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The Capitulation Problem for certain Number Fields

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§1. Abstract

We study the capitulation problem for certain number fields of degree 3, 4, and 6.

(I) Capitulation of the 2-ideal classes of $\mathbb{Q}(\sqrt{d}, i)$ (by A. AZIZI)

Let $d \in \mathbb{N}$, $i = \sqrt{-1}$, $\mathbf{k} = \mathbb{Q}(\sqrt{d}, i)$, $\mathbf{k}_1^{(2)}$ be the Hilbert 2-class field of $\mathbf{k}$, $\mathbf{k}_2^{(2)}$ be the Hilbert 2-class field of $\mathbf{k}_1^{(2)}$, $C_{\mathbf{k},2}$ be the 2-component of the ideal class group of $\mathbf{k}$ and G_2 the Galois group of $\mathbf{k}_2^{(2)}/\mathbf{k}$. We suppose that $C_{\mathbf{k},2}$ is of type $(2, 2)$; then $\mathbf{k}_1^{(2)}$ contains three extensions $F_i/\mathbf{k}$, $i = 1, 2, 3$. The aim of this section is to study the capitulation of the 2-ideal classes in F_i, $i = 1, 2, 3$, and to determine the structure of G_2.

(II) On the capitulation of the 3-ideal classes of a cubic cyclic field (by M. AYADI)

Let k be a cubic cyclic field over $\mathbb{Q}$, and $\mathbf{k}_1^{(3)}$ the Hilbert 3-class field of $\mathbf{k}$. If the class number of $\mathbf{k}$ is exactly divisible by 9, then its 3-ideal class group is of type $(3, 3)$, and $\mathbf{k}_1^{(3)}$ contains four cubic extensions $\mathbf{K}_i/\mathbf{k}$ in which we study the capitulation problem for the 3-ideal classes of $\mathbf{k}$.

(III) On the capitulation of the 3-ideal classes of the normal closure of a pure cubic field (by M. C. ISMAILI)

Let $\Gamma = \mathbb{Q}(\sqrt[3]{n})$ be a pure cubic field, $\mathbf{k} = \mathbb{Q}(\sqrt[3]{n}, j)$ its normal closure ($j = e^{\frac{2i\pi}{3}}$), $\mathbf{k}_1^{(3)}$ the Hilbert 3-class field of $\mathbf{k}$, and let $S_{\mathbf{k}}$ be the 3-ideal class group of $\mathbf{k}$. When $S_{\mathbf{k}}$ is of type $(3, 3)$, we study the

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capitulation of the 3-ideal classes of $S_{\mathbf{k}}$ in the four intermediate extensions of $\mathbf{k}_1^{(3)}/\mathbf{k}$, and we show that if the class number of Γ is divisible by 9, then we have some necessary conditions on n. We have also some informations about the unit group of $\mathbf{k}$ in some cases.

§2. Intoduction

Let $\mathbf{k}$ be a number field of finite degree over $\mathbb{Q}$ and $C_{\mathbf{k}}$ be the class group of $\mathbf{k}$. Let $\mathbf{F}$ be an unramified extension of $\mathbf{k}$ of finite degree and let $O_{\mathbf{F}}$ be its ring of integers. We say that an ideal $\mathcal{A}$ (or the ideal class of $\mathcal{A}$) of $\mathbf{k}$ capitulates in $\mathbf{F}$ if it becomes principal in $\mathbf{F}$, i.e., if $\mathcal{A}O_{\mathbf{F}}$ is principal in $\mathbf{F}$. The Hilbert class field $\mathbf{k}_1$ of $\mathbf{k}$ is the maximal abelian unramified extension of $\mathbf{k}$. Let p be a prime number; the Hilbert p-class field $\mathbf{k}_1^{(p)}$ of $\mathbf{k}$ is the maximal abelian unramified extension of $\mathbf{k}$ such that $[\mathbf{k}_1^{(p)} : \mathbf{k}] = p^n$ for some integer n. The first important result on capitulation was conjectured by D. Hilbert and proved by E. Artin and P. Furtwängler. It deals with the case $\mathbf{F} = \mathbf{k}_1$.

Theorem 2.1 (Principal ideal theorem). *Let $\mathbf{k}_1$ be the Hilbert class field of $\mathbf{k}$. Then every ideal of $\mathbf{k}$ capitulates in $\mathbf{k}_1$.*

The principal ideal theorem was generalized by Tannaka and Terada to the next one. Let $\mathbf{k}_0$ be a subfield of $\mathbf{k}$ such that $\mathbf{k}/\mathbf{k}_0$ is abelian and let $(\mathbf{k}/\mathbf{k}_0)^*$ be the relative genus field of $\mathbf{k}/\mathbf{k}_0$.

Theorem 2.2 (Tannaka–Terada). *If $\mathbf{k}/\mathbf{k}_0$ is cyclic, then every ambiguous ideal class of $\mathbf{k}/\mathbf{k}_0$ is principal in $(\mathbf{k}/\mathbf{k}_0)^*$.*

The case where $\mathbf{F}/\mathbf{k}$ is a cyclic extension of prime degree was studied by D. Hilbert in his Theorem 94:

Theorem 2.3 (Theorem 94). *Let $\mathbf{F}/\mathbf{k}$ be a cyclic extension of prime degree. Then there exists at least one class (not trivial) in $\mathbf{k}$ which capitulates in $\mathbf{F}$.*

We find in the proof of Theorem 94 this result:
Let σ be a generator of the Galois group of $\mathbf{F}/\mathbf{k}$ and $N_{\mathbf{F}/\mathbf{k}}$ be the norm of $\mathbf{F}/\mathbf{k}$. Let $E_{\mathbf{L}}$ be the unit group of the field $\mathbf{L}$. Let $E_{\mathbf{F}}^*$ be the group of units of norm 1 in $\mathbf{F}/\mathbf{k}$. Then the group of classes of $\mathbf{k}$ which capitulates in $\mathbf{F}$ is isomorphic to the quotient group $E_{\mathbf{F}}^*/E_{\mathbf{F}}^{1-\sigma} = H^1(E_{\mathbf{F}})$, the cohomology group of $G = \langle\sigma\rangle$ acting on the group $E_{\mathbf{F}}$.

