Advanced Studies in Pure Mathematics 30, 2001
Class Field Theory – Its Centenary and Prospect
pp. 31–62

# How Hasse was led to the Theory of Quadratic Forms, the Local-Global Principle, the Theory of the Norm Residue Symbol, the Reciprocity Laws, and to Class Field Theory

**Günther Frei**

## §1. Representation by Rational Quadratic Forms over $\mathbb{Q}$.

1. Hasse began his studies during the First World War on the 27th of September 1917 at the University in Kiel, where Otto Toeplitz was his principal teacher. After the war Hasse moved to Göttingen where he registered at the Georg-August University on the 16th of December 1918. At that time Göttingen was the center of mathematical research, not only in Germany but worldwide. The three main chairs for pure mathematics were occupied by Hilbert, Hecke (when Hecke left to Hamburg in 1920, Courant, who was Extraordinarius since 1918, became Hecke's successor) and Landau (see [Scha-1990]). Emmy Noether was Extraordinaria (associate professor). When Hecke, Hasse's most influential teacher, was appointed to the newly founded University of Hamburg in the spring of 1920, Hasse decided to leave Göttingen. He exmatriculated on the 23rd of March 1920 and moved to Marburg in order to study under Kurt Hensel the theory of $p$-adic numbers, introduced by Hensel in a short note in 1897 (see [He-1897]). This decision was taken after Hasse had acquired, while still in Göttingen, Hensel's book "*Zahlentheorie*" (see [He-1913)] on the 20th of March 1920. In this book Hensel developed in more detail the theory of $p$-adic numbers for the rational numbers $\mathbb{Q}$. He had already presented a thorough introduction to algebraic number theory and $p$-adic and, more generally, to $\pi$-adic numbers, for an algebraic number field $K$ with respect to a prime divisor $\mathfrak{p}$ dividing $\pi \in K$ exactly to the first power, in the book "*Theorie der algebraischen Zahlen*" (see [He-1908]).

Received July 12, 1998.
Revised December 9, 1998.

2. Hasse registered at the University of Marburg on the 11th of May 1920 in order to acquire the last two out of a total of eight semesters required before he was allowed to graduate as a Doctor of Philosophy. Already at the end of May 1920 Hensel suggested to Hasse, as a research subject for the doctoral dissertation, to continue the investigations begun by Hensel in the last chapter (Zwölftes Kapitel) of his book "Zahlentheorie" (see [He-1913]) on the conditions under which an integer or a rational number can be represented by a binary (see [He-1913], Zwölftes Kapitel, §3 and §7) or ternary (see [He-1913], Zwölftes Kapitel, §4) rational quadratic form (i.e. with rational coefficients) over the rational numbers $\mathbb{Q}$. Hasse was to examine whether the necessary conditions, given by Hensel by means of $p$-adic numbers (see [He-1913], pp. 312-314 and pp. 326-336) for the representation of an integer (or a rational number) $m$ by a rational diagonal[1] binary quadratic form $f(x,y) = ax^2 + by^2$ or a rational diagonal ternary quadratic form $f(x,y) = ax^2 + by^2 + cz^2$ with rational numbers $x,\ y,\ z \in \mathbb{Q}$, are also sufficient and he was to look for analogous conditions for quaternary quadratic forms.

3. Already on the 26th of May 1921 Hasse graduated as "Doctor Philosophiae" (Ph. D.) with the thesis entitled "*Zur Theorie der quadratischen Formen, insbesondere ihrer Darstellbarkeitseigenschaften im Bereich der rationalen Zahlen und ihrer Einteilung in Geschlechter*", where Hasse solved the problem posed by Hensel completely. Not only did he solve it for ternary and quaternary rational quadratic forms but for any rational quadratic form in $n$ variables over the rational numbers.

The clue for the solution was suggested to Hasse by Hensel (see [Ha-1975], Volume I, pp. VIII-IX) on a postcard dated 2nd of October 1920. Hasse soon discovered that the solution for binary quadratic forms is given by a reduction principle going back to Lagrange. He found it in the lectures on number theory by Dirichlet, edited by Dedekind, "*Vorlesungen über Zahlentheorie*", Vierte Auflage, Vieweg, Braunschweig, 1894 (see [DD-1894], §157, in particular p. 428). It is possible that Hasse was led to this reference by a remark made by Minkowski in [Mi-1890], p. 13 (Ges. Abh., Bd. 1, p. 227).

4. Dedekind says there on p. 422, §156 and on p. 428, §157, that

**Theorem 1.** *A necessary condition for the existence of a proper solution $x,\ y,\ z \in \mathbb{Z}$, that is with $x,\ y,\ z$ relatively prime, to the equation $ax^2 + by^2 + cz^2 = 0$ (with $a,\ b,\ c \in \mathbb{Z}$, square free and relatively prime)*

[1] Any rational quadratic form is rationally equivalent to a diagonal quadratic form (see [He-1913], p. 296).

*is that $-bc, -ca, -ab$ are quadratic residues for $a$, $b$, $c$ respectively and that $a$, $b$, $c$ cannot all have the same sign,*

and then continues on p. 428, §157 by saying that, because of a reduction principle going back to Lagrange one can prove that

**Theorem 2.** *The conditions in Theorem* 1 *are also sufficient.*

Namely, he writes

"Mit Hülfe einer Reductionsmethode, welche im Wesentlichen von *Lagrange* herrührt, lässt sich nun wirklich beweisen, dass also folgender Satz besteht:

*Sind $a$, $b$, $c$ drei von Null verschiedene und durch kein Quadrat theilbare relative Primzahlen, welche nicht alle dasselbe Vorzeichen haben, und sind die Zahlen $-bc$, $-ca$, $-ab$ resp. quadratische Reste der Zahlen $a$, $b$, $c$, so ist die Gleichung $ax^2 + by^2 + cz^2 = 0$ eigentlich lösbar.*"

Dedekind refers to

- Lagrange: *Sur la solution des problèmes indéterminés du second degré.* Mém. de l'Acad. de Berlin, tome XXIII, 1769. (Œuvres de Lagrange, tome II, 1868, p. 375).
- Lagrange: *Additions aux Éléments d'Algèbre par L. Euler*, §. V.
- Legendre: *Théorie des Nombres*, 3me édition, tome I, §§. III, IV.
- Gauss: *Disquisitiones Arithmeticae*, artt. 294, 295.

As a matter of fact, this Theorem by Lagrange was the cornerstone for Legendre's proof of the quadratic reciprocity law (see [Fr-1994], pp. 73-74 and compare also with Hensel's proof in [He-1913], pp. 324-325). However, Legendre's proof was correct, as Gauss pointed out (see [Ga-1801], art. 151 and artt. 296-297), only in two out of eight cases. In the other six cases Legendre made use of the theorem (proved only later by Dirichlet in 1837) that an arithmetic progression prime to a given modulus contains at least one prime number (see [Fr-1994], p.74).

Dedekind makes use of this theorem in order to prove Gauss' fundamental theorem stating that each proper class of proper integral binary quadratic forms of a given discriminant $d$ lying in the principal genus is the square of a proper class (of binary quadratic forms) of discriminant $d$.

5. Dedekind's proof of Theorem 2, based on Lagrange's reduction, runs by induction on what Dedekind calls the *index*. If $f(x, y) = ax^2 + by^2 + cz^2$ is the given quadratic form for which one can suppose that $|a| \leq |b| \leq |c|$, then the one of the three numbers out of $|ab|$, $|ca|$, $|bc|$ which lies between the other two is called the index $J$ of $f$, that is $J = |ca|$. Dedekind first shows that the theorem is correct if the index $J$ is equal to 1, because then $|a| = |b| = |c| = 1$. Hence $x = y = 1$, $z = 0$

gives a solution if, for instance, $a$ and $b$ have different signs. Thereafter he proceeds as follows.

If $J \geq 2$, then the condition that $-ab$ is a quadratic residue of $c$ implies $ar^2 + b = cC$ for integers $r$ and $C$ with $|r| \leq \frac{1}{2}|c|$ and $|C| < J$. If $a'$ is the greatest common divisor of $ar^2, b$ and $cC$ (if $C \neq 0$), we put $b' := \frac{ab}{a'}$ and $c' := \frac{C}{a'\gamma^2}$, where $\gamma^2$ is the biggest square contained in $\frac{C}{a'}$. Then $a', b', c'$ satisfy all the conditions of Theorem 1, but the corresponding equation $a'x'^2 + b'y'^2 + c'z'^2 = 0$ has an index $J'$ which is strictly smaller than $J$, hence by induction admits a solution $x', y', z' \in \mathbb{Z}$. From this solution one can construct a solution $x, y, z \in \mathbb{Z}$ of the original equation $ax^2 + by^2 + cz^2 = 0$.

6. When Hasse wrote to Hensel that he found the solution to the problem by means of a reduction going back to Lagrange and presented by Dedekind in §157 of the fourth edition (1894) of the lectures on number theory by Dirichlet and augmented by Dedekind, but that he could not see any connection with the $p$-adic numbers, Hensel wrote back on this postcard (see [Ha-1975], Volume I, pp. IX), given here in a free translation:

He [that is Hensel] always thought that there is the following underlying question. If one knows that an analytic function has a rational character [i.e. is rational, that is admits a Laurent series with finite principal part] at each place, then it is a rational function. If one knows that a number has a rational character [i.e. is rational, that is admits a $p$-adic expansion with finite principal part] for each finite prime $p$ and for the infinite prime $p_\infty$, then this does not imply that it is a rational number. How does this have to be completed?

Hensel had already began to examine this question at the end of his book of 1913 (see [He-1913], Zwölftes Kapitel, §7, pp. 336-337).

This remark made Hasse realize that the conditions in Lagrange's theorem (Theorem 1), namely that

*$-bc$, $-ca$, $-ab$ are quadratic residues for $a$, $b$, $c$ respectively and that $a$, $b$, $c$ cannot all have the same sign, where $a$, $b$, $c$ are square free integers relatively prime to each other,*

can be interpreted as follows with the help of Hensel's criteria given in Hensel's book of 1913 for the representation of 0 by a ternary diagonal quadratic form in the field of $p$-adic numbers (see [He-1913], Zwölftes Kapitel, §4, in particular p. 312) :

*The ternary quadratic form $f(x, y) = ax^2 + by^2 + cz^2$ represents 0 (non trivially) over the $p$-adic numbers for each prime $p$.*

The criteria actually given in [He-1913], p. 312 has the following form:

**Theorem 3.** *The ternary quadratic form* $f(x, y) = ax^2 + by^2 + cz^2$ *with rational numbers* $a$, $b$, $c$ *not all of even , resp. of odd order, represents* 0 *(non trivially) over the p-adic numbers for an odd prime p if and only if at least one of the three Legendre symbols*

$$(\frac{-ab}{p}),\ (\frac{-bc}{p}),\ (\frac{-ca}{p})$$

*is equal to one.*

A similar criterion is given for the prime $p = 2$.

