Advanced Studies in Pure Mathematics 36, 2002
Algebraic Geometry 2000, Azumino
pp. 115–143

Refined Cycle Maps

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Abstract.

We explain the theory of refined cycle maps associated to arithmetic mixed sheaves. This includes the case of arithmetic mixed Hodge structures, and is closely related to work of Asakura, Beilinson, Bloch, Green, Griffiths, Müller-Stach, Murre, Voisin and others.

Introduction

One of the most fundamental problems in the theory of algebraic cycles would be Beilinson's conjecture on mixed motives [4], which predicts the bijectivity of the cycle map

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^{2p}_{D^b\mathcal{MM}(X)}(\mathbb{Q}_X, \mathbb{Q}_X(p)) \tag{0.1}$$

for any smooth projective variety X over a field k. Here $\mathrm{CH}^p(X)_{\mathbb{Q}}$ is the Chow group of codimension p algebraic cycles modulo rational equivalence on X with $\mathbb{Q}$-coefficients, and $D^b\mathcal{MM}(X)$ is the bounded derived category of the (conjectural) abelian category of mixed motivic sheaves on X. By the adjoint relation for the structure morphism $a_X : X \to \mathrm{Spec}\, k$, the conjecture would be equivalent to the bijectivity of

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^{2p}_{D^b\mathcal{MM}(\mathrm{Spec}\, k)}(\mathbb{Q}_{\mathrm{Spec}\, k}, (a_X)_*\mathbb{Q}_X(p)), \tag{0.2}$$

because $\mathbb{Q}_X$ should be the pull-back by a_X of the constant object $\mathbb{Q}_{\mathrm{Spec}\, k}$ on $\mathrm{Spec}\, k$. It is known that this conjecture implies many other important conjectures on algebraic cycles, such as those of Murre [36], [37], and Bloch [7].

In the case when k is embeddable into $\mathbb{C}$ (e.g. if k is a number field or $\mathbb{C}$), a natural question would be whether $\mathcal{MM}(\mathrm{Spec}\, k)$ is close to the category of $\mathcal{M}_{\mathrm{SR}}(\mathrm{Spec}\, k)$, the category of *systems of realizations,* which was introduced by Deligne [20] (see also [21], [30]). It is expected that

Received April 2, 2001.
2000 *Mathematics Subject Classification*: 14C30, 32S35. Keywords: mixed Hodge structure, mixed Hodge module, algebraic cycle, Abel-Jacobi map.

the essential image of the natural functor $\mathcal{MM}(\operatorname{Spec} k) \to \mathcal{M}_{\mathrm{SR}}(\operatorname{Spec} k)$ would be quite close to the full subcategory $\mathcal{M}_{\mathrm{SR}}(\operatorname{Spec} k)^{\mathrm{go}}$ consisting of objects of geometric origin (which was introduced in [6] for l-adic sheaves). So the first test of the conjecture would be whether the cycle map

$$(0.3) \qquad cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^{2p}_{D^b\mathcal{M}_{\mathrm{SR}}(\operatorname{Spec} k)^{\mathrm{go}}}(\mathbb{Q}_{\operatorname{Spec} k}, (a_X)_*\mathbb{Q}_X(p))$$

is bijective. We can easily show that the surjectivity of (0.3) is equivalent to the algebraicity of absolute Hodge cycles [20] on any smooth projective k-varieties. See [45], [48]. Thus the surjectivity is essentially reduced to the Hodge conjecture, which is not easy to prove as well-known. (However, during an attempt to solve the conjecture we obtained a germ of a new idea by trying to restrict the Leray spectral sequence to the generic fiber of a morphism [46].)

Since the category of mixed motives should be universal as far as cohomology is concerned, we define the category of systems of realizations as an approximation by endowing the cohomology group with as much structure as possible. However, it has been realized by many people that for a complex algebraic variety (where $k = \mathbb{C}$), its cohomology group has more structure. This was first observed by M. Green, P. Griffiths, and C. Voisin in the study of the image of the Abel-Jacobi map for a generic hypersurface, where we have to use the fact that a complex algebraic variety is actually defined over a finitely generated subring of $\mathbb{C}$. See [25], [26], [27], [55], [56]. (This fact was also essential for the theory of mod p reduction of l-adic sheaves [6].) Then, using the models over the finitely generated subrings of $\mathbb{C}$, it is natural to define the category of *arithmetic mixed Hodge Modules* (or more generally, *arithmetic mixed sheaves*) on a complex algebraic variety X as the inductive limit of the categories of mixed Hodge Modules (or mixed sheaves) on the models of X. In the case the ground field k is a finitely generated field over $\mathbb{Q}$, this was already considered in [46] and [47], 1.9 inspired by the arguments in the Appendix to Lect. 1 of [7], and it is enough to take further the inductive limit over k. In particular, we get for the case $X = \operatorname{Spec}\mathbb{C}$ the category of arithmetic mixed Hodge structures, which is a refinement of mixed Hodge structures (see also [1] where the theory of mixed sheaves [47] is also used in an essential way). It seems that the term "arithmetic mixed Hodge structure" was first used in the work of Green and Griffiths [27], where the $\mathbb{Q}$-structure was not considered because they were mostly interested in infinitesimal variations of Hodge structures.

The main point of the theory of arithmetic mixed sheaves is that the injectivity of the refined cycle map of $\mathrm{CH}^2(X)_{\mathbb{Q}}$ for a smooth projective complex algebraic variety X can be reduced to the injectivity of the Abel-Jacobi map for codimension two cycles on smooth projective models of X. See (4.2). This shows that an additional hypothesis in [1] is unnecessary. See also [26], [27], [28]. The result can be extended to some case of higher Chow cycles as in (4.5) . However, it should be noted that this category is too big, and the forgetful functor to the category of mixed Hodge structures is not fully faithful (see [51], 2.5). We would have to restrict to the subcategory consisting of objects of "geometric origin" for the study of the surjectivity of the cycle map although it does not seem to cause a big problem for the injectivity. Note that our theory applies also to the category of objects of geometric origin, because it is equivalent to the limit of the category of objects of geometric origin on the models of X. The injectivity of the Abel-Jacobi map for varieties over number fields has been conjectured by Beilinson [3] and Bloch (see also [9]), and is one of the most interesting problems in this area. See also [28].

To show that the obtained new category is really better than the usual one, we can prove that, restricting the refined cycle map to the kernel of the Griffiths' Abel-Jacobi map for codimension two cycles, its image is infinite dimensional if X has a nontrivial global two form. See (4.1). This was inspired by [58], and is a consequence of Bloch's diagonal argument in [7] combined with Murre's result on the Albanese motive [36]. (Some special case is treated in [1] using a different method.) A similar assertion can be proved also for higher Chow groups. See (4.4).

I would like to thank L. Barbieri-Viale and A. Rosenschon for useful comments and good questions. I also thank the referee for useful comments.

We review the theory of mixed Hodge Modules and cycle maps in Sections 1 and 2. We define the category of arithmetic mixed sheaves in Sect. 3, and states the main results in Sect. 4. Some examples are given in Sect. 5.

§1. Mixed Hodge Modules

1.1. $\mathcal{D}$-Modules. Let X be a smooth algebraic variety over a filed k of characteristic zero. Then we have the sheaf of algebraic linear differential operators $\mathcal{D}_{X/k}$ which has the increasing filtration F by the order of operators. Since $\mathcal{D}_{X/k} = \mathcal{D}_{X/k'}$ for a finite extension $k' \subset k$, we will write $\mathcal{D}_X$ for $\mathcal{D}_{X/k}$ in the sequel.

Let (M, F) be a coherent filtered $\mathcal{D}_X$-Module. Here we assume always that F is exhaustive and $F_pM = 0$ for $p \ll 0$. Then the coherence of (M, F) means that $\mathrm{Gr}^F M\,(:= \oplus_p \mathrm{Gr}_p^F M)$ is coherent over $\mathrm{Gr}^F \mathcal{D}_X$.

We say that (M, F) is *holonomic* if it is coherent and $\dim \operatorname{supp} \operatorname{Gr}^F M = \dim X$, where $\operatorname{supp} \operatorname{Gr}^F M$ is a subvariety of the cotangent space of X which is naturally isomorphic to $\operatorname{Spec} \operatorname{Gr}^F \mathcal{D}_X$. We say that a $\mathcal{D}_X$-Module is holonomic if it is a coherent $\mathcal{D}_X$-Module and has locally a filtration F such that (M, F) is holonomic. The category of coherent (or holonomic) filtered $\mathcal{D}_X$-Modules will be denoted by $MF_{\mathrm{coh}}(\mathcal{D}_X)$ (or $MF_{\mathrm{hol}}(\mathcal{D}_X)$). Forgetting the filtration, we have $M_{\mathrm{coh}}(\mathcal{D}_X), M_{\mathrm{hol}}(\mathcal{D}_X)$ similarly.

If $k = \mathbb{C}$ and A is a subfield of $\mathbb{C}$, let $\operatorname{Perv}(X, A)$ denote the abelian category of perverse sheaves on X^{an} with A-coefficients and with algebraic stratifications. See [6]. Then we have the de Rham functor

$$\mathrm{DR} : M_{\mathrm{hol}}(\mathcal{D}_X) \to \operatorname{Perv}(X, \mathbb{C}) \tag{1.1.1}$$

defined by $M \to \Omega_X^{\dim X} \otimes_{\mathcal{D}_X}^{\mathbf{L}} M$ (using a natural projective resolution of the right $\mathcal{D}_X$-Module $\Omega_X^{\dim X}$). See e.g. [11]. This functor is exact and faithful.

1.2. Direct images. Let $f : X \to Y$ be a proper morphism of smooth k-varieties. Then the cohomological direct image functor $H^j f_* : MF_{\mathrm{coh}}(\mathcal{D}_X) \to MF_{\mathrm{coh}}(\mathcal{D}_Y)$ is defined by factorizing f by $X \to X \times_k Y \to Y$, where the first morphism is the embedding by the graph of f, and the second is the projection.

If $i : X \to Y$ is a closed embedding of codimension d, let $(y_1, \dots, y_n)$ be a local coordinate system of Y (i.e. it defines an étale map of an open subvariety of Y to $\mathbb{A}^n$) such that $X = \{y_1 = \cdots = y_d = 0\}$. Let $\partial_j = \partial/\partial y_j$ so that $\mathcal{D}_Y$ is locally identified with $\mathcal{O}_Y \otimes_{\mathbb{C}} \mathbb{C}[\partial_1, \dots, \partial_n]$. Then the direct image $i_* M$ is locally isomorphic to $M \otimes_{\mathbb{C}} \mathbb{C}[\partial_1, \dots, \partial_d]$ using the coordinates (see [11]), and the filtration F on $i_* M$ is defined by

$$F_p i_* M = \sum_{\nu \in \mathbb{N}^d} F_{p-|\nu|-d} M \otimes \partial^\nu, \tag{1.2.1}$$

where $\partial^\nu = \prod_{1 \le k \le d} \partial_k^{\nu_k}$ for $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}^d$. (This is independent of the choice of the coordinates.)