With this result and other results on cohomology, we have:

Theorem 2.4. *Let $\mathbf{F}/\mathbf{k}$ be a cyclic extension of prime degree. Then the number of classes which capitulate in $\mathbf{F}/\mathbf{k}$ is equal to $[\mathbf{F} : \mathbf{k}][E_{\mathbf{k}} : N_{\mathbf{F}/\mathbf{k}}(E_{\mathbf{F}})]$.*

The case where $\mathbf{F}/\mathbf{k}$ is an abelian extension was treated by H. Suzuki who has proved Miyake's conjecture: *In an abelian extension $\mathbf{F}/\mathbf{k}$ the number of classes of $\mathbf{k}$ which capitulate in $\mathbf{F}$ is a multiple of $[\mathbf{F} : \mathbf{k}]$.*

Let p be a prime number and let $\mathbf{k}_1^{(p)}$ (resp. $\mathbf{k}_2^{(p)}$) be the Hilbert p-class field of $\mathbf{k}$ (resp. of $\mathbf{k}_1^{(p)}$). If $\mathbf{L}$ is a subfield of $\mathbf{k}_1$ and $\mathcal{A}$ is an ideal class of $\mathbf{k}$ whose order is equal to p^m for some integer m. Then $\mathcal{A}$ capitulates in $\mathbf{L}$ if and only if $\mathcal{A}$ capitulates in $\mathbf{L} \cap \mathbf{k}_1^{(p)}$. So we study only the capitulation of classes whose order is equal to p^m in the subfields of $\mathbf{k}_1^{(p)}$, and since the capitulation problem is solved when $\mathbf{k}_1^{(p)}/\mathbf{k}$ is cyclic, we study only the cases where $\mathbf{k}_1^{(p)}/\mathbf{k}$ is not cyclic.

For more details see [Mi - 89], [Su - 91], [CF - 91], [Ism - 92], [Az - 93], [Ay - 95] and [Az - 97].

§3. Capitulation of the 2-ideal classes of some biquadratic fields

Let $\mathbf{k}$ be a number field such that the 2-component $C_{\mathbf{k},2}$ of $C_{\mathbf{k}}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let G_2 be the Galois group of $\mathbf{k}_2^{(2)}/\mathbf{k}$. By class field theory, $Gal(\mathbf{k}_1^{(2)}/\mathbf{k}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then $\mathbf{k}_1^{(2)}$ contains three quadratic extensions of $\mathbf{k}$ denoted by $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$. Under these conditions, Kisilevsky [Ki-76] proved the following.

Theorem 3.1. *Let $\mathbf{k}$ be such that $C_{\mathbf{k},2} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then we have three types of capitulation:*

Type 1: *The four classes of $C_{\mathbf{k},2}$ capitulate in each extension $\mathbf{F}_i$, $i = 1, 2, 3$. This is possible if and only if $\mathbf{k}_1^{(2)} = \mathbf{k}_2^{(2)}$.*

Type 2: *The four classes of $C_{\mathbf{k},2}$ capitulate only in one extension among the three extensions $\mathbf{F}_i$, $i = 1, 2, 3$. In this case the group G_2 is dihedral.*

Type 3: *Only two classes capitulate in each extension $\mathbf{F}_i$, $i = 1, 2, 3$. In this case the group G_2 is semi-dihedral or quaternionic.*

In this section, we suppose that $\mathbf{k} = \mathbb{Q}(\sqrt{d}, i)$ where $d \in \mathbb{N}$ is such that $C_{\mathbf{k},2} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and we study the capitulation problem in the extensions $\mathbf{F}_i/\mathbf{k}$, $i = 1, 2, 3$.

$$\begin{tikzcd}
& \mathbf{k}_2^{(2)} \arrow[d, dash] & \\
& \mathbf{k}_1^{(2)} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbf{F}_1 \arrow[dr, dash] & \mathbf{F}_2 \arrow[d, dash] & \mathbf{F}_3 \arrow[dl, dash] \\
& \mathbf{k} \arrow[d, dash] & \\
& \mathbb{Q} &
\end{tikzcd}$$

Diagram 1

The first step is to study the structure of $C_{\mathbf{k},2}$. Using genus theory, the class number formula for biquadratic fields, Kaplan's results on the 2-part of the class number for quadratic number fields and other results, we can prove

