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On Parities of Relative Class Numbers of certain CM-Extensions

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Abstract.

Let k/F and k'/F be CM-extensions, $K = kk'$ and K_+ the maximal totally real subfield of K. It holds $h^-(k) \mid h^-(K)$ if k/F is unramified at all finite primes or K_+/F is unramified. Hence, $h^-(k)$ is an obstacle that prevents $h^-(K)$ from being 1. On the other hand, analytic class number formula implies the class number relation $h^-(K) = h^-(k)h^-(k')/c(K/F)$, where $c(K/F)$ is an integer determined by units. Consistency of the first assertion and the class number relation is guaranteed by $h^-(k')$. As a reason of the consistency, the parity equality $h^-(k) \equiv h^-(k') \pmod 2$ is formulated under the situation of the first assertion. (See Theorems 1 and 2.) Non-trivial Examples (§§3.2) and a proof of the parity equality are given. The tool behind the first assertion and indices related with the class number relation are discussed in detail.

§1. Introduction

A finite extension of $\mathbb{Q}$ is called a number field. The number $h(L)$ of ideal classes of a number field L is called the class number of L. When k is a totally imaginary quadratic extension of a totally real number field F, the extension k/F is called a CM-extension and k is called a CM-field. The subfield F is identified as the maximal totally real subfield of k, which is denoted by k_+. Hence, the ratio $h^-(k) = h(k)/h(F)$ is determined by the CM-field k. We call $h^-(k)$ the relative class number of a CM-field k.

An important property of relative class numbers comes from class field theory. We denote by $\mathcal{H}(L)$ the Hilbert class field of a number field L, i.e., the maximal unramified abelian extension of L. Then, we have $h(L) = [\mathcal{H}(L) : L]$ by class field theory. Let k be a CM-field. Since $\mathcal{H}(k_+)/k_+$ is a totally real normal extension and since the quadratic

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extension k/k_+ is ramified at an infinite prime, we have $[k\mathcal{H}(k_+) : k] = [\mathcal{H}(k_+) : k_+]$. Since $\mathcal{H}(k_+)/k_+$ is an unramified abelian extension, so is $k\mathcal{H}(k_+)/k$. Hence, $k\mathcal{H}(k_+) \subset \mathcal{H}(k)$ holds. The mentioned equality of degrees now implies

$$h^-(k) = [\mathcal{H}(k) : k\mathcal{H}(k_+)]. \tag{1}$$

In particular, $h^-(k)$ is an integer.

We shall prove the following two theorems on parity equality:

Theorem 1. *Let k/F and k'/F be CM-extensions. Assume that k/F and k'/F are unramified at all finite primes. Then, the following parity equality holds:*

$$h^-(k) \equiv h^-(k') \pmod 2.$$

Theorem 2. *Let k/F and k'/F be CM-extensions. Set $K = kk'$. Assume that K_+/F is unramified. Then, the following equivalence holds:*

$$h^-(k) \equiv h^-(k') \pmod 2.$$

As examples of §§3.2 will show, the interesting cases indeed exist. In some cases, exponents of 2 in $h^-(k)$ and $h^-(k')$ are equal. (See Examples 35 and 37.) In some cases, they are different. (See Examples 28, 29 and 38.)

The background of these Theorems is a problem of divisibility of relative class numbers in relation with inclusion of CM-fields, i.e., a problem of obstacle for class number one. Our Theorems 1 and 2 came up from investigation in the tools for approaching that problem.

Let k and K be CM-fields satisfying $k \subset K$. Hasse proved that $h^-(k)$ divides $h^-(K)$ multiplied by 2 to a suitable exponent when $K/\mathbb{Q}$ is abelian [1]. (Dependence of the exponent on K and k was mysterious.) Hirabayashi and Yoshino calculated relative class numbers of imaginary abelian number fields whose conductors are less than or equal to 200 and placed them in Hasse's diagrams [2, 3]. In the diagrams, Horie observed $h^-(k) \mid 4\, h^-(K)$. He proved his observation true under the assumption that $K/\mathbb{Q}$ is abelian [5, Theorem 1]. He also gave an example of a pair $k = \mathbb{Q}(\sqrt{-4 \cdot -3 \cdot -7})$ and $K = \mathbb{Q}(\sqrt{-4}, \sqrt{-3}, \sqrt{-7})$ with $h^-(k) = 4$, $h^-(K) = 1$ and $h^-(K)/h^-(k) = 1/4$, which illustrates that the assertion of the following Theorem 3 is best possible.

In [11], we proved a generalization of his theorem:

Theorem 3. *Let $k \subset K$ be CM-fields. Then, $h^-(k)$ divides $4\, h^-(K)$.*

Hence, $h^-(k)/4$ is an obstacle that prevents $h^-(K)$ from being 1.

The basic ideas of proofs of [5] and [11] are totally different. Horie used well-known factorization of relative class number of imaginary abelian number fields into product of generalized Bernoulli numbers, and regrouped factors for proving his theorem. However, we do not even assume K/k to be normal in Theorem 3. Therefore, a completely different method is required for proving it. We combined three algebraic tools, which are listed below:

1. Group Theoretic Tool;
2. Field Theoretic Tool;
3. Class Number Relation.

We shall briefly explain the three tools, and define the strategy of this paper.

Group Theoretic Tool We denote by $\mathcal{C}(L)$ the ideal class group of a number field L. When k is a CM-field, we denote by $\iota_k : \mathcal{C}(k_+) \longrightarrow \mathcal{C}(k)$ the natural lift of ideal classes. We have

$$\kappa(k)\, h^-(k) \;=\; \#\operatorname{coker} \iota_k$$

where $\kappa(k)$ denotes the order of $\ker \iota_k$ and $\#A$ denotes the cardinality of a given set A. As we shall review in Lemma 17, $\kappa(k)$ is either 1 or 2. Hence, $\operatorname{coker} \iota_k$ has much information on $h^-(k)$.

Exploiting information from $\operatorname{coker} \iota_k$, we proved several statements. One of such statements is on a delicate relation on $h^-(k)$ with a property of k_+:

Lemma 4 (Lemma 26 of [11]). *Let k be a CM-field and r the 2-rank of the strict class group of k_+. Set $u(k) = 2$ if k/k_+ is unramified at all finite primes and $u(k) = 1$ otherwise. Then, $2^r/\,\kappa(k)u(k)$ is an integer, and divides $h^-(k)$.*

Let r' be the 2-rank of the class group of k_+. Then, $2^{r'}$ divides $2^r/u(k)$. Hence, Lemma 4 implies the well-known divisibility: $2^{r'}/\,\kappa(k) \mid h^-(k)$. (See [14, Theorem 10.12].) However, Lemma 4 turned out useful for dealing with combined structure of unit group and class group.

If K is a CM-field satisfying $k \subset K$, the norm map $N : \operatorname{coker} \iota_K \longrightarrow \operatorname{coker} \iota_k$ is well-defined. If the order of the cokernel of N is either 1 or 2, it follows easily that $h^-(k)$ divides $4\,h^-(K)$. Investigating the structure of the cokernel of N, we obtained a nicer statement:

Proposition 5 (Corollary 28 of [11]). *Let $k \subset K$ be CM-fields. Assume K does not contain a bicyclic biquadratic extension of k. Then, $h^-(k)$ divides $4\,h^-(K)$.*

Field Theoretic Tool When k is a CM-field, we denote by $\mathcal{H}^0(k)$ the maximal CM-field that is contained in $\mathcal{H}(k)$. We have the following:

Lemma 6. *Let k be a CM-field. Then, $\mathcal{H}^0(k)$ is well-defined and the extension $\mathcal{H}^0(k)/k_+$ is abelian.*

An illustration of a relation of this concept with relative class numbers is the following:

Lemma 7. *Let $k \subset K$ be CM-fields. Then, the degree $[\mathcal{H}(k) : \mathcal{H}^0(k)]$ divides $h^-(K)$. Therefore, $h^-(k)$ divides $[\mathcal{H}^0(k) : k\mathcal{H}(k_+)]$ $h^-(K)$.*

Extending this idea, we proved the following two Propositions: (See Propositions 22 and 23 of [11].)

Proposition 8. *Let $k \subset K$ be CM-fields. Assume k/k_+ is unramified at all finite primes. Then, $h^-(k)$ divides $h^-(K)$.*

Proposition 9. *Let $k \subset K$ be CM-fields. Assume K_+/k_+ is unramified. Then, $h^-(k)$ divides $h^-(K)$.*

Relation of unramifiedness and divisibility of relative class numbers will be examined in more detail in §§2.1.

Class Number Relation When L is a number field, $W(L)$ denotes the group of roots of unity of L, $w(L)$ the order of $W(L)$ and $E(L)$ the group of units of L. When k is a CM-field, $Q(k)$ denotes Hasse's unit index $[E(k) : W(k)E(k_+)]$. Property of $Q(k)$ and the aforementioned index $\kappa(k)$ shall be reviewed in §§2.2, in which the notion of Viéte ideal will be recognized.

Let $k \subset K$ be CM-fields and assume that K/k is quadratic. Set $F = k_+$. Then, K_+/F is also quadratic. Hence, K/F is bicyclic biquadratic. Therefore, K/F contains a CM-extension k'/F other than k/F. Analytic class number formula implies the following class number relation:

$$h^-(K) = \frac{1}{c(K/F)} h^-(k)\, h^-(k') \tag{2}$$

where

$$c(K/F) = \frac{w(k)w(k')}{w(K)} \frac{Q(k)Q(k')}{Q(K)}. \tag{3}$$

In the current paper, we call $c(K/F)$ the denominator constant of class number relation for K/F. It is known that the fist factor in the right hand side of (3) is either 1 or 2. (See Lemma 23.) It is also known that a Hasse's unit index is also either 1 or 2. (See Lemma 17.) Hence, it looks as if $c(K/F)$ can take all values from $\frac{1}{2}, 1, 2, 4, 8$.