If $q : X \times_k Y \to Y$ is the projection, we have the relative de Rham complex $\mathrm{DR}_{X \times Y/Y}(M, F)$ such that $F_p \mathrm{DR}_{X \times Y/Y}(M)^j = \Omega_{X \times Y/Y}^{j+\dim X} \otimes_{\mathcal{O}} F_{p+j} M$. Then the cohomological direct image is defined to be the cohomology sheaf of the filtered direct image of $\mathrm{DR}_{X \times Y/Y}(M, F)$ by q. (Note that the filtration on the direct image complex is not necessarily strict, and we take the induced filtration on the cohomology sheaf.)

It is known that holonomic filtered $\mathcal{D}$-Modules are stable by the cohomological direct image under a proper morphism, and the cohomological direct image functors for (filtered) $\mathcal{D}$-Modules and perverse sheaves are compatible with each other via the functor (1.1.1), see e.g. [11], [43].

1.3. Mixed Hodge Modules on complex algebraic varieties. Let X be a smooth complex algebraic variety. We say that $M = ((M_{\mathcal{D}}, F),\ M_{\mathbb{Q}},\ W,\ \alpha)$ is a bifiltered holonomic $\mathcal{D}_X$-Module with rational structure, if $(M_{\mathcal{D}}, F)$ is a holonomic filtered $\mathcal{D}_X$-Module, $M_{\mathbb{Q}}$ is a perverse sheaf with rational coefficients, α is a comparison isomorphism $\mathrm{DR}(M_{\mathcal{D}}) = M_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ in $\mathrm{Perv}(X, \mathbb{C})$, and W is a pair of finite increasing filtrations on $M_{\mathcal{D}}$ and $M_{\mathbb{Q}}$ compatible with α. Here $\mathrm{Perv}(X, A)$ for $A = \mathbb{Q}$ or $\mathbb{C}$ is the full subcategory of $\mathrm{Perv}(X^{\mathrm{an}}, A)$ (see [6]) consisting of perverse sheaves with algebraic stratifications, and similarly for $D_c^b(X, A)$. A mixed Hodge Module on X is a bifiltered holonomic $\mathcal{D}_X$-Module M with rational structure satisfying the following conditions:

The first condition is that mixed Hodge Modules are defined Zariski-locally. The second is that, restricting a mixed Hodge Module M to an open smooth subvariety Z of $\operatorname{supp} M$ on which K is a (shifted) local system, it is isomorphic to the direct image by the closed immersion $Z \to X \setminus (\operatorname{supp} M \setminus Z)$ of an admissible variation of mixed Hodge structure in the sense of [32], [54] (see also (1.4) below) up to a shift of the weight filtration. Here the converse is also true, and an admissible variation of mixed Hodge structure is a mixed Hodge Module ([44], 3.27). The last condition claims that mixed Hodge Modules are locally obtained inductively by gluing mixed Hodge Modules supported on a divisor and admissible variations of mixed Hodge structures on a smooth closed subvariety in the complement on the divisor:

Let g be a function on X such that the restriction of a bifiltered holonomic $\mathcal{D}_X$-Module with rational structure M to the complement U of $g^{-1}(0)$ is the direct image of an admissible variation of mixed Hodge structure M' on a smooth variety Y by the closed immersion $i : Y \to U$. Let $\psi_{g,1}$ and $\varphi_{g,1}$ be the nearby and vanishing cycle functors with unipotent monodromy [18]. Then the condition is that the nearby and vanishing cycles $\psi_{g,1}M, \varphi_{g,1}M$ are well defined and the obtained $M'' := \varphi_{g,1}M$ is a mixed Hodge Module supported on $g^{-1}(0)$. (The first condition consists of the compatibility of the three filtrations F, W, V and the existence of the relative monodromy filtration, see [44], 2.2.)

Here we have canonical morphisms of mixed Hodge Modules

$$\mathrm{can} : \psi_{g,1} i_* M' \to M'', \quad \mathrm{Var} : M'' \to \psi_{g,1} i_* M'(-1),$$

satisfying the gluing condition

$$\text{Var} \circ \text{can} = N, \tag{1.3.1}$$

where $N = \log T_u$ with $T = T_s T_u$ the Jordan decomposition of the monodromy T. (Here (-1) denotes the Tate twist.) We can show that M is uniquely determined by $(M', M'', \text{can}, \text{Var})$ satisfying the gluing condition (1.3.1). More precisely, we have an equivalence between the category of mixed Hodge Modules on X and the category of $(M', M'', \text{can}, \text{Var})$ satisfying (1.3.1). See [44], 2.28. Furthermore, the corresponding M is uniquely determined by using the above condition on the well-definedness of the nearby and vanishing cycle functors. See [44], 2.8.

We denote by $\text{MHM}(X)$ the category of mixed Hodge Modules on X. It is an abelian category such that any morphism is strictly compatible with the Hodge and weight filtrations F, W in the strong sense. See [43], 5.1.14.

Now let X be a (singular) complex algebraic variety. We consider the category $\text{LE}(X)$ whose objects are closed embeddings $U \to V$ where U is an open subvariety of X and V is smooth. The morphisms are pairs of morphisms between U and between V making a commutative diagram. Here the morphisms of U are assumed to be compatible with the inclusions to X. For $\{U \to V\} \in \text{LE}(X)$, let $\text{MHM}_U(V)$ denote the category of mixed Hodge Modules on V supported on U. Then a mixed Hodge Module on X is a collection of mixed Hodge Modules $M_{U \to V} \in \text{MHM}_U(V)$ for $\{U \to V\} \in \text{LE}(X)$ (which is called the representative of M for $\{U \to V\}$) together with isomorphisms

$$v_* M_{U \to V}|_{V' \setminus (U' \setminus U)} = M_{U' \to V'}|_{V' \setminus (U' \setminus U)}$$

for $(u, v) : \{U \to V\} \to \{U' \to V'\}$ satisfying the usual cocycle condition. (We can define the category of filtered $\mathcal{D}$-Modules on X similarly.)

In the case X is smooth, we can show that this definition is equivalent to the previous one (i.e. we have naturally an equivalence of categories).

Actually, to define a mixed Hodge Module on a singular X, it is not necessary to define $M_{U \to V}$ for all $\{U \to V\}$; it is enough to do so for $\{U \to V\}$ such that the U cover X, but the gluing morphisms are defined by using the closed embeddings $U \cap U' \to V \times V'$.

1.4. Admissible variation of mixed Hodge structure. Let X be a smooth complex algebraic variety, and $\overline{X}$ a smooth compactification of X such that the complement $D := \overline{X} \setminus X$ is a divisor with normal crossings. Let $M = ((M_{\mathcal{O}}; F, W), (M_{\mathbb{Q}}, W))$ be a variation of mixed

Hodge structure on X, where $M_{\mathcal{O}}$ is the underlying $\mathcal{O}_X$-Module with the integrable connection ∇ having regular singularity at infinity, and $M_{\mathbb{Q}}$ is the underlying $\mathbb{Q}$-local system with an isomorphism $\mathrm{Ker}\,\nabla^{\mathrm{an}} = M_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. By [32], [54], M is admissible if the graded pieces $\mathrm{Gr}_k^W M$ are polarizable variations of Hodge structures with quasi-unipotent local monodromies, and furthermore the following conditions are satisfied:

In the case the local monodromies around D are all unipotent, let $\overline{M}_{\mathcal{O}}$ be Deligne's extension of $M_{\mathcal{O}}$ (see [17]). Then
(i) The filtrations F, W on M are extended to $\overline{M}_{\mathcal{O}}$ so that $\mathrm{Gr}_F^p \mathrm{Gr}_k^W \overline{M}_{\mathcal{O}}$ is a locally free $\mathcal{O}_{\overline{X}}$-Module for any p, k.
(ii) The relative monodromy filtration exists for the local monodromy around each irreducible component of D.

Here we can replace $\overline{M}_{\mathcal{O}}$ with its restriction to $\overline{X} \setminus \mathrm{Sing}\, D$, i.e. it is enough to consider the conditions around the smooth points of D, see [32].

In general, the above conditions should be satisfied for the pull-back of M by a dominant morphism such that the local monodromies of the pull-back are unipotent. More precisely, the condition for admissible variation is analytic local on a compactification of X, and it is enough to take locally a ramified cover of a polydisk as usual. Here we can consider $M_{\mathcal{O}}^{\mathrm{an}}$ instead of $M_{\mathcal{O}}$, because $M_{\mathcal{O}}$ is uniquely determined by $M_{\mathcal{O}}^{\mathrm{an}}$ due to the regularity and GAGA.

1.5. Mixed Hodge Modules on algebraic varieties. Let k be a subfield of $\mathbb{C}$, and X a smooth k-variety. Let $X_{\mathbb{C}} = X \otimes_k \mathbb{C}$. Then a mixed Hodge Module M on X consists of $((M_{\mathcal{D}}, F), W), (M_{\mathbb{Q}}, W)$ and α such that $(((M_{\mathcal{O}} \otimes_k \mathbb{C}, F), W), (M_{\mathbb{Q}}, W), \alpha)$ is a mixed Hodge Module on $X_{\mathbb{C}}$, where $(M_{\mathcal{O}}, F) \in MF_{\mathrm{hol}}(\mathcal{D}_X), M_{\mathbb{Q}} \in \mathrm{Perv}(X_{\mathbb{C}}, \mathbb{Q})$ with a finite increasing filtration W, and α is a comparison isomorphism $\mathrm{DR}(M \otimes_k \mathbb{C}) = K \otimes_{\mathbb{Q}} \mathbb{C}$ in $\mathrm{Perv}(X_{\mathbb{C}}, \mathbb{C})$ which is compatible with W. Here we also assume that polarizations on the graded pieces $\mathrm{Gr}_k^W M$ are defined over k, i.e., they are induced by isomorphisms of filtered $\mathcal{D}_X$-Modules $\mathrm{Gr}_k^W (M_{\mathcal{D}}, F)(k) \simeq \mathbb{D}\mathrm{Gr}_k^W (M_{\mathcal{D}}, F)$ compatible with pairings of perverse sheaves, where $\mathbb{D}$ denotes the dual. This is necessary to assure that the graded pieces are semisimple.

