return to the proof of 6.1.2. Let $\{S_j\}_{j \in J}$ be a directed inductive system obtained by Sublemma 6.1.3. We apply the descent lemma 6.1.1 with $A = C[[X_1, \dots, X_n]]$, $A_j = S_j$, $I = fA$ and $I' = m'$. Then there is an index $j_0 \in J$ and subsystems $\{I_j\}_{j\in J,\ j\geq j_0}$ and $\{I'_j\}_{j\in J,\ j\geq j_0}$ satisfying conditions a') and b') of 6.1.1. Put $\mathcal{O}_j = A_j/I_j$, $\mathcal{J}_j = I'_j/I_j$ for $j \geq j_0$. We have an inductive system $\{\mathcal{O}_j\}_{j\in J, j\geq j_0}$ and $\{\mathcal{J}_j\}_{j\in J, j\geq j_0}$ such that

a) $\mathcal{O}_j$ is a regular local ring.
b) $\mathcal{J}_j$ is an ideal of $\mathcal{O}_j$, and defines a regular subscheme of codimension $\dim \mathcal{O}$.
c) $\mathcal{O} \simeq \varinjlim_{j\in J, j\geq j_0} \mathcal{O}_j$, $\mathcal{J}_j\mathcal{O} = m$.

By condition c),

$$H^q(\operatorname{Spec} \mathcal{O} \setminus V(m), \Lambda) \simeq \varinjlim_{j\in J, j\geq j_0} H^q(\operatorname{Spec} \mathcal{O}_j \setminus V(\mathcal{J}_j), \Lambda),$$

for any $q \in \mathbf{Z}$. $\mathrm{cl}(V(\mathcal{J}_j)) \in H^{2d}_{V(\mathcal{J}_j)}(\operatorname{Spec} \mathcal{O}_j, \Lambda(n)) = H^{2d-1}(\operatorname{Spec} \mathcal{O}_j \setminus V(\mathcal{J}_j), \Lambda(n))$ is mapped to $\mathrm{cl}(V(m))$. So the purity for $(\operatorname{Spec} \mathcal{O}_j,\ V(\mathcal{J}_j))$ implies the punctual purity for $(\mathcal{O}, m)$. Q.E.D.

Proposition 6.1.4. *Assume that the purity is true for any closed immersion $Y \to X$ of arithmetic regular schemes over $\mathbf{Z}$. Then the purity is true for any regular strict local ring of arithmetic type over a Cohen ring.*

Proof. Since any Cohen ring C is absolutely unramified, it is a directed inductive limit of subrings which are regular and essentially of finite type over $\mathbf{Z}$ ([A]). The rest of the limit argument is treated similarly as in the proof of Proposition 6.1.2 using the descent lemma 6.1.1. We omit the details. Q.E.D.

Corollary 6.1.5. *Assume that the puntual purity is true for any pair $(\mathcal{O}, m)$ of a regular strict local ring $\mathcal{O}$ of arithmetic type over $\mathbf{Z}$ and the maximal ideal m. Then the absolute purity conjecture is true.*

§7. Proof of purity: K-theory

We prove the punctual purity for a regular strict local ring of arithmetic type over $\mathbf{Z}$ by the method of local Lefschetz pencils, using induction on the dimension. Our vanishing theorem 5.2.1, which is proved purely by an étale cohomological method using the local duality and a local affine Lefschetz theorem, is not enough to prove the purity. In addition to the information obtained from 5.2.1, we use the relationship between étale cohomology and étale K-theory to get further vanishing.