Theorem 3.2. *Let Q be the Hasse unit index of $\mathbf{k}$ and let $C_{\mathbf{k},2}$ be the 2-component of the class group of $\mathbf{k}$. Let $\mathbf{k}^{(*)}$ be the genus field of $\mathbf{k}$. Then the group $C_{\mathbf{k},2}$ is of type $(2,2)$ if and only if one of the next cases occurs:*
(1) $d = 2pq$, $p \equiv -q \equiv 1 \pmod 4$, *at least two of the three symbols* $\left(\frac{p}{q}\right), \left(\frac{2}{p}\right), \left(\frac{2}{q}\right)$ *are* -1 *and* Q *is* 1, *in which case,* $\mathbf{k}^{(*)} = \mathbf{k}_1^{(2)} = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{2}, i)$;
(2) $d = 2q_1q_2$, $q_1 \equiv q_2 \equiv -1 \pmod 4$, $\left(\frac{q_1}{q_2}\right) = -\left(\frac{q_2}{q_1}\right) = 1$, $\left(\frac{2}{q_1}\right) = -\left(\frac{2}{q_2}\right) = 1$ *and* $Q = 1$, *in which case,* $\mathbf{k}^{(*)} = \mathbf{k}_1^{(2)} = \mathbb{Q}(\sqrt{q_1}, \sqrt{q_2}, \sqrt{2}, i)$;
(3) $d = p_1p_2$, $p_1 \equiv 1 \pmod 8$, $p_2 \equiv 5 \pmod 8$ *and* $\left(\frac{p_1}{p_2}\right) = -1$, *in which case,* $\mathbf{k}^{(*)} = \mathbf{k}(\sqrt{p_1}) \neq \mathbf{k}_1^{(2)}$;
(4) $d = pq$, $p \equiv 1 \pmod 8$, $q \equiv -1 \pmod 4$, $\left(\frac{p}{q}\right) = -1$ *and* $Q = 2$, *in which case,* $\mathbf{k}^{(*)} = \mathbf{k}(\sqrt{p}) \neq \mathbf{k}_1^{(2)}$.

Remarks 3.1. If $\mathbf{k}^{(*)} \neq \mathbf{k}_1^{(2)}$, we set $\mathbf{F}_1 = \mathbf{k}^{(*)} = \mathbf{k}(\sqrt{p})$ where $p \equiv 1 \pmod 8$, $\mathbf{F}_2 = \mathbf{k}(\sqrt{a+bi})$ and $\mathbf{F}_3 = \mathbf{k}(\sqrt{a-bi})$ where a and b are two integers such that $p = a^2 + b^2$, $a \equiv 1 \pmod 4$ and $b \equiv 0 \pmod 4$.

In order to determine the number of ideal classes which capitulate in $\mathbf{F}_i/\mathbf{k}$, $i = 1, 2, 3$, we have to determine the unit group of each $\mathbf{F}_i$, $i = 1, 2, 3$, where $\mathbf{F}_i$ is a composite of three quadratic fields. So using the previous results and others, we obtain the next solution of the capitulation problem.

Theorem 3.3. *Let $C_{\mathbf{F}_i,2}$ be the 2-component of the class group of $\mathbf{F}_i$ and let $j_i : C_{\mathbf{k},2} \longrightarrow C_{\mathbf{F}_i,2}$ be the canonical homomorphism.*
(1) *If $\mathbf{k}^{(*)} = \mathbf{k}_1^{(2)}$, then $\mathbf{k}_1^{(2)} \neq \mathbf{k}_2^{(2)}$, $G_2 \simeq Q_m$ or $S_m\,(m > 3)$ and $|ker\, j_i| = 2$ for $i = 1,2,3$* (capitulation type 3), *where Q_m and S_m are respectively the group of quaternions and the semi-dihedral group of order 2^m.*
(2) *Let $\mathbf{k}^{(*)} \neq \mathbf{k}_1^{(2)}$. Then $|ker\, j_1| = 4$. Moreover,*
(a) *If d is divisible by a prime $q \equiv -1 \pmod 4$ and $p \neq x^2 + 32y^2$, then $\mathbf{k}_2^{(2)} = \mathbf{k}_1^{(2)}$, $G_2 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $|ker\, j_i| = 4$ for $i = 1,2,3$* (capitulation type 1);
(b) *If d is not divisible by a prime $q \equiv -1 \pmod 4$ or if $p = x^2 + 32y^2$, then $\mathbf{k}_2^{(2)} \neq \mathbf{k}_1^{(2)}$, $G_2 \simeq D_m (m \geq 3)$, the dihedral group of order 2^m, and $|ker\, j_i| = 2$ for $i = 2,3$* (capitulation type 2).

For more details see [Az - 93] and [Az - 97].

Numerical Examples.

Values of d	Capitulation types
$17 \cdot 7,\ 17 \cdot 5,\ 73 \cdot 7$	type 1
$41 \cdot 13,\ 41 \cdot 7$	type 2
$2 \cdot 3 \cdot 7,\ 2 \cdot 3 \cdot 5,\ 2 \cdot 5 \cdot 7,\ 2 \cdot 13 \cdot 3,\ 2 \cdot 5 \cdot 11$	type 3

Table 1

§4. On the capitulation of the 3-ideal classes of a cubic cyclic field

Let $\mathbf{k}$ be a cubic cyclic field over $\mathbb{Q}$ whose class number is exactly divisible by 9. Let $\mathbf{k}_1^{(3)}$ be its Hilbert 3-class field and let $\mathbf{k}^{(*)}$ be its absolute genus field. Then the 3-ideal class group of $\mathbf{k}$ is of type $(3,3)$, and $\mathbf{k}_1^{(3)}/\mathbf{k}$ contains four subfields $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ and $\mathbf{K}_4$. We want to study the capitulation problem of the 3-ideal classe of $\mathbf{k}$.

For the details of all the proofs and results given in this section see [Ay-95].

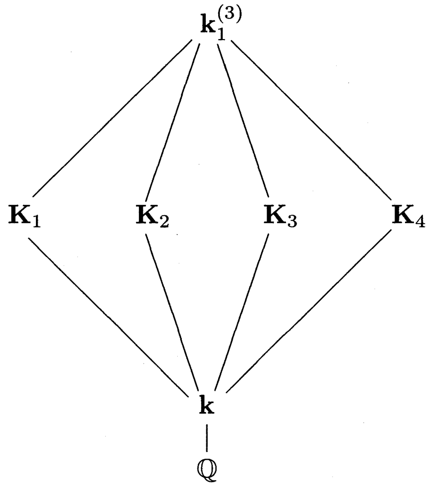

Diagram 2

We have to distinguish two cases.