We will denote by $\mathrm{MHM}(X/k)$ the category of mixed Hodge Modules on X/k. This is an abelian category such that every morphism is strictly compatible with F, W in the strong sense. We have naturally the forgetful functors

$$\mathrm{MHM}(X/k) \to \mathrm{MHM}(X_{\mathbb{C}}) \to \mathrm{Perv}(X_{\mathbb{C}}, \mathbb{Q}),$$

which induce

$$D^b\mathrm{MHM}(X/k) \to D^b\mathrm{MHM}(X_{\mathbb{C}}) \to D^b_c(X_{\mathbb{C}}, \mathbb{Q}), \tag{1.5.1}$$

using the canonical functor $D^b\mathrm{Perv}(X_{\mathbb{C}}, \mathbb{Q}) \to D^b_c(X_{\mathbb{C}}, \mathbb{Q})$ in [6].

We can define similarly the notion of admissible variation of mixed Hodge structures on X (i.e., the underlying filtered $\mathcal{O}$-Modules and polarizations are defined over X.) Then we see that mixed Hodge Modules on X are obtained locally by gluing mixed Hodge Modules supported on a divisor and admissible variations of mixed Hodge structure on a smooth closed subvariety in the complement of the divisor as in (1.3).

1.6. Mixed sheaves. In this paper we consider more generally the category of mixed sheaves $\mathcal{M}(X/k)$ in the sense of [47]. However, to simplify the explanation, we assume in this paper that $\mathcal{M}(X/k)$ is either $\mathrm{MHM}(X/k)$ defined above or the category $\mathcal{M}_{\mathrm{SR}}(X/k)$ consisting of *systems of realizations* $((M_{\mathcal{D}}; F, W), (M_\sigma, W), (M_l, W))$, where $(M_{\mathcal{D}}, F)$ is a holonomic filtered $\mathcal{D}$-Module on X endowed with a finite filtration W, (M_σ, W) for an embedding $\sigma : k \to \mathbb{C}$ is a filtered perverse sheaf on $(X \otimes_{k,\sigma} \mathbb{C})^{\mathrm{an}}$ with $\mathbb{Q}$-coefficients, and (M_l, W) for a prime number l is a filtered perverse l-adic sheaf on $X_{\overline{k}} := X \otimes_k \overline{k}$ with $\mathbb{Q}_l$-coefficients which has a continuous action of the Galois group of $\overline{k}/k$ (i.e. the action is lifted to perverse sheaves with $\mathbb{Z}_l$-coefficients). Furthermore these are endowed with comparison isomorphisms

$$\mathrm{DR}((M_{\mathcal{D}}, W) \otimes_{k,\sigma} \mathbb{C}) = (M_\sigma, W) \otimes_{\mathbb{Q}} \mathbb{C}, \quad \epsilon^* i^*_{\overline{\sigma}}(M_l, W) = (M_\sigma, W) \otimes_{\mathbb{Q}} \mathbb{Q}_l$$

for an extension $\overline{\sigma} : \overline{k} \to \mathbb{C}$ of σ (in a compatible way with the action of $\mathrm{Gal}(\overline{k}/k)$, see [30]). Here $A = \mathbb{Q}, \overline{k}$ is the algebraic closure of k in $\mathbb{C}$, and $i_{\overline{\sigma}}$: $X_{\mathbb{C}} \to X_{\overline{k}}$ is the canonical morphism. (See [6] for ϵ^*.) In the case $X = \mathrm{Spec}\, k$, $\mathcal{M}_{\mathrm{SR}}(\mathrm{Spec}\, k)$ coincides with the category of systems of realizations introduced by Deligne [20], [21] (and this formulation is due to Jannsen [30]).

For $\mathcal{M}(X/k) = \mathrm{MHM}(X/k)$ or $\mathcal{M}_{\mathrm{SR}}(X/k)$, there exists canonically the base change functor

$$\mathcal{M}(X/k) \to \mathcal{M}(X \otimes_k k'/k') \tag{1.6.1}$$

for a finite extension $k \subset k'$ in a compatible way with the cohomological direct image and pull-back and also with dual and external product, etc. There is also the (canonically defined) forgetful functor

$$\mathcal{M}(X/k) \to \mathrm{MHM}(X/k), \tag{1.6.2}$$

which is compatible with the standard functors as above.

If X/k is smooth and purely d-dimensional, we have the constant object $\mathbb{Q}_{X/k}$ in $D^b\mathcal{M}(X/k)$ which actually belongs to $\mathcal{M}(X/k)[-d]$ by the definition of the perverse sheaf [6]. We have also the Tate twist $\mathbb{Q}_{X/k}(j)$ for $j \in \mathbb{Z}$ (using the cohomology of $\mathbb{P}^{-j}$ for j negative, and taking the dual for j positive).

1.7. Theorem. *For a morphism f of algebraic varieties over k, there exist canonical functors f_*, $f_!$, f^*, $f^!$, $\mathbb{D}$, $\otimes$, $\mathcal{H}om$, etc. between the derived categories $D^b\mathcal{M}(X/k)$ in a compatible way with the corresponding functors between the derived categories $D^b\mathrm{MHM}(X/k)$, $D^b\mathrm{MHM}(X_{\mathbb{C}})$ or $D^b_c(X_{\mathbb{C}}, \mathbb{Q})$ via the functors* (1.5.1) *and* (1.6.2).

Proof. This follows from the same argument as in [44]. See also [47]. Indeed, using Beilinson's resolution, the stability by the direct image is reduced to the one by the cohomological direct image for an affine morphism [5]. (If f is quasi-projective, this is especially simple by taking two sets of affine coverings of X associated with general hyperplane sections and using the co-Cech and Cech complexes.) The pull-backs are defined to be the adjoint functors of the direct images. For the existence, we may assume f is either a closed embedding i or a projection p. In the former case, the assertion is reduced to the full faithfulness of the direct image

$$i_* : D^b\mathcal{M}(X/k, \mathbb{Q}) \to D^b\mathcal{M}(Y/k, \mathbb{Q}),$$

which is shown by using the functor ξ_g in [44], 2.22. In the latter case it is enough to show the existence of $a_X^*\mathbb{Q}_{\mathrm{Spec}\,k/k}$ (using the duality and the external product). But this is represented by any complex $M^\bullet$ having a morphism $\mathbb{Q}_{\mathrm{Spec}\,k/k} \to (a_{X/k})_*M^\bullet$ in $D^b\mathcal{M}(X/k)$ such that the image of $M^\bullet$ in $D^b_c(X_{\mathbb{C}}, \mathbb{Q})$ is isomorphic to $\mathbb{Q}_{X_{\mathbb{C}}}$ and the image of the morphism is identified with the canonical morphism $\mathbb{Q} \to (a_{X_{\mathbb{C}}})_*\mathbb{Q}_{X_{\mathbb{C}}}$. So it exists locally on open subsets which are embeddable into smooth varieties, and we can glue them by using the adjoint morphism for the inclusion of open subvarieties, see [44], 4.4. Q.E.D.

1.8. Definition. For a k-variety X with structure morphism $a_{X/k} : X \to \mathrm{Spec}\,k$, we define

(1.8.1)
$$\mathbb{Q}_{X/k}(j) = a_{X/k}^*\mathbb{Q}_{\mathrm{Spec}\,k/k}(j), \quad H^j(X/k, \mathbb{Q}(j)) = H^j(a_{X/k})_*\mathbb{Q}_{X/k}(j).$$

We omit $/k$ in the case $k = \mathbb{C}$.

1.9. Decomposition of the direct images. If X is smooth proper over k, we have a noncanonical isomorphism

$$(a_{X/k})_*\mathbb{Q}_{X/k} \simeq \oplus_j H^j(X/k, \mathbb{Q})[-j] \quad \text{in } D^b\mathcal{M}(\mathrm{Spec}\,k/k). \tag{1.9.1}$$

See e.g. [44], 4.5.3.

§2. Cycle Map and Geometric Origin

2.1. Cycle map. Let X be a smooth k-variety. We define an analogue of Deligne cohomology by

$$H^i_{\mathcal{D}}(X/k, \mathbb{Q}(j)) = \mathrm{Ext}^i(\mathbb{Q}_{X/k}, \mathbb{Q}_{X/k}(j)) \\ \big(= \mathrm{Ext}^i(\mathbb{Q}_k, (a_{X/k})_*\mathbb{Q}_{X/k}(j))\big), \tag{2.1.1}$$

where the extension groups are taken in the derived category of $\mathcal{M}(X/k)$ or $\mathcal{M}(\mathrm{Spec}\, k/k)$, and the second isomorphism follows from the adjoint relation between the direct image and the pull-back by $a_{X/k}$. (In the case $k = \mathbb{C}$, we will often omit $/k$ to simplify the notation.)

Let $\mathrm{CH}^p(X)_{\mathbb{Q}}$ be the Chow group consisting of codimension p cycles modulo rational equivalence on X with rational coefficients. Then we have naturally the cycle map

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X/k, \mathbb{Q}(p)). \tag{2.1.2}$$

If a cycle ζ is represented by an irreducible closed subvariety Z, then $cl(\zeta)$ is defined to be the composition of

$$\mathbb{Q}_{X/k} \to \mathbb{Q}_{Z/k} \to \mathrm{IC}_{Z/k}\mathbb{Q}[-d_{Z/k}]$$

with its dual, by using the dualities

$$\mathbb{D}(\mathbb{Q}_{X/k}) = \mathbb{Q}_{X/k}(d_{X/k})[2d_{X/k}], \quad \mathbb{D}(\mathrm{IC}_{Z/k}\mathbb{Q}) = \mathrm{IC}_{Z/k}\mathbb{Q}(d_{Z/k}),$$

where $\mathrm{IC}_{Z/k}\mathbb{Q}$ is the intersection complex, and $d_{X/k} = \dim X/k$. See [44], 4.5.15. We can show that the cycle map is compatible with the pushdown and the pull-back of cycles. (where a morphism is assumed to be proper in the case of pushdown.) See [45, II].

The composition of (2.1.2) with the natural projection

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Hom}\big(\mathbb{Q}_k, H^{2p}(X/k, \mathbb{Q}(p))\big)$$

is the usual cycle map. Let $\mathrm{CH}^p_{\mathrm{hom}}(X)_{\mathbb{Q}}$ be its kernel (which consists of homologically equivalent to zero cycles). Then (2.1.2) induces a generalized Abel-Jacobi map over k:

$$\mathrm{CH}^p_{\mathrm{hom}}(X)_{\mathbb{Q}} \to J^p(X/k)_{\mathbb{Q}} := \mathrm{Ext}^1\big(\mathbb{Q}_k, H^{2p-1}(X/k, \mathbb{Q}(p))\big), \tag{2.1.3}$$

where Ext^1 is taken in $\mathcal{M}(\mathrm{Spec}\, k/k)$.