Lemmermeyer gave a new formula for $c(K/F)$ in [8]:

$$c(K/F) = \frac{2^{1+v}}{[E(K) : E(k)E(k')E(K_+)]} \tag{4}$$

where $v = 1$ if the both of k and k' are obtained by adjoining to F square roots of units in F or $v = 0$ otherwise. This formula and the original formula imply $c(K/F) \neq 8$. Indeed, we shall see

$$c(K/F) \in \{1, 2, \dots, 2^{1+v}\} \tag{5}$$

in Lemma 24. One consequence is the following:

Proposition 10. *Let $k \subset K$ be CM-fields. Assume that K/k is quadratic. Then, $h^-(k)$ divides $4\,h^-(K)$.*

The class number relation was also used in the critical part of our proof for Theorem 3.

Cooperation of Tools We illustrate importance and cooperation of the three tools explained by giving a sketch of a proof of Theorem 3, in which the three tools cooperate covering different areas.

Let $k \subset K$ be CM-fields. Let M be a maximal intermediate field of K/k such that $h^-(k)$ divides $h^-(M)$. It suffice to show $h^-(M) \mid 4\,h^-(K)$.

The first step is an application of the group theoretic tool. If K/M does not contain a bicyclic biquadratic extension of M, Proposition 5 implies $h^-(M) \mid 4\ h^-(K)$. We turn to the other case: we assume that K/M contains a bicyclic biquadratic extension of M.

The second step is the most involved one. It is a search for a quadratic extension M' of M in K such that $h^-(k)$ divides $h^-(M')$. Tools for the search are Proposition 9 and the class number relation (2) together with (5). Lemma 4 is also applied to avoid certain obstacle coming from delicate structure of the strict class group of M_+. It was shown that the search will be successful unless the following three conditions hold:

(a) The strict class number of M_+ is odd;
(b) The CM-field K contains three CM-extensions $M_1/M_+, M_2/M_+, M_3/M_+$ of prime power conductor over M_+;
(c) And $M \subset M_1 M_2 M_3$.

On the other hand, the choice of M implies that the search cannot be successful. Hence conditions (a), (b) and (c) hold. Then, the genus theory, possibly combined with Lemma 7, applied to M/M_+ implies the desired result.

Competition of Tools We have seen that the field theoretic tool and the class number relation play important roles in an interesting problem of divisibility of relative class numbers. In certain situation, these tools can be used independently to study divisibility in the same pair of relative class numbers. A delicate competition of the two tools, which arises in such a situation, motivated the current work.

Let k/F and k'/F be distinct CM-extensions. Set $K = kk'$. The class number relation (2) is equivalent to

$$\frac{h^-(K)}{h^-(k)} = \frac{h^-(k')}{c(K/F)}. \tag{6}$$

Assertion (5) implies that the ratio in the right hand side lies in $\frac{1}{4}\mathbb{Z}$. This shows the ratio in the left hand side to be an integer when $h^-(k')$ is a multiple of 4 or $c(K/F)$ is 1. However, Examples 11, 12, 13 and 14 show that the ratio may not be an integer in some cases.

On the other hand, Propositions 8 and 9 assert

$$\frac{h^-(K)}{h^-(k)} \in \mathbb{Z} \tag{7}$$

under certain situation.

The class number relation and the field theoretic tool with their own coverages are competing. Their own coverages intersect at a small area of Integrality (7). It looks as if the two tools contradict in the intersection: (6) suggest a non-trivial denominator while (7) exclude a non-trivial denominator. An effort, i.e., Theorems 1 and 2, to fill in the seeming gap is the current work.

Strategy As our target is comparison of the field theoretic tool and the class number relation, we need a review of the two tools. Our interest on the latter is in the denominator constant. We shall discuss properties and complicatedness of the denominator constant. For this purpose, we need a review of properties of indices related with CM-fields, in which we shall recognize the notion of Viéte ideals. We shall put them in the bottom-up order in §2: we shall review the field theoretic tool in §§2.1; indices in §§2.2; and the denominator constant in §§2.3. (Note that some part of §§2.2 are added for interpreting Examples.) In §3, we shall give an answer to the problem of the previous paragraph: we shall identify the problem with a Suspicion in §§3.1 and formulate the Suspicion as the parity equality of Theorems 1 and 2; we give non-trivial example of the problem in §§3.2; and prove Theorems 1 and 2 in §§3.3.

§2. Detail of the Competing Tools

We shall review detail of the two competing tools (the field theoretic tool and the class number relation) out of the aforementioned three tools. We shall firstly review the field theoretic tools in §§2.1. We shall secondly review indices related with CM-extensions in §§2.2, in which the notion of Viéte ideal shall be proposed. We will lastly apply §§2.2 to the denominator constant of class number relation in §§2.3.

2.1. Field Theoretic Tool

We shall prove Lemmas 6, 7, and Propositions 8 and 9 for illustration of the concept of $\mathcal{H}^0(L)$. We shall also give examples that illustrate relation of unramifiedness and divisibility of relative class numbers. We shall lastly illustrate another competition of tools (the group theoretic tool and the field theoretic tool) by giving a proof of a well-known lemma through a use of $\mathcal{H}^0(L)$.

Let L be a CM-field. Then, the non-trivial conjugation of L/L_+ is called the complex conjugation of L.

Proof of Lemma 6. Let k be a CM-field and M the maximal totally real subfield of $\mathcal{H}(k)$. Then, it is obvious that kM is a CM-field. We show that kM is the maximal CM-field that is contained in $\mathcal{H}(k)$, i.e., $\mathcal{H}^0(k) = kM$.

Let L be a CM-field contained in $\mathcal{H}(k)$. Choose $\delta \in k$ such that $k = k_+(\sqrt{-\delta})$ and $\delta' \in L$ such that $L = L_+(\sqrt{-\delta'})$. Then, δ and δ' are totally positive. On the other hand, $k_+, L_+ \subset M$ follows from the choice of M. Hence, we have $\delta, \delta' \in M$. Hence, $M(\sqrt{\delta\delta'})$ is totally real. On the other hand, $M(\sqrt{\delta\delta'}) \subset \mathcal{H}(k)$ follows from $k, L \subset \mathcal{H}(k)$. The maximality of M now implies $M(\sqrt{\delta\delta'}) = M$, i.e., $\delta\delta' \in (M^\times)^2$. Hence, $LM = kM$ holds. We get $L \subset kM$ as desired.

Since the first assertion of the lemma is established, we show the second assertion. By class field theory, normality of k/k_+ implies normality of $\mathcal{H}(k)/k_+$. Hence, the choice of M implies normality of M/k_+. On the other hand, M/k_+ is disjoint with k/k_+. Therefore, we get an isomorphism $\mathrm{Gal}(kM/k) \simeq \mathrm{Gal}(M/k_+)$. The left hand side is abelian by the choice of M. Hence, the right hand side is also abelian. By composing abelian extensions, we get an abelian extension kM/k_+. Since $\mathcal{H}^0(k)$ is constructed as kM, we now conclude $\mathcal{H}^0(k)/k_+$ is abelian. □

Proof of Lemma 7. Let $k \subset K$ be CM-fields.

Step 1. (Composition): By class field theory, normality of k/k_+ implies normality of $\mathcal{H}(k)/k_+$. Noting that $k_+ \subset \mathcal{H}(K_+)$, we get normality of $\mathcal{H}(k)\mathcal{H}(K_+)/\mathcal{H}(K_+)$.
Step 2. (Galois Action): Let σ be the complex conjugation of $K\mathcal{H}(K_+)$. Since $k \subset K$ is totally imaginary, comparison of degrees implies K $\mathcal{H}(K_+) = k\mathcal{H}(K_+)$, and hence $K\mathcal{H}(K_+) \subset \mathcal{H}(k)\mathcal{H}(K_+)$. Therefore, Step 1 implies that σ extends to an element σ' of $\mathrm{Gal}(\mathcal{H}(k)\mathcal{H}(K_+)/\mathcal{H}(K_+))$.
Step 3. (Intersection): By the choice of σ', we see that σ' preserves $K\mathcal{H}(K_+)$. On the other hand, σ' belongs to $\mathrm{Aut}(\mathcal{H}(k)\mathcal{H}(K_+)/k_+)$. Noting that $\mathcal{H}(k)/k_+$ is normal, we see that σ' preserves $\mathcal{H}(k)$. Therefore, σ' preserves the intersection $\mathcal{H}(k) \cap K\mathcal{H}(K_+)$. We denote by $\sigma'' \in \mathrm{Gal}(\mathcal{H}(k) \cap K\mathcal{H}(K_+)/k_+)$ the restriction of σ' to $\mathcal{H}(k) \cap K\mathcal{H}(K_+)$. Then, σ'' turns out to be a restriction of σ. Hence, we get $\sigma''^2 = 1$. On the other hand, $\mathcal{H}(k) \cap K\mathcal{H}(K_+)$ contains k, on which σ acts nontrivially. Therefore, $\sigma'' \neq 1$ holds. Let M be the fixed field of σ''. Then, $\mathcal{H}(k) \cap K\mathcal{H}(K_+)/M$ is quadratic. We get $\mathcal{H}(k) \cap K\mathcal{H}(K_+) = kM$. Since k is totally imaginary and M, which is fixed by σ, is totally real, this identity implies that $\mathcal{H}(k) \cap K\mathcal{H}(K_+)$ is a CM-field. By definition of $\mathcal{H}^0(k)$, we now get $\mathcal{H}(k) \cap K\mathcal{H}(K_+) \subset \mathcal{H}^0(k)$.
Step 4. (Tower): By class field theory and Step 3, we get the following tower:

$$\begin{aligned} k\mathcal{H}(k_+) \subset K\mathcal{H}(K_+) \cap \mathcal{H}(k) &\subset \mathcal{H}^0(k) \\ &\subset \mathcal{H}(k) \subset K\mathcal{H}(K_+)\mathcal{H}(k) \subset \mathcal{H}(K). \end{aligned}$$

We see $[\mathcal{H}(k) : \mathcal{H}^0(k)]$ divides $[\mathcal{H}(k) : K\mathcal{H}(K_+) \cap \mathcal{H}(k)]$. The latter degree equals $[K\mathcal{H}(K_+)\mathcal{H}(k) : K\mathcal{H}(K_+)]$, which divides $[\mathcal{H}(K) : K\mathcal{H}(K_+)]$. Recalling (1), we get $[\mathcal{H}(k) : \mathcal{H}^0(k)] \mid h^-(K)$, which is the first assertion of the Lemma. Recalling (1) again, we get $h^-(k) = [\mathcal{H}(k) : \mathcal{H}^0(k)][\mathcal{H}^0(k) : k\mathcal{H}(k_+)] \mid [\mathcal{H}^0(k) : k\mathcal{H}(k_+)]h^-(K)$. □

Proof of Proposition 8. Since k/k_+ is unramified at all finite primes, class field theory implies that $\mathcal{H}(k)/k_+$ is unramified at all finite primes. Hence, $\mathcal{H}^0(k)_+/k_+$ is unramified. By Lemma 6, we see $\mathcal{H}^0(k)_+ \subset \mathcal{H}(k_+)$. We now have $\mathcal{H}^0(k) = k\mathcal{H}^0(k)_+ \subset k\mathcal{H}(k_+)$. Since the reverse inclusion is already shown, we get $k\mathcal{H}(k_+) = \mathcal{H}^0(k)$. Now, Lemma 7 implies the proposition. □

Proof of Proposition 9. We use the following fact that is shown in Step 4 of the proof for Lemma 7: $[\mathcal{H}(k) : K\mathcal{H}(K_+) \cap \mathcal{H}(k)] \mid h^-(K)$. We also use the tower shown in the same step.