Hence Theorem 1 and Theorem 2 can be given the following form, which, according to Dedekind's proof, is now obtained by means of Lagrange's reduction (see [Ha-1923a], (II) Fundamentalsatz, pp. 130-131):

**Theorem 4.** *The ternary quadratic form* $f(x, y) = ax^2 + by^2 + cz^2$ *represents* 0 *(non trivially) over the rational numbers* $\mathbb{Q}$ *if and only if it represents* 0 *(non trivially) over the p-adic numbers* $\mathbb{Q}(p)$ *for each prime* $p$ *(finite and infinite).*

This is how Hasse was led to the general *Local-Global-Principle*, called *Fundamental Theorem* by Hasse, for the representation by quadratic forms (see [Ha-1923a], (II) Fundamentalsatz, p. 130):

**Theorem 5.** *A rational number* $m \in \mathbb{Q}$ *is represented by a rational quadratic form* $f$ *over the rational numbers* $\mathbb{Q}$ *if and only if* $m$ *is represented by* $f$ *over all p-adic fields (i.e. completions)* $\mathbb{Q}(p)$ *for* $p$ *finite and infinite.*

The solution of Hensel's representation problem was thus reduced to
(1) the proof of Theorem 5
and to
(2) give criteria for the representation of $m \in \mathbb{Q}$ by $f$ in $\mathbb{Q}(p)$ for each prime $p$ (finite and infinite).

7. The results of Hasse's thesis were published in Crelle's Journal (Journal für die reine und angewandte Mathematik), of which Hensel was the chief editor, under the title "*Über die Darstellbarkeit von Zahlen durch quadratische Formen im Körper der rationalen Zahlen*" (Crelle 152 (1923), 129-148; see [Ha-1923a]).

There Hasse first remarks that the representation of a rational number $m \in \mathbb{Q}$ by a quadratic form can be reduced to a representation of 0 by another quadratic form (see [Ha-1923a], (IV), p. 133 and also [He-1913], Zwölftes Kapitel, §5, p. 314 and §6, p. 325):

**Theorem 6.** *A rational number $m \in \mathbb{Q}$, $m \neq 0$, is represented by a rational n-ary quadratic form $f = f(x_1, ..., x_n)$ over the rational numbers $\mathbb{Q}$, resp. over the field of p-adic numbers $\mathbb{Q}(p)$, if and only if 0 is represented (non trivially) by the $n+1$-ary quadratic form $F = f(x_1, ..., x_n) - mx^2$ over $\mathbb{Q}$, resp. over $\mathbb{Q}(p)$.*

Then Hasse interprets Hensel's results for binary quadratic forms (see [Ha-1923a], Satz 4, p. 135; see also [He-1913], p. 312):

**Theorem 7.** *A ternary rational quadratic form $f$ represents 0 over $\mathbb{Q}(p)$ if and only if a diagonal form $f_0 = a_1x^2 + a_2y^2 + a_3z^2$ rationally equivalent to $f$ has the trivial symbol, now called Hasse symbol:*

$$c_p(f) = c_p(f_0) := (\frac{-a_1a_2, -a_1a_3}{p}) = +1.$$

The symbol $(\frac{b,a}{p})$, now called (*quadratic*) *Hilbert symbol*, was introduced by Hilbert in his 'Zahlbericht' (see [Hi-1897], §64) as a norm symbol modulo $p$. But later it was defined by Hensel with the help of his $p$-adic numbers (see [He-1913], Zwölftes Kapitel, §5, p. 315):

**Definition 8.**

$$(\frac{b,a}{p}) = \begin{cases} +1 & \text{if } b = x^2 - ay^2 \text{ is solvable with } x, y \in \mathbb{Q}(p), \\ -1 & \text{otherwise.} \end{cases}$$

In the case of a binary quadratic form Hensel called $c_p := (\frac{,d}{p})$ the *character with respect to p* of the binary quadratic form $f = ax^2 + bxy + cy^2$ with discriminant $d := b^2 - 4ac$ (see [He-1913], Zwölftes Kapitel, §7, p. 327), a term taken from Gauss' theory of binary (integral) quadratic forms (see [Ga-1801], art. 230 or also [Fr-1979]).

The notation $c_p$ was chosen by Hasse probably following Minkowski who introduced the notation $C_p$ to denote the Minkowski invariant for rational quadratic forms. This invariant $C_p$ takes on the values +1 or -1 for each prime $p$ and it is invariant under rational equivalence (see [Mi-1890], p. 6; Ges. Abh., Bd. 1, p. 220). Hasse was always very careful in choosing his notations.

For the Hasse symbol one has the *Product Theorem* (see [Ha-1923a], formula (4.), p. 135 and also [He-1913], Zwölftes Kapitel, §6, pp. 321-322) and compare with [Mi-1890], formula (7), p. 18; Ges. Abh., Bd. 1, p. 232):

**Theorem 9.**

$$\prod_p c_p(f) = 1$$

*the product taken over all primes (finite and infinite).*

This property is due to *Hilbert's Reciprocity Law* for the Hilbert symbol (see [Hi-1897], §69, Hilfssatz 14 and [He-1913], Zwölftes Kapitel, §6, pp. 321-322):

**Theorem 10.**

$$\prod_p \left(\frac{b, a}{p}\right) = 1$$

*the product taken over all primes (finite and infinite).*

Then Hasse derives the *Local-Global-Principle* for the representation of 0 by a ternary rational quadratic form (*Fundamentalsatz*) and hence solves Hensel's problem for binary rational quadratic forms (see [Ha-1923a], Satz 6, p. 135):

**Theorem 11.** *A ternary rational quadratic form $f$ represents* 0 *(non trivially) over the rational numbers $\mathbb{Q}$ if and only if $f$ represents* 0 *(non trivially) over the p-adic numbers $\mathbb{Q}(p)$ for all primes $p$ (finite and infinite), that is if and only if $c_p(f) = 1$ for all $p$.*

For the proof Hasse follows Dedekind's proof of Theorem 2 step by step, that is he uses the reduction procedure of Lagrange.

Then Hasse shows that this Local-Global-Principle for ternary rational quadratic forms with respect to the representation of 0 is true for any $n$-ary rational quadratic form (see [Ha-1923a], Satz 14, p. 142 and Satz 21, p. 146).

## §2. Equivalence of Rational Quadratic Forms over $\mathbb{Q}$.

1. Already on the 8th of December 1921 Hasse was in a position to hand in his habilitation paper which was dedicated to the problem of equivalence of rational quadratic forms over the rational numbers. He was led to this problem by his thesis where he already had to consider equivalence of quadratic forms over $\mathbb{Q}$ and over $\mathbb{Q}(p)$ (see [Ha-1923a]).

In the thesis he also already found some necessary conditions for the equivalence of two quadratic forms over $\mathbb{Q}$ and over $\mathbb{Q}(p)$ (e.g. [Ha-1923a] §§4,5,7,14 and Tabellen 1) and 2)). In order to determine the equivalence over $\mathbb{Q}$ for two rational quadratic forms, Hasse again made use of Hensel's $p$-adic numbers and discovered another *Local-Global-Principle* (see [Ha-1923b], (II.), p. 208):

**Theorem 12.** *Two rational quadratic forms $f$ and $g$ are equivalent over the rational numbers $\mathbb{Q}$ if and only if they are equivalent over the $p$-adic numbers $\mathbb{Q}(p)$ for all primes $p$ (finite and infinite).*

Hence the rational equivalence of two rational quadratic forms was reduced to
(1) the proof of Theorem 12
and to
(2) give criteria or invariants for the equivalence of two rational quadratic forms $f$ and $g$ over $\mathbb{Q}(p)$ for all primes $p$ (finite and infinite).

2. Inspired by Minkowski's work on integral quadratic forms (see [Ha-1923b], pp. 206-208 and [Mi-1890], Ges. Abh., Bd. 1, p. 219ff.) and starting from his own results presented in his thesis, Hasse was able to give a full set of invariants for the equivalence of rational quadratic forms over $\mathbb{Q}(p)$. To the already known invariants, i.e. the number of variables $n$, the discriminant $d$ and the Sylvester index $c_{p_\infty} = J$ of a rational quadratic form $f$, Hasse added the now so called *Hasse symbol* $c_p(f)$ (see [Ha-1923b], pp. 216-217 and [O'M-1963], p. 167) to obtain a full set of invariants. This Hasse symbol is a product of Hilbert symbols (see [Ha-1923b], formula (23.), p. 217) and hence is expressible in terms of $p$-adic numbers because of Hensel's general definition (Definition 8) of the Hilbert symbol by means of $p$-adic numbers. It satisfies the *Product Law* (see [Ha-1923b], formula (30.), p. 219):

**Theorem 13.**

$$\prod_p c_p(f) = 1,$$

*the product taken over all primes (finite and infinite).*

These results were published in Volume 152 (1923) in Crelle's Journal, pp. 205-224 (see [Ha-1923b]), in the same volume which contained already the results of his thesis (pp. 129-148).

## §3. Rational Quadratic Forms over a Number Field $K$. Quadratic Norm Residue Symbol in $K$.

1. In the following Volume 153 (1924), pp. 12-43, Hasse extended his discoveries on the necessary and sufficient conditions for the representation of a rational number by a rational quadratic form over $\mathbb{Q}$ and the equivalence of rational quadratic forms over $\mathbb{Q}$, first to the more general case of symmetric matrices with coefficients in $\mathbb{Q}$ (see [Ha-1924a]), and then in two other papers in the same volume (pp. 113-130 and pp. 158-162) to the case where the ground field $\mathbb{Q}$ is replaced by any algebraic number field $K$ (see [Ha-1924c] and [Ha-1924d]).

2. In the last two papers Hensel's $\mathfrak{p}$-adic numbers for a number field $K$ and the Local-Global-Principle again turn up as the fundamental tool for the investigations. In addition, completely new tools had to be introduced, namely the theory of Hilbert and Furtwängler on *Hilbert's quadratic norm residue symbol* $(\frac{\beta,\alpha}{\mathfrak{p}})$ in a number field $K$, Weber's generalization (1897) to number fields $K$ of Dirichlet's theorem on the existence of primes in arithmetic progressions and the (*General*) *Quadratic Reciprocity Law* by *Hilbert* in a number field $K$, that is

**Theorem 14.**

$$\prod_{\mathfrak{p}}(\frac{\beta,\alpha}{\mathfrak{p}}) = 1$$

*where* $\mathfrak{p}$ *runs over all prime divisors in* $K$ *(finite and infinite).*

They play a central and crucial rôle in Hasse's paper on the representation theory of quadratic forms in a number field $K$ (see [Ha-1924c], in particular p. 114).

Hasse needs the quadratic Hilbert reciprocity law in $K$ in order to establish the Local-Global-Principle for the representation of 0 by a *quaternary* rational quadratic form over $K$.