Let $\mathrm{CH}^p(X,m)_{\mathbb{Q}}$ be Bloch's higher Chow group with rational coefficients of a smooth k-variety X [10]. By [47] we have the cycle map

$$(2.1.4)\qquad cl : \mathrm{CH}^p(X,m)_{\mathbb{Q}} \to H_{\mathcal{D}}^{2p-m}(X/k,\mathbb{Q}(p)).$$

If X is smooth proper over k and $m > 0$, then this cycle map induces a generalized Abel-Jacobi map over k:

$$(2.1.5)\qquad \mathrm{CH}^p(X,m)_{\mathbb{Q}} \to \mathrm{Ext}^1\big(\mathbb{Q}_k, H^{2p-m-1}(X/k,\mathbb{Q}(p))\big),$$

because the Leray spectral sequence for $H_{\mathcal{D}}^{2p-m}(X/k,\mathbb{Q}(p))$ degenerates at E_2 by (1.9.1), and $\mathrm{Hom}\big(\mathbb{Q}_k, H^{2p-m}(X/k,\mathbb{Q}(p))\big) = 0$.

2.2. Griffiths' Abel-Jacobi map. If $k = \mathbb{C}$ and $\mathcal{M}(X) = \mathrm{MHM}(X)$, then $H_{\mathcal{D}}^i(X,\mathbb{Q}(j))$ for a smooth projective variety X coincides with Deligne cohomology in the usual sense ([22], [23]), and the cycle map (2.1.2) coincides with Deligne's cycle map. In particular, (2.1.3) coincides with Griffiths' Abel-Jacobi map

$$(2.2.1)\qquad \mathrm{CH}^p_{\mathrm{hom}}(X) \to J^p(X) \big(= \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{2p-1}(X,\mathbb{Z}(p)))\big)$$

tensored with $\mathbb{Q}$, where $J^p(X)$ is the Griffiths intermediate Jacobian [29], and the last isomorphism follows from [12].

2.3. Injectivity of the Abel-Jacobi map. It is expected that higher extension groups Ext^i $(i > 1)$ should vanish in the (conjectural) category of mixed motives over a number field. Since the category of systems of realizations is an approximation of the category of mixed motives, it is interesting whether the Abel-Jacobi map (2.1.3) is injective in the case k is a number field. Actually Beilinson conjectures the injectivity of the composition of (2.1.3) with the natural morphism:

$$(2.3.1)\qquad \mathrm{CH}^p_{\mathrm{hom}}(X)_{\mathbb{Q}} \to J^p(X/k)_{\mathbb{Q}} \to J^p(X_{\mathbb{C}})_{\mathbb{Q}}$$

at least if we restrict it to the subgroup $\mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}}$ consisting of algebraically equivalent to zero cycles [3]. Since the image of $\mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}}$ by (2.1.3) is contained in the algebraic part of the Jacobian, and the restriction of (2.3.1) to $\mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}}$ is defined algebraically (3.10), it would be natural to conjecture the injectivity of (2.3.1) for the algebraically equivalent to zero cycles. However, it may be better to conjecture the injectivity of (2.1.3) in general.

2.4. Geometric origin. We denote by $\mathcal{M}(X/k)^{\mathrm{go}}$ the full subcategory of $\mathcal{M}(X/k)$ consisting of objects of geometric origin (see [6] for the case of perverse sheaves). This is by definition the smallest full subcategory of $\mathcal{M}(X/k)$ which is stable by the standard cohomological functors

$H^j f_*, H^j f_!, H^j f^*, H^j f^!$, etc. and also by subquotients in $\mathcal{M}(X/k)$, and contains the constant object $\mathbb{Q}_{\mathrm{Spec}\, k/k}$ for $X = \mathrm{Spec}\, k$. (This satisfies the axiom of mixed sheaves, see [47], 7.1.) Actually, it is enough to assume the stability by the cohomological direct images and pull-backs, because the nearby and vanishing cycle functors are expressed by using the direct images and pull-backs in the same way as in [18] (see e.g. [47], 5.7), and the stability by dual and external product follows from the compatibility with those functors. (More precisely we have (2.5) below.)

We define $H^i_{\mathcal{D}}(X/k, \mathbb{Q}(j))^{\mathrm{go}}$ as in (2.1.1) with $\mathcal{M}(X/k)$ replaced by $\mathcal{M}(X/k)^{\mathrm{go}}$. Note that the natural morphism

(2.4.1) $$H^i_{\mathcal{D}}(X/k, \mathbb{Q}(j))^{\mathrm{go}} \to H^i_{\mathcal{D}}(X/k, \mathbb{Q}(j))$$

is not injective in general. We have the cycle map

(2.4.2) $$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X/k, \mathbb{Q}(p))^{\mathrm{go}}$$

factorizing (2.1.2). Let

$$J^p(X/k)_{\mathbb{Q}} = \mathrm{Ext}^1_{\mathcal{M}}\big(\mathbb{Q}, H^{2p-1}(X/k, \mathbb{Q}(p))\big),$$
$$J^p(X/k)^{\mathrm{go}}_{\mathbb{Q}} = \mathrm{Ext}^1_{\mathcal{M}^{\mathrm{go}}}\big(\mathbb{Q}, H^{2p-1}(X/k, \mathbb{Q}(p))\big),$$

where $\mathcal{M}$ and $\mathcal{M}^{\mathrm{go}}$ mean $\mathcal{M}(\mathrm{Spec}\, k/k)$ and $\mathcal{M}(\mathrm{Spec}\, k/k)^{\mathrm{go}}$. We have a canonical injection

(2.4.3) $$J^p(X/k)^{\mathrm{go}}_{\mathbb{Q}} \to J^p(X/k)_{\mathbb{Q}}.$$

(If $k = \mathbb{C}$ and $\mathcal{M}(X) = \mathrm{MHM}(X)$, we omit $/k$.)

We can show the following:

2.5. Proposition. *For $\mathcal{M} \in \mathcal{M}(X/k)$, it is of geometric origin if and only if for any point of X, there exist an open neighborhood U, a closed embedding $i : U \to Z$, a quasi-projective morphism $\pi : Y \to Z$, and a divisor D on Y such that $i_*\mathcal{M}|_U$ is isomorphic to a subquotient of $H^j\pi_*j_!\mathbb{Q}_{(Y\setminus D)/k}$ in $\mathcal{M}(Z/k)$. Here $j : Y \setminus D \to Y$ denotes the inclusion morphism, and we may assume that D is a divisor with normal crossings on Y.*

(See [47] and also [45, I].)

2.6. Theorem. *Assume $\mathcal{M}(X/k) = \mathcal{M}_{\mathrm{SR}}(X/k)$ in* (1.6). *Then the following assertions are equivalent:*
(i) The cycle map (2.4.2) *is surjective for any smooth projective variety X over k.*

(*ii*) *Absolute Hodge cycles on any smooth projective varieties over k are algebraic.*

If $k = \mathbb{C}$ *and* $\mathcal{M}(X) = \mathrm{MHM}(X)$, *then the equivalence holds with absolute Hodge cycles replaced by Hodge cycles, and these are further equivalent to*:
(*iii*) *The images of* (2.2.1) *and* (2.4.3) *coincide for any smooth complex projective varieties.*

(See [45, I] and [48].)

2.7. Remark. By (2.6), the surjectivity of the cycle map (2.4) is reduced to the algebraicity of absolute Hodge cycles, and the latter is easily reduced to the usual Hodge conjecture. To show the last conjecture, it would be natural to consider a morphism of a variety (which is birational to X) to another variety of dimension ≥ 2 (e.g. by taking Lefschetz pencils successively). In [46] we tried to restrict the Leray spectral sequence to each fiber using extension groups. Here the category of usual mixed Hodge structures is not good enough because of the vanishing of higher extension groups. But it is also unclear whether the full subcategory of objects of geometric origin is useful, since their higher extension groups are very difficult to calculate, although they are not expected to vanish.

Another big problem in this attempt is that, even if we could get cycles on fibers, it is not clear whether they come from one cycle on the total space. The difficulty comes from the fact that the higher extension classes do not form a section of some geometric object over the base space as in the case of normal functions.

To solve the last problem it would be natural to consider the generic fiber of the morphism and try to find an argument corresponding to "spreading out" of algebraic cycles (as in [7], p. 1.20). Then we get the idea of taking the inductive limit of the category of mixed sheaves on the pull-back of nonempty open subvarieties of the base space. See [46] and [47], 1.9. Unfortunately, this idea did not work well for the problem mentioned above, because we do not yet have a good category of mixed sheaves (which is strong enough to solve the problem). However, seeing earlier work of M. Green [25] and C. Voisin [55], we notice that the cohomology of a complex algebraic variety has more structure than expressed in the systems of realizations, and find that the converse of the above argument would be possible. A complex algebraic variety X has a model of finite type over a subfield k of $\mathbb{C}$ having a morphism to another integral k-variety whose geometric generic fiber over $\operatorname{Spec}\mathbb{C}$ is isomorphic to X. Then it is natural to consider the inductive limit of mixed sheaves on the models of X. See also [1] and especially [56],

p. 194. (Note that a similar idea was essentially used in the theory of mod p reduction for perverse sheaves [6].) Thus we get the notion of arithmetic mixed sheaf which will be explained in the next section.

§3. Arithmetic Mixed Sheaves

3.1. Construction. Let k, K be subfields of $\mathbb{C}$ such that $k \subset K$. Then for a K-variety X, there exists a finitely generated k-subalgebra R of K such that X is defined over R, i.e., there is an R-scheme X_R of finite type such that $X = X_R \otimes_R K$. For a finitely generated smooth k-subalgebra R' of K containing R, let $X_{R'} = X_R \otimes_R R', S' = \operatorname{Spec} R', d_{R'} = \dim_k S'$, and let $k_{R'}$ be the algebraic closure of k in R'. Then $k_{R'}$ coincides with the algebraic closure of k in the function field $k(S')$ of S' (because S' is normal), and $k_{R'}$ is a finite extension of k_R. Furthermore, $S/k_{R'}$ is geometrically irreducible and $S'_{\mathbb{C}} := S' \otimes_{k_{R'}} \mathbb{C}$ is connected. We define

$$\mathcal{M}(X/K)_{\langle k \rangle} = \varinjlim \mathcal{M}(X_{R'}/k_{R'})[-d_{R'}],$$

where the inductive limit is taken over R' as above. Here we denote by $\mathcal{M}(X_{R'}/k_{R'})[-d_{R'}]$ the category of mixed sheaves $\mathcal{M}(X_{R'}/k_{R'})$ shifted by $-d_{R'}$ in the derived category. Note that the shift is necessary due to the normalization of perverse sheaves in [6] which implies that perverse sheaves are stable by the usual pull-back f^* under a smooth morphism f up to the shift of complexes by the relative dimension.