Let $M = (K\mathcal{H}(K_+) \cap \mathcal{H}(k))_+$. Then, $M \subset \mathcal{H}(K_+)$ follows. Since K_+/k_+ is unramified, so is $\mathcal{H}(K_+)/k_+$. Therefore, M/k_+ is also unramified. On the other hand, we have $M/k_+ \subset \mathcal{H}^0(k)$. By Lemma 6, we see M/k_+ is abelian. Therefore, we get $M \subset \mathcal{H}(k_+)$ and hence $K\mathcal{H}(K_+) \cap \mathcal{H}(k) \subset k\mathcal{H}(k_+)$. Since the reverse inclusion is already shown in the tower, we get $K\mathcal{H}(K_+) \cap \mathcal{H}(k) = k\mathcal{H}(k_+)$. Now, the divisibility at the beginning of this proof implies that $h^-(k) = [\mathcal{H}(k) : k\mathcal{H}(k_+)]$ divides $h^-(K)$ □

Examples in §§3.2 will illustrate the truth of Proposition 8 and Proposition 9, i.e., $h^-(k) \mid h^-(K)$ holds when k/k_+ is unramified at all finite primes or when K_+/k_+ is unramified. In some cases, however, $h^-(k)$ fail to divide $h^-(K)$ while one or two of K/k and K/K_+ are unramified at all finite primes:

Example 11. *Let $k = \mathbb{Q}(\sqrt{-4 \cdot 5})$ and $K = k(\sqrt{-4})$. Then, $h^-(k) = 2$ and $h^-(K) = 1$ hold. Hence, $h^-(K)/h^-(k) = 1/2 \notin \mathbb{Z}$. Note that K/k is unramified.*

Example 12. *Let $k = \mathbb{Q}(\sqrt{-4 \cdot -3 \cdot -7})$ and $K = k(\sqrt{-3})$. Then, $h^-(k) = 4$ and $h^-(K) = 2$ hold. Hence, $h^-(K)/h^-(k) = 1/2 \notin \mathbb{Z}$. Note that K/k is unramified.*

Example 13. *Let $k = \mathbb{Q}(\sqrt{-4 \cdot -3 \cdot -7})$ and $K = k(\sqrt{-11})$. Then, $h^-(k) = 4$ and $h^-(K) = 2$ hold. Hence, $h^-(K)/h^-(k) = 1/2 \notin \mathbb{Z}$. Note that K/K_+ is unramified at all finite primes. ($K_+ = \mathbb{Q}(\sqrt{-4 \cdot -3 \cdot -7 \cdot -11})$.)*

Example 14. *Let $F = \mathbb{Q}(\sqrt{-4 \cdot -7})$. Then, $h(F) = 1$. Let $k = F(\sqrt{-3})$ and $k' = F(\sqrt{-4})$. Then, we have $w(k) = 2 \cdot 3$, $w(k') = 4$, $Q(k) = 1$, $Q(k') = 2$, $\kappa(k) = \kappa(k') = 1$, $h^-(k) = 2$ and $h^-(k') = 1$. Here, k/F is ramified above (2) and k'/F is unramified at all finite primes. Set $K = kk'$. Then, we have $K_+ = \mathbb{Q}(\sqrt{-4 \cdot -3}, \sqrt{-4 \cdot -7})$, $h(K_+) = 1$, $w(K) = 4 \cdot 3$, $Q(K) = 2$, $\kappa(K) = 1$, $c(K/F) = 2$ and $h^-(K) = 1$. Hence, $h^-(K)/h^-(k) = 1/2 \notin \mathbb{Z}$. Note that K/k is unramified and K/K_+ is unramified at all finite primes.*

In Examples 11, 12 and 14, $\mathcal{H}^0(k) \cap k\mathcal{H}(K_+)/k$ is quadratic. This fact and the field theoretic method just explained imply $h^-(k) \mid 2h^-(K)$. (See Step 4 of the proof for Lemma 7.) However, $\mathcal{H}^0(k) \cap k\mathcal{H}(K_+)/k$ is quartic in Example 13. Hence, the field theoretic method only explains $h^-(k) \mid 4\, h^-(K)$.

We denote by $E^+(F)$ the group of totally positive units of a number field F. The following Lemma is well-known:

Lemma 15. *Let k/F be a CM-extension. Let r'' be the 2-rank of $E^+(F)/E(F)^2$. Assume that k/F is ramified at some finite prime. Then, $2^{r''}$ divides $h^-(k)$.*

Remark. This Lemma can be deduced from Lemma 4. For illustration of another competition of tools, we review an alternative proof.

Proof of Lemma 15. Let H be the maximal abelian extension of F that is unramified at all finite primes. Then, $kH \subset \mathcal{H}(k)$ holds. Therefore, $[kH : k\mathcal{H}(F)]$ divides $h^-(k) = [\mathcal{H}(k) : k\mathcal{H}(F)]$. It suffices to show $[kH : k\mathcal{H}(F)] = 2^{r''}$. The quadratic extension k/F is disjoint with H/F since k/F is ramified at some prime. Further, H/F is normal. Therefore, we get $[kH : k] = [H : F]$. Similarly, we get $[k\mathcal{H}(F) : k] = [\mathcal{H}(F) : F]$. Hence, we have $[kH : k\mathcal{H}(F)] = [H : \mathcal{H}(F)]$. By class field theory, we have $[H : \mathcal{H}(F)] = 2^{r''}$. □

2.2. Indices Related with CM-Extensions

This subsection will be devoted to a review of the standard theory on Hasse's unit indices of CM-fields and capitulation kernel of CM-extensions since those objects are basic obstacles closely related with the denominator constants. There are nice descriptions in [1] and [9]. We follow the latter reference. We also propose to recognize the notion of Viéte ideal. The result shall be used for controlling the denominator constant of class number relation. It is also used for interpreting Examples.

Hasse's unit indices of CM-fields and capitulation in CM-extensions are information on certain delicate structure of ambiguous ideals:

Definition 16. *Let k/F be a CM-extension. An ideal of k is called ambiguous if it is invariant under* $\mathrm{Gal}(k/F)$. *The group of generators of ambiguous principal ideals is denoted by $A(k)$. The group of elements of k that generate ideals of F is denoted by $P(k/F)$.*

We are interested in the order $\kappa(k)$ of the kernel of the natural lift $\iota_k : \mathcal{C}(F) \longrightarrow \mathcal{C}(k)$ of ideal classes. It is obvious that $\kappa(k)$ equals the index $[P(k/F) : F^\times E(k)]$. We now characterize indices $Q(k)$ and κ_k:

Lemma 17. *Let k/F be a CM-field and σ the complex conjugation of k. Then, we have the following inclusions:*

$$W(k)^2 = W(k)^{1-\sigma} \subset E(k)^{1-\sigma} \subset P(k/F)^{1-\sigma} \subset A(k)^{1-\sigma} = W(k).$$

Moreover, Hasse's unit index $Q(k) = [E(k) : W(k)E(F)]$ and $\kappa(k) = [P(k/F) : F^\times E(k)]$ are characterized by

$$Q(k) = [E(k)^{1-\sigma} : W(k)^{1-\sigma}]; \quad \kappa(k) = [P(k/F)^{1-\sigma} : E(k)^{1-\sigma}].$$

Therefore, $Q(k)\kappa(k)$ *divides* 2. *We also have*

$$\frac{2}{Q(k)\kappa(k)} = [A(k) : P(k/F)].$$

Proof. Let $N_{k/F}$ be the norm map from $k^\times$ to $F^\times$ and $o(k)$ the ring of integers of k. We firstly recall Kronecker's characterization of $W(k)$:

$$W(k) = o(k) \cap \ker N_{k/F}.$$

The inclusion $W(k) \subset o(k) \cap \ker N_{k/F}$ is obvious. The reverse inclusion is the heart of Kronecker's Theorem: an algebraic integer α is necessarily a root of unity if all archimedean valuation of powers of α are uniformly bounded.

We secondly prove the first assertion. The mentioned characterization of $W(k)$ implies the first identity. The three inclusions follow from $W(k) \subset E(k) \subset P(k/F) \subset A(k)$. The last identity is proven by verification of inclusions in the both directions: The group $A(k)^{1-\sigma}$ obviously lies in $\ker N_{k/F}$. It also lies in $E(k) \subset o(k)$. Hence, we get $A(k)^{1-\sigma} \subset W(k)$ by the characterization of $W(k)$. We turn to the reverse inclusion. By Hilbert 90 and the characterization of $W(k)$, any $\xi \in W(k)$ is of the form $\alpha^{1-\sigma}$ for some $\alpha \in k$. We get $\alpha^\sigma = \xi^{-1}\alpha$ and hence $\alpha \in A(k)$. It follows $\xi \in A(k)^{1-\sigma}$. Now, we see $W(k) \subset A(k)^{1-\sigma}$.