7.1. Localization in K-theory

For a noetherian scheme X and $q \in \mathbf{Z}$, $K_q(X)$ denotes the K-group of X made from the category of locally free coherent sheaves of $\mathcal{O}_X$-modules on X. We fix a prime ℓ which is invertible on X and an integer $\nu > 0$. We consider the mod ℓ^ν K-theory $(K/\ell^\nu)_q(X)$. In order that it has a good product structure, we assume either $\ell > 3$ or $\ell = 3$ and $\nu > 1$, or $\ell = 2$ and $\nu > 2$. We have the long exact sequence

$$\cdots \to K_q(X) \xrightarrow{\ell^\nu} K_q(X) \to (K/\ell^\nu)_q(X) \to K_{q-1}(X) \xrightarrow{\ell^\nu} K_{q-1}(X) \to \cdots$$

Lemma 7.1.1. *Let $\mathcal{O}$ be a henselian regular local ring, and f be an element in $m \setminus m^2$. For $q \in \mathbf{Z}$, there is a canonical isomorphism*

$$(K/\ell^\nu)_q(\mathcal{O}) \oplus (K/\ell^\nu)_{q-1}(\mathcal{O}) \xrightarrow{\simeq} (K/\ell^\nu)_q(\mathcal{O}[f^{-1}]).$$
$$(x, y) \mapsto \mathrm{res}_q\, x + [f] \cup \mathrm{res}_{q-1}\, y$$

Here $\mathrm{res}_q : (K/\ell^\nu)_q(\mathrm{Spec}\,\mathcal{O}) \to (K/\ell^\nu)_q(\mathrm{Spec}\,\mathcal{O}[f^{-1}])$ *is the restriction map, and* $[f]$ *is the image of* $f \in \mathcal{O}[f^{-1}]^\times$ *under* $K_1(\mathcal{O}[f^{-1}]) \to (K/\ell^\nu)_1(\mathcal{O}[f^{-1}])$.

Proof. By the rigidity theorem of Gabber ([Ga]),

$$(K/\ell^\nu)_q(\mathcal{O}) \overset{\mathrm{res}'_q}{\xrightarrow{\simeq}} (K/\ell^\nu)_q(\mathcal{O}/f\mathcal{O})$$

for any $q \in \mathbf{Z}$ by the restriction map. By the excision sequence in K-theory using the regularity of $\mathcal{O}$ and $\mathcal{O}/f\mathcal{O}$, we have an exact sequence

$$\begin{aligned}\cdots \to &(K/\ell^\nu)_q(\mathcal{O}/f\mathcal{O}) \to (K/\ell^\nu)_q(\mathcal{O})\\ &\xrightarrow{\mathrm{res}_q} (K/\ell^\nu)_q(\mathcal{O}[f^{-1}]) \xrightarrow{\delta_q} (K/\ell^\nu)_{q-1}(\mathcal{O}/f\mathcal{O}) \to \cdots .\end{aligned}$$

Note that $\delta_q([f] \cup \mathrm{res}_{q-1}\, y) = \mathrm{res}'_{q-1}\, y$, and hence the composition of

$$(K/\ell^\nu)_{q-1}(\mathcal{O}) \xrightarrow{[f]\cup} (K/\ell^\nu)_q(\mathcal{O}[f^{-1}]) \xrightarrow{\delta_q} (K/\ell^\nu)_{q-1}(\mathcal{O}/f\mathcal{O})$$

is an isomorphism. Especially, δ_{q+1} is surjective for any q, showing that res_q is injective, and the claim follows. Q.E.D.

Proposition 7.1.2. *Let $\mathcal{O}$ be a regular strict local ring, and f be an element in $m \setminus m^2$. For a prime ℓ which is invertible in $\mathcal{O}$ and $\nu, q \in \mathbf{Z}$, $\nu \geq 0$,*

$$(K/\ell^\nu)_q(\mathcal{O}[f^{-1}])[\beta^{-1}] \xleftarrow{\simeq} \begin{cases} (K/\ell^\nu)_0(\mathcal{O}[f^{-1}])(\frac{q}{2}) & (\textit{if } q \textit{ is even}) \\ (K/\ell^\nu)_1(\mathcal{O}[f^{-1}])(\frac{q-1}{2}) & (\textit{if } q \textit{ is odd}). \end{cases}$$

Here $\beta \in (K/\ell^\nu)_2(\mathcal{O}[f^{-1}])$ is a Bott element.