More precisely, we define the order relation $R' < R''$ by the inclusion $R' \subset R''$ together with the smoothness of R'' over R'. Then we have natural functors

$$\begin{aligned}\mathcal{M}(X_{R'}/k_{R'})[-d_{R'}] &\to \mathcal{M}(X_{R'} \otimes_{k_{R'}} k_{R''}/k_{R''})[-d_{R'}] \\ &\to \mathcal{M}(X_{R''}/k_{R''})[-d_{R''}],\end{aligned} \tag{3.1.1}$$

where the first comes from (1.6.1). Note that $R' \otimes_{k_{R'}} k_{R''} \to R''$ is injective because $k_{R'}$ is algebraically closed in the fraction field of R'.

We have the canonical functors

$$\mathcal{M}(X_{R'}/k_{R'})[-d_{R'}] \to \mathrm{MHM}(X_{\mathbb{C}}) \tag{3.1.2}$$

compatible with (3.1.1), because $X_{\mathbb{C}}$ $(:= X \otimes_K \mathbb{C})$ is identified with the closed fiber of $X_{R'} \otimes_{k_{R'}} \mathbb{C}$ over the closed point of $S'_{\mathbb{C}}$ defined by the inclusion $R' \subset \mathbb{C}$. So we get the canonical functor

$$\iota : \mathcal{M}(X/K)_{\langle k \rangle} \to \mathrm{MHM}(X_{\mathbb{C}}) \tag{3.1.3}$$

compatible with (3.1.1), (3.1.2). (We will omit $/K$ if $K = \mathbb{C}$.)

We denote by H^i the (shifted) cohomology functor from the derived categories of $\mathcal{M}(X_{R'}/k_{R'})$ and $\mathcal{M}(X/K)_{\langle k\rangle}$ to $\mathcal{M}(X_{R'}/k_{R'})[-d_{R'}]$ and $\mathcal{M}(X/K)_{\langle k\rangle}$ respectively. It is compatible with (3.1.1–3).

In the case $X = \mathbb{C}$, we define

$$\mathcal{M}_{K,\langle k\rangle} = \mathcal{M}(\operatorname{Spec} K/K)_{\langle k\rangle},$$

and it will be denoted by $\mathcal{M}_{\langle k\rangle}$ if $K = \mathbb{C}$. In the case $\mathcal{M}(X) = \mathrm{MHM}(X)$, it is denoted by $\mathrm{MHS}_{K,\langle k\rangle}$, and by $\mathrm{MHS}_{\langle k\rangle}$ when $K = \mathbb{C}$.

We have the canonical functor

$$\iota : \mathcal{M}_{K,\langle k\rangle} \to \mathrm{MHS}, \tag{3.1.4}$$

where the target is the category of graded-polarizable mixed $\mathbb{Q}$-Hodge structures in the usual sense [16].

For $j \in \mathbb{Z}$, we have $\mathbb{Q}_{K,\langle k\rangle}(j) \in \mathcal{M}_{K,\langle k\rangle}$ which is represented by a constant variation of Hodge structure of type $(-j,-j)$ as usual. This is denoted by $\mathbb{Q}_{\langle k\rangle}(j)$ if $K = \mathbb{C}$.

3.2. Remarks. (i) The extension groups in $\mathcal{M}_{K,\langle k\rangle}$ are too big, and the natural functor (3.1.4) is not fully faithful, see [51], 2.5 (ii).

(ii) Let $\mathrm{MHS}_{\nabla/\overline{k}}$ denote the category of mixed Hodge structures whose $\mathbb{C}$-part is endowed with an integrable connection ∇ over $\overline{k}$. Then the functor (3.1.4) factors through $\mathrm{MHS}_{\nabla/\overline{k}}$, and $\mathrm{MHS}_{\langle k\rangle}$ is a full subcategory of $\mathrm{MHS}_{\nabla/\overline{k}}$.

(iii) If K contains $\overline{k}$, we have equivalences of categories

$$\mathrm{MHM}(X/K)_{\langle k\rangle} \to \mathrm{MHM}(X/K)_{\langle\overline{k}\rangle}, \quad \mathrm{MHS}_{K,\langle k\rangle} \to \mathrm{MHS}_{K,\langle\overline{k}\rangle}.$$

induced by $\mathrm{MHM}(X_{R'}/k_{R'}) \to \mathrm{MHM}(X_{R'} \otimes_{k_{R'}} \overline{k}/\overline{k})$. See [51], 2.8.

3.3. Theorem. *The category $\mathcal{M}(X/K)_{\langle k\rangle}$ is an abelian category, and there exist canonical functors f_*, $f_!$, f^*, $f^!$, $\mathbb{D}$, $\otimes$, $\mathcal{H}om$, etc. between the derived categories $D^b\mathcal{M}(X/K)_{\langle k\rangle}$ in a compatible way with the functor ι.*

Proof. This follows from (1.7) by using an analogue of the generic base change theorem in [19]. Q.E.D.

3.4. Refined cycle map. For a K-variety X with structure morphism $a_{X/K} : X \to \operatorname{Spec} K$, we define

$$\mathbb{Q}_{X/K,\langle k\rangle}(j) = a_{X/K}^*\mathbb{Q}_{K,\langle k\rangle}(j) \in D^b\mathcal{M}(X/K)_{\langle k\rangle},$$
$$H^i(X/K,\mathbb{Q}_{\langle k\rangle}(j)) = H^i(a_{X/K})_*\mathbb{Q}_{X/K,\langle k\rangle}(j) \in \mathcal{M}_{K,\langle k\rangle}.$$

The latter is represented by $H^i\pi_*\mathbb{Q}_{X_R/k_R}(j)$ for a model $\pi : X_R \to S = \operatorname{Spec} R$ of X. We define an analogue of Deligne cohomology by

$$\begin{aligned} H^i_{\mathcal{D}}(X/K, \mathbb{Q}_{\langle k\rangle}(j)) &= \operatorname{Ext}^i(\mathbb{Q}_{K,\langle k\rangle}, (a_{X/K})_*\mathbb{Q}_{X/K,\langle k\rangle}(j)) \\ &\quad \big(= \operatorname{Ext}^i(\mathbb{Q}_{X/K,\langle k\rangle}, \mathbb{Q}_{X/K,\langle k\rangle}(j))\big), \end{aligned}$$

which is isomorphic to the inductive limit of

$$H^i_{\mathcal{D}}(X_R/k_R, \mathbb{Q}(j)) = \operatorname{Ext}^i(\mathbb{Q}_{X_R/k_R}, \mathbb{Q}_{X_R/k_R}(j)).$$

If $K = \mathbb{C}$, we omit $/K$ or K, to simplify the notation.

If X/K is smooth, we have the refined cycle map

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X/K, \mathbb{Q}_{\langle k\rangle}(p)) \tag{3.4.1}$$

by taking the inductive limit of the cycle map in (2.1.2) for models X_R/k_R of X:

$$cl_R : \mathrm{CH}^p(X_R)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X_R/k_R, \mathbb{Q}_{k_R}(p)). \tag{3.4.2}$$

This means that the cycle map is defined by taking models of cycles.

For smooth K-varieties X, Y, let

$$C^i(X,Y)_{\mathbb{Q}} = \oplus_j \mathrm{CH}^{i+\dim X_j}(X_j \times_K Y)_{\mathbb{Q}},$$

where the X_j are the irreducible components of X. Then the cycle map induces

(3.4.3)
$$\begin{aligned} C^i(X,Y)_{\mathbb{Q}} &\to \operatorname{Ext}^{2i+2\dim X}(\mathbb{Q}_{X\times Y/K,\langle k\rangle}, \mathbb{Q}_{X\times Y/K,\langle k\rangle}(i+\dim X)) \\ &= \operatorname{Hom}((a_{X/K})_*\mathbb{Q}_{X/K,\langle k\rangle}, (a_{Y/K})_*\mathbb{Q}_{Y/K,\langle k\rangle}(i)[2i]), \end{aligned}$$

where X is assumed connected. This is compatible with the composition of correspondences, see [45], II, 3.3. In particular, the action of $C^i(X,Y)_{\mathbb{Q}}$ on the Chow groups corresponds by the cycle map to the composition of morphisms with the image of (3.4.3), i.e. for $\Gamma \in C^i(X,Y)_{\mathbb{Q}}$, we have the commutative diagram:

(3.4.4)
$$\begin{array}{ccc} \mathrm{CH}^p(X)_{\mathbb{Q}} & \xrightarrow{cl_*} & \operatorname{Hom}(\mathbb{Q}_{K,\langle k\rangle}, (a_{X/K})_*\mathbb{Q}_{X/K,\langle k\rangle}(p)[2p]) \\ \downarrow{\scriptstyle \Gamma_*} & & \downarrow{\scriptstyle \Gamma_*} \\ \mathrm{CH}^{p+i}(Y)_{\mathbb{Q}} & \xrightarrow{cl_*} & \operatorname{Hom}(\mathbb{Q}_{K,\langle k\rangle}, (a_{Y/K})_*\mathbb{Q}_{Y/K,\langle k\rangle}(p+i)[2p+2i]) \end{array}$$

We have similarly the refined cycle map for the higher Chow groups

$$cl : \mathrm{CH}^p(X,m)_{\mathbb{Q}} \to H^{2p-m}_{\mathcal{D}}(X/K, \mathbb{Q}_{\langle k\rangle}(p)) \tag{3.4.5}$$

as the limit of (2.1.4).

3.5. Indecomposable higher Chow groups. Let $m = 1$. Then an element of $\mathrm{CH}^p(X, 1)$ is represented by $\sum_i (Z_i, g_i)$ where the Z_i are integral closed subvarieties of codimension $p-1$ in X and the g_i are nonzero rational functions on Z_i such that $\sum_i \operatorname{div} g_i = 0$ in X. In particular, we have a well-defined morphism

$$\mathrm{CH}^{p-1}(X)_{\mathbb{Q}} \otimes_{\mathbb{Z}} k^* \to \mathrm{CH}^p(X,1)_{\mathbb{Q}}. \tag{3.5.1}$$

Its image and cokernel are denoted by $\mathrm{CH}^p_{\mathrm{dec}}(X,1)_{\mathbb{Q}}$ and $\mathrm{CH}^p_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$ respectively. Their elements are called decomposable and indecomposable higher cycles respectively.