We thirdly prove the second assertion. By the first assertion, the map $1-\sigma$ induces a homomorphism

$$\phi_{E(k)} : E(k)/W(k)E(F) \longrightarrow W(k)/W(k)^{1-\sigma}$$

of quotients. Let $\eta \in E(k)$ satisfy $\eta W(k)E(F) \in \ker \phi_{E(k)}$, i.e., $\eta^{1-\sigma} = \xi^{1-\sigma}$ for some $\xi \in W(k)$. Then, η/ξ is invariant under σ and hence lies in $E(F)$. It follows $\eta \in W(k)E(F)$. Therefore, $\phi_{E(k)}$ is injective. Noting that $\operatorname{Im}\phi_{E(k)} = E(k)^{1-\sigma}/W(k)^{1-\sigma}$, we get the first identity.

By the first assertion, the map $1-\sigma$ induces a homomorphism

$$\phi_{P(k/F)} : P(k/F)/F^\times E(k) \longrightarrow W(k)/E(k)^{1-\sigma}$$

of quotients. Let $\alpha \in P(k/F)$ satisfy $\alpha F^\times E(k) \in \ker \phi_{P(k/F)}$, i.e., $\alpha^{1-\sigma} = \eta^{1-\sigma}$ for some $\eta \in E(k)$. Then, α/η is invariant under σ and hence lies in $F^\times$. It follows $\alpha \in F^\times E(k)$. Therefore, $\phi_{P(k/F)}$ is injective. Noting that $\operatorname{Im}\phi_{P(k/F)} = P(k/F)^{1-\sigma}/E(k)^{1-\sigma}$, we get the second identity.

We nextly deduce the third assertion from the first two assertions.

We lastly prove the last assertion. By the first assertion, the map $1-\sigma$ induces a surjective homomorphism

$$\phi_{A(k)} : A(k)/P(k/F) \longrightarrow W(k)/P(k/F)^{1-\sigma}$$

of quotients. Let $\alpha \in A(k)$ satisfy $\alpha P(k/F) \in \ker \phi_{A(k)}$, i.e., $\alpha^{1-\sigma} = \gamma^{1-\sigma}$ for some $\gamma \in P(k/F)$. Then, α/γ is invariant under σ and hence lies in $F^{\times}$. It follows $\alpha \in P(k/F)$. Therefore, $\phi_{P(k/F)}$ is an injection and hence is an isomorphism. Identity $[A(k) : P(k/F)] = [A(k)^{1-\sigma} : P(k/F)^{1-\sigma}]$ follows. The first two assertions and this identity imply the last assertion. □

Lemma 18. *Let $k \subset K$ be CM-fields. Then we have $w(k) \mid w(K)$, $w(k)Q(k) \mid w(K)Q(K)$ and $w(k)Q(k)\kappa(k) \mid w(K)Q(K)\kappa(K)$.*

Proof. The first assertion is obvious. The second and the third assertions are proven in a similar way. Hence, we give a proof for the third assertion.

Since $w(k) = 2\,[W(k)^{1-\sigma} : 1]$, the second assertion of Lemma 17 implies $\kappa(k)Q(k)w(k) = 2\,[P(k/k_+)^{1-\sigma} : 1]$. Similarly, we can obtain $\kappa(K)Q(K)\ w(K) = 2\,[P(K/K_+)^{1-\sigma} : 1]$. On the other hand, we obviously have $P(k/k_+) \subset P(K/K_+)$. Hence, $[P(k/k_+)^{1-\sigma} : 1]$ divides $[P(K/K_+)^{1-\sigma} : 1]$. The desired assertion follows immediately. □

Remark. If $w(K)/w(k)$ is odd, the latter two divisibilities of Lemma 17 imply $Q(k) \mid Q(K)$ and $Q(k)\kappa(k) \mid Q(K)\kappa(K)$. If further we have $Q(k) = 2$, we get $Q(K) = 2$. Hence, we can sometimes calculate a Hasse's unit index of a CM-field through calculation of a Hasse's unit index of a smaller CM-field. However, $Q(K)/Q(k) = 1/2$ sometimes happens when $w(K)/w(k)$ is even. Lenstra's example in the preface to 1985-edition of [1] (see also [9]) is

Example 19. *Let $F = Q(\sqrt{8 \cdot 17})$. Then, $h(F) = 2$. Let $k = F(\sqrt{-4})$ and $k' = F(\sqrt{-8})$. Then, we have $w(k) = 4$, $w(k') = Q(k) = Q(k') = 2$, $\kappa(k) = \kappa(k') = 1$ and $h^-(k) = h^-(k') = 4$. Set $K = kk'$. We have $K_+ = \mathbb{Q}(\sqrt{8}, \sqrt{17})$, $h(K_+) = 1$. (K_+/F is unramified.) We have $w(K) = 8, Q(K) = 1$, $\kappa(K) = 1$, $\tau(K/F) = 2$, $c(K/F) = 4$ and $h^-(K) = 4$. Therefore, $h^-(k)$ and $h^-(k')$ divide $h^-(K)$. Note that $Q(K)/Q(k) = Q(K)/Q(k') = 1/2$ holds.*

See Hirabayashi and Yoshino [4] for further discussion and examples.

Determination of indices $Q(k)$ and $\kappa(k)$ is relatively easy if k does not contain $\sqrt{-1}$. However, it becomes delicate if k contains $\sqrt{-1}$. Therefore, we prepare a tool for dealing with CM-fields which contain $\sqrt{-1}$.

Definition 20. *We define Viéte numbers $V_0, V_2, V_3, \dots$ by*

$$V_0 = 2; \quad V_{i+1} = 2 + \sqrt{V_i} \quad (i = 0, 1, 2, \dots). \tag{8}$$

The Viéte index I of a number field L is the maximal index i such that $V_i \in L$. The ideal $\mathcal{V} = (V_I)$ is called the Viéte ideal of L.

Remark. Viéte's historical formula for π is

$$\frac{2}{\pi} = \sqrt{\frac{V_0}{4}}\sqrt{\frac{V_1}{4}}\sqrt{\frac{V_2}{4}}\cdots$$

with square roots taken in positive real numbers. (See e.g. [7, p. 251].)

Viéte numbers are algebraic integers. We see that $\mathbb{Q}(V_i, \sqrt{-1})$ is the 2^{i+2}-th cyclotomic field. Or more precisely, $V_i = (1+\zeta)(1+\zeta^{-1})$ holds for some 2^{i+2}-th root ζ of unity. The Viéte ideal $\mathcal{V}$ is characterized by $\mathcal{V} = \left((1+\zeta)\left(1+\zeta^{-1}\right)\right)$ for a generator ζ of the 2-part of $W(L(\sqrt{-1}))$.

With notion of Viéte ideals, we determine $Q(k)$ and $\kappa(k)$ of CM-fields k:

Lemma 21. *Let k/F be a CM-extension. Choose $\delta \in F$ such that $k = F(\sqrt{-\delta})$. If $k \neq F(\sqrt{-1})$, indices $Q(k)$ and $\kappa(k)$ are determined as follows:*

	$Q(k)$	$\kappa(k)$	*condition;*
(i)	1	1	*if (δ) is not a square of any ideal of F;*
(ii)	1	2	*if (δ) is a square of a non-principal ideal of F;*
(iii)	2	1	*if (δ) is a square of a principal ideal of F.*

If $k = F(\sqrt{-1})$, indices $Q(k)$ and $\kappa(k)$ are determined as follows:

	$Q(k)$	$\kappa(k)$	*condition;*
(iv)	1	1	*if $\mathcal{V}$ is not a square of any ideal of F;*
(v)	1	2	*if $\mathcal{V}$ is a square of a non-principal ideal of F;*
(vi)	2	1	*if $\mathcal{V}$ is a square of a principal ideal of F.*

where $\mathcal{V}$ is the Viéte ideal of F.

Proof. We denote the complex conjugation of k by σ.

Case (i): We assume (δ) is not a square of any ideal of F. Then, $\sqrt{-\delta} \in A(k)$ and $\sqrt{-\delta} \notin P(k/F)$ hold. By the last statement of Lemma 17, we conclude $Q(k)\kappa(k) = 1$.

Case (ii): We assume (δ) is a square of a non-principal ideal of F. Obviously, $\kappa(k) > 1$ follows. By the third assertion of Lemma 17, we conclude $\kappa(k) = 2$ and $Q(k) = 1$.

Case (iii): We assume $k \neq F(\sqrt{-1})$ and (δ) is a square of a principal ideal (β) in F with $\beta \in F^\times$. Then, $\varepsilon = \delta/\beta^2$ is a unit in F and $\sqrt{-\varepsilon} = \sqrt{-\delta}/\beta \in k$ holds. Let ζ be a generator of $W(k)$ and set $\eta =$

$\zeta\sqrt{-\varepsilon}$. Then, we get $\eta^{1-\sigma} = -\zeta^2$. Since $\sqrt{-1} \notin k$, $(\eta^{1-\sigma})^2 = \zeta^4$ generates the subgroup $W(k)^2$ of index 2 in $W(k)$. Since $\xi^2 \in W(k)^2$ and $-1 \notin W(k)^2$, the unit $\eta^{1-\sigma} = -\zeta^2$ does not belong to $W(k)^2$. We get $[E(k)^{1-\sigma} : W(k)^2] > 1$. By the second assertion of Lemma 17, we conclude $Q(k) = 2$ and $\kappa(k) = 1$.

We now assume $k = F(\sqrt{-1})$. Let ζ be the generator of the 2-part of $W(k)$ and ξ the generator of the odd-part of $W(k)$. Then $\xi^{1-\sigma} = \xi^2$ is also a generator of the odd-part of $W(k)$. On the other hand, $(1+\zeta)^{1-\sigma} = \zeta$ generates the 2-part of $W(k)$. Therefore, $\alpha^{1-\sigma}$ generates $W(k)$ for $\alpha = (1+\zeta)\xi$. On the other hand, $(\alpha^2) = ((1+\zeta)(1+\zeta^{-1})) = \mathcal{V}$ holds. (Recall comment after Definition 20.)

Case (iv): In addition to $k = F(\sqrt{-1})$, we assume that $\mathcal{V}$ is not a square of any ideal of F. The fact $(\alpha)^2 = \mathcal{V}$ and the current assumption imply $[A(k) : P(k/F)] > 1$. By the last assertion of Lemma 17, we conclude $Q(k) = \kappa(k) = 1$.