In the case of the representation of 0 by a *ternary* rational quadratic form over $K$ the Local-Global-Principle is obtained from the following (norm) theorem of Furtwängler (see [Ha-1924c], §5, p. 122 and [Fu-1913], Satz 118, p. 429):

**Theorem 15.** *Let* $\alpha, \beta$ *be two integers in a number field* $K$. *If*

$$(\frac{\beta,\alpha}{\mathfrak{p}}) = 1$$

*for all prime divisors* $\mathfrak{p}$ *in* $K$ *(finite and infinite), then* $\beta$ *is a relative norm of a number in* $K(\sqrt{\alpha})$.

This theorem is the very last theorem in the series of three papers by Furtwängler on the reciprocity laws (see [Fu-1909], [Fu-1912], [Fu-1913]). Furtwängler deduced it by following the footsteps of Gauss (see [Ga-1801], artt. 262, 286, 287), namely from the Hilbert quadratic reciprocity law in number fields $K$ via the theorem on the existence of genera and the generalization to $K$ of what we consider today as one of the main theorems of class field theory of quadratic extensions. It asserts that each (proper) class of (proper integral) quadratic forms in the principal genus (with given discriminant) is the square of a (proper) class.

As for the criteria for the representability in a local field $K(\mathfrak{p})$ of 0 by a binary, ternary or quaternary quadratic form Hasse heavily builds on his paper [Ha-1924b], his joint paper with Hensel [HH-1923] as well as on Hensel's paper [He-1922] on the explicit description of the norm residue symbol $(\frac{\beta,\alpha}{\mathfrak{p}})$.

3. So we see that Hasse's generalization of his theory of rational quadratic forms over $\mathbb{Q}$ to quadratic forms over a number field $K$ naturally led him to look for an explicit determination of the (quadratic) norm residue symbol in $K$, to study the quadratic Hilbert reciprocity law in $K$ and to make use of Weber's generalization to number fields $K$ of Dirichlet's theorem on primes in arithmetic progressions. All these theorems are closely related to class field theory over $K$.

In fact, Furtwängler already in 1909 stressed the close connection between the reciprocity law in $K$ and the theory of the (Hilbert) class field over $K$ (see [Fu-1909], p. 5) and he made the interesting remark (see [Fu-1909], p. 2) that it was the quadratic reciprocity law in number fields $K$ whose class number is divisible by 2 that led Hilbert to sketch a *general theory of class fields for relatively abelian number fields* (see [Hi-1902]) on the basis of his theory of relatively quadratic fields (see [Hi-1899]).

So it seems that *Takagi* must have misunderstood Hilbert's intention when he says that Hilbert misled him on how to develop class field theory (see [Ka-1977], p. 4).[2]

4. Let us add what Hasse himself has to say in his paper [Ha-1924c] on p. 114 (the number field is called $k$ instead of $K$):

"Meine Entwicklungen fußen vor allem auf den Sätzen über das quadratische *Hilbert*sche Normenrestsymbol $(\frac{\alpha,\beta}{\mathfrak{p}})$, die von *Hilbert-Furtwängler* in deren Arbeiten über die Reziprozitätsgesetze und Klassenkörper erhalten sind und neuerdings von Herrn *Hensel* und mir

[2] I would like to thank Pierre Kaplan for bringing Takagi's complaint to my attention.

auf Grund der *Hensel*schen Methoden in der algebraischen Zahlentheorie von anderen Grundlagen ausgehend behandelt und erweitert werden. Insbesondere lege ich das allgemeine quadratische Reziprozitätsgesetz in der *Hilbert*schen Fassung

$$\prod_{\mathfrak{p}}\left(\frac{\alpha,\beta}{\mathfrak{p}}\right)=1$$

zugrunde, das zum Beweis meines Prinzips (I.)[3] verwendet wird, ebenso zu demselben Zweck den auf algebraische Körper verallgemeinerten Satz von den Primzahlen in einer arithmetischen Reihe, also die Tatsache, daß in jeder Idealklasse im allgemeinsten Sinne eines algebraischen Zahlkörpers unendlich viele Primideale vorhanden sind. Die Notwendigkeit der Verwendung der in diesen beiden Sätzen steckenden transzendenten[4] Methoden zum Beweis des Prinzips (I.) im Gegensatz zu allen übrigen, rein arithmetischen Entwicklungen dieser Arbeit scheint mir in der Natur der Sache zu liegen. Es soll aus der Möglichkeit gewisser Beziehungen für jeden einzelnen Primteiler $\mathfrak{p}$ von $k$ auf das Bestehen dieser Beziehungen in $k$ selbst, d. h. für die *Gesamtheit* aller $\mathfrak{p}$ geschlossen werden. Es ist daher natürlich, daß hierbei Betrachtungen über die "Dichtigkeit" von Primteilern $\mathfrak{p}$ gewisser Eigenschaften hineinspielen, wie sie doch den genannten transzendenten Beweisen eigentümlich sind."

## §4. $l$-th Degree Norm Residue Symbol in a Number Field $K$.

1. Almost on the same day when Hasse handed in his habilitation paper, Hensel had finished his article on norm residues in general relatively abelian number fields which we already mentioned (see [He-1922]), namely on the 5th of December 1921. Hilbert had developed this theory in the case where the ground field $K$ is an algebraic number field and the corresponding (relative) field extension $L/K$ is a quadratic or a Kummer extension. In this paper, Hensel extended Hilbert's theory to the case where $L/K$ is any finite abelian extension of algebraic number fields (see [He-1922]).

2. Hensel first gives a new definition of a norm residue (see [He-1922], p. 2):

**Definition 16.** Let $L/K$ be a finite algebraic extension of algebraic number fields, $\mathfrak{p}$ a prime divisor in $K$ and $\mathfrak{P}$ a prime divisor of $\mathfrak{p}$

[3] that is, the Local-Global-Principle
[4] that is, analytic

lying in $L$. Let $K(\mathfrak{p})$ be the completion of $K$ with respect to $\mathfrak{p}$ and $L(\mathfrak{P})$ the completion of $L$ with respect to $\mathfrak{P}$.

Then a number $\beta \in K \subseteq K(\mathfrak{p})$ is called a *norm residue* of $L$ with respect to $\mathfrak{p}$, if there exists a number $B \in L(\mathfrak{P})$ such that $\beta = N(B)$, where $N = N_{L(\mathfrak{P})/K(\mathfrak{p})}$ denotes the norm from $L(\mathfrak{P})$ to $K(\mathfrak{p})$.

This definition coincides with the definition given by Hilbert for a norm residue in relatively quadratic extensions $L/K$ or in a *Kummer extension* $L/K$, so called by Hilbert if $K = \mathbb{Q}(\zeta)$, where $\zeta$ is a primitive $l$-th root of unity for a prime $l$ and $L = K(\sqrt[l]{\alpha})$, where $\alpha$ is an algebraic integer in $K$ which is not the $l$-th power of another integer in $K$ (see [Hi-1899], §7 and [Hi-1897], §129):

**Definition 17.** Let $L/K$ be a finite algebraic extension of algebraic number fields, $\mathfrak{p}$ a prime in $K$, $\mathfrak{o}_K$ the ring of integers in $K$ and $\mathfrak{o}_L$ the ring of integers in $L$.

Then an integer $\beta \in \mathfrak{o}_K$ is called a *norm residue* of the field $L$ with respect to $\mathfrak{p}$, if for any power $\mathfrak{p}^n$, $n \in \mathbb{N}$, of $\mathfrak{p}$ there exists an integer $B \in \mathfrak{o}_L$ such that $\beta \equiv N(B)$ modulo $\mathfrak{p}^n$, where $N = N_{L/K}$ denotes the norm from $L$ to $K$.

3. Then Hensel determines the norm residues with respect to a prime divisor $\mathfrak{p}$ in the case where $K$ contains the $l$-th roots of unity for a prime $l$ and $L = K(\sqrt[l]{\alpha})$ for an integer $\alpha \in K$ which is not the $l$-th power of another integer in $K$ and $\mathfrak{p}$ does not divide $l$. This is done by means of well chosen, what we will call (*multiplicative*) *Hensel bases*, but what Hensel calls *fundamental systems* (*for the multiplicative representation*) or *multiplicative fundamental systems* (see [He-1922], p. 5 and [He-1916], pp. 205-206 and also [HH-1923], formula (1), p. 264 and also [Ha-1969], §15, 1.) for the local fields $K(\mathfrak{p})$ and $L(\mathfrak{P})$ (see [He-1922], §2, in particular equations (7) and (7a)).

For this field extension $K(\sqrt[l]{\alpha})/K$ Hensel also defines explicitly a norm residue symbol

$$\left\{\frac{\beta, \alpha}{\mathfrak{p}}\right\} = \zeta^{\lambda}$$

where $\beta \in K \subseteq K(\mathfrak{p})$, $\zeta$ is a primitive $l$-th root of unity and $\lambda$ is a rational integer obtained in terms of the exponents of $\alpha$ and $\beta$ with respect to a multiplicative Hensel basis properly chosen in $K(\mathfrak{p})$, or more precisely, as a $2 \times 2$ determinant of the first two exponents, namely the order number $e$ and the index $f$ (see [Ha-1924b], p. 76), of $\alpha$ and of $\beta$ if $l \neq 2$, and a modification of the determinant if $l = 2$ (see [He-1922], equation (11), p. 9).

For this explicit norm residue symbol Hensel can establish what he calls the *Permutation Law* (see [He-1922], equation (12), p. 10 and compare with [He-1913], formula (6), p. 316):

**Theorem 18.**

$$\{\frac{\beta,\alpha}{\mathfrak{p}}\}\cdot\{\frac{\alpha,\beta}{\mathfrak{p}}\}=1.$$

and the *Decomposition Law* (see [He-1922], equation (12), p. 10 and compare with [He-1913], formula ($12^a$), p. 318):

**Theorem 19.**

$$\{\frac{\beta_1,\alpha}{\mathfrak{p}}\}\{\frac{\beta_2,\alpha}{\mathfrak{p}}\}=\{\frac{\beta_1\beta_2,\alpha}{\mathfrak{p}}\}$$

*and*

$$\{\frac{\beta,\alpha_1}{\mathfrak{p}}\}\{\frac{\beta,\alpha_2}{\mathfrak{p}}\}=\{\frac{\beta,\alpha_1\alpha_2}{\mathfrak{p}}\}.$$

It follows from Hensel's definition of the symbol $\{\frac{\beta,\alpha}{\mathfrak{p}}\}$ that the following property holds (see [He-1922], Theorem (C), p. 6):

**Theorem 20.**

$$\{\frac{\beta,\alpha}{\mathfrak{p}}\}=1$$

*if and only if $\beta$ is a norm residue of $L=K(\sqrt[l]{\alpha})$ with respect to $\mathfrak{p}$.*

Hence Theorem 18 implies that $\beta$ is a norm residue of $K(\sqrt[l]{\alpha})$ with respect to $\mathfrak{p}$ if and only if $\alpha$ is a norm residue of $K(\sqrt[l]{\beta})$ with respect to $\mathfrak{p}$ (see [He-1922], p. 10).