3.6. Leray Filtration. We have the Leray spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i\big(\mathbb{Q}_k, H^j(X/K, \mathbb{Q}_{\langle k\rangle}(p))\big) \Rightarrow H^{i+j}_{\mathcal{D}}(X/K, \mathbb{Q}_{\langle k\rangle}(p)),$$

which degenerates at E_2. Indeed, by the decomposition theorem (see e.g. [44], 4.5.3), we have a noncanonical isomorphism

$$(a_{X/K})_* a^*_{X/K} \mathbb{Q}_{K,\langle k\rangle} \simeq \oplus_j H^j(X/K, \mathbb{Q}_{\langle k\rangle})[-j] \quad \text{in } D^b\mathcal{M}_{K,\langle k\rangle}. \tag{3.6.1}$$

We denote by F_L the associated filtration on $H^{i+j}_{\mathcal{D}}(X/K, \mathbb{Q}_{\langle k\rangle}(p))$, and also the filtration on $\mathrm{CH}^p(X)_{\mathbb{Q}}$ induced by the cycle map (3.4.1). This means that $F_L^{r+1}\mathrm{CH}^p(X)_{\mathbb{Q}}$ is the kernel of

$$cl : F_L^r \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^r\big(\mathbb{Q}_{K,\langle k\rangle}, H^{2p-r}(X/K, \mathbb{Q}_{\langle k\rangle}(p)\big),$$

and the cycle map induces injective morphisms

$$\mathrm{Gr}^r_{F_L} cl : \mathrm{Gr}^r_{F_L} \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^r\big(\mathbb{Q}_{K,\langle k\rangle}, H^{2p-r}(X/K, \mathbb{Q}_{\langle k\rangle}(p))\big).$$

3.7. Remark. By definition, $F_L^1\mathrm{CH}^p(X)_{\mathbb{Q}}$ coincides with the subgroup $\mathrm{CH}^p_{\mathrm{hom}}(X)_{\mathbb{Q}}$ consisting of cohomologically equivalent to zero cycles. For $p = 2$, let $\mathrm{CH}^p_{\mathrm{AJ}}(X)_{\mathbb{Q}}$ denote the kernel of the Abel-Jacobi map. Then

$$F_L^2\mathrm{CH}^p(X)_{\mathbb{Q}} \subset \mathrm{CH}^p_{\mathrm{AJ}}(X)_{\mathbb{Q}}. \tag{3.7.1}$$

Indeed, for a model $\pi : X_R \to S = \operatorname{Spec} R$ of X, we have a commutative diagram

$$\begin{array}{ccc}
F_L^1\mathrm{CH}^p(X_R)_{\mathbb{Q}} & \longrightarrow & \mathrm{Ext}^1\big(\mathbb{Q}_{S,\langle k\rangle}, H^{2p-1}\pi_*\mathbb{Q}_{X_R/k_R}(p)\big) \\
\big\downarrow & & \big\downarrow \\
F_L^1\mathrm{CH}^p(X_{\mathbb{C}})_{\mathbb{Q}} & \longrightarrow & \mathrm{Ext}^1\big(\mathbb{Q}, H^{2p-1}(X_{\mathbb{C}}, \mathbb{Q}(p))\big),
\end{array}$$

where the vertical morphisms are induced by $X_{\mathbb{C}} \to X_R$ which is given by the inclusion $R \to \mathbb{C}$. For $p = \dim X$, the equality holds in (3.7.1). This follows from Murre's Chow-Künneth decomposition [36] by using the compatibility of the action of a correspondence (3.4.4), see e.g. [51], 3.6. This can be generalized to algebraically equivalent to zero cycles of any codimension, see (3.9).

We have $\mathrm{Gr}^r_{F_L}\mathrm{CH}^p(X)_{\mathbb{Q}} = 0$ for $r > p$ using (3.4.4), see [45], II. It seems that the filtration F_L coincides with a new filtration of M. Green which was explained in the conference [28]. It is expected that F_L gives a conjectural filtration of Beilinson [3] and Bloch (see also [7], [31]).

3.8. Murre's filtration. Assume that a smooth projective variety X admits the Chow-Künneth decomposition in the sense of Murre [36], [37]. Then $\mathrm{CH}^p(X)_{\mathbb{Q}}$ has Murre's filtration F_M. (See also [31].) For F_L as in (3.6), we can show (see [51], 4.9):

$$(3.8.1) \qquad F_M \subset F_L \quad \text{and} \quad F_M = F_L \mod \cap_i F_L^i.$$

In particular, $F_M = F_L$ if the cycle map (3.4.1) for X is injective. The injectivity of (3.4.1) can be used for the construction of the Chow-Künneth decomposition.

In [52], Shuji Saito has constructed a filtration F_{Sh} on $\mathrm{CH}^p(X)_{\mathbb{Q}}$ by induction. If we modify slightly his definition or assume the standard conjectures, we can show that his filtration is contained in F_L and they coincide in the case where the Künneth components of the diagonal are algebraic and the cycle map (3.4.1) is injective for the given X (see [51], 4.9, and also [52, II] where we assume the last two hypotheses for any smooth projective varieties).

3.9. Proposition. *Let* $\mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}}$ *denote the subgroup consisting of algebraically equivalent to zero cycles. Then we have*

$$(3.9.1) \qquad F_L^2\mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}} = \mathrm{CH}^p_{\mathrm{AJ}}(X)_{\mathbb{Q}} \cap \mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}}.$$

Proof. Let $J^p(X)_{\mathrm{alg}}$ be the image of $\mathrm{CH}^p_{\mathrm{alg}}(X)$ by the Abel-Jacobi map to $J^p(X)$. Then it has a structure of abelian variety. Furthermore, if X and a cycle ζ are defined over a subfield K of $\mathbb{C}$ which is finitely generated over k, then so are $J^p(X)_{\mathrm{alg}}$ and the image of ζ by the Abel-Jacobi map (enlarging K if necessary), see (3.10) below. Since the functor (3.1.3) is induced by the inclusion $R' \to \mathbb{C}$ which gives a geometric generic point of $\mathrm{Spec}\, R'$, we get the assertion. Q.E.D.

3.10. Algebraic part of the intermediate Jacobian. Let Y be a closed subvariety of X with pure codimension $p-1$, and $\widetilde{Y} \to Y$ a resolution of singularities. We have the Gysin morphism

$$H^1(\widetilde{Y}, \mathbb{Z})(1-p) \to H^{2p-1}(X, \mathbb{Z}). \tag{3.10.1}$$

We assume that Y is sufficiently large so that the image is maximal dimensional. Then $J^p(X)_{\mathrm{alg}}$ coincides with the image of the induced morphism (see (2.2.1)):

$$J^1(\widetilde{Y}) \to J^p(X). \tag{3.10.2}$$

Indeed, for $\zeta \in \mathrm{CH}^p_{\mathrm{alg}}(X)$, there is a one-parameter family $\{\zeta_s\}$ over a connected curve S such that ζ is the difference of ζ_{s_1} and ζ_{s_0} for $s_0, s_1 \in S$, and we may assume that Y contains the supports of ζ_s, because the images of (3.10.1–2) do not change by enlarging Y by the above assumption on Y. More precisely, $\{\zeta_s\}$ comes from a cycle Γ on $S \times X$, and ζ belongs to the image of the composition of a correspondence $\Gamma' \in \mathrm{CH}^1(S \times \widetilde{Y})$ and the pushforward by $\widetilde{Y} \to X$, where Γ' is obtained by pulling back $\Gamma \in \mathrm{CH}^1(S \times Y)$.

In particular, $J^p(X)_{\mathrm{alg}}$ has a structure of abelian variety as a quotient of $\mathrm{Pic}(\widetilde{Y})^0$. (See also [2]). If X and ζ are defined over K, we may assume that so are Y and $\widetilde{Y}$ by enlarging K if necessary. Then the quotient of $\mathrm{Pic}(\widetilde{Y})^0$ is also defined over K. Indeed, the quotient corresponds to a quotient system of realizations of $H^1(\widetilde{Y}_K/K, \mathbb{Z})$, which is defined by using the image of (3.10.1). Here $\widetilde{Y}_K$ is a model of $\widetilde{Y}$ over K, and $H^1(\widetilde{Y}_K/K, \mathbb{Z})$ is a system of realizations with integral coefficients, which is defined in a similar way to (1.6) and (1.8). So the assertion follows because the divisor class is defined in $\mathrm{Pic}(\widetilde{Y}_K)^0$.

3.11. Universal ind-abelian quotient. We can give a more precise description of $\mathrm{CH}^p_{\mathrm{alg}}(X)$ related to [35], [42], when the base field k is an algebraically closed field of characteristic 0. By the above argument, we have

$$\mathrm{CH}^p_{\mathrm{alg}}(X) = \varinjlim \mathrm{CH}^1_{\mathrm{alg}}(Y), \tag{3.11.1}$$

where the inductive limit is taken over closed subvarieties Y of pure codimension $p-1$, and $\mathrm{CH}^1_{\mathrm{alg}}(Y)$ is the image of $\mathrm{CH}^1_{\mathrm{alg}}(\widetilde{Y}) = \mathrm{Pic}(\widetilde{Y})^0$ with $\pi : \widetilde{Y} \to Y$ a resolution of singularities. As is well-known, $\mathrm{Pic}(\widetilde{Y})^0$ is the group of k-valued points of the Picard variety. The latter will be denoted by P_Y, because it is independent of the choice of $\widetilde{Y}$. It is the product of P_{Y_i} for the irreducible components Y_i of Y.

Let Λ be the set of closed subvarieties Y of pure codimension $p-1$ in X. It has a natural ordering by the inclusion relation. For $Y, Y' \in \Lambda$

such that $Y < Y'$, there is a natural morphism of abelian varieties $\lambda_{Y,Y'} : P_Y \to P_{Y'}$. So we get an inductive system of abelian varieties $\{P_Y\}$. We say that an inductive subsystem of closed group subschemes $\{P'_Y\}$ of $\{P_Y\}$ is strict if $\lambda_{Y,Y'}^{-1}(P'_{Y'}) = P'_Y$ for any $Y < Y'$. For example, a regular morphism to an abelian variety [42] defines a strict subsystem by taking the kernel. Let $\{P'_Y\}$ be the minimal strict inductive subsystem of closed group subschemes $\{P_Y\}$ such that $P'_Y(k)$ contains the kernel of $P_Y(k) \to \mathrm{CH}^p_{\mathrm{alg}}(X)$ for any Y. Let $\overline{P}_Y = P_Y/P'_Y$. Then $\{\overline{P}_Y\}$ is an inductive system with injective transition morphisms. Let

$$A^p(X)^{\mathrm{ab}} = \varinjlim \overline{P}_Y(k). \tag{3.11.2}$$

This is called the universal ind-abelian quotient of $\mathrm{CH}^p_{\mathrm{alg}}(X)$. Clearly the ind-object $\{\overline{P}_Y\}$ has the universal property for the regular morphisms to abelian varieties (see [42]). By Murre [35], $A^p(X)^{\mathrm{ab}}$ is an abelian variety if $p = 2$. (Indeed, $\dim \overline{P}_Y$ is bounded by using (19) of loc. cit. and taking an abelian subvariety of P_Y whose intersection with P'_Y is finite.) It coincides with the algebraic part of the intermediate Jacobian if furthermore $k = \mathbb{C}$.