Case (v): In addition to $k = F(\sqrt{-1})$, we assume that $\mathcal{V}$ is a square of a non-principal ideal of F. Then, α generates an ideal of F. Hence, we get $\kappa(k) > 1$. By the third assertion of Lemma 17, we conclude $Q(k) = 1$ and $\kappa(k) = 2$.

Case (vi): In addition to $k = F(\sqrt{-1})$, we assume that $\mathcal{V}$ is a square of a principal ideal (β) in F with $\beta \in F$. We see that α/β is a unit in k and $(\alpha/\beta)^{1-\sigma} = \alpha^{1-\sigma}$ generates $W(k)$. By the second assertion of Lemma 17, we conclude $Q(k) = 2$ and $\kappa(k) = 1$.

We determined $Q(k)$ and $\kappa(k)$ in all cases. □

The following lemma is also well-known and is useful for calculation.

Lemma 22. *Let k be a CM-field. If $E^+(k_+) = E(k_+)^2$, we have $Q(k) = 2$. If $h(k_+)$ is odd, we have $\kappa(k) = 2$.*

Proof. Let $F = k_+$. We prove contrapositive of the assertions.

Assume $Q(k) = 2$. Then, Case (iii) or (vi) of Lemma 21 holds: In Case (iii), there is an element β of F and $\eta \in E^+(F)$ such that $\delta = \eta\beta^2$. Hence, we have $F(\sqrt{-\eta}) = k \neq F(\sqrt{-1})$. Hence, we get $\eta \in E^+(F) - E(F)^2$ and hence $E^+(k_+) \neq E(k_+)^2$ In Case (vi), there is an element β of F and $\eta \in E^+(F)$ such that $V_i = \eta\beta^2$, where I denotes the Viéte index of F. By definition of Viéte index, V_i is not a square in F. Hence, we get $\eta \in E^+(F) - E(F)^2$ and hence $E^+(k_+) \neq E(k_+)^2$. A proof of the first assertion completes.

Assume $\kappa(k) = 2$. Then, Case (ii) or (v) of Lemma 21 holds. In either case, there is a non-principal ideal of F whose square is principal. Therefore, $h(F)$ is even. A proof of the second assertion completes. □

Lemmata of this subsection is silently used for calculation of examples through out the current paper.

2.3. Denominator Constant of Class Number Relation

We shall prove that the denominator constant of class number relation belongs to $\{1, 2, 4\}$. We also investigate the real delicacy of combination of indices by several examples.

Lemma 23. *Let k/F and k'/F be distinct CM-extensions. Denote by i the Viéte index of F. Set $\tau(K/F) = 2$ if $K = F(\sqrt{-1}, \sqrt{V_i})$ and $\tau(K/F) = 1$ otherwise. Then, we have*

$$[W(K) : W(k)W(k')] = \tau(K/F).$$

Moreover, we have

$$\frac{w(k)w(k')}{w(K)} = \frac{2}{\tau(K/F)}.$$

Proof. We firstly reduce the second assertion to the first assertion. We note that $k \cap k' = F$ implies $W(k) \cap W(k') = W(F) = \{\pm 1\}$. Therefore, we have $W(k)W(k')/\{\pm 1\} \simeq W(k)/\{\pm 1\} \times W(k')/\{\pm 1\}$. In particular, we get $2\,\#(W(k)W(k')) = w(k)w(k')$. Therefore, the second assertion is reduced to the first assertion.

We now prove the first assertion. Let ρ be the non-trivial conjugation of K/k and σ the complex conjugation of K. Then, $\rho\sigma$ becomes the non-trivial conjugation of K/k'. We consider the maps $\psi : \Xi \in W(K) \longmapsto (\Xi^{1+\rho}, \Xi^{1+\rho\sigma}) \in W(k) \times W(k')$ and $\varphi : (\xi, \xi') \in W(k) \times W(k') \longmapsto \xi\xi' \in W(K)$. Identities $\xi^{1+\rho} = \xi^2$, $\xi^{1+\rho\sigma} = N_{k/F}\xi = 1$, ${\xi'}^{1+\rho} = N_{k'/F}\xi = 1$, and ${\xi'}^{1+\rho\sigma} = \xi'^2$ imply $\psi\varphi(\xi, \xi') = (\xi^2, {\xi'}^2)$. On the other hand, we have $\varphi\psi(\Xi) = \Xi^{2+\rho(1+\sigma)} = \Xi^2$. Therefore, ψ and φ induces isomorphisms between the odd-parts of $W(k) \times W(k')$ and $W(K)$. (They do not necessarily give a pair of inverse isomorphisms.) Since ψ factors through $W(k)W(k')$, we see that the odd-parts of $W(k)W(k')$ and $W(K)$ are identical. Comparison of the 2-part of $W(k)W(k')$ and $W(K)$ is left.

If $\sqrt{-1} \notin K$, then the 2-parts of $W(k), W(k')$ and $W(K)$ are all identical to $\{\pm 1\}$. Therefore the 2-part of $W(k)W(k')$ and $W(K)$ agrees. On the other hand, $\tau(K/F) = 1$ holds in this situation. Therefore, we get $[W(K) : W(k)W(k')] = 1 = \tau(K/F)$.

We now assume $\sqrt{-1} \in K$. We assume $\sqrt{-1} \in k$ without loss of generality. The 2-part of $W(k)$ is generated by a 2^{i+2}-th root of unity. The 2-part of $W(k')$ is $\{\pm 1\}$. Hence, the 2-part of $W(k)W(k')$ is generated by a 2^{i+2}-th root of unity. Let I be the Viéte index of K_+.

Then, the 2-part of $W(K)$ is generated by a 2^{I+2}-th root of unity. Under our situation, $\tau(K/F) = 2^{I-i}$. Comparison of the order of 2-parts of $W(k)W(k')$ and $W(K)$ now implies $[W(K) : W(k)W(k')] = \tau(K/F)$. Our proof for the first assertion completes. □

We are now ready to prove the following lemma.

Lemma 24. *Let k/F and k'/F be distinct CM-extensions. Then, we have*

$$[E(K) : E(k)E(k')E(K_+)] = 1 \text{ or } 2.$$

Moreover, the denominator constant $c(K/F)$ of class number relation satisfy

$$c(K/F) \in \{1, 2, 4\}.$$

If not both of k and k' are obtained by adjoining square roots of units in F to F, the denominator constant $c(K/F)$ satisfies

$$c(K/F) \in \{1, 2\}.$$

Proof. We firstly reduce the second and the third assertions to the first assertion. By (4), we have

$$c(K/F) = 2^{1+v}/[E(K) : E(k)E(k')E(K_+)]$$

with $v \in \{0, 1\}$. Hence, the second assertion is reduced to the first assertion. The condition of the third assertion implies $v = 0$. Hence, the third assertion is reduced to the first assertion.

We now prove the first assertion. We have the following inclusions:

$$W(k)W(k')E(K_+) \subset W(K)E(K_+) \subset E(K).$$

Identity $[E(K) : W(k)W(k')E(K_+)] = \tau(K/F)Q(K)$ follows. On the other hand, we have the following inclusions:

$$W(k)W(k')E(K_+) \subset E(k)E(k')E(K_+) \subset E(K).$$

Therefore, the index $[E(K) : E(k)E(k')E(K_+)]$ divides $\tau(K/F)Q(K)$. The assertion of the theorem follows if $\tau(K/F)$ or $Q(K)$ is 1. (Recall that Lemma 17 and 23 imply $\tau(K/F), Q(K) \in \{1, 2\}$.)

We now assume $\tau(K/F) = Q(K) = 2$. Let i be the Viéte index of F. Under the current assumption, $K_+ = F(\sqrt{V_i})$ holds by Lemma 23. Therefore, $i+1$ is the Viéte index of K_+. By Lemma 21, $Q(K) = 2$ implies that the Viéte ideal (V_{i+1}) of K_+ is a square of a principal ideal of K_+. Choose $\alpha \in K_+$ such that $(V_{i+1}) = (\alpha)^2$. Taking norm to F, we get $(V_i) = (N_{K_+/F}V_{i+1}) = (N_{K_+/F}\,\alpha)^2$. By Lemma 21, we get $Q(k) = 2$.

Let ζ be a generator of the 2-part G of $W(k)$. Set $\beta = N_{K_+/F}\,\alpha$ and $\gamma = (1+\zeta)/\beta$. Then, $\gamma \in E(k)$ holds. On the other hand, $\beta/\sqrt{V_i} \in E(K_+)$ follows from $K_+ = F(\sqrt{V_i})$. Set $\xi = \gamma \cdot \beta/\sqrt{V_i}$. Then, $\xi \in E(k)E(K_+)$ holds. On the other hand $\xi = (1+\zeta)/\sqrt{V_i}$ generates the 2-part of $W(K)$. Therefore, $W(K)E(K_+) \subset E(k)W(k')E(K_+)$ follows. Hence, we get $W(K)E(K_+) \subset E(k)E(k')E(K_+)$. Now, we see $[E(K) : E(k)E(k')E(K_+)]$ divides $[E(K) : W(K)E(K_+)] = Q(K) = 2$. □

In the proof, the following fact became apparent:

Lemma 25. *Let k'/F be a CM-extension other than $F(\sqrt{-1})$. Set $k = F(\sqrt{-1})$ and $K = kk'$. Then, the following implications hold.*

- $\tau(K/F) = Q(K) = 2 \implies Q(k) = 2;$
- $\tau(K/F) = Q(k) = 2 \implies Q(k') = 2;$
- $\tau(K/F) = Q(k') = 2 \implies Q(k) = 2;$
- $[E^+(F) : E(F)^2] = Q(k) = Q(k') = 2 \implies \tau(K/F) = 2.$

Here, $E^+(F)$ denotes the group of totally positive units of F.