In the case where $K$ is the $l$-th cyclotomic field $K=\mathbb{Q}(\zeta_l)$ Hilbert had already introduced in §131 of his Zahlbericht [Hi-1897] an explicit norm residue symbol $\{\frac{\beta,\alpha}{\mathfrak{q}}\}$ for any prime divisor $\mathfrak{q}$ in $K$ as a certain $l$-th root of unity, and he had shown in §133 for a prime divisor $\mathfrak{q}$ not dividing $l$ (and also in certain cases if $\mathfrak{q}$ divides $l$) that this Hilbert norm residue symbol is equal to 1 if and only if $\beta\in K$ is a norm residue of $L=K(\sqrt[l]{\alpha})$ with respect to $\mathfrak{q}$ (see [Hi-1897], Satz 151). It follows immediately from Hilbert's definition of this Hilbert norm residue symbol that it satisfies the Permutation Law and the Decomposition Law (see [Hi-1897], §131, formulae (80) and (83)).

At the end of his paper Hensel mentions (see [He-1922], p. 10) that one gets the same results also in the most difficult but also most important and essential case where $\mathfrak{p}$ is a divisor of $l$, and he announces that he will treat this case in a second paper.

4. This second paper was presented to the Mathematische Annalen on the 1st of May 1923 (see [HH-1923]). However, it appears as a joint paper of Hensel and Hasse and is a result of joint discussions, as the authors mention at the very beginning of their article. So it becomes already clear, and this is what we will see in more detail, that Hasse's occupation with the norm residue symbol for a number field $K$, first in the quadratic case and then for odd prime degrees $l$, began in the year 1922, very probably under the influence of Hensel. The quadratic norm residue symbol was, in fact, needed for his theory of representing numbers by quadratic forms in the local fields $K(\mathfrak{p})$, which we discussed before(see [Ha-1924c]).

Soon thereafter, or even at the same time, Hasse must have become aware (maybe also under the influence of Hensel) of the importance of the (quadratic) reciprocity law for the norm residue symbol in the form given by Hilbert. We have seen that this law played a crucial rôle in his paper on the representation by quadratic forms in algebraic number fields, published in Crelle's Journal 153 (1924) (see [Ha-1924c]).

5. In that year 1922 Hasse obtained his habilitation on the 28th of February at the University of Marburg and then had to do his "Vorbereitungszeit" (preparation period for teaching at colleges, required for future college teachers) and pass the pedagogical examination for the "Lehramt an höheren Schulen", which he did on the 6th of July 1922. On the 25th of September he was awarded a research grant (Forschungsstipendium) of 308 000 Mark from the "Notgemeinschaft für die deutsche Wissenschaft" for one year "zur Fortführung der Arbeiten auf dem Gebiete der höheren Zahlentheorie", and already for the autumn semester 1922/23 he was offered a paid lectureship (bezahlter Lehrauftrag) for Geometry at the University of Kiel. This offer was arranged by his former teacher in Kiel, Otto Toeplitz, whom Hasse had met at the annual meeting of the DMV (Deutsche Mathematiker-Vereinigung), held in Leipzig from the 17th until the 24th of September 1922. It was Hensel who suggested to Hasse that he accompany him to this meeting. There Hasse must also have met Emil Artin (probably for the first time), who delivered a lecture on quasi-ergodic geodesic orbits in the complex upper half plane $\mathcal{H}$ (or more precisely on $\mathcal{H}/SL(2,\mathbb{Z})$) with respect to the Poincaré metric, entitled "Über einen Fall von geodätischen Linien mit quasiergodischem Verlauf" (see also [Ar-1924b]).

6. At any rate, by the first of May 1923 Hasse was fully in possession of the theory of norm residues as developed by Hilbert (see [Hi-1899]) and Hensel (see [He-1922]) and the reciprocity laws for the

$l$-th power and $l^n$-th power residues as developed by Hilbert (see [Hi-1899]) and Furtwängler (see [Fu-1904], [Fu-1909], [Fu-1912], [Fu-1913]). He must also already have known Takagi's long and fundamental paper on class field theory of 1920 "*Über eine Theorie des relativ Abel'schen Zahlkörpers*" (see [Ta-1920]), since all these papers are explicitly mentioned in a footnote on p. 262 and then explicitly referred to on p. 263 of the joint publication [HH-1923].

We will now make these assertions more precise. Let us begin with what the authors have to say on the pages 262-263 in [HH-1923] (the extension $L/K$ is now called $K/k$ and $\mathfrak{p}$ and $\mathfrak{l}$ denote prime divisors in $k$):

"Während die Resultate für den in N.R.[5] behandelten Fall eines zu $l$ primen $\mathfrak{p}$ im wesentlichen mit den Sätzen der Hilbert-Furtwänglerschen Theorie[6] übereinstimmen und sich von jenen nur durch die u. E. naturgemäßere Behandlungsweise unterscheiden, was schon in der viel einfacheren Definition des Normenrestcharakters (N.R., S. 2)[7] deutlich zum Ausdruck kommt, werden die Ergebnisse dieser Arbeit erheblich über die entsprechenden Resultate Hilberts und Furtwänglers hinausgehen. Das dortige Hauptresultat, das in dem Satz enthalten ist[8]:

*Geht $\mathfrak{l}$ nicht in der Relativdiskriminante von $K$ auf, so sind alle zu $\mathfrak{l}$ primen Zahlen von $k$ Normenreste von $K$ nach $\mathfrak{l}$. Im anderen Falle bilden die zu $\mathfrak{l}$ primen Normenreste von $K$ nach $\mathfrak{l}$ eine Untergruppe vom Index $l$ aller zu $\mathfrak{l}$ primen Restklassen nach jedem genügend hohen Modul $\mathfrak{l}^g$ (es genügt stets $g \geq \frac{el}{l-1}$, wenn $l$ genau durch $\mathfrak{l}^e$ teilbar),*

gibt nämlich nur eine Abzählung der zu $\mathfrak{l}$ primen Normenreste. Wir werden hier erstens das Resultat in vollster Allgemeinheit, d. h. auch für zu $\mathfrak{l}$ nicht prime Zahlen erhalten und zweitens die betreffende Untergruppe der Normenreste genau angeben.

Das erhaltene Ergebnis wird von dem jüngeren von uns in einigen weiteren Arbeiten zur Aufstellung einer systematischen Theorie der quadratischen Formen in einem algebraischen Körper $k$,[9] sowie zu bemerkenswerten Verallgemeinerungen der bekannten Furtwänglerschen

---

[5] that is, in [He-1922]

[6] Here Hasse and Hensel refer to [Hi-1899], [Fu-1904], [Fu-1909], [Fu-1912], [Fu-1913] and [Ta-1920].

[7] that is, in [He-1922], p. 2

[8] Here Hasse and Hensel refer explicitly to [Fu-1904], p. 47 and to [Ta-1920], Satz 9, p. 28.

[9] published in [Ha-1924c]

Reziprozitätsgesetze für $l$-te Potenzreste in $k$ auf nichtprimäre Zahlen verwendet werden[10]."

and further down on p. 267:

"Die hier angewandte Methode[11] eröffnet ferner die Möglichkeit, zu einer in der Hilbert-Furtwänglerschen Theorie keinen Platz findenden, *direkten* Definition und expliziten Formel für das den Normenrestcharakter von $\beta$ in bezug auf $k(\sqrt[l]{\alpha})$ ausdrückende Hilbertsche Normenrestsymbol $(\frac{\beta,\alpha}{\mathfrak{l}})$ zu gelangen, und so eine neue Grundlage für die Behandlung des allgemeinsten Reziprozitätsgesetzes für die $l$-ten Potenzreste in $k$ zu schaffen, worauf der jüngere von uns in einigen weiteren Arbeiten einzugehen gedenkt."[12]

7. We can gain some more insight on the sequence of events that led Hasse to the norm residue symbols and to the reciprocity laws from the most interesting correspondence between Kurt Hensel and Helmut Hasse.[13] It is fascinating to watch Hasse progressing in a very short period to the center of number theory, starting out from the theory of quadratic forms, proceeding to the theory of norm residues and then to the laws of reciprocity and finally ending up in class field theory.

On the 2nd of February 1923 Hasse wrote from Kiel, where he held a lectureship since October 1922, that he had just settled the theory of the norm residue symbol for a prime divisor $\mathfrak{l}$ dividing the prime degree $l = [K : k]$, where $K = k(\sqrt[l]{\alpha})$, for the moment for $l = 2$, in the way they discussed when they met last time, that is, during the Christmas vacation 1922[14], namely to give an explicit expression for the norm residue symbol by means of a Hensel basis:

"Zu meiner großen Freude kann ich Ihnen heute mitteilen, daß mir soeben gelungen ist, die Theorie des Normenrestsymbols für einen Primteiler $\mathfrak{l}$ in dem in unserer letzten Besprechung formulierten Sinn zu einem befriedigenden Abschluss zu bringen. Da ich überzeugt bin, daß Sie an diesem Resultat eine ebenso große Freude haben werden, als ich selber, schreibe ich Ihnen sogleich, kaum 5 Minuten nach dem letzten Federstrich. — Es handelte sich darum, auch im Falle eines Primteilers $\mathfrak{l}$

[10]published in [Ha-1924f]

[11]that is, the method of the systematic use of Hensel bases for $k$ and $K$, or more precisely for $k(\mathfrak{p})$ and $K(\mathfrak{P})$, where $\mathfrak{P}$ in $K$ is a divisor of $\mathfrak{p}$ in $k$,

[12]published in [Ha-1924b], [Ha-1924e], [Ha-1925c] and [Ha-1924f].

[13]This correspondence is kept, according to Hasse's will, at the University Library in Göttingen.

[14]see the postcard from Hasse to Hensel of 10.2.1923.

eine Darstellung des Normensymbols anzugeben, die seinen Wert unmittelbar aus der Exponentialdarstellung zu erschließen gestattet. Ich habe die Untersuchung vorläufig für den Fall $l = 2$, d. h. des quadratischen Symbols durchgeführt, da mir für ungerades $l$ die von Ihnen gefundenen Resultate noch nicht in dem erforderlichen Maße zur Verfügung standen."