§4. Main Results

In this section X is assumed to be a smooth complex projective variety. The first result is the nontriviality of the refined cycle map restricted to the kernel of the Abel-Jacobi map, which is inspired by Voisin's result [58] (see also [1]):

4.1. Theorem. *The image of* $\mathrm{Gr}^2_{F_L} cl$ *in* (3.6) *for* $p = \dim X$ *is an infinite dimensional* $\mathbb{Q}$*-vector space if* $\Gamma(X, \Omega^2_X) \neq 0$. (*See* [51], 4.4.)

This follows from Bloch's diagonal argument combined with Murre's Chow-Künneth decomposition for the Albanese motive. If $\Gamma(X, \Omega^1_X) = 0$, the latter is not necessary, and the argument is rather simple. It is remarked by the referee that a similar assertion holds for the cycle map to the arithmetic de Rham cohomology $H^{2p}_{\mathrm{DR}}(X/k)$ of X which is isomorphic to the inductive limit of the de Rham cohomology $H^{2p}_{\mathrm{DR}}(X_R/k)$ over the models X_R/R of $X/\mathbb{C}$.

We say that a smooth (resp. smooth projective) k-variety Y is a *k-smooth* (resp. *k-smooth projective*) *model* of a complex algebraic variety X, if Y has a morphism to an integral k-variety whose geometric generic fiber over $\operatorname{Spec} \mathbb{C}$ is isomorphic to X. The main point in the theory of arithmetic mixed sheaves is that the injectivity of the refined cycle map can be reduced to that of the Abel-Jacobi map for smooth projective models over number fields (2.1.3). (This shows that an additional assumption in [1] is unnecessary.)

4.2. Theorem. *Assume k is a number field. Then the cycle map* $\mathrm{CH}^p(X)_{\mathbb{Q}} \to H_{\mathcal{D}}^{2p}(X, \mathbb{Q}_{\langle k \rangle}(p))$ *in* (3.4.1) *is injective if the above Abel-Jacobi map* (2.1.3) *over k_Y is injective for any k-smooth models Y of X, where k_Y is the algebraic closure of k in $\Gamma(Y, \mathcal{O}_Y)$. In the case $p = 2$, the last assumption is reduced to the same injectivity for any k-smooth projective models Y of X.* (*See* [51], 4.6.)

We can describe the image of the usual cycle map to Deligne cohomology assuming the Hodge conjecture, if $H^{2p-1}(X, \mathbb{Q}_{\langle k \rangle})$ is *global section-free* in the sense that the local system on $S_{\mathbb{C}}$ underlying the representative of $H^{2p-1}(X, \mathbb{Q}_{\langle k \rangle})$ on any $S = \operatorname{Spec} R$ has no nontrivial global sections.

4.3. Proposition. *If $H^{2p-1}(X, \mathbb{Q}_{\langle k \rangle})$ is global section-free in the above sense, and the Hodge conjecture for codimension p cycles holds for any k-smooth projective models of X, then the image of the usual cycle map to Deligne cohomology*

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H_{\mathcal{D}}^{2p}(X, \mathbb{Q}(p)) \tag{4.3.1}$$

coincides with $\mathrm{Im}\big(H_{\mathcal{D}}^{2p}(X, \mathbb{Q}_{\langle k \rangle}(p)) \to H_{\mathcal{D}}^{2p}(X, \mathbb{Q}(p))\big)$, *and similarly for the Abel-Jacobi map with Deligne cohomology replaced by the intermediate Jacobian.* (*See* [51], 4.1.)

We can show some assertions for higher Chow groups corresponding to (4.1–2). Recall that the coniveau filtration $N^pH^i(X, \mathbb{Q})$ for a smooth proper variety X is defined to be the kernel of $H^i(X, \mathbb{Q}) \to H^i(U, \mathbb{Q})$ for a sufficiently small open subvariety U of X such that $\dim X \setminus U \le p$.

4.4. Theorem. *Assume $N^{p-2}H^{2p-3}(X, \mathbb{Q}) \ne 0$. Then for any positive integer m, there exist $\zeta_i \in \mathrm{CH}_{\mathrm{hom}}^{p-1}(X)_{\mathbb{Q}}$ for $1 \le i \le m$ together with a finitely generated subfield K of $\mathbb{C}$ such that for any complex numbers $\alpha_1, \dots, \alpha_m$ not algebraic over K, the images of $\zeta_i \otimes \alpha_i$ for $1 \le i \le m$ by the composition of* (3.5.1) *and* (3.4.5) *are linearly independent over $\mathbb{Q}$. See* [51], 5.2.

4.5. Theorem. *Assume k is a number field. Then the cycle map* (3.4.5) *for $p = 2, m = 1$ is injective if the generalized Abel-Jacobi map* (2.1.5) *for the same p, m over k_Y is injective for any k-smooth projective models Y of X, where k_Y is as in* (4.2). (*See* [51], 5.3.)

Related to the countability of the indecomposable higher Chow group $\mathrm{CH}_{\mathrm{ind}}^2(X, 1)_{\mathbb{Q}}$ in (3.5), we have

4.6. Theorem. *If the cycle map* (3.4.5) *for* X *is injective for* $p = 2, m = 1$, *then it induces an injective morphism* (*see* [51], 5.7) :
(4.6.1)
$$\mathrm{CH}^2_{\mathrm{ind}}(X,1)_{\mathbb{Q}} \to \mathrm{Ext}^1\big(\mathbb{Q}_{\langle k\rangle}, H^2(X,\mathbb{Q}_{\langle k\rangle}(2))/N^1H^2(X,\mathbb{Q}_{\langle k\rangle}(2))\big).$$

For the image of (4.6.1), we can show

4.7. Proposition. *Assume* k *is a number field. Then* (4.6.1) *is surjective if* $H^2(X,\mathbb{Q}_{\langle k\rangle})/\ N^1H^2(X,\mathbb{Q}_{\langle k\rangle})$ *is global section-free as in* (4.3) *and if the Abel-Jacobi map* (2.1.3) *is injective for codimension* 2 *cycles on any* k*-smooth projective models of* X. (*See* [51], 5.11.)

4.8. Remark. By (4.6), Voisin's conjecture on the countability of $\mathrm{CH}^2_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$ (see [57]) can be reduced to the injectivity of (2.1.5) (i.e. to the hypothesis of (4.5)), because we have an analogue of the rigidity argument due to Beilinson [3] and Müller-Stach [34] as follows:

4.9. Proposition. *For a smooth complex projective variety* X, *let* $\widetilde{N}^iH^{2i}(X,\mathbb{Q}_{\langle k\rangle})$ *be the maximal subobject of* $H^{2i}(X,\mathbb{Q}_{\langle k\rangle})$ *which is isomorphic to a direct sum of copies of* $\mathbb{Q}_{\langle k\rangle}(-i)$. *Then the image of the morphism*
(4.9.1)
$$\mathrm{CH}^p(X,1)_{\mathbb{Q}} \to \mathrm{Ext}^1(\mathbb{Q}_{\langle k\rangle}, H^{2p-2}(X,\mathbb{Q}_{\langle k\rangle}(p))/\widetilde{N}^{p-1}H^{2p-2}(X,\mathbb{Q}_{\langle k\rangle}(p)))$$

induced by the cycle map is countable. (*See* [51], 5.9.)

4.10. Remark. We have the reduced higher Abel-Jacobi map

$$\mathrm{CH}^p_{\mathrm{ind}}(X,1)_{\mathbb{Q}} \to \mathrm{Ext}^1\big(\mathbb{Q}, (H^{2p-2}(X,\mathbb{Q})/\mathrm{Hdg}^{p-1}(X))(p)\big) \tag{4.10.1}$$

in the usual sense, where $\mathrm{Hdg}^{p-1}(X)$ denotes the group of Hodge cycles. This is an analogue of (4.6.1). By A. Beilinson [3] and M. Levine [33], it can be described quite explicitly by using currents in a similar way to Griffiths' Abel-Jacobi map (see also [24], [34]). This is generalized to the nonproper smooth case [50]. It is not easy to construct nontrivial indecomposable higher cycles (see [13], [14], [15], [24], [34], [57], etc.) nor nontrivial elements in the image of (4.6.1). An example of a nontrivial higher cycle whose support is the one point compactification of $\mathbb{C}^*$ is given in [50]. This is also an example such that its image by (4.10.1) is not contained in the image of $F^1H^2(X,\mathbb{C})$ (see [15] for another example). It was originally considered in order to find an indecomposable higher cycle on a self-product of an elliptic curve of CM type, where the cycle map to real Deligne cohomology used in [24] does not work.

By [38], [40], the kernel of (4.10.1) is isomorphic to

$$\mathrm{Coker}(K_2(\mathbb{C}(X))_{\mathbb{Q}} \to \varinjlim \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^2(U,\mathbb{Q})(2))), \tag{4.10.2}$$

where the morphism is given by $d\log\wedge d\log$ at the level of integral logarithmic forms, and the inductive limit is taken over nonempty open subvarieties U of X. This isomorphism follows easily from the localization sequence of mixed Hodge structures together with the fact that the residue of $d\log f \wedge d\log g$ coincides with the logarithmic differential of the tame symbol of $\{f, g\}$ up to sign. It is conjectured by Beilinson that (4.10.2) should vanish.

§5. Examples

By (4.2) and (4.5), some major problems are reduced to the injectivity of the generalized Abel-Jacobi maps (2.1.3) and (2.1.5), and this is the crucial point in the theory of arithmetic mixed sheaves. However, it is not easy to verify, for example, the injectivity of (2.1.3) even for a surface X unless $p_g(X) = 0$. The problem seems to be of arithmetic nature, because this injectivity does not hold unless the ground field k is a number field. We try to illustrate the difficulty of the problem in the following example.

5.1. Product of elliptic curves. Let E_i be an elliptic curve with the origin O_i defined over a number field k for $i = 1, 2$. Let $X = E_1 \times E_2$ where the subscript k is omitted to simplify the notation because the base change by $k \to \mathbb{C}$ is not used here. The choice of the origin gives a double cover $E_i \to \mathbb{P}^1$ ramified over four points $\Sigma_i \subset \mathbb{P}^1$, which contain the image of the origin.