First case: $[\mathbf{k}^{(*)} : \mathbf{k}] = 3$. It turns out that this is equivalent to each of the following conditions:
- $Gal(\mathbf{k}_1^{(3)}/\mathbb{Q})$ is not abelian;
- $\mathbf{k}^{(*)} = \mathbf{K}_i$ for some $i \in \{1, 2, 3, 4\}$;
- Exactly two distinct prime numbers p and q are ramified in $\mathbf{k}$.

Second case: $[\mathbf{k}^{(*)} : \mathbf{k}] = 9$. This is equivalent to each of the following conditions:
- $Gal(\mathbf{k}_1^{(3)}/\mathbb{Q})$ is abelian;
- $\mathbf{k}^{(*)} = \mathbf{k}_1^{(3)}$;
- Exactly three distinct prime numbers p, q and r are ramified in $\mathbf{k}$.

(A) Case where $[\mathbf{k}^{(*)} : \mathbf{k}] = 3$

In this case, exactly two prime numbers p and q are ramified in $\mathbf{k}$, and there exists another unique cubic cyclic field denoted by $\tilde{\mathbf{k}}$ having the same conductor as $\mathbf{k}$. Denote by $h_{\mathbf{k}}$ (resp. by $h_{\tilde{\mathbf{k}}}$) the class number of $\mathbf{k}$ (resp. of $\tilde{\mathbf{k}}$).

Theorem 4.1. *Let* $\mathbf{k}$ *be a cubic cyclic field of conductor divisible only by* p *and* q. *Then*

$$9 || h_{\mathbf{k}} \Leftrightarrow 9 || h_{\tilde{\mathbf{k}}}.$$

If $9 || h_{\mathbf{k}}$, *then* $\mathbf{k}$ *and* $\tilde{\mathbf{k}}$ *have the same Hilbert* 3*-class field.*

Let σ be a generator of $Gal(\mathbf{k}/\mathbb{Q})$ and let $\delta = \sigma - 1$. From class field theory we know that $\tilde{\mathbf{k}}_1^{(3)}$ corresponds to S^{δ^2} and that S^{δ^2} is trivial, where $S = Gal(\mathbf{k}_1^{(3)}/\mathbf{k})$. Moreover the group of ambiguous classes is of order 3, and generated by the classes $[\mathcal{P}]$ and $[\mathcal{Q}]$, where $\mathcal{P}$ and $\mathcal{Q}$ are the prime ideals of $\mathbf{k}$ lying above p and q. We have of course $[\mathcal{P}]^n[\mathcal{Q}]^m = 1$ for some $n, m \in \{0, 1, 2\}$ and $(n, m) \neq (0, 0)$; the nontrivial relation $[\mathcal{P}]^n[\mathcal{Q}]^m = 1$ is obtained by calculating a constant of Parry denoted $b_{\mathbf{k}}$. Here $b_{\mathbf{k}} = p^n q^m$ is caculated from a fundamental unit of $\mathbf{k}$ (generating over $\mathbb{Z}[\sigma]$ the unit group of $\mathbf{k}$) and its irreducible polynomial (see [Pa-90]).

Theorem 4.2. *Let $\mathcal{P}$, $\mathcal{Q}$ (resp. $\tilde{\mathcal{P}}$, $\tilde{\mathcal{Q}}$) be the prime ideals of $\mathbf{k}$ (resp. of $\tilde{\mathbf{k}}$) lying above p and q. Then the following assertions are true:*
(1) $\forall n, m \in \mathbb{N}$; $[\mathcal{P}]^n[\mathcal{Q}]^m = 1 \Leftrightarrow [\tilde{\mathcal{P}}]^n[\tilde{\mathcal{Q}}]^m = 1$.
(2) $[\mathcal{P}] = 1$ or $[\mathcal{Q}] = 1 \Leftrightarrow 9||h_{\mathbf{k}^{(*)}}$. Saying this, is equivalent to: $[\mathcal{P}] \neq 1$ and $[\mathcal{Q}] \neq 1 \Leftrightarrow 3||h_{\mathbf{k}^{(*)}}$.

The fact that the prime $\mathcal{P}$ (resp. $\tilde{\mathcal{P}}$) is inert in $\mathbf{k}^{(*)}/\mathbf{k}$ (resp. in $\mathbf{k}^{(*)}/\tilde{\mathbf{k}}$) and that $\mathbf{k}_1^{(3)} = \tilde{\mathbf{k}}_1^{(3)}$, we get that the Artin maps $(\mathbf{k}_1^{(3)}/\mathbf{k}, \mathcal{P})$, $(\tilde{\mathbf{k}}_1^{(3)}/\tilde{\mathbf{k}}, \tilde{\mathcal{P}})$, and $(\mathbf{k}_1^{(3)}/\mathbf{k}^*, \mathcal{P}^*)$ are equal, where $\mathcal{P}^*$ is a prime in $\mathbf{k}^{(*)}$ lying above p; so we obtain (1). The fact that the 3-class number of $\mathbf{k}^{(*)}$ is equal to 3 or 9 is obtained by using a formula giving $h_{\mathbf{k}^{(*)}}$ where $\mathbf{k}^{(*)}$ is considered as the composite of cubic cyclic fields, so the assertion (2) is proved by calculating some unit index involving Parry's constant (see [Pa-90]).