8. From [Ha-1924b] we can see more precisely what Hasse was doing in the case $l = 2$, while following Hensel's idea to deduce the value of the norm residue symbol directly from the exponents of a well chosen multiplicative presentation of $\alpha$ and $\beta$ with respect to a Hensel basis in $k(\mathfrak{l})$. He gets:

$$(\frac{\alpha, \beta}{\mathfrak{l}}) = (-1)^L,$$

where $L$ is a symmetric bilinear form (or — what amounts to the same, since $l = 2$ — a skew-symmetric bilinear form) of the exponents of $\alpha$ and $\beta$ with respect to a Hensel basis in $k(\mathfrak{l})$ (see [Ha-1924b], Hauptsatz, p. 77). The existence of such a form is deduced directly from the decomposition and permutation law for the quadratic norm residue symbol $(\frac{\alpha,\beta}{\mathfrak{l}})$, due to the fact that in the quadratic case the latter simply takes the form $(\frac{\alpha,\beta}{\mathfrak{l}}) = (\frac{\beta,\alpha}{\mathfrak{l}})$ (see [Ha-1924b], Satz 1, p. 80).

As for the case $l \neq 2$, $l$ prime, Hasse says in that same letter that the conclusions are the same, except for the crucial permutation law for which he would like to have a more direct proof. By this Hasse meant a proof which does not make use of the $l$-th degree Hilbert reciprocity law in $k$. Thereby he was alluding to Furtwängler who was building on the Hilbert reciprocity law in $k$ in order to define the $l$-th degree norm residue symbol $(\frac{\alpha,\beta}{\mathfrak{l}})$ for a prime divisor $\mathfrak{l}$ dividing the degree $l$ and to deduce the permutation law for it from the fact that the definition and the permutation law were already established for the symbols $(\frac{\alpha,\beta}{\mathfrak{p}})$ with prime divisors $\mathfrak{p}$ not dividing $l$ (see [Fu-1904], §14, p. 37). This direct proof was given a little later in [Ha-1925a] in a rather roundabout way.

9. In [Ha-1924e] Hasse accomplished the task of deducing, similarly as in the case $l = 2$, the existence of a unique skew symmetric bilinear form $L$ modulo $l$ from the decomposition law

$$(\frac{\alpha_1, \beta}{\mathfrak{l}})(\frac{\alpha_2, \beta}{\mathfrak{l}}) = (\frac{\alpha_1\alpha_2, \beta}{\mathfrak{l}})$$

and the permutation law

$$(\frac{\alpha, \beta}{\mathfrak{l}})(\frac{\beta, \alpha}{\mathfrak{l}}) = 1$$

such that the norm residue symbol $(\frac{\alpha,\beta}{\mathfrak{l}})$ can be expressed as

$$(\frac{\alpha,\beta}{\mathfrak{l}}) = (\frac{\beta,\alpha}{\mathfrak{l}})^{-1} = \zeta^L$$

for a fixed primitive $l$-th root of unity $\zeta$ (see [Ha-1924e], Satz 3, p. 186).

The skew-symmetric bilinear form $L$ is again given in terms of the exponents of $\alpha$ and $\beta$ with respect to a Hensel basis in $k(\mathfrak{l})$ (see [Ha-1924e], Satz 1, p. 186).

10. The values of $(\frac{\alpha,\beta}{\mathfrak{p}}) = \zeta^L$ for a prime $\mathfrak{p}$ not dividing $l$ and $(\frac{\alpha,\beta}{\mathfrak{l}}) = \zeta'^{L'}$ for a prime $\mathfrak{l}$ dividing $l$ are uniquely determined, except for the choice of $\zeta$ and $\zeta'$. In the case where $\mathfrak{p}$ does not divide $l$, $\zeta$ can be normalized in a natural way by the choice of an appropriate Hensel basis, but in the case where $\mathfrak{l}$ divides $l$, this is not possible for $\zeta'$. However, Hasse is able to obtain a normalization also in this case by postulating the Hilbert reciprocity law in $k$ for degree $l$:

$$\prod_{\mathfrak{q}} (\frac{\alpha,\beta}{\mathfrak{q}}) = 1$$

for *all* $\alpha, \beta \in k$, where the product is taken over all pime divisors $\mathfrak{q}$ in $k$ (see [Ha-1924e], pp. 189-191).

Hence this normalization is subject to the validity of the Hilbert reciprocity law for the prime degree $l$, for which, in fact, proofs had already been given by Furtwängler (see [Fu-1912], Satz 16, p. 385) and by Takagi (see [Ta-1922], §11; or [Ta-1973], p. 209) by transcendental (i.e. analytic) means.[15]

11. Later Hasse went back again and again to this fundamental problem of giving an explicit and canonical description of the norm residue symbol, e.g. [Ha-1930] and [Ha-1933], where he introduced the norm residue symbol as a fundamental 2-cycle and herewith prepared the way for the cohomological formulation of the Artin-Tate reciprocity law and the fundamental theorems of class field theory.

12. These results on the explicit determination of the $l$-th degree norm residue symbol enabled Hasse to give strong improvements of the explicit reciprocity laws for $l$-th powers as established by Hilbert and Furtwängler (see [Ha-1924f]).

[15] Hasse in [Ha-1924e] on p. 191 gives an incorrect title in his reference to Takagi.

13. Hasse then continues in still the same letter of the 2nd of February 1923 referring to the planned but not yet published papers [Ha-1925a] and [Ha-1924e] we just discussed:

"Ich werde mir diese Übertragung[16] noch einmal gründlich überlegen, kann aber vorläufig nicht ausführlich vorgehen, da mir die Grundlagen, nämlich Ihre Entwicklungen über diesen Fall noch nicht ganz im Kopf sind. Ich würde mich sehr freuen, wenn ich bei Gelegenheit meiner Anwesenheit in Marburg im März ausführlich mit Ihnen über den Fall sprechen könnte.

Es wäre mir sehr lieb, wenn es sich machen ließe, daß mein heutiges Resultat vielleicht noch im Anschluß an Ihre demnächst erscheinende Arbeit über dasselbe Problem veröffentlicht werden könnte[17]. Sollten Sie damit einverstanden sein, so möchte ich mir die Bitte erlauben, daß Sie mit der Einsendung Ihres Manuskriptes – ich meine von Ihnen gehört zu haben, daß Sie es bei Springer[18] drucken lassen wollen – warten, bis zu meinem Kommen nach Marburg, damit ich mir noch die nötigen Angaben betr. Rückverweisung auf Stellen Ihrer Arbeit holen kann. Ich würde dann selbstverständlich unmittelbar nach Semesterschluß, also etwa am 5. März in Marburg sein, wo sich übrigens auch meine Braut seit einigen Tagen wieder eingefunden hat.

Die Fortsetzung meiner Habilitationsschrift, die bisher noch nicht zum Druck gegeben ist, habe ich nunmehr fertig gestellt, und darf sie Ihnen wohl bei dieser Gelegenheit dann noch mitbringen, ebenso eine oder zwei weitere Arbeiten über quadr. Formen in algebraischen Körpern, die ich nächster [?] Tage fertigstellen werde. – "

Hasse then finishes his letter by saying:

"In der Hoffnung Ihnen mit der Mitteilung meines Resultats eine Freude gemacht zu haben, möchte ich Ihnen gleichzeitig meinen herzlichsten Dank aussprechen für Ihre so schöne und wertvolle Anregung, mich mit diesen Fragen zu beschäftigen. Ich war die letzten Tage wie im Fieber dabei, und meine große Freude über den glücklichen Erfolg können Sie sich kaum vorstellen. Die Zahlentheorie birgt doch wahrlich die schönsten Schätze in der Mathematik!"

14. From this letter we can infer that, indeed, Hasse's occupation with the theory of norm residues was initiated by Hensel, that Hasse and Hensel planned to publish their results on the norm residues for the case

[16] namely from $l = 2$ to $l \neq 2$

[17] Hasse's result appeared in [Ha-1924b] and Hensel's paper was published as the joint paper [HH-1923].

[18] that is, in the Mathematische Annalen

when the prime divisor $\mathfrak{l}$ in $k$ divides the prime degree $l = [k(\sqrt[l]{\alpha}) : k]$ in two separate papers in the Mathematische Annalen, but that they must have decided, when they met in Marburg in the beginning of March 1923, to publish a joint paper (see the letter of 21st of March 1923).

15. This letter of the 2nd of February 1923 was followed by a long series of letters and postcards from Hasse to Hensel written in short intervals and announcing a whole series of new discoveries, testifying that Hasse had reached an extremely productive period[19].

On a postcard dated the 10th of February 1923, Hasse first refers to Hensel's investigations on the computation of the norm residue symbol in the case $\mathfrak{p} = \mathfrak{l}$ by means of a Hensel basis, a problem they discussed during the Christmas vacation.

On a postcard dated from the 14th of February 1923, Hasse announces that he believes that he can now prove the permutation law for the norm residue symbol $(\frac{\alpha,\beta}{\mathfrak{l}})$ with $\mathfrak{l}$ dividing $l$ for an odd prime $l$ (see [Ha-1925a]):

"den Vertauschungssatz für ungerades $l$ glaube ich jetzt beweisen zu können. Ich muss es in den nächsten Tagen mal mit meinen neuartigen Gedanken probieren."

## §5. Explicit Reciprocity Laws

1. Then two days later Hasse begins to study the explicit reciprocity laws to which he was probably led by the article of Furtwängler [Fu-1904], where Furtwängler deduces the general reciprocity law for the $l$-th power residues and the two complementary laws for *primary* and *hyperprimary* numbers in a number field $k$ from the Hilbert reciprocity law for norm residues of degree $l$ in $k$ at the end of his article (see [Fu-1904], §17, p. 47).

First Hasse reports in a letter (Kiel, 16th of February 1923) on a new discovery about the quadratic Hilbert symbol $(\frac{-1,-1}{\mathfrak{l}})$ where $\mathfrak{l}$ is a prime divisor in $k$ dividing 2:

"Schon wieder kann ich Ihnen ein schönes Resultat mitteilen, das mir gestern und heute zugefallen ist. Meine Bemühungen richteten sich zunächst auf das spezielle Hilbertsche Symbol $(\frac{-1,-1}{\mathfrak{l}})$ in einem beliebigen algebraischen Körper $k$, wenn $\mathfrak{l}$ ein Teiler der 2 ist. Dieses Symbol interessierte mich besonders, da es schon bei der Reduktion der Form $L$

[19] 6 of the 24 papers published in Volume 153 (1924) and 5 out of 24 papers published in Volume 154 (1925) of Crelle's Journal were written by Hasse.

eine gewisse Rolle spielte, dann aber auch in der Theorie der quadratischen Formen eine Rolle spielt, wie Sie sich wohl aus meiner Habilitationsschrift entsinnen, wo die Symbole $(\frac{-1,-1}{p})$ in den Invarianten häufig vorkommen."

Hasse there determines the value $(\frac{-1,-1}{\mathfrak{l}})$ explicitly with Hensel's method of using fundamental multiplicative systems (Hensel bases) for the ground field $k$ and for the corresponding extension field $K$ and obtains:

$$(\frac{-1,-1}{\mathfrak{l}}) = (-1)^{ef}$$

where $e$ is the ramification order (ramification index) and $f$ the degree of (the residue field of) $\mathfrak{l}$ in $k$. This value is compatible with the quadratic Hilbert reciprocity law in $k$:

$$\prod_{\mathfrak{q}}(\frac{-1,-1}{\mathfrak{q}}) = 1$$

(over all primes $\mathfrak{q}$, finite and infinite).