Let P_i be k-valued point of E_i, and $\zeta_i = [P_i] - [-P_i]$ for $i = 1, 2$. Put $\zeta = \zeta_1 \times \zeta_2$. Then it belongs to the kernel of the Abel-Jacobi map (i.e. of the Albanese map in this case). So we have to show that a multiple of ζ is rationally equivalent to zero. Consider the involution σ of X defined by $x \to -x$. Then ζ is invariant by this involution, and is identified with a cycle of $X' = X/\sigma$. This X' is a double covering of $S := \mathbb{P}^1 \times \mathbb{P}^1$ ramified over the divisor $D = \Sigma_1 \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \Sigma_2$, and has 16 ordinary double points as well-known. Let τ be the involution associated with the double covering. Then ζ is a τ-anti-invariant cycle, i.e. $\tau^*\zeta = -\zeta$. (Note that any cycle with rational coefficients on X' coincides with a τ-anti-invariant cycle modulo τ-invariant cycles, and τ-invariant cycles are essentially trivial modulo rational equivalence.)

If we consider a curve which is invariant by τ, the description of a rational function on it is complicated. So we try to find a curve C on S together with a rational function g on S such that C can be lifted to a curve C' on X' which is birational to C, and the divisor of the pull-back g' of g to C' coincides with a multiple of ζ in X'. Since S is a self-product of $\mathbb{P}^1$, a curve on S is described explicitly by using an equation. However, it is not easy to express the condition on birational lifting of the curve. Let x, y be affine coordinates of $\mathbb{A}^2$, which is the complement of two irreducible components of D in $\mathbb{P}^1 \times \mathbb{P}^1$. For $i = 1, 2$, let f_i be the defining equation of $\Sigma_i \cap \mathbb{A}^1$ which is a polynomial of degree 3. Then the restriction of X' over $\mathbb{A}^2$ is given by $z^2 = f_1(x)f_2(y)$, and an example of a liftable curve by $g_1(x,y)^2 = f_1(x)f_2(y)g_2(x,y)^2$, where $g_1(x,y), g_2(x,y)$ are polynomials with no common factors. However, how to choose a rational function on C is still a nontrivial problem. If we take $x - c$ for $c \in k$ such that $\{x = c\}$ is not contained in Σ_1, then we see that $[Q] - [O]$ and hence $[Q] - [\tau Q]$ are rationally equivalent to zero, where Q is the sum of the points (counted with multiplicity) in the intersection of C' with the elliptic curve $\{c\} \times E_2$ (the sum is taken in the elliptic curve E_2), and O is any of the points in the inverse image of D which are rationally equivalent to each other. This Q depends only on the bidegrees of g_1, g_2, because the parameter spaces are rational. In particular, we get only a countable number of Q for each c. It is unclear whether Q is a nontorsion point of E_2.

5.2. Higher cycles on an elliptic curve. Since the higher extension groups should vanish in the category of (conjectural) mixed motives over a number field, it is expected that a higher cycle of the form $\zeta \otimes \alpha$ for $\zeta \in \mathrm{CH}^1_{\mathrm{hom}}(X)$ and $\alpha \in k^*$ (see (3.5)) vanishes in $\mathrm{CH}^2(X,1)_{\mathbb{Q}}$ if X is smooth proper over a number field k (see [8], [39]).

In the case of an elliptic curve E with origin O defined over k, take $P \in E(k)$, and let $P_m = mP \in E(k)$ for $m = 0, 1, 2, 3$. Then we have a rational function f on E such that $\operatorname{div} f = 2[P_1] - [P_0] - [P_2]$. Let T_Q denote the translation by $Q \in E(k)$, and $g = T^*_{-P} f$. Considering the tame symbol of $\{cf^3, cg^3\}$ for an appropriate $c \in k^*$, we see that $([P_3] - [P_0]) \otimes \alpha$ vanishes in $\mathrm{CH}^2(X,1)$ for some $\alpha \in k^*$. We can verify that $\alpha \neq 1$ for a general P as follows.

We take the Weierstrass equation $y^2 = x^3 + Ax + B$ so that the origin of E is the point at infinity. Then f is given by the function $T^*_{-P}(x - a)$ where $a = x(P)$ (the value of x at P). Consider the function $(x - a)^2 T^*_{-P} x$. This can be extended to a function on a neighborhood of P. We denote its value at P by $h(P)$. Then $h(P) = h(-P)$ because $x(P) = x(-P)$ and $h(P) = \lim_{Q \to P}(x(Q) - x(P))^2 x(Q - P)$. Furthermore, the above c is given by $h(P)^{-1}$, and α by $x(2P)^9 h(P)^{-3}$. It is easy to see that the last function of P goes to the infinity when P approaches to a point P' such that $2P' = O$. (As remarked by the referee, the above $h(P)$ coincides with $y(P)^2$ by using the Weierstrass $\mathfrak{p}$-function. He also notes that we get a similar result by calculating simply the tame symbol of $\{f,g\}$ and using the well-definedness of $\mathrm{CH}^1(X)\otimes\mathbb{C}\to\mathrm{CH}^2(X,1)$, because the tame symbol is bilinear.)

With the notation of (5.1), a cycle ζ of the form $\zeta_1\times\zeta_2$ with $\zeta_i\in\mathrm{CH}^1(E_i)$ is called decomposable. Assume that the ζ_i are homologically equivalent to zero. Then ζ should be rationally equivalent to zero if the ground field k is a number field. But the situation is rather complicated in general. We explain here a special case where the cycle is detected by the refined cycle map.

5.3. Strictly decomposable cycles. Let X_1 and X_2 be smooth complex projective varieties defined over a subfield k of $\mathbb{C}$, i.e. there exist smooth projective k-varieties $X_{i,k}$ such that $X_i=X_{i,k}\otimes_k\mathbb{C}$. Set $X=X_1\times X_2, X_k=X_{1,k}\times_k X_{2,k}$. We say that a cycle ζ on X is strictly decomposable if there exist subfields K_i of $\mathbb{C}$ finitely generated over k, together with cycles ζ_i on $X_{i,K_i}:=X_{i,k}\otimes_k K_i$ for $i=1,2$ such that the algebraic closure k' of k in K_i is independent of i, the canonical morphism $K_1\otimes_{k'}K_2\to\mathbb{C}$ is injective, and ζ coincides with the base change of the cycle $\zeta_1\times_{k'}\zeta_2$ on $X_{1,K_1}\times_{k'}X_{2,K_2}=X_{k'}\otimes_{k'}(K_1\otimes_{k'}K_2)$ by $K_1\otimes_{k'}K_2\to\mathbb{C}$. Here we may assume $k'=k$, replacing k if necessary. We say that a strictly decomposable cycle is of bicodimension (p_1,p_2) if $\operatorname{codim}\zeta_i=p_i$. Put $p=p_1+p_2$.

Let R_i be a finitely generated smooth k-subalgebra of K_i such that the fraction field is K_i, and ζ_i is defined over R_i. Set $S_i=\operatorname{Spec}R_i\otimes_k\mathbb{C}$. Let $\xi_i^j\in H^{2p_i-j}(X_i,\mathbb{Q})\otimes H^j(S_i,\mathbb{Q})(p_i)$ be the Künneth components of the cycle class of $\zeta_i\otimes_k\mathbb{C}$ in $H^{2p_i}(X_i\times S_i,\mathbb{Q}(p_i))$. Let $\widetilde{M_i}$ be the pull-back of

$$M_i:=H^{2p_i-1}(X_i,\mathbb{Q})(p_i)$$

by $a_{S_i}:S_i\to\operatorname{Spec}\mathbb{C}$. If $\xi_i^0=0$, then $\zeta_i\in F_L^1\mathrm{CH}^{p_i}(X_{i,k}\otimes_k K_i)_{\mathbb{Q}}$, and $\mathrm{Gr}_{F_L}^1 cl_{R_i}(\zeta_i)$ (see (3.4.2)) gives

$$\widetilde{\xi_i^1}\in\operatorname{Ext}^1(\mathbb{Q}_{S_i},\widetilde{M_i}).$$

By the Leray spectral sequence, we have an exact sequence

$$0\to\operatorname{Ext}^1(\mathbb{Q},M_i)\to\operatorname{Ext}^1(\mathbb{Q}_{S_i},\widetilde{M_i})\to\operatorname{Hom}(\mathbb{Q},H^1(S_i,\mathbb{Q})\otimes M_i),$$

where the first morphism is given by the pull-back by a_{S_i}. Note that ξ_i^1 coincides with the image of $\widetilde{\xi_i^1}$ in the last term. If $\xi_i^1=0$, then $\widetilde{\xi_i^1}$ comes from $\operatorname{Ext}^1(\mathbb{Q},M_i)$, and we may assume $R_i=k$ as long as $\mathrm{Gr}_{F_L}^1 cl_{R_i}(\zeta_i)$ is concerned (e.g. if $p_i=1$).

We say that a strictly decomposable cycle ζ is *degenerate*, if either $\xi_i^0=\xi_i^1=0$ for both i, or $\xi_i^0=\widetilde{\xi_i^1}=0$ for some i. In the case $p_i=1$, it is expected that a degenerate ζ is rationally equivalent to zero by Beilinson's conjecture (2.3) when k is a number field. However, if ζ is nondegenerate, we can detect it by the refined cycle map as follows. (This is by joint work with A. Rosenschon [41].)

5.4. Theorem. *With the above notation, let ζ be a strictly decomposable cycle of codimension p. Then $cl(\zeta) \neq 0$ in the notation of* (3.6). *More precisely, $\zeta \in F_L^r\mathrm{CH}^p(X)_{\mathbb{Q}}$ and $\mathrm{Gr}^r_{F_L} cl(\zeta) \neq 0$ if the number of the i such that $\xi_i^0 = 0$ is r.*

5.5. Remark. The assertion (5.4) was first considered in the case both ξ_1^1 and ξ_2^1 are nonzero, in order to show the nonvanishing of the composition of certain extension classes (see [49]). A. Rosenschon studied Nori's construction of a cycle [53] in the case of a self-product of an elliptic curve without complex multiplication, and obtained a special case of (5.4). Then these two were generalized to (5.4), see [41]. This can be extended to the higher cycle case (loc. cit.) As an application, we can show that $\mathrm{CH}^{p+1}_{\mathrm{ind}}(X_1 \times X_2, 1)_{\mathbb{Q}}$ is uncountable if $\Gamma(X_1, \Omega^1_{X_1}) \neq 0$ and the reduced higher Abel-Jacobi map (see (4.10.1)) for X_2 is not zero. This is a generalization of a result of Gordon and Lewis [24] (see also [14]).

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