Theorem 4.3. (1) *All the 3-ideal classes capitulate in each of the four intermediate fields of $\mathbf{k}_1^{(3)}/\mathbf{k}$ if and only if $3||h_{\mathbf{k}^{(*)}}$. In this case, $\mathbf{k}_1^{(3)} = \mathbf{k}_n^{(3)}$ for each $n \geq 2$.*
(2) Let $\mathbf{L}$ be a subextension of $\mathbf{k}_1^{(3)}$ which is cubic over $\mathbf{k}$. Then only the ambiguous ideal classes capitulate in $\mathbf{L}$ if and only if $9||h_{\mathbf{k}^{(*)}}$. In this case, $\mathbf{k}_2^{(3)} = \mathbf{k}_n^{(3)}$ for each $n \geq 3$.

The first assertion is obvious. For the second, the unit index in the extension $\mathbf{k}^{(*)}/\mathbf{k}$ is 1, so only the three ambiguous classes capitulate in $\mathbf{k}^{(*)}$ (see [Fr-93] and [Ja-88]); we use the fact that the group $Gal(\mathbf{k}_2^{(3)}/\mathbf{k})$ has two generators and we prove that $Gal(\mathbf{k}_2^{(3)}/\mathbf{k})$ is metacyclic of order 27 (see [Bl-58]); so the conclusion is obtained via the transfer for groups of order 27. See [Mi-89] for more information on transfer and [Ne-67] for all the different groups of order 27.

Numerical Examples.

$f_{\mathbf{k}}$	$b_{\mathbf{k}}$
$657 = 9 \cdot 73$	$(3)^2(73)$
$1267 = 7 \cdot 181$	$(7)(181)$
$2439 = 9 \cdot 271$	(271)
$5971 = 7 \cdot 853$	(853)

(B) Case where $[\mathbf{k}^{(*)} : \mathbf{k}] = 9$

In this case, $\mathbf{k}_1^{(3)} = \mathbf{k}^{(*)}$ and exactly three prime numbers p, q and r are ramified in $\mathbf{k}$; there are exactly three other cubic cyclic fields having the same conductor as $\mathbf{k}$. Using the cubic symbol, G. Gras distinguished 13 different situations (see [Gr-73]). We solved the capitulation problem for four of them, namely under the following equivalent conditions:
Let p, q, r be distinct prime numbers $\equiv 1 \pmod 3$, and allow p to be equal to 3; the Hilbert 3-class field of each cubic cyclic field of conductor dividing $(pqr)^2$ is equal to its absolute genus field.
Under these conditions we have:

Theorem 4.4. *If $\mathbf{k}$ is a cubic cyclic field of conductor pqr (or $9qr$ if $p = 3$) and if $9 || h_{\mathbf{k}}$, then all the 3-ideal classes capitulate in each of the four intermediate fields of $\mathbf{k}_1^{(3)}/\mathbf{k}$.*

By using Parry's constant and some unit index (see [Pa-90]), we prove that the 3-class number of each bicubic bicyclic field in $\mathbf{k}_1^{(3)}/\mathbf{k}$ is equal to 3.

Numerical examples. Suppose that $\mathbf{k}$ is a cubic cyclic field with conductor $f_{\mathbf{k}} \leq 16000$. Then $\mathbf{k}$ satisfies the last theorem if and only if $f_{\mathbf{k}} \in \{819, 1197, 1729, 1953, 2223, 2331, 2709, 2821, 2843, 3627, 3913, 4221, 4329, 5031, 5301, 5551, 5719\}$.

§5. On the capitulation of the 3-ideal classes of the normal closure of a pure cubic field

Let $\Gamma = \mathbb{Q}(\sqrt[3]{n})$ be a pure cubic field with class number h_Γ, $\mathbf{k} = \mathbb{Q}(\sqrt[3]{n}, j)$ its normal closure ($j = e^{\frac{2i\pi}{3}}$), $\mathbf{k}_1^{(3)}$ the Hilbert 3-class field of $\mathbf{k}$, and $S_{\mathbf{k}}$ the 3-ideal class group of $\mathbf{k}$. Suppose that $E_{\mathbf{k}}$ is the group of units of $\mathbf{k}$, E_0 the subgroup of $E_{\mathbf{k}}$ generated by the units of all proper subfields of $\mathbf{k}$, and $u = [E_{\mathbf{k}} : E_0]$. Let $\mathrm{Gal}(\mathbf{k}/\mathbb{Q}) = \langle \sigma, \tau \rangle$, $\mathrm{Gal}(\mathbf{k}/\mathbf{k}_0) = \langle \sigma \rangle$, $\mathrm{Gal}(\mathbf{k}/\Gamma) = \langle \tau \rangle$, where $\sigma^3 = \tau^2 = 1, \sigma\tau = \tau\sigma^2$ and $\sigma^2\tau = \tau\sigma$.
The relation between the class number $h_{\mathbf{k}}$ of $\mathbf{k}$ and the class number h_Γ of Γ is given by $h_{\mathbf{k}} = h_\Gamma^2 \frac{u}{3}$ (see [B-C-71]).

Proposition 5.1. (1) $S_{\mathbf{k}} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \Leftrightarrow 3$ *divides exactly* h_Γ *and* $u = 3$.
(2) *If* $S_{\mathbf{k}}$ *is of rank* 2 *and if* 3 *exactly divides* h_Γ, *then* $u = 3$, *whereupon* $S_{\mathbf{k}} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

The study of the structure of $S_{\mathbf{k}}$ and its rank is based on Gerth's results in [Ge-75], [Ge-76] and [Ger-76].