2. More generally Hasse is able to determine the symbol $(\frac{\alpha,-1}{\mathfrak{l}})$, for $\alpha \in k$ and $\alpha \equiv 1 \pmod 2$, explicitly with the help of the local trace $S_{\mathfrak{l}}$ of $\frac{\alpha-1}{2}$ in $k(\mathfrak{l})$, that is the trace from $k(\mathfrak{l})$ to $\mathbb{Q}(2)$:

$$(\frac{\alpha,-1}{\mathfrak{l}}) = (-1)^{S_{\mathfrak{l}}(\frac{\alpha-1}{2})}$$

and then obtains a part of the first complementary law of the quadratic reciprocity in $k$, namely

$$(\frac{-1}{\alpha}) = (-1)^{S(\frac{\alpha-1}{2})}$$

where $S$ denotes the global trace in $k$, that is the trace from $k$ to $\mathbb{Q}$. He then expresses the hope that he will obtain in a similar way the whole quadratic reciprocity law in $k$, e.g. the analogy of the second complementary law for algebraic number fields $k$.

His results appeared as special cases published in [Ha-1924f] (for $(\frac{-1,-1}{\mathfrak{l}})$ see [Ha-1924f], p. 201 and also [Ha-1924c], p. 127; for $(\frac{\alpha,-1}{\mathfrak{l}})$ and $(\frac{-1}{\alpha})$ see [Ha-1924f], (4a.), p. 194, where the more general case $(\frac{\alpha,\zeta}{\mathfrak{l}})$ with $\mathfrak{l}$ a divisor of any given prime $l$ and $\zeta$ a primitive $l$-th root of unity is treated).

3. Already on the next day Hasse writes enthusiastically on a postcard (Kiel, 17.2.1923) that he was able to extend quite considerably his

special results on $(\frac{-1,-1}{\mathfrak{l}})$ and $(\frac{\alpha,-1}{\mathfrak{l}})$ towards a general quadratic reciprocity law for any number field $k$ and to establish the full analogy with the case of the rational number field $\mathbb{Q}$ by applying Hensel's method of making use of a Hensel basis in the corresponding local field $k(\mathfrak{l})$:

"Ich scheine in einer sehr glücklichen Epoche meines Lebens angelangt zu sein, denn ganz plötzlich öffnen sich mir tausend Tore, durch die neue, schöne Erkenntnisse einströmen. Im Anschluß an mein spezielles Resultat über $(\frac{-1,-1}{\mathfrak{l}})$ und $(\frac{\alpha,-1}{\mathfrak{l}})$ habe ich nunmehr mit derselben, d. h. mit *Ihren* schönen Fundamentalsystemen für die mult[iplikative] Darstellung, das gesamte quadratische Reziprozitätsgesetz für beliebige algebraische Zahlkörper sehr wesentlich erweitert und nun erst die volle Analogie mit dem rationalen Zahlkörper hergestellt."

and he ends:

"Ich erzähle Ihnen bald ausführlich, welch' einfache Beweise dieser Sätze sich mit Ihren so handlichen Fundamentalsystemen geben lassen. Mit Stolz und Dank sehe ich zu Ihnen als dem Schöpfer dieser Methoden auf."

4. Eleven days later, on the 28th of February 1923, he reports that he succeeded in extending his results on the quadratic reciprocity law to the reciprocity law for the $l$-th power residues in a field $k$, just before he leaves to Hamburg, where he is going to present a lecture about his new results on the reciprocity laws:

"Eben vor meiner Abreise noch die erfreuliche Mitteilung, daß das Reziprozitätsgesetz für $l$-te Potenzreste sich ebenso schön erweitern läßt, wie ich es Ihnen neulich für die quadratischen Reste schrieb. Näheres bald mündlich. Ich komme am 6. März nach Marburg und würde mich sehr freuen, wenn Sie mich (am besten durch Nachricht an meine Schwiegermutter) wissen ließen, wann ich Sie zu Hause antreffe. Morgen trage ich in Hamburg meine Resultate über die Reziprozitätsgesetze vor."

It must have been on this occasion (1st of March 1923) that Hasse and Artin began to discuss the reciprocity laws which eventually led to two joint papers "*Über den zweiten Ergänzungssatz zum Reziprozitätsgesetz der $l$-ten Potenzreste im Körper $k_\zeta$ der $l$-ten Einheitswurzeln in Oberkörpern von $k_\zeta$*" (Crelle 154 (1925), 143-148) (see [AH-1925]), to which the first letters of their correspondence, beginning on the 9th of July 1923, were dedicated, and "*Die beiden Ergänzungssätze zum Reziprozitätsgesetz der $l^n$-ten Potenzreste im Körper der $l^n$-ten Einheitswurzeln*" (Abh. Math. Sem. Hamburg 6 (1928), 146-162) (see [AH-1928]).

5. From the publication [Ha-1924f], Hasse's very first paper dedicated to the explicit reciprocity laws, written up on the 30th of June 1923 and presented to Crelle's Journal, we can see that Hasse found the following form of the general reciprocity law and the two complementary laws for the $l$-th power residues in any number field $k$ containing the $l$-th roots of unity for a prime number $l$ and in cases where $\alpha$ and $\beta$ are *non-primary* numbers in $k$. The laws were obtained by making systematic use of the explicit formulae for the norm residue symbols, derived by Hensel and Hasse with the help of well chosen Hensel bases in the corresponding local fields. They are completely analogous to the quadratic reciprocity laws in $\mathbb{Q}$ (see [Ha-1924f], p.194):

**Theorem 21.** *If $k$ is a number field containing the $l$-th roots of unity for a prime number $l$ and $\alpha, \beta \in k$, then*

(1) $(\frac{\alpha}{\beta})(\frac{\beta}{\alpha})^{-1} = \zeta^{S(\frac{\alpha-1}{l}\cdot\frac{\beta-1}{\lambda_0})}$, *if* $(\alpha, \beta) = 1$ *and* $\alpha \equiv 1 \bmod l$, $\beta \equiv 1 \bmod \mathfrak{l}_0$

(2) $(\frac{\zeta}{\alpha}) = \zeta^{S(\frac{\alpha-1}{l})}$, *if* $\alpha \equiv 1 \bmod l$

(3) $(\frac{l}{\alpha}) = \zeta^{S(\frac{\alpha-1}{l\lambda_0})}$, *if* $\alpha \equiv 1 \bmod l\mathfrak{l}_0$,

*where* $(-)$ *denotes the Legendre symbol generalized to the $l$-th power residues in $k$, $S$ denotes the trace from $k$ to $\mathbb{Q}$, $\zeta$ is a fixed primitive $l$-th root of unity, $\lambda_0 = 1 - \zeta$ and $\mathfrak{l}_0 = (1-\zeta) = (\lambda_0)$ is the corresponding principal divisor in $\mathbb{Q}(\zeta) \subseteq k$.*

In his proof of Theorem 21, Hasse starts out from the Hilbert reciprocity law for norm residues of degree $l$ in $k$, namely

$$\prod_{\mathfrak{q}}(\frac{\alpha, \beta}{\mathfrak{q}}) = 1,$$

proved by Furtwängler in 1904 and 1912 (see [Fu-1904], Satz 47, p. 46 and [Fu-1912], p. 347), in order to deduce the three relations

(1') $(\frac{\alpha}{\beta})(\frac{\beta}{\alpha})^{-1} = \prod_{i=1}^{t}(\frac{\beta,\alpha}{\mathfrak{l}_i})$, if $\alpha, \beta$ and $l$ are prime to each other;

(2') $(\frac{\varepsilon}{\alpha}) = \prod_{i=1}^{t}(\frac{\alpha,\varepsilon}{\mathfrak{l}_i})$, if $\alpha$ is prime to $l$ and $(\varepsilon) = \mathfrak{a}^l$, where $\mathfrak{a}$ is an ideal in $k$;

(3') $(\frac{\lambda}{\alpha}) = \prod_{i=1}^{t}(\frac{\alpha,\lambda}{\mathfrak{l}_i})$, if $\alpha$ is prime to $l$ and $(\lambda) = \prod_{i=1}^{t} \mathfrak{l}_i^{a_i}\mathfrak{a}^l$ where $l = \mathfrak{l}_1^{e_1} \ldots \mathfrak{l}_t^{e_t}$ is the prime decomposition of $l$ in $k$.

Then Hasse evaluates the norm residue symbols on the right hand side by means of his explicit formula for the $l$-th degree norm residue symbol $(\frac{\alpha,\gamma}{\mathfrak{l}_i})$ for a prime divisor $\mathfrak{l}_i$ of $l$.

6. In the case of the second complementary law (3') Hasse also makes use of an argument he learnt from Artin, when $l = 2, 3$, an argument he

generalized in collaboration with Artin to the case where $l$ is any prime degree.

He remarks on p. 204:

"Herr *E. Artin* teilte mir kürzlich ein einfaches Rekursionsverfahren mit, das gestattet, im Falle des quadratischen Reziprozitätsgesetzes im rationalen Körper und des kubischen im Kreiskörper $k_\varrho$ der dritten Einheitswurzeln den zweiten Ergänzungssatz aus dem allgemeinen Gesetz zu erschließen, und somit andeutet, daß der in der *Hilbert-Furtwängler*schen Theorie stets besonders schwer zu beweisende zweite Ergänzungssatz auch im allgemeinen Falle als nicht tieferliegend anzusehen ist, als das allgemeine Reziprozitätsgesetz und der erste Ergänzungssatz. In einer gemeinsamen Besprechung konnten wir dann dies Verfahren auf den Kreiskörper $k_\zeta$ der $l$-ten Einheitswurzeln übertragen."

From this remark we can conjecture that it must have been after Hasse's lecture on the reciprocity laws in Hamburg that Artin realized that the second complementary law in the cases $l = 2, 3$ can be deduced from the general reciprocity law (for $l$-th powers), that is from (1), and that in the following discussion they were able to extend Artin's idea to any prime number $l$. Hasse gave an account of this idea in [Ha-1924f], pp. 204-207.

In the simplest case, where $l = 2$ and $k = \mathbb{Q}$, Artin's argument runs as follows (p. 204):[20]

Suppose that

$$\left(\frac{-1}{a}\right) = (-1)^{\frac{a-1}{2}}$$

for an odd positive integer $a$, and that

$$\left(\frac{a}{b}\right) = \left(\frac{b}{a}\right),$$

with $a, b$ relatively prime, where $b$ is a positive odd integer and $a$ is primary, that is $a \equiv 1$ modulo 4.