When $[E^+(F) : E(F)^2] = Q(k) = Q(k') = 2$, equality $Q(K) = 2$ is possible but not necessary. An example of $Q(K) = 1$ is Example 19. Two examples of $Q(K) = 2$ are below:

Example 26. *Let $F = \mathbb{Q}(\sqrt{-8\cdot -3})$. Then, $h(F) = 1$. The Viéte index i of F is 0 and $(V_i) = (2+\sqrt{6})^2$. Let $k = F(\sqrt{-4})$ and $k' = F(\sqrt{-3})$. Then, we have $w(k) = 4$, $w(k') = 2\cdot 3$, $Q(k) = Q(k') = 2$, $\kappa(k) = \kappa(k') = 1$, $h^-(k) = 2$ and $h^-(k') = 1$. Here, k/F is ramified above (2) and k'/F is unramified at all finite primes. Set $K = kk'$. The Viéte index I of K_+ is 1 and $(V_I) = ((1+\sqrt{2}+\sqrt{3})/\sqrt{2})^2$. We have $K_+ = F(\sqrt{8})$, $h(K_+) = 1$, $w(K) = 8\cdot 3$, $Q(K) = 2$, $\kappa(K) = 1$, $\tau(K/F) = c(K/F) = 2$ and $h^-(K) = 1$. Therefore, $h^-(k')$ divides $h^-(K)$.*

Example 27. *Let $F = \mathbb{Q}(\sqrt{-8\cdot -7})$. Then, $h(F) = 1$. The Viéte index i of F is 0 and $(V_i) = (4+\sqrt{14})^2$. Let $k = F(\sqrt{-4})$ and $k' = F(\sqrt{-8})$. Then, we have $w(k) = 4$, $w(k') = Q(k) = 2$, $Q(k') = 2$, $\kappa(k) = \kappa(k') = 1$, $h^-(k) = 4$ and $h^-(k') = 1$. Here, k/F is ramified above (2) and k'/F is unramified at all finite primes. Set $K = kk'$. We have $h(K_+) = 1$. The Viéte index I of K_+ is 1 and $(V_I) = ((1+\sqrt{2}+\sqrt{7})/\sqrt{2})^2$. We have $K_+ = F(\sqrt{8})$, $h(K_+) = 1$, $w(K) = 8$, $Q(K) = 2$,*

$\kappa(K) = 1$, $\tau(K/F) = c(K/F) = 2$ and $h^-(K) = 2$. Therefore, $h^-(k')$ divides $h^-(K)$.

In Examples 26 and 27, the Viéte ideal of K_+ is a square of a principal ideal of K_+ which is not a lift of any ideal of F. In some cases, the Viéte ideal of K_+ is a square of a lift of an ideal of F:

Example 28. *Let $F = \mathbb{Q}(\sqrt{-8 \cdot -3}, \sqrt{-4 \cdot -11})$. Then, we have $h(F) = 2$. The Viéte index i of F is 0. We have $(V_i) = (2+\sqrt{6})^2 = (3+\sqrt{11})^2 = (8+\sqrt{66})^2$. The prime ideal (2) ramifies totally in $F/\mathbb{Q}$ and the prime ideal of F above (2) is generated by $2/(1+\sqrt{6}+\sqrt{11})$. Let $k = F(\sqrt{-4})$ and $k' = F(\sqrt{-8})$. Then, we have $w(k) = 4$, $w(k') = 2 \cdot 3$, $Q(k) = Q(k') = 2$, $\kappa(k) = \kappa(k') = 1$, $h^-(k) = 4$ and $h^-(k') = 2$. Here, k/F and k'/F are unramified at all finite primes. Set $K = kk'$. Then, we have $K_+ = F(\sqrt{8})$. The Viéte index I of K_+ is 1. We have $(V_I) = (2/(1+\sqrt{6}+\sqrt{11}))^2$. We have $h(K_+) = 1$, $w(K) = 8 \cdot 3$, $Q(K) = 2$, $\kappa(K) = 1$, $\tau(K/F) = c(K/F) = 2$ and $h^-(K) = 4$. Therefore, $h^-(k)$ and $h^-(k')$ divide $h^-(K)$.*

Example 29. *Let $F = \mathbb{Q}(\sqrt{-8 \cdot -3}, \sqrt{-4 \cdot -7})$. Then, we have $h(F) = 2$. The Viéte index i of F is 0. We have $(V_i) = (2+\sqrt{6})^2 = (3+\sqrt{7})^2$. However, the prime ideal of $\mathbb{Q}(\sqrt{8 \cdot -3 \cdot -7})$ above (2) is non-principal as it is verified by use of Legendre symbol. The prime ideal (2) ramifies totally in $F/\mathbb{Q}$ but the prime ideal of F above (2) is non-principal. (Note that its norm to $\mathbb{Q}(\sqrt{8 \cdot -3 \cdot -7})$ is non-principal.) Let $k = F(\sqrt{-4})$ and $k' = F(\sqrt{-8})$. Then, we have $w(k) = 4$, $w(k') = 2 \cdot 3$, $Q(k) = Q(k') = 2$, $\kappa(k) = \kappa(k') = 1$, $h^-(k) = 2$ and $h^-(k') = 4$. Here, k/F and k'/F are unramified at all finite primes. Set $K = kk'$. Then, we have $K_+ = F(\sqrt{8})$. The Viéte index I of K_+ is 1. We have $(V_I) = ((1+\sqrt{2}+\sqrt{3})/\sqrt{2})^2 = ((1+\sqrt{2}+\sqrt{7})/\sqrt{2})^2$. (The prime ideal of F above (2) capitulates in K_+/F.) We have $h(K_+) = 1$, $w(K) = 8 \cdot 3$, $Q(K) = 2$, $\kappa(K) = 1$, $\tau(K/F) = c(K/F) = 2$ and $h^-(K) = 4$. Therefore, $h^-(k)$ and $h^-(k')$ divide $h^-(K)$.*

(Proof of $h(K_+) = 1$ for these two examples is in [13]. Since K_+/F is unramified, this implies $h(F) = 2$ by class field theory.)

We cannot infer $\tau(K/F) = 2$ from $Q(k) = Q(k') = 2$ and $k = k_+(\sqrt{-1})$ alone, although $\tau(K/F) = 2$ is possible as Examples 19, 26 and 27 show.

Example 30. *Let $F = \mathbb{Q}(\sqrt{8}, \sqrt{-8 \cdot -7})$. Then, we have $h(F) = 1$. We also have $[E^+(F) : E(F)^2] = 4$. The Viéte index i of F is 1. We have $(V_i) = ((1+\sqrt{2}+\sqrt{7})/\sqrt{2})^2$. Let $k = F(\sqrt{-4})$ and $k' = F\left(\sqrt{-(3+\sqrt{7})(2+\sqrt{2})/2}\right)$. Then, we have $w(k) = 8$, $w(k') =$*

$Q(k) = Q(k') = 2$, $\kappa(k) = \kappa(k') = 1$, $h^-(k) = 2$ and $h^-(k') = 4$. Here, k/F is unramified at all finite primes and k'/F is ramified above (2). Set $K = kk'$. Then, K_+/F is ramified above (2). The Viéte index I of K_+ equals i. We have $h(K_+) = 1$, $w(K) = 8$, $Q(K) = 2$, $\kappa(K) = 1$, $\tau(K/F) = 1$, $c(K/F) = 4$ and $h^-(K) = 2$. Therefore, $h^-(k)$ divides $h^-(K)$.

Below is slightly difficult part of calculation of Example 30:
Data of F: The group $E(F)$ is generated by -1, $1+\sqrt{2}$, $(4+\sqrt{14})/\sqrt{2}$, $(3+\sqrt{7})/\sqrt{2}$. Therefore, $E^+(F)$ is generated by $(1+\sqrt{2})^2$, $(1+\sqrt{2})(4+\sqrt{14})/\sqrt{2}$, $(1+\sqrt{2})(3+\sqrt{7})/\sqrt{2}$. Hence have $[E^+(F) : E(F)^2] = 4$.
A quartic subfield of k': Let $\eta = \sqrt{-(3+\sqrt{7})(2+\sqrt{2})/2}$ and $\eta' \neq \pm\eta$ a conjugate of η over $\mathbb{Q}(\sqrt{14})$. Then, we have $(\eta\eta')^2 = 1$. Since $\pm\eta'$ are conjugate integers of η over $\mathbb{Q}(\sqrt{14})$, we get that $\eta'' = 1/\eta$ is a conjugate of η over $\mathbb{Q}(\sqrt{14})$. It is easy to verify $(\eta+\eta'')^2 = -4-\sqrt{14}$ and $(\eta-\eta'')^2 = -8-\sqrt{14} = (-4+\sqrt{14})(3+\sqrt{14})^2$. Therefore, k' contains the normal closure of $L = \mathbb{Q}\left(\sqrt{-4-\sqrt{14}}\right)$. Comparing degrees, we see that k' is the normal closure of L. Since the maximal abelian subfield of L is $\mathbb{Q}(\sqrt{14})$, we have $w(L) = 2$. Since $\sqrt{-4-\sqrt{14}}$ belongs to $A(L) - P(L/F)$, the last assertion of Lemma 17 implies $Q(L) = \kappa(L) = 1$. We have $h^-(L) = 2$. (See [10, p 1143].)
Data of k': Since k' is obtained by composing conjugate fields of L, class number relation (2) and (3) imply $h^-(k') = w(k')Q(k')h^-(L)^2/4 = w(k')Q(k')$. Since the maximal abelian subfield of k' is $\mathbb{Q}(\sqrt{8}, \sqrt{-8\cdot -7})$, we have $w(k') = 2$. On the other hand, $k'/\mathbb{Q}$ is non-abelian while $k/\mathbb{Q}$ is abelian. Hence, $k' \neq k$ follows. Moreover, η is a unit. These two points imply $Q(k') = 2$ by Lemma 21. We now see $h^-(k') = 4$. Pari (ver. 2.06) confirms $h(k') = h(\mathbb{Q}[X]/(X^8 + 12X^6 + 24X^4 + 12X^2 + 1)) = 1$.
Data of K_+: Since the maximal abelian subfield of K_+ is F, the Viéte index I of K_+ equals i. Pari (ver. 2.06) computes $h(K_+) = h(\mathbb{Q}[X]/(X^8 - 12X^6 + 24X^4 - 12X^2 + 1)) = 1$.
Data of K: The previous assertion implies $Q(K) = Q(k) = 2$ and $\tau(K/F) = 1$. Since the maximal abelian subfield of K is k, we have $w(K) = w(k) = 8$. Now, we have enough data to calculate $c(K/F) = 4$ and $h^-(K) = 2$.