The action of the Galois group of $\mathbf{k}/\mathbb{Q}$ on $S_{\mathbf{k}}$ and genus theory allow us to distinguish three different cases (see [Ism-92]). We let $k_0 = \mathbb{Q}(j)$ and we define $(\mathbf{k}/\mathbf{k}_0)^*$ to be the relative genus field of $\mathbf{k}$ over $\mathbf{k}_0$. Then
(1) $\mathbf{k}$ is of type I if $(\mathbf{k}/\mathbf{k}_0)^* = \mathbf{k}\Gamma_1$, where Γ_1 is the Hilbert 3-class field of Γ;
(2) $\mathbf{k}$ is of type II if $(\mathbf{k}/\mathbf{k}_0)^* \neq \mathbf{k}\Gamma_1$ and $(\mathbf{k}/\mathbf{k}_0)^*$ is a proper subfield of $\mathbf{k}_1^{(3)}$;
(3) $\mathbf{k}$ is of type III if $(\mathbf{k}/\mathbf{k}_0)^* = \mathbf{k}_1^{(3)}$.

When the 3-group $S_{\mathbf{k}}$ is of type (3,3), it has 4 subgroups of order 3, denoted by H_j, $1 \leq j \leq 4$. Let $\mathbf{K}_j$ be the intermediate extension of $\mathbf{k}_1/\mathbf{k}$, corresponding by class field theory to H_j. As each $\mathbf{K}_j$ is cyclic of order 3 over $\mathbf{k}$, there is at least one subgroup of order 3 of $S_{\mathbf{k}}$, i.e., at least one H_l for some $l \in \{1,2,3,4\}$, which capitulates in $\mathbf{K}_j$ (by Hilbert's theorem 94).

Definition 5.1. Let S_j be a generator of H_j $(1 \leq j \leq 4)$ corresponding to $\mathbf{K}_j$. For $1 \leq j \leq 4$, let $i_j \in \{0,1,2,3,4\}$. We say that the capitulation is of type (i_1, i_2, i_3, i_4) to mean the following:
(1) when $i_j \in \{1,2,3,4\}$, then only the class S_{i_j} and its powers capitulate in $\mathbf{K}_j$;
(2) when $i_j = 0$, then all the 3-classes capitulate in $\mathbf{K}_j$.

Suppose that $\mathbf{k}$ is of type I; we show (see [Ism-92]) that $S_{\mathbf{k}} = \{\mathcal{A}^{r+s\sigma} \,|\, 0 \leq r,s \leq 2\}$ where $\mathcal{A}$ is such that $\mathcal{A}^\tau = \mathcal{A}$. The four subgroups of $S_{\mathbf{k}}$ are given by: $H_1 = \langle \mathcal{A} \rangle$, $H_2 = \langle \mathcal{A}^\sigma \rangle$, $H_3 = \langle \mathcal{A}^{1+\sigma} \rangle$, and $H_4 = \langle \mathcal{A}^{\sigma-1} \rangle$ which corresponds to $\mathbf{K}_4 = (\mathbf{k}/\mathbf{k}_0)^*$.

Theorem 5.1. *Let p and q be prime numbers and let* $u = [E_{\mathbf{k}} : E_0]$.
(1) *If* $\mathbf{k}$ *is of type I, then the possible forms of* n *(where* $\Gamma = \mathbb{Q}(\sqrt[3]{n})$*) are*

- (i) $n = p^{e_1}$, $p \equiv 1 \pmod 3$ *with* $e_1 \in \{1,2\}$;
- (ii) $n = 3^e p^{e_1}$, $p \equiv 4$ *or* $7 \pmod 9$ *with* $e, e_1 \in \{1,2\}$;
- (iii) $n = p^e q^{e_1} \equiv \pm 1 \pmod 9$, p *or* $-q \equiv 4$ *or* $7 \pmod 9$ *and* $e, e_1 \in \{1,2\}$.

(2) *Let* $n \in \mathbb{N}$ *be as in* (ii) (*resp.* (iii)), *let* $\left(\frac{3}{p}\right)_3 \neq 1$ (*resp.* $\left(\frac{q}{p}\right)_3 \neq 1$)) *and assume* $3 || h_\Gamma$. *Then* $u = 1$, $S_{\mathbf{k}}$ *is cyclic of order* 3 *and* $E_{\mathbf{k}} = \langle \varepsilon, \varepsilon^\sigma, -j \rangle$, *where* ε *is the fundamental unit of* Γ.

Theorem 5.2. (1) *All the* 3*-classes capitulate in* $\mathbf{K}_4 = \mathbf{k}\Gamma_1 = (\mathbf{k}/\mathbf{k}_0)^*$.
(2) *The numbers of* 3*-classes capitulating in* $\mathbf{K}_1$, $\mathbf{K}_2$ *and* $\mathbf{K}_3$ *are the same. More precisely, the possible capitulation types are* $(0,0,0,0)$, $(1,2,3,0)$ *or* $(4,4,4,0)$.

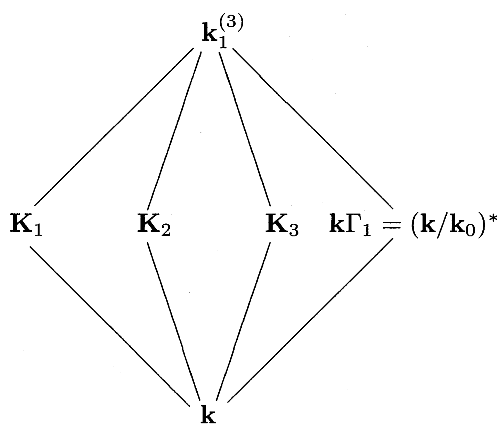

Diagram 3

Suppose that $\mathbf{k}$ is of type II; we show (see [Ism-92]) that the four cubic fields $\mathbf{K}_i$ are given as follows: $\mathbf{K}_1 = (\mathbf{k}/\mathbf{k}_0)^*$ which corresponds by class field theory to $H_1 = S_{\mathbf{k}}^{(\sigma)} = <\mathcal{A}>$, $\mathbf{K}_2 = \mathbf{k}\Gamma_1''$, $\mathbf{K}_3 = \mathbf{k}\Gamma_1'$ and $\mathbf{K}_4 = \mathbf{k}\Gamma_1$, where Γ_1 (resp. Γ_1', Γ_1'') is the Hilbert 3-class field of Γ (resp. of the two other cubic fields Γ', Γ'' contained in $\mathbf{k}$).