[20] *Franz Lemmermeyer* kindly informed me that Julius König in an article on the quadratic reciprocity law, entitled "Das Reciprocitätsgesetz in der Theorie der quadratischen Reste" [Acta Math. 22 (1899), 181-192], writes that Kronecker applied this argument in his lectures. Furthermore, that this argument has been used bei Dintzl in the biquadratic case, "Über den zweiten Ergänzungssatz des biquadratischen Reziprozitätsgesetzes" [Monatshefte Math. 10 (1899), 88-96], as well as in the cubic case, [Monatshefte Math. 10 (1899), 303-306].

Then for any odd $a$:

$$\left(\frac{2}{a}\right) = \left(\frac{-1}{a}\right)\left(\frac{-2}{a}\right) = (-1)^{\frac{a-1}{2}}\left(\frac{a-2}{a}\right) = \\ (-1)^{\frac{a-1}{2}}\left(\frac{a}{a-2}\right) = (-1)^{\frac{a-1}{2}}\left(\frac{2}{a-2}\right),$$

since one of $a, a-2$ is primary.

Hence by iteration

$$\left(\frac{2}{a}\right) = (-1)^{\frac{a-1}{2}+\frac{a-3}{2}+\ldots+1}\left(\frac{2}{1}\right) = (-1)^{\frac{a^2-1}{8}}.$$

7. From all we can see, there is no indication that Artin had any special interest in the explicit reciprocity law before that lecture delivered by Hasse in Hamburg on the 1st of March 1923, and it might well be that it was Hasse's talk in Hamburg that stimulated Artin's interest in the reciprocity laws.

Artin had done his doctoral dissertation in 1921 under the direction of Gustav Herglotz in Leipzig on quadratic extensions over function fields whose constants lie in a finite field, in other words, on hyperelliptic curves over finite fields (see [Ar-1924a]). He was following up investigations started by Dedekind on so-called higher congruences (see [De-1857]; Crelle 54 (1857), 1-26). In his dissertation Artin systematically developed the arithmetic of quadratic extensions of function fields over finite fields. The center of his study was the class number formula which led him to introduce the $\zeta$-function of function fields and to formulate and conjecture the Riemann hypothesis for these fields. Thus function fields and $\zeta$-functions were Artin's primary interest at that time.

The question, studied in the summer 1921, of whether the $\zeta$-function $\zeta_k$ of a field $k$ divides the $\zeta$-function $\zeta_K$ of the field $K$ if $K/k$ is an algebraic extension of number fields, led Artin to introduce, in July 1923, his $L$-series in the case where $K/k$ is a Galois extension (see [Ar-1923a] and [Ar-1923b]). The comparison of these $L$-series with Weber's $L$-series, in the case that $K/k$ is abelian, led him to conjecture what is now called the Artin reciprocity law (see [Ar-1923b]). At that time (1923), Artin was able to verify this conjecture only in some special cases. It was only in July 1927 that he succeeded in giving a general proof of his conjecture (see [Ar-1927]), after a paper by Čebotarev on density of primes, published in 1925 in the Mathematische Annalen 95 (p. 191), had provided him with a key idea.

8. After Hasse had come to Marburg on the 6th of March 1923 to visit Hensel and to discuss the norm residue theory for a prime divisor $\mathfrak{l}$

dividing the degree $l = [k(\sqrt[l]{\alpha}) : k]$, Hasse and Hensel must have decided to publish their findings not in two separate papers, each signed by their respective author, but jointly in one paper. Hasse took over the task to write up the article as can be seen from a letter from Hasse to Hensel on the 21st of March 1923:

"Beiliegend übersende ich Ihnen das nunmehr fertiggestellte Manuskript unserer gemeinsamen Arbeit und möchte Sie freundlichst bitten, es noch einmal genau durchzulesen. Ich habe es mit großer Liebe und Sorgfalt bis ins Einzelne durchdacht und eigentlich jedes Wort und jeden Passūs einer reiflichen Erwägung unterzogen, wie es wohl am besten sei, zu schreiben. So glaube ich nunmehr eine nach allen Seiten hin ausgeglichene und passende Form gefunden zu haben und bin gespannt, Ihr Urteil darüber zu vernehmen.

[...]

- daß die Arbeit in *gemeinsamen Besprechungen* zwischen uns entstanden ist, können wir wohl schreiben. Wenn auch jeder von uns wesentliche Punkte allein gefunden hat, so ist doch die jetzt vorliegende Form als eine schöne Durchdringung unserer beiderseitigen Ideen anzusehen. Ich für mein Teil kann nur versichern, daß mein Anteil zum größten Teil Ihren wertvollen Anregungen bei unseren Besprechungen Weihnachten (ja auch schon vorigen Sommer) und kürzlich zu verstehen ist.

[...]

Was nun die Bezeichnungen anbelangt, so haben mir diese am meisten Nachdenken verursacht. Ich weiss, wie sehr eine gute Bezeichnungsweise die Verständlichkeit und Beliebtheit einer Arbeit zu heben vermag.

[...]

Ich habe das Manuskript, bis auf einige bei der Korrektur noch einzufügende Zitate gleich druckfertig gemacht. Wenn Sie es für gut befinden, schicken Sie es wohl bald ein. Mir liegt sehr an einem baldigen Erscheinen, da eben alle weiteren Arbeiten von mir darauf fußen.

[...]

Zu weiteren mathematischen Überlegungen bin ich noch nicht gekommen, und werde (wohl?) auch bis Mitte Sommer wenig machen können. Denn jetzt bin ich ohne Literatur und [dar]an meine Vorlesungen für den Sommer [zu] präparieren. Pfingsten wird meine Hochzeit sein. Nichtsdestotrotz behalte ich unsere großen Ziele, vor allem einen vernünftigen Beweis des Reziprozitätsgesetzes dauernd im Auge."

On the 31st of March 1923 Hasse is able to announce that he proved the permutation and decomposition law for $\mathfrak{l}$ a divisor of $l$ by means of group theory (subgroups of the norm residues) (see [Ha-1925a]).

In a letter of 21st of April 1923 Hasse says that he is planning to prepare an exposition of Takagi's class field theory for a course:

"Außerdem habe ich gerade die Ausarbeitung eines Kollegs über die Klassenkörpertheorie von Takagi vor, die ich mit unseren Methoden sehr schön und einfach darstellen kann."

and further down:

"Ich habe in den letzten Tagen auf verschiedenen allabendlichen Spazierfahrten per Rad einmal ganz gründlich über das allgemeine Normenrestproblem nachgedacht, dessen Grundlegung Sie kürzlich gaben. Zunächst etwas Spezielles: Unsere gemeinsame Arbeit setzt die Existenz der $l$-ten Einheitswurzeln in $k$ voraus, gibt also keine vollständige Theorie (mit dem früheren für $\mathfrak{p}$ zusammen) des Normenrestproblems für *reine, absolut zyklische Körper von Primzahlgrad $l$.* Nun habe ich in der Takagischen Arbeit die ganz entsprechende Untersuchung für *ganz beliebige* relativ zyklische Körper von Primzahlgrad gefunden, also auch solche, die durch *nicht reine* Gleichungen definiert werden, z. B. den kubischen Körper, wenn er Galoissch ist, etz. Takagi's Methoden sind nicht sehr schön, [...]

Seine Methoden *sind* $\lambda$-adisch und es ist eine Kleinigkeit, wie Sie aus obiger Stichprobe sehen, sie zu unseren Zielen zu vervollständigen, d. h. mittels multiplikativer Normalform das kritische Element zu charakterisieren und so die Anwendung auf höhere Reziprozitätsgesetze in der Art, wie ich sie vorhabe, vorzubereiten. –

Die Takagischen Resultate stellen sich also als der erste Schritt zu der von Ihnen beabsichtigten Verallgemeinerung des Normenrestproblems auf beliebige Relativkörper dar."

Hasse then sketches how to proceed in the general case of an arbitrary (relative) extension $K/k$ in oder to solve the norm residue problem, that is to determine whether an element in $k$ is a norm residue or not and to determine the subgroup of norm residues and its index. He proposes to study the local structure, i. e. to analyze stepwise the tower of fields "climbing" from $k(\mathfrak{l})$ over the inertial, the first and the higher ramification fields up to $K(\mathfrak{L})$, where $\mathfrak{L}$ is a prime divisor of $\mathfrak{l}$ lying in $K$.

In a letter, dated 23rd of April 1923, Hasse comes back to his ideas for a very general norm residue theory in arbitrary (relative) extensions $K/k$ and for 'the most general' reciprocity law in $K/k$:

"In meinem Briefe vom Sonnabend vergaß ich noch zu sagen, daß ich nach meinen "überfliegenden" Überlegungen das "allgemeinste Normenrestsymbol" für außerordentlich wertvoll halte. Ich glaube, daß eine ganz neue Sorte von Reziprozitätsgesetzen auf diese Weise erschlossen werden kann, von denen dann die bisherigen nur Spezialfälle sind. Als oberstes Ziel möchte ich den Nachweis von

$$\prod_{\mathfrak{P}} \{\frac{\gamma}{\mathfrak{P}}\} = 1$$

hinstellen, was dann das "allgemeinste" Reziprozitätsgesetz für beliebige Relativkörper darstellen würde. Durch Sätze über die Auswertung von $\{\frac{\gamma}{\mathfrak{P}}\}$ würden vermutlich Reziprozitätsgesetze über die $m$-ten Potenzreste ($m$ beliebige ganze Zahl), aber noch vielmehr, nämlich Sätze über "nicht Galoissche Kongruenzreste" folgen, die sich vermutlich in Reziprozitäten zwischen den Relativdiskriminanten äußern und womöglich zu schönen Diskriminantensätzen überhaupt führen. Dieses Ziel ist aber noch in weiter Ferne. Denn dazu ist außer der Bestimmung des Symbols $\{\frac{\gamma}{\mathfrak{P}}\}$ noch ein ganz entsprechendes Gebäude für *beliebige* Körper notwendig, wie das Hilbert-Furtwängler-Takagische für Abelsche Körper."

It is not clear whether Hasse actually taught the planned course on Takagi's class field theory, but he did write up an exposition of that theory, partly in order to prepare himself for the study of the norm residue theory and the reciprocity laws in non-Galois extensions $K/k$. He gave a report on this exposition at the meeting of the German Mathematical Society (DMV) in Danzig in the autumn of 1925. It was Hilbert who asked him to deliver this report, and it was this report which gave rise to Hasse's famous report (Bericht) on class field theory which was to have an enormous impact on the later development (see [Ha-1926]).

I would like to thank my friend *Cornelius Greither* for a linguistic improvement of the text.

## References

[Ar-1923a] Artin, Emil, *Über die Zetafunktionen gewisser algebraischer Zahlkörper*, Math. Ann., **89** (1923), pp. 147–156.

[Ar-1923b] Artin, Emil, *Über eine neue Art von L-Reihen*, Abh. Math. Sem. Hamburg, **3** (1923), pp. 89–108.