When $k = F(\sqrt{-4})$, $Q(k) = 2$, $Q(k') = 1$ and $\tau(K/F) = 1$ hold, Lemma 18 implies $Q(K) = 2$ and hence $c(K/F) = 2$. However, the situation is again complicated when $k = F(\sqrt{-4})$, $Q(k) = 1$, $Q(k') = 2$ and $\tau(K/F) = 1$. There is an example of $Q(K) = 1$ ($c(K/F) = 4$) and examples of $Q(K) = 2$ ($c(K/F) = 2$.)

Example 31. *Let* $F = \mathbb{Q}(\sqrt{-3\cdot 5\cdot -7})$*. Then, we have* $h(F) = 2$*. The Viéte index* i *of* F *is* 0*. The Viéte ideal* (V_i) *of* F *is not a square of any ideal of* F*. Let* $\varepsilon = 41+4\sqrt{105}$*. Then, we have* $(10+\sqrt{105})^2 = 5\varepsilon$*. Let* $k = F(\sqrt{-4})$ *and* $k' = F(\sqrt{-4\cdot 5}) = F(\sqrt{-\varepsilon})$*. Then, we have* $w(k) = 4$, $w(k') = 2$, $Q(k) = 1$, $Q(k') = 2$, $\kappa(k) = \kappa(k') = 1$, $h^-(k) = 4$ *and* $h^-(k') = 8$*. Here,* k/F *and* k'/F *are unramified above* (2)*. Set* $K = kk'$*. Then,* $K_+/F = F(\sqrt{5})/F$ *is unramified. The Viéte index* I *of* K_+ *is* 0*. The Viéte ideal* (V_I) *of* F *is not a square of any ideal of* F*. We have* $h(K_+) = 1$, $w(K) = 4$, $Q(K) = \kappa(K) = \tau(K/F) = 1$, $c(K/F) = 4$ *and* $h^-(K) = 8$*. Therefore,* $h^-(k)$ *and* $h^-(k')$ *divide* $h^-(K)$*.*

Example 32. *Let* $F = \mathbb{Q}(\sqrt{-4\cdot -3\cdot 5})$*. Then, we have* $h(F) = 2$*. The Viéte index* i *of* F *is* 0*. The Viéte ideal* (V_i) *of* F *is a square of a non-principal ideal of* F*. Let* $\varepsilon = 4+\sqrt{15}$*. Then, we have* $(3+\sqrt{15})^2 = 6\varepsilon$*. Let* $k = F(\sqrt{-4})$ *and* $k' = F(\sqrt{8\cdot -3}) = F(\sqrt{-\varepsilon})$*. Then, we have* $w(k) = 4$, $w(k') = 2$, $Q(k) = 1$, $Q(k') = 2$, $\kappa(k) = 2$, $\kappa(k') = 1$, $h^-(k) = 1$ *and* $h^-(k') = 4$*. Here,* k/F *is unramified at all finite primes and* k'/F *is unramified above* (2)*. Set* $K = kk'$*. Then, we have* $K_+/F = F(\sqrt{-8\cdot -3})/F$ *is ramified above* (2)*. The Viéte index* I *of* K_+ *is* 0*. We have* $(V_I) = (4+\sqrt{6})^2$*. We have* $h(K_+) = 2$, $w(K) = 4$, $Q(K) = 2$, $\kappa(K) = \tau(K/F) = 1$, $c(K/F) = 2$ *and* $h^-(K) = 2$*. Therefore,* $h^-(k)$ *divides* $h^-(K)$*.*

Example 33. *Let* $F = \mathbb{Q}(\sqrt{-8\cdot -3\cdot 5})$*. Then, we have* $h(F) = 2$*. The Viéte index* i *of* F *is* 0*. The Viéte ideal* (V_i) *of* F *is a square of a non-principal ideal of* F*. Let* $\varepsilon = 11+2\sqrt{30}$*. Then, we have* $(5+\sqrt{30})^2 = 5\varepsilon$*. Let* $k = F(\sqrt{-4})$ *and* $k' = F(\sqrt{-4\cdot 5}) = F(\sqrt{-\varepsilon})$*. Then, we have* $w(k) = 4$, $w(k') = 2$, $Q(k) = 1$, $Q(k') = 2$, $\kappa(k) = 2$, $\kappa(k') = 1$, $h^-(k) = 2$ *and* $h^-(k') = 4$*. Here,* k/F *and* k'/F *are unramified above* (2)*. Set* $K = kk'$*. Then, we have* $K_+ = F(\sqrt{5})$*. Then,* K_+/F *is unramified. The Viéte index* I *of* K_+ *is* 0*. We have* $(V_I) = (2+\sqrt{6})^2$*. We have* $h(K_+) = 1$, $w(K) = 4$, $Q(K) = 2$, $\kappa(K) = \tau(K/F) = 1$, $c(K/F) = 2$ *and* $h^-(K) = 4$*. Therefore,* $h^-(k)$ *and* $h^-(k')$ *divide* $h^-(K)$*.*

§3. Consistency of the Two Competing Tools

We shall firstly formulate the essential part of the proposed problem of consistency in §§3.1. The formulation will be the parity equality of Theorems 1 and 2. We shall secondly give non-trivial examples of parity equality in §§3.2. We shall lastly prove parity equality in §§3.3.

3.1. Parity Equality as Consistency

We shall formulate the essential part of the proposed problem of consistency.

Let k/F and k'/F be distinct CM-extensions. Set $K = kk'$. We display Identity (6) and (5):

$$\frac{h^-(k')}{c(K/F)} = \frac{h^-(K)}{h^-(k)}; \tag{9}$$

$$c(K/F) \in \{1, 2, \ldots, 2^{1+v}\} \tag{10}$$

where v is 1 if k and k' are obtained by adjoining square roots of units to F and 0 otherwise.

Identity (9) together with (10) suggests $h^-(K)/h^-(k)$ can have non-trivial denominator. Indeed, it does have non-trivial denominator in some cases as Examples 11, 12, 13 and 14 show.

On the other hand, Propositions 8 and 9 implies (7), i.e.,

$$\frac{h^-(K)}{h^-(k)} \in \mathbb{Z} \tag{11}$$

under the situation

(A) k/F is unramified at all finite primes or
(B) K_+/F is unramified.

This looks contradicting the mentioned suggestion. We analyze delicate relation of (9) and (11).

1. If $h^-(k)$ is odd, comparison of denominators in the both sides of (9) implies (11).

2. If $h^-(k)$ is even under situation (A) or (B), we need either $2 \mid h^-(k')$ or $c(K/F) = 1$ for consistency of (9) and (11). Indeed, $c(K/F)$ is often 2. Therefore, we are lead to

> **Suspicion:** Some principle forces $h^-(k')$ to be even when $h^-(k)$ is even under situation (A) or (B).

Of course, possibility of $c(K/F) = 4$ poses a further difficult problem. (See Example 19.) However, Suspicion explains some part of consistency of (11) with (9).

3. Under situation (A), we have $E^+(k_+) \neq E(k_+)^2$. If k'/F is ramified at some finite prime under situation (A), $h^-(k')$ is even (by Lemma 15) and Suspicion is explained.

4. Therefore, interesting cases are (A′) and (B), where (A′) is the following situation:

(A′) k/F and k'/F are unramified at all finite primes.

Since situations (A′) and (B) have symmetry with respect to exchange of k and k', Suspicion is formulated as the following equivalence

$$h^-(k) \text{ is even } \Longleftrightarrow h^-(k') \text{ is even}$$

under situation (A′) or (B). The equivalence is stated as parity equality in Theorems 1 and 2.

In conclusion, Theorems 1 and 2 are interpretation of some part of a delicate competition and consistency of the field theoretic tool and class number relation.

3.2. Examples of Parity Equality

We shall give delicate examples for Theorems 1, 2, Propositions 8 and 9. The examples shall illustrate that the formulated problem indeed makes sense.

We begin with Theorem 1 and Proposition 8.

Example 34. *Let $F = \mathbb{Q}(\sqrt{-4\cdot -3\cdot 5})$. Then, $h(F) = 2$. The Viéte index i of F is 0. The Viéte ideal (V_i) is a square of a non-principal ideal of F. Let $k = F(\sqrt{-4})$ and $k' = F(\sqrt{-3})$. Then, we have $w(k) = 4, w(k') = 2\cdot 3$, $Q(k) = Q(k') = 1, \kappa(k) = \kappa(k') = 2$ and $h^-(k) = h^-(k') = 1$. Moreover, k/F and k'/F are unramified at all finite primes. Set $K = kk'$. The Viéte index I of K_+ equals i. We have $(V_I) = (1+\sqrt{3})^2$. We have $K_+ = F(\sqrt{5})$, $h(K_+) = 1$, $w(K) = 4\cdot 3$, $Q(K) = 2$, $\kappa(K) = 1$, $\tau(K/F) = 1$, $c(K/F) = 1$ and $h^-(K) = 1$. Therefore $h^-(k)$ and $h^-(k')$ divide $h^-(K)$.*

Example 35. *Let $F = \mathbb{Q}(\sqrt{-4\cdot -3\cdot 17})$. Then, $h(F) = 2$. The Viéte index i of F is 0. We have $(V_i) = (7+\sqrt{51})^2$. Let $k = F(\sqrt{-4})$ and $k' = F(\sqrt{-3})$. Then, we have $w(k) = 4$, $w(k') = 2\cdot 3$, $Q(k) = 2$, $Q(k') = 1$, $\kappa(k) = 1$, $\kappa(k') = 2$, and $h^-(k) = h^-(k') = 2$. Moreover, k/F and k'/F are unramified at all finite primes. Set $K = kk'$. The Viéte index I of K_+ equals i. We have $K_+ = F(\sqrt{17})$, $h(K_+) = 1$, $w(K) = 4\cdot 3$, $Q(K) = 2$, $\kappa(K) = 1$, $\tau(K/F) = 1$, $c(K/F) = 2$ and $h^-(K) = 2$. Therefore $h^-(k)$ and $h^-(k')$ divide $h^-(K)$.*

The CM-field k' in Example 35 shows that the order of 2 in $h^-(k')$ can be greater than the lower bound imposed by Lemma 4.