$$\begin{tikzcd}[column sep=small]
 & & & \mathbf{k}_1^{(3)} \arrow[dlll, dash] \arrow[dl, dash] \arrow[dr, dash] \arrow[drrr, dash] & & & \\
(\mathbf{k}/\mathbf{k}_0)^* \arrow[drrr, dash] & & \mathbf{k}\Gamma_1'' \arrow[dr, dash] & & \mathbf{k}\Gamma_1' \arrow[dl, dash] & & \mathbf{k}\Gamma_1 \arrow[dlll, dash] \\
 & & & \mathbf{k} & & &
\end{tikzcd}$$

Diagram 4

Theorem 5.3. (1) *The class $\mathcal{A}$ capitulates in the four cubic extensions $\mathbf{K}_i$, $1 \leq i \leq 4$.*
(2) *The numbers of 3-classes capitulating in $\mathbf{K}_2$, $\mathbf{K}_3$ and $\mathbf{K}_4$ are the same. More precisely, the possible capitulation types are $(0,0,0,0)$, $(0,1,1,1)$, $(1,0,0,0)$ or $(1,1,1,1)$.*

Theorem 5.4. *Let q_i be prime numbers $\equiv -1 \pmod 3$.*
(1) *If the field $\mathbf{k}$ is of type II, then the possible forms of n (where $\Gamma = \mathbb{Q}(\sqrt[3]{n})$) are*

- (i) $n = 3^e q_1^{e_1}$ *with* $q_1 \equiv -1 \pmod 9$ *and* $e, e_1 \in \{1,2\}$;
- (ii) $n = q_1^{e_1} q_2^{e_2}$ *with* $q_1 \equiv q_2 \equiv -1 \pmod 9$ *and* $e_1, e_2 \in \{1,2\}$;
- (iii) $n = 3^e q_1^{e_1} q_2^{e_2}$ *with* q_1 *or* $q_2 \equiv 2$ *or* $5 \pmod 9$, $e_1, e_2 \in \{1,2\}$, $e \in \{0,1,2\}$ *and* $n \not\equiv \pm 1 \pmod 9$;
- (iv) $n = q_1^{e_1} q_2^{e_2} q_3^{e_3}$ *with* q_1 *or* q_2 *or* $q_3 \equiv 2$ *or* $5 \pmod 9$, $n \equiv \pm 1 \pmod 9$ *and* $e_1, e_2, e_3 \in \{1,2\}$.

(2) *If the integer n has one of the four forms of* (1) *and if* $3 \| h_\Gamma$, *then the index $u = 3$, whereupon $\mathbf{k}$ is of type II.*
(3) *The normal closure $\mathbf{k}$ of $\Gamma = \mathbb{Q}(\sqrt[3]{n})$ is of type II if and only if n has one of the four forms of* (1) *and* $3 \| h_\Gamma$.

Suppose finally that $\mathbf{k}$ is of type III. Then we have the following.

Theorem 5.5. *Let p, q, q_1 and q_2 be prime numbers such that $p \equiv -q \equiv -q_1 \equiv -q_2 \equiv 1 \pmod 3$. The normal closure $\mathbf{k} = \mathbb{Q}(j, \sqrt[3]{n})$ of $\Gamma = \mathbb{Q}(\sqrt[3]{n})$ is of type III if and only if $3 \| h_\Gamma$, and n has one of the following forms*:

(i) $n = 3^e p^{e_1}$ *with* $p \equiv 1 \pmod 9$ *and* $e, e_1 \in \{1, 2\}$;
(ii) $n = q^e p^{e_1}$ *with* $-q \equiv p \equiv 1 \pmod 9$ *and* $e, e_1 \in \{1, 2\}$;
(iii) $n = p^e q_1^{e_1} q_2^{e_2}$ *with* p *or* $-q_1$ *or* $-q_2 \equiv 4$ *or* $7 \pmod 9$, $n \equiv \pm 1 \pmod 9$ *and* $e, e_1, e_2 \in \{1, 2\}$;
(iv) $n = 3^e p^{e_1} q^{e_2}$ *with* p *or* $-q \equiv 4$ *or* $7 \pmod 9$, $e \in \{0, 1, 2\}$, $e_1, e_2 \in \{1, 2\}$ *and* $n \not\equiv \pm 1 \pmod 9$.

Let us remark that $\mathbf{k}$ is of type III means that $(\mathbf{k}/\mathbf{k}_0)^* = \mathbf{k}_1^{(3)}$; in this case for $\forall \mathcal{A} \in S_{\mathbf{k}}$ we have $\mathcal{A}^\sigma = \mathcal{A}$, i.e., all the 3-classes are ambiguous classes.

When n has one of the four forms of the last theorem, and if $p \equiv -q \equiv -q_1 \equiv -q_2 \equiv 1 \pmod 3$, we have $p = \pi_1\pi_2$, $-q = \pi$, $-q_1 = \pi_3$ and $-q_2 = \pi_4$, where π, π_i $(1 \leq i \leq 4)$ are prime integers of $\mathbf{k}_0$; we also have $3\mathcal{O}_{\mathbf{k}_0} = (\lambda)^2$ with $\lambda = 1 - j$. We denote respectively by P_1, P_2, Q, Q_1, Q_2 and I the prime ideal of $\mathbf{k}$ lying above π_1, π_2, π, π_3, π_4 and λ. We summarize in the next theorem most of the results concerning the capitulation problem when $\mathbf{k}$ is of type III.