Proof. Let $k = \mathcal{O}/m$ be the residue field of $\mathcal{O}$. By a theorem of Suslin ([Sus]), $(K/\ell^\nu)_q(k)$ is zero for q odd, and is canonically isomorphic to $\mathbf{Z}/\ell^\nu(i)$ for $q = 2i \geq 0$. Hence the localization by a Bott element induces $(K/\ell^\nu)_q(k) \xrightarrow{\simeq} (K/\ell^\nu)_q(k)[\beta^{-1}]$ for $q \geq 0$. By the rigidity theorem of Gabber ([Ga]),

$$(K/\ell^\nu)_q(\mathcal{O}) \xrightarrow{\simeq} (K/\ell^\nu)_q(k)$$

by the restriction map for any q, and hence $(K/\ell^\nu)_q(\mathcal{O}) \xrightarrow{\simeq} (K/\ell^\nu)_q(\mathcal{O})[\beta^{-1}]$ for $q \geq 0$. Proposition 7.1.2 follows from 7.1.1 since the étale K-theory is mod 2-periodic. Q.E.D.

7.2. Etale K-theory: Conclusion of the proof of purity

Let X be a separated regular scheme having a finite Krull dimension. Assume the following three conditions on X and the prime ℓ.

a) ℓ is invertible on X, $\sqrt{-1}$ is contained in $\Gamma(X, \mathcal{O}_X)$ if $\ell = 2$.
b) Every residue field of X admits a finite Tate-Tsen filtration.
c) There is a uniform bound for the ℓ-cohomological dimensions of the residue fields.

Under these assumptions, R. Thomason constructs the following spectral sequence strongly converging to the étale K-theory of X, which is an analogue of the Atiyah-Hirzebruch spectral sequence for the topological K-theory of topological spaces ([Thom1], a more detailed account is found in [Jar]).

$$E_2^{p,q} = \begin{cases} H^p(X, \mathbf{Z}/\ell^\nu(i)) & (q = -2i), \\ 0 & (q \text{ is odd}) \end{cases} \Longrightarrow (K/\ell^\nu)_{-p-q}(X)[\beta^{-1}].$$

Here β is a Bott element. If a primitive ℓ^ν-th root of unity is contained in $\Gamma(X, \mathcal{O}_X)$, one can choose β from $(K/\ell^\nu)_2(X)$.

Let $\mathcal{O}$ be a regular strict local ring of arithmetic type over $\mathbf{Z}$, and f be an element in $m \setminus m^2$. Then the assumptions a)–c) are satisfied for $\operatorname{Spec} \mathcal{O}[f^{-1}]$, and hence the spectral sequence

$$(*) \qquad E_2^{p,q} = \begin{cases} H^p(\operatorname{Spec}\ \mathcal{O}[f^{-1}], \mathbf{Z}/\ell^\nu(i)) & (q = -2i), \\ 0 & (q \text{ is odd}) \end{cases}$$
$$\Longrightarrow (K/\ell^\nu)_{-p-q}(\mathcal{O}[f^{-1}])[\beta^{-1}]$$

exists by the theory of Thomason.

Theorem 7.2.1. *Let $\mathcal{O}$ be a regular strict local ring of arithmetic type over $\mathbf{Z}$, and f be an element in $m \setminus m^2$. Assume that the punctual purity is true for any regular strict local ring of arithmetic type over $\mathbf{Z}$ of dimension strictly less than* $\dim \mathcal{O}$. *Then the spectral sequence* $(*)$ *degenerates at E_2.*

Proof. We may assume that $d = \dim \mathcal{O} \geq 1$. By the vanishing theorem 5.2.1, all columns in the E_2-term vanish except for $p = 0, 1, d-1, d$. It follows that any differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ for $r \geq 2$ on the spectral sequence vanishes except possibly for $p = 0, 1$.