[Ar-1924a] Artin, Emil, *Quadratische Körper im Gebiete der höheren Kongruenzen I, II*, Math. Zeitschrift, **19** (1924), pp. 153–246.

[Ar-1924b] Artin, Emil, *Ein mechanisches System mit quasiergodischen Bahnen*, Abh. Math. Sem. Hamburg, (1924), pp. 170–175.

[Ar-1927] Artin, Emil, *Beweis des allgemeinen Reziprozitätsgesetzes*, Abh. Math. Sem. Hamburg, **5** (1927), pp. 353–363.

[AH-1925] Artin, Emil and Hasse, Helmut, *Über den zweiten Ergänzungssatz zum Reziprozitätsgesetz der l-ten Potenzreste im Körper*

$k_\zeta$ *der l-ten Einheitswurzeln und in Oberkörpern von* $k_\zeta$, J. reine angew. Math., **154** (1925), pp. 143–148.

[AH-1928] Artin, Emil and Hasse, Helmut, *Die beiden Ergänzungssätze zum Reziprozitätsgesetz der* $l^n$*-ten Potenzreste im Körper der* $l^n$*-ten Einheitswurzeln*, Abh. Math. Sem. Hamburg, **6** (1928), pp. 146–162.

[De-1857] Dedekind, Richard, *Abriß einer Theorie der höheren Kongruenzen in bezug auf einen reellen Primzahl-Modulus*, J. reine angew. Math., **54** (1857), pp. 1–26.

[DD-1894] Dirichlet, Lejeune; Dedekind, Richard, "Vorlesungen über Zahlentheorie", Vierte Auflage, Vieweg, Braunschweig, 1894.

[Fr-1979] Frei, Günther, *On the Development of the Genus of Quadratic Forms*, Annales des Sciences Mathématiques du Québec, **3** (1979), pp. 5–62.

[Fr-1994] Frei, Günther, "*The Reciprocity Law from Euler to Eisenstein*; in the Intersection of History and Mathematics" (Editors: Sasaki Ch., Sugiura M., Dauben J.W.) pp. 67–88, Birkhäuser, Basel, 1994.

[Fu-1904] Furtwängler, Philipp, *Über die Reziprozitätsgesetze zwischen l-ten Potenzresten in algebraischen Zahlkörpern, wenn l eine ungerade Primzahl bedeutet*, Math. Ann., **58** (1904), pp. 1–50. Slightly changed reprint of an article in Göttinger Abhandlungen 1902.

[Fu-1909] Furtwängler, Philipp, *Die Reziprozitätsgesetze für Potenzreste mit Primzahlexponenten in algebraischen Zahlkörpern, I*, Math. Ann., **67** (1909), pp. 1–31.

[Fu-1912] Furtwängler, Philipp, *Die Reziprozitätsgesetze für Potenzreste mit Primzahlexponenten in algebraischen Zahlkörpern, II*, Math. Ann., **72** (1912), pp. 346–386.

[Fu-1913] Furtwängler, Philipp, *Die Reziprozitätsgesetze für Potenzreste mit Primzahlexponenten in algebraischen Zahlkörpern, III*, Math. Ann., **74** (1913), pp. 413–429.

[Ha-1923a] Hasse, Helmut, *Über die Darstellbarkeit von Zahlen durch quadratische Formen im Körper der rationalen Zahlen*, J. reine angew. Math., **152** (1923), pp. 129–148.

[Ha-1923b] Hasse, Helmut, *Über die Äquivalenz quadratischer Formen im Körper der rationalen Zahlen*, J. reine angew. Math., **152** (1923), pp. 205–224.

[Ha-1924a] Hasse, Helmut, *Symmetrische Matrizen im Körper der rationalen Zahlen*, J. reine angew. Math., **153** (1924), pp. 12–43.

[Ha-1924b] Hasse, Helmut, *Zur Theorie des quadratischen Hilbertschen Normenrestsymbols in algebraischen Körpern*, J. reine angew. Math., **153** (1924), pp. 76–93.

[Ha-1924c] Hasse, Helmut, *Darstellbarkeit von Zahlen durch quadratische Formen in einem beliebigen algebraischen Zahlkörper*, J. reine angew. Math., **153** (1924), pp. 113–130.

[Ha-1924d] Hasse, Helmut, *Äquivalenz quadratischer Formen in einem beliebigen algebraischen Zahlkörper*, J. reine angew. Math., **153** (1924), pp. 158–162.

[Ha-1924e] Hasse, Helmut, *Zur Theorie des Hilbertschen Normenrestsymbols in algebraischen Körpern*, J. reine angew. Math., **153** (1924), pp. 184–191.

[Ha-1924f] Hasse, Helmut, *Das allgemeine Reziprozitätsgesetz und seine Ergänzungssätze in beliebigen algebraischen Zahlkörpern für gewisse, nicht-primäre Zahlen*, J. reine angew. Math., **153** (1924), pp. 192–207.

[Ha-1925a] Hasse, Helmut, *Direkter Beweis des Zerlegungs und Vertauschungssatzes für das Hilbertsche Normenrestsymbol in einem algebraischen Zahlkörper im Falle eines Primteilers $\mathfrak{l}$ des Relativgrades $l$*, J. reine angew. Math., **154** (1925), pp. 20–35.

[Ha-1925b] Hasse, Helmut, *Über das allgemeine Reziprozitätsgesetz der $l$-ten Potenzreste im Körper $k_\zeta$ der $l$-ten Einheitswurzeln und in Oberkörpern von $k_\zeta$*, J. reine angew. Math., **154** (1925), pp. 96–109.

[Ha-1925c] Hasse, Helmut, *Zur Theorie des Hilbertschen Normenrestsymbols in algebraischen Körpern*, J. reine angew. Math., **154** (1925), pp. 174–177.

[Ha-1925d] Hasse, Helmut, *Das allgemeine Reziprozitätsgesetz der $l$-ten Potenzreste für beliebige, zu $l$ prime Zahlen in gewissen Oberkörpern des Körpers der $l$-ten Einheitswurzeln*, J. reine angew. Math., **154** (1925), pp. 199–214.

[Ha-1925e] Hasse, Helmut, *Der zweite Ergänzungssatz zum Reziprozitätsgesetz der $l$-ten Potenzreste für beliebige zu $l$ prime Zahlen in gewissen Oberkörpern des Körpers der $l$-ten Einheitswurzeln*, J. reine angew. Math., **154** (1925), pp. 215–218.

[Ha-1926] Hasse, Helmut, *Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper, Teil I, Teil Ia*, Jahresbericht der D.M.-V., **35** (1926), **36** (1927).

[Ha-1930] Hasse, Helmut, *Neue Begründung und Verallgemeinerung der Theorie des Normenrestsymbols*, J. reine angew. Math., **162** (1930), pp. 134–144.

[Ha-1933] Hasse, Helmut, *Die Struktur der R. Brauerschen Algebrenklassengruppe über einem algebraischen Zahlkörper. Insbesondere Begründung der Theorie des Normenrestsymbols und die Herleitung des Reziprozitätsgesetzes mit nichtkommutativen Hilfsmitteln*, Math. Annalen, **107** (1933), pp. 731–760.

[Ha-1969] Hasse, Helmut, "Zahlentheorie", 3. Aufl., Akademie-Verlag, Berlin, 1969.

[Ha-1975] Hasse, Helmut, "Mathematische Abhandlungen", herausgegeben von Heinrich Wolfgang Leopoldt und Peter Roquette, Walter de Gruyter, Berlin, 1975.

[HH-1923] Hasse, Helmut and Hensel, Kurt, *Über die Normenreste eines relativ-zyklischen Körpers vom Primzahlgrad $l$ nach einem Primteiler $\mathfrak{l}$ von $l$*, Math. Ann., **90** (1923), pp. 262–278.

[He-1897] Hensel, Kurt, *Über eine neue Begründung der Theorie der algebraischen Zahlen*, Jahresbericht der D.M.-V, **6** (1897), pp. 83–88.

[He-1908] Hensel, Kurt, "Theorie der algebraischen Zahlen", Teubner, Leipzig und Berlin, 1908.

[He-1913] Hensel, Kurt, "Zahlentheorie", Göschen, Leipzig und Berlin, 1913.

[He-1916] Hensel, Kurt, *Die multiplikative Darstellung der algebraischen Zahlen für den Bereich eines beliebigen Primteilers*, J. reine angew. Math., **146** (1916), pp. 189–215.

[He-1922] Hensel, Kurt, *Über die Normenreste in den allgemeinsten relativ-abelschen Zahlkörpern*, Math. Ann., **85** (1922), pp. 1–10.

[Hi-1897] Hilbert, David, *Die Theorie der algebraischen Zahlkörper*, Jahresbericht der D.M.-V, **4** (1897), pp. 175–546.

[Hi-1899] Hilbert, David, *Über die Theorie des relativ-quadratischen Zahlkörpers*, Math. Ann., **51** (1899), pp. 1–127.

[Hi-1902] Hilbert, David, *Über die Theorie der relativ-Abelschen Zahlkörper*, Acta Math., **26** (1902), pp. 99–132. Slightly changed reprint of an article in Nachrichten der K. Ges. der Wiss. zu Göttingen 1898.

[Ka-1977] Kaplan, Pierre, "Teiji Takagi (1875-1960) et la découverte de la théorie du corps de classes", Typoscript, Tokyo, 1977.

[Mi-1890] Minkowski, Hermann, *Über die Bedingungen, unter welchen zwei quadratische Formen mit rationalen Koeffizienten ineinander rational transformiert werden können*, (Auszug aus einem von Herrn H. Minkowski in Bonn an Herrn Adolf Hurwitz gerichteten Briefe). J. reine angew. Math., **106** (1890), 5–26; Ges. Abh., Bd. 1, pp. 219–239.

[O'M-1963] O'Meara, Timothy O., "Introduction to Quadratic Forms", Springer-Verlag, New York, 1963.

[Scha-1990] Scharlau, Winfried (editor), "Mathematische Institute in Deutschland 1800-1945" DMV, Vieweg, Braunschweig/Wiesbaden, 1990.

[Ta-1920] Takagi, Teiji, *Über eine Theorie des relativ Abel'schen Zahlkörpers*, Journ. of Coll. of Science, Univ. of Tokyo, **41**, Art 9, 1920, pp. 1–133.

[Ta-1922] Takagi, Teiji, *Über das Reciprocitätsgesetz in einem beliebigen algebraischen Zahlkörper*, Journ. of Coll. of Science, Univ. of Tokyo, **44**, Art 5, 1922, pp. 1–50.

[Ta-1973] Takagi, Teiji, "Collected Papers", Iwanami Shoten, Tokyo, 1973.

*Département de mathématiques et de statistique*
*Université Laval*
*Ste-Foy, Québec, G1K 7P4*
*Canada*
*E-mail address*: gfrei@mat.ulaval.ca