Theorem 5.6. *Suppose that the normal closure* $\mathbf{k} = \mathbb{Q}(\sqrt[3]{n}, j)$ *of* $\Gamma = \mathbb{Q}(\sqrt[3]{n})$ *is of type III.*
(A) *If* n *has one of the four forms of last theorem with the property that the prime number* $p = \pi_1\pi_2$ *dividing* n *satisfies* $p \equiv 1 \pmod 9$, *or if* n *has the fourth form with* $p \not\equiv 1 \pmod 9$ *and* $(n, 3) = 1$, *then we have the following:*
(1) $\mathbf{k}_1^{(3)} = \mathbf{k}(\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2})$, P_1P_2 *is not a principal ideal in* $\mathbf{k}$ *and* $S_{\mathbf{k}} = \langle [P_1P_2], [P_1] \rangle$.
(2) $\mathbf{K}_1 = \mathbf{k}(\sqrt[3]{\pi_1\pi_2})$, $\mathbf{K}_2 = \mathbf{k}(\sqrt[3]{\pi_2})$, $\mathbf{K}_3 = \mathbf{k}(\sqrt[3]{\pi_1})$ *and* $\mathbf{K}_4 = \mathbf{k}\Gamma_1 = \mathbf{k}(\sqrt[3]{\pi_1\pi_2^2})$.
(3) $[P_1P_2]$ *capitulates in* $\mathbf{K}_1$, $[P_2]$ *capitulates in* $\mathbf{K}_2$, $[P_1]$ *capitules in* $\mathbf{K}_3$ *and all the* 3*-classes capitulate in* $\mathbf{K}_4$.
(4) *The possible capitulation types are* $(0,0,0,0)$, $(1,3,2,0)$, $(0,3,2,0)$ *or* $(1,0,0,0)$.
(B) *If* n *has the form of* (iii) *with* $p \not\equiv 1 \pmod 9$ *or* n *has the form of* (iv) *with* $3 \mid n$, *then all the* 3*-classes capitulate in* $\mathbf{K}_4$ *and we have the following capitulation types depending on some conditions on the ideals* Q, I, Q_1 *and* Q_2 :
(a) $(0,4,4,0)$, $(1,4,4,0)$, $(4,4,4,0)$, $(1,0,0,0)$ *or* $(4,0,0,0)$;
(b) $(0,0,0,0)$;
$(0,3,2,0)$ *or* $(0,2,3,0)$;
$(1,0,0,0)$;
$(1,3,2,0)$ *or* $(1,2,3,0)$.

Theorem 5.7. *Let h_Γ be the class number of the pure cubic field $\Gamma = \mathbb{Q}(\sqrt[3]{n})$. If $n = c^e p^{e_1}$, where $c = 3$ or q, and p, q are prime numbers such that $p \equiv -q \equiv 1 \pmod 9$ and $e, e_1 \in \{1, 2\}$, then*

$$\left(\frac{c}{p}\right)_3 = 1 \Rightarrow 3^2 | \, h_\Gamma.$$

When n has the form (iii) or the form (iv), we prove seven other similar results. Each time, we construct, under certain conditions, a natural integer c such that:

$$\left(\frac{c}{p}\right)_3 = 1 \Rightarrow 3^2 | \, h_\Gamma.$$

The proof of all the results given in this section can be found in [Ism-92]. In this work we used also the arithmetic properties of a pure cubic field (see [De-00]), Kummer theory and the cubic symbol (see [I-R-82]). For the following numerical examples we used the tables given in [B-87] and [B-W-Z-71].

Numerical Examples.
(1) For $p \in \{61, 67, 103, 151\}$ we have $\mathbf{k} = \mathbb{Q}(j, \sqrt[3]{p})$ is of type I.
(2)

n	h_Γ	$\mathbf{k} = \mathbb{Q}(j, \sqrt[3]{n})$
$3 \cdot 17 = 51$	3	type II
$3^2 \cdot 17 = 153$	$9 = 3^2$	$S_\mathbf{k} \cong C_9 \times C_9$
$3 \cdot 53 = 159$	3	type II
$3^2 \cdot 53 = 477$	$9 = 3^2$	$S_\mathbf{k} \cong C_9 \times C_9$
$3 \cdot 71 = 213$	$21 = 7 \cdot 3$	type II
$3^2 \cdot 71 = 639$	$18 = 2 \cdot 3^2$	$S_\mathbf{k} \cong C_9 \times C_9$
$3 \cdot 89 = 267$	$15 = 5 \cdot 3$	type II
$3^2 \cdot 89 = 801$	$6 = 3 \cdot 2$	type II
$3 \cdot 107 = 321$	$9 = 3^2$	$S_\mathbf{k} \cong C_9 \times C_9$
$3^2 \cdot 107 = 963$	3	type II

Table 2

(3) For each integer n in the next table, $\mathbf{k} = \mathbb{Q}(j, \sqrt[3]{n})$ is of type III.

n	h_Γ	n	h_Γ
$3 \cdot 19 = 57$	6	$3^2 \cdot 19 = 171$	6
$3 \cdot 37 = 111$	3	$3^2 \cdot 37 = 333$	3
$3 \cdot 109 = 327$	12	$3^2 \cdot 109 = 981$	3
$3 \cdot 127 = 381$	12	$3 \cdot 163 = 489$	3
$3 \cdot 181 = 543$	3	$3 \cdot 199 = 597$	3

Table 3

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