We turn to Theorem 2 and Proposition 9.

Example 36. *Let $F = \mathbb{Q}(\sqrt{8\cdot 5})$. Then, $h(F) = 2$. The Viéte index of F is 0. The Viéte ideal (V_i) is a square of a non-principal ideal of F. Let $k = F(\sqrt{-4})$ and $k' = F(\sqrt{-8})$. Then, we have $w(k) = 4$,*

$w(k') = 2$, $Q(k) = Q(k') = 1$, $\kappa(k) = \kappa(k') = 2$ and $h^-(k) = h^-(k') = 1$. Extensions k/F and k'/F are ramified above (2). Set $K = kk'$. Then, $K_+/F = F(\sqrt{8})/F$ is unramified. The Viéte index I of K_+ is 1. The Viéte ideal (V_I) is not a square of any ideal of K_+. We have $h(K_+) = 1$, $w(K) = 8$, $Q(K) = 1$, $\kappa(K) = 1$, $\tau(K/F) = 2$, $c(K/F) = 1$, and $h^-(K) = 1$. Therefore $h^-(k)$ and $h^-(k')$ divide $h^-(K)$.

Example 37. *Let $F = \mathbb{Q}(\sqrt{8\cdot 5})$. Then, $h(F) = 2$. Let $k = F(\sqrt{-3})$ and $k' = F(\sqrt{8\cdot -3})$. Then, we have $w(k) = 2\cdot 3$, $w(k') = 2$, $Q(k) = Q(k') = \kappa(k) = \kappa(k') = 1$ and $h^-(k) = h^-(k') = 2$. Extensions k/F and k'/F are ramified above (3). Set $K = kk'$. Then, $K_+/F = F(\sqrt{8})/F$ is unramified. We have $h(K_+) = 1$, $w(K) = 2\cdot 3$, $Q(K) = 1$, $\kappa(K) = 1$, $\tau(K/F) = 1$, $c(K/F) = 2$, and $h^-(K) = 2$. Therefore $h^-(k)$ and $h^-(k')$ divide $h^-(K)$.*

Example 38. *Let $F = \mathbb{Q}(\sqrt{8\cdot 5})$. Then, $h(F) = 2$. Let $k = F(\sqrt{-7})$ and $k' = F(\sqrt{8\cdot -7})$. Then, we have $w(k) = w(k') = 2$, $Q(k) = Q(k') = \kappa(k) = \kappa(k') = 1$, $h^-(k) = 2$ and $h^-(k') = 4$. Extensions k/F and k'/F are ramified above (7). Set $K = kk'$. Then, $K_+/F = F(\sqrt{8})/F$ is unramified. We have $h(K_+) = 1$, $w(K) = 2$, $Q(K) = 1$, $\kappa(K) = 1$, $\tau(K/F) = 1$, $c(K/F) = 2$, and $h^-(K) = 4$. Therefore $h^-(k)$ and $h^-(k')$ divide $h^-(K)$.*

Example 19 is also an example of Theorem 2 and Proposition 9.

We have seen non-trivial examples of Theorems 1, 2, Propositions 8 and 9.

3.3. Proof of Parity Equality

We shall firstly reduce Theorem 1 to Theorem 2 and then prove Theorem 2.

Proof of Theorem 1. CM-extensions k/F and k'/F are unramified at all finite primes. Then, K/F is unramified at all finite primes. Hence, K_+/F is unramified at all finite primes. On the other hand K_+/F is unramified at the infinite primes since K_+ is totally real. Therefore, K_+/F is unramified. Theorem 2 now implies the desired equivalence. □

Proof of Theorem 2. We introduce some notation and reformulate the assertion. We denote by $\mathcal{H}^{(2)}(L)$ the maximal 2-extension of L in $\mathcal{H}(L)$ for a number field L. Since $\mathcal{H}(L)/L$ is abelian, the order of $[\mathcal{H}(L) : \mathcal{H}^{(2)}(L)]$ is always odd. When L is a CM-field (i.e., L_+ makes sense), the ratio $[\mathcal{H}(L) : L\mathcal{H}(L_+)]/[\mathcal{H}^{(2)}(L) : L\mathcal{H}^{(2)}(L_+)]$ is odd. Therefore, the parity of $[\mathcal{H}(L) : L\mathcal{H}(L_+)]$ and that of $[\mathcal{H}^{(2)}(L) : L\mathcal{H}^{(2)}(L_+)]$ are identical. By the identification (1), the former index is $h^-(L)$. On the other hand, $2 \mid [\mathcal{H}^{(2)}(L) : L\mathcal{H}^{(2)}(L_+)]$ is equivalent to $[\mathcal{H}^{(2)}(L) : L\mathcal{H}^{(2)}(L_+)] > 1$. Therefore, we get:

$$(12) \qquad 2 \mid h^-(L) \iff [\mathcal{H}^{(2)}(L) : L\mathcal{H}^{(2)}(L_+)] > 1.$$

The assertion of the Theorem is now equivalent to

$$(13) \qquad [\mathcal{H}^{(2)}(k) : k\mathcal{H}^{(2)}(F)] > 1 \iff [\mathcal{H}^{(2)}(k') : k'\mathcal{H}^{(2)}(F)] > 1.$$

By symmetry, it suffice to prove the implication from left to right. We assume the left hand side and prove the right hand side in several steps:
Step 1. (Isolation of essential case): If $k = k'$, our conclusion is trivial. Therefore, we assume $k \neq k'$. By class field theory, unramifiedness of the quadratic extensionK_+/F implies that the 2-rank of $\mathcal{C}(F)$ is positive. If the 2-rank of $\mathcal{C}(F)$ is greater than 1, Lemma 4 implies that $h^-(k')$ is even, which is equivalent to our conclusion through (12). We now assume that the 2-rank $\mathcal{C}(F)$ is 1. By class field theory, $\mathcal{H}^{(2)}(F)/F$ is a non-trivial cyclic extension.
Step 2. (Construction of extension): Since the quadratic extension K_+ $/F$ is unramified, we have $k' \subset K = kK_+ \subset k\mathcal{H}^{(2)}(F)$. Inclusion $k'\mathcal{H}^{(2)}(F) \subset k\mathcal{H}^{(2)}(F)$ follows. By symmetry, we get the reverse inclusion and hence $k'\mathcal{H}^{(2)}(F) = k\mathcal{H}^{(2)}(F)$. Hence, our assumption implies $[\mathcal{H}^{(2)}(k) : k'\ \mathcal{H}^{(2)}(F)] > 1$.
Step 3. (Unramifiedness): On the other hand, $\mathcal{H}^{(2)}(k)/K$ is unramified since K is an intermediate field of an unramified extension $\mathcal{H}^{(2)}(k)/k$. The extension $K/k' = k'K_+/k'$ is also unramified since K_+/F is unramified. Therefore, $\mathcal{H}^{(2)}(k)/k'$ is unramified.
Step 4. (Galois property): Since k/F is normal, class field theory implies normality of $\mathcal{H}^{(2)}(k)/F$. It follows that $\mathcal{H}^{(2)}(k)/k' = k'\mathcal{H}^{(2)}(k)$ $/k'F$ is also normal. On the other hand, $k'\mathcal{H}^{(2)}(F)/k'$ is cyclic since $\mathcal{H}^{(2)}(F)/F$ is cyclic by Step 1.
Step 5. (Abelian extension): Let $G = \mathrm{Gal}(\mathcal{H}^{(2)}(k)/k')$. Let H be the maximal abelian extension of k' in $\mathcal{H}^{(2)}(k)$. It turns out $[H : k'\mathcal{H}^{(2)}(F)]$ > 1. Suppose contrary $H = k'\mathcal{H}^{(2)}(F)$. Then, the maximal abelian quotient of G is cyclic. Hence, Burnside Basis Theorem implies that G is cyclic. (See e.g. [12, Theorem 1.16 (p. 92)] for Burnside Basis Theorem.) Hence $\mathcal{H}^{(2)}(k)/k'$ is abelian, i.e., $H = \mathcal{H}^{(2)}(k)$ holds. Now, the conclusion of Step 2 contradicts the supposition on H. By contradiction, we see $[H : k'\mathcal{H}^{(2)}(F)] > 1$.
Step 6. (Class Field): On the other hand, Step 3 and the definition of H implies $H \subset \mathcal{H}^{(2)}(k')$. Therefore, $[\mathcal{H}^{(2)}(k') : k'\mathcal{H}^{(2)}(F)] > 1$ follows. □

§4. Conclusion

We reviewed Horie's theorem on divisibility of relative class numbers, i.e., a theorem on an obstacle for class number one. It was explained that a generalization of Horie's theorem has been proven by cooperation of three tools: the group theoretic tool, the field theoretic tool and class number relation. A certain competition of the latter two tools was explained. The competition arose when a pair of distinct CM-extensions k/F and k'/F with $K = kk'$ are in one of the following situations: k/F is unramified at all finite primes or; K_+/F is unramified. The second tool gave apparently stronger obstacle for class number one. In §1, the reason for consistency of application of the two tools, i.e., for integrality of the ratio $h^-(k')/c(K/F)$, is asked.

The two tools were discussed in detail in §2 before analysis of the problem.

Suspicion was responsibility of $h^-(k')$ for consistency and hence for the obstacle. (See §§3.1.) Suspicion was formulated as parity equality of Theorems 1 and 2. The parity equality and the real problem of consistency were illustrated by an example in §§3.2. The two theorems were proven by the field theoretic tool in §§3.3. Unfortunately, the proof was one-sided. Hence, it was delicate if the consistency was really explained. However, responsibility of $h^-(k')$ for the obstacle to class number one was established. It was also confirmed by examples.

A further problem is caused by the possibility of $c(K/F) = 4$. Indeed, Examples 19 and 31 show that $c(K/F) = 4$ sometimes happen in the situation of Theorem 2. As Example 19 of §§2.2 and Examples 26 through 33 of §§2.3 show, the value of $c(K/F)$ is hard to understand. Therefore, Theorems 1 and 2 constitute a meaningful answer to the problem of consistency although they might not constitute the perfect answer.

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