Put $U = \operatorname{Spec} \mathcal{O}[f^{-1}]$. For $s \in \mathbf{Z}$, let $\{\mathrm{Fil}^t(K/\ell^\nu)_s(U)[\beta^{-1}]\}_{t \in \mathbf{Z}}$ be the decreasing filtration on $(K/\ell^\nu)_s(U)[\beta^{-1}]$ obtained by the spectral sequence. For $i \in \mathbf{Z}$, we have the edge homomorphism

$$\begin{aligned} e^{0,-2i} : (K/\ell^\nu)_{2i}(U)[\beta^{-1}] &= \mathrm{Fil}^0(K/\ell^\nu)_{2i}(U)[\beta^{-1}] \\ &\to E_2^{0,-2i} = H^0(U, \mathbf{Z}/\ell^\nu(i)). \end{aligned}$$

Since $\mathrm{Gr}^0_{\mathrm{Fil}}(K/\ell^\nu)_{2i-1}(U)[\beta^{-1}] = E_\infty^{0,1-2i} = 0$, we have a homomorphism

$$\begin{aligned} \tilde{e}^{1,-2i} : (K/\ell^\nu)_{2i-1}(U)[\beta^{-1}] &= \mathrm{Fil}^1(K/\ell^\nu)_{2i-1}(U)[\beta^{-1}] \\ &\to E_2^{1,-2i} = H^1(U, \mathbf{Z}/\ell^\nu(i)). \end{aligned}$$

To show the differential $d_r^{p,q}$ on $E_r^{p,q}$ $(r \geq 2)$ is zero for $p = 0, 1$, it suffices to prove the following lemma since it implies that $E_2^{p,q} = E_\infty^{p,q}$ for $p = 0, 1$.

Lemma 7.2.2. $e^{0,-2i}$ *and* $\tilde{e}^{1,-2i}$ *are isomorphisms for $i \in \mathbf{Z}$.*

Proof. We choose a Bott element β of degree 2. The product with β induces an isomorphism of spectral sequences $E_r^{p,q} \xrightarrow{\simeq} E_r^{p,q-2}$ of degree -2, so we may assume that $i = 0$. By Proposition 7.1.2, it suffices to see the two maps

$$\begin{aligned} e^{0,0} &: (K/\ell^\nu)_0(U) \xrightarrow{\simeq} (K/\ell^\nu)_0(U)[\beta^{-1}] \to H^0(U, \mathbf{Z}/\ell^\nu) \\ \tilde{e}^{1,0} &: (K/\ell^\nu)_1(U) \xrightarrow{\simeq} (K/\ell^\nu)_1(U)[\beta^{-1}] \to H^1(U, \mathbf{Z}/\ell^\nu(1)) \end{aligned}$$

are isomorphisms. For $e^{0,0}$, this is clear.

$$\tilde{e}^{1,0} : \mathcal{O}(U)^\times/(\mathcal{O}(U)^\times)^{\ell^\nu} = (K/\ell^\nu)_1(U) \to H^1(U, \mathbf{Z}/\ell^\nu(1))$$

is identified with the map obtained by the Kummer theory. The claim now follows by $H^1(U, \mathbb{G}_m) = 0$.

Corollary 7.2.3. $E_\infty^{p,q} = 0$ *if* $p > 1$.

Proof. This is clear since $\mathrm{Fil}^t(K/\ell^\nu)_s(U)[\beta^{-1}] = 0$ for $t \geq 1$ if s is even, for $t \geq 2$ if s is odd by Lemma 7.2.2. Q.E.D.

7.2.4. Now we can complete the proof of the absolute purity theorem 2.1.1.

By Corollary 6.1.5, it suffices to prove the punctual purity for any regular strict local ring $\mathcal{O}$ of arithmetic type over $\mathbf{Z}$. We may assume that the coefficient ring Λ is $\mathbf{Z}/\ell^\nu$, where ℓ is a prime which is invertible in $\mathcal{O}$ and $\nu \geq 0$, and that $\nu > 1$ if $\ell = 3$ and $\nu > 2$ if $\ell = 2$. We use induction on $d = \dim \mathcal{O}$. If $d = 0$, the claim is obvious. Assume $d \geq 1$. By our induction hypothesis, the punctual purity is true for any regular strict local ring of arithmetic type over $\mathbf{Z}$ of dimension strictly less than d. We choose an element $f \in m \setminus m^2$. The assumption of theorem 7.2.1 is satisfied, and hence the spectral sequence

$$\begin{aligned} E_2^{p,q} &= H^p(\operatorname{Spec} \mathcal{O}[f^{-1}],\, \mathbf{Z}/\ell^\nu(i)) \\ &\Longrightarrow (K/\ell^\nu)_{-p-q}(\mathcal{O}[f^{-1}])[\beta^{-1}] \qquad (q = -2i). \end{aligned}$$

degenerates at E_2. $E_2^{p,q} = E_\infty^{p,q}$ for any $p, q \in \mathbf{Z}$. For $p \geq 2$, $E_\infty^{p,q} = 0$ by Corollary 7.2.3, and hence

$$H^p(\operatorname{Spec} \mathcal{O}[f^{-1}],\ \mathbf{Z}/\ell^\nu) = 0$$

for $p \geq 2$. This implies the purity for $(\operatorname{Spec} \mathcal{O},\ V(f))$, and the punctual purity for $(\mathcal{O}, m)$ by Proposition 3.1.2. Q.E.D.

§8. Consequences

Here we list some consequences of the absolute purity theorem.

Consequence. *Let S be a noetherian regular scheme of dimension at most one. Let n be an integer which is invertible on S, and let $\Lambda = \mathbf{Z}/n$. If X is a regular S-scheme of finite type, the dualizing complex K_X normalized as in §4 satisfies*

$$\Lambda(\delta_X)[2\delta_X] \xrightarrow{\sim} K_X.$$

Especially, Λ is a dualizing complex of X.

Consequence (semi-purity). *Let X be a noetherian regular scheme, let n be an integer which is invertible on S, and let $\Lambda = \mathbf{Z}/n$. If $Y \overset{i}{\hookrightarrow} X$ is a closed immersion of codimension $\geq c$,*

$$R^q i^! \Lambda = 0 \quad (q < 2c).$$

Consequence. *Let X be a noetherian regular scheme, let Y be a divisor on X with simple normal crossings, and let $j : X \setminus Y \hookrightarrow X$ be the inclusion. Let n be an integer which is invertible on X, and let $\Lambda = \mathbf{Z}/n$. Then the canonical maps give isomorphisms*

$$R^q j_* \Lambda \simeq \wedge^q R^1 j_* \Lambda \quad \textit{for all } q \in \mathbf{Z},$$

$$R^1 j_* \Lambda \simeq \bigoplus_{i \in I} (\iota_{Y_i})_* \Lambda(-1)$$

where $(Y_i)_{i \in I}$ is the set of irreducible components of Y, and ι_{Y_i} is the inclusion map $Y_i \hookrightarrow X$ for $i \in I$.

Here is a generalization of the above claim in terms of logarithmic schemes (see [Il1] for log étale cohomology).

Consequence (absolute local acyclicity). *Let X be an fs log-scheme such that the underlying scheme X^{cl} is noetherian. Assume that X is log-regular. Let U be the open subset of X defined by*

$$U = \{x \in X; (M_X/\mathcal{O}_X^\times)_{\bar{x}} = \{1\}\},$$

i.e., U is the maximal open set where the log-structure is trivial. Let n be an integer which is invertible on X, and let $\Lambda = \mathbf{Z}/n$. For a Λ-smooth sheaf $\mathcal{F}$ on $X_{\mathrm{et}}^{\log}$,

$$\mathcal{F} \xrightarrow{\simeq} Rj_* j^* \mathcal{F}$$

holds. Here j is the morphism of topoi $U_{\mathrm{et}}^{\mathrm{cl}} = U_{\mathrm{et}}^{\log} \to X_{\mathrm{et}}^{\log}$.

Here is a conditional result which follows from the absolute purity.

Consequence. *Assume that the resolution of singularities of quasi-excellent schemes is true. Let S be a quasi-compact quasi-excellent scheme, and $f : X \to S$ be a finite type morphism. For an integer $n \geq 1$ which is invertible on S, let $\Lambda = \mathbf{Z}/n$. Then for a Λ-constructible sheaf $\mathcal{F}$ on X, $R^q f_* \mathcal{F}$ is Λ-constructible for any $q \in \mathbf{Z}$, and vanishes except for a finite number of q's, i.e., the finiteness theorem is true for quasi-excellent schemes.*

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