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Geometry of cuspidal sextics and their dual curves

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§1. Introduction

Let C be a given irreducible plane curve of degree n defined by $f(x,y) = 0$ where $f(x,y)$ is an irreducible polynomial. C is called a *torus curve of type* (p,q) if $p,q|n$ and $f(x,y)$ is written as $f(x,y) = f_{n/p}(x,y)^p + f_{n/q}(x,y)^q$ for some polynomials $f_{n/p}, f_{n/q}$ of degree n/p and n/q respectively. This terminology is due to Kulikov, [K2]. Torus curves have been studied by many authors ([Z], [O1],[K2], [D],[T]).

In the process of studying Zariski pairs in the moduli of plane curves of degree 6 with 3 cusps of type $y^4 - x^3 = 0$, we have observed that *there exist two irreducible components* $\mathcal{N}_{3,1}/PSL(3,\mathbf{C})$ *and* $\mathcal{N}_{3,2}/PSL(3,\mathbf{C})$ *which corresponds to torus curves and non-torus curves respectively* (Lemma 25). *Their dual curves are sextics with six cusps and three nodes.* Starting from this observation, we study the moduli space of sextic with 6 cusps and 3 nodes which we denote by $\mathcal{M}$ and the moduli of their dual curves. It turns out that $\mathcal{M}$ has a beautiful symmetry. The "regular part" (=Plücker curves) of $\mathcal{M}$ is stable by the dual curve operation and the moduli of 3 (3,4)-cuspidal sextics $\mathcal{N}_3$ is on the "boundary" of $\mathcal{M}$ in a nice way (Theorem 18). By the dual operation, this moduli is isomorphic to a "singular" stratum $\mathcal{M}_3$ of $\mathcal{M}$, which consists of 6 cuspidal 3 nodal sextics with 3 flexes of order 2. The moduli space $\mathcal{M}$ is a disjoint union of torus curves and non-torus curves. The generic Alexander polynomial $\Delta_C(t)$ of $\mathbf{P}^2 - C$ is determined by the type of C. Namely if C is a torus curve, $\Delta_C(t) = t^2 - t + 1$ and $\pi_1(\mathbf{P}^2 - C) = \mathbf{Z}_2 * \mathbf{Z}_3$, while for non-torus curve C, $\Delta_C(t) = 1$. Moreover we show that *the dual curve C^* is a torus curve if and only if C is a torus curve.* This is striking, as it implies also that the topology of the complement is preserved by the dual operation for a torus curve in $\mathcal{M}$.

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This paper is composed as follows. In section 2, we study dual curves and their singularities. We show a simple lemma which enable us to compute the defining polynomials of the dual curves explicitly (Lemma 4) and then we introduce a stratification, which is called *flex stratification*, in the local moduli space of a germ of singularity. This stratification enjoys the following property. The defect of a singularity to the number of flexes is constant on each stratum and the image of a stratum by the dual map is again a stratum. Thus the topological structure of the dual singularity is also constant along a stratum (Theorem 14).

In section 3, we study the moduli space $\mathcal{M}$ and other moduli spaces which appear on the canonical stratification of the "closure" $\widehat{\mathcal{M}}$ of $\mathcal{M}$ (Theorem 18). In section 4, we compute the moduli space of sextics with 3 (3,4)-cusps. In sections 5 and 6, we compute the fundamental groups of the complements of 3 (3,4)-cuspidal sextics of torus type and non-torus type. In section 7, we give a new Zariski triple of plane curves of degree 12 with 12 (3,4)-cusps, as an application of Theorem 18.

§2. Dual curves

In this section, we first recall some basic properties of the dual curves. For general references, refer to [W1], [B-K] and [N]. We will also present several new results on dual curves which will be used in later sections (Lemma 4, Theorem 14).

Let C be an irreducible plane curve in $\mathbf{P}^2$ and let $F(X,Y,Z)=0$ be an irreducible polynomial which defines C and let $f(x,y)=F(x,y,1)$. Here X,Y,Z are homogeneous coordinates and (x,y) are affine coordinates given by $x=X/Z, y=Y/Z$. At a simple point $P=(\alpha,\beta)\in C\cap \mathbf{C}^2$, the tangent line T_PC is given by $\frac{\partial f}{\partial x}(\alpha,\beta)(x-\alpha)+\frac{\partial f}{\partial y}(\alpha,\beta)(y-\alpha)=0$.

Let $\mathbf{P}^{*2}$ be the dual projective space with homogeneous coordinates U,V,W. The dual curve C^* of C is the closure of the image of the mapping $P\mapsto T_PC$ for the regular points $P\in C$. More explicitly it is given by $(X,Y,Z)\mapsto (U,V,W)$ where $U=F_X,\ V=F_Y$ and $W=F_Z$. Thus a defining homogeneous polynomial of C^*, denoted by $G(U,V,W)$, can be obtained by eliminating X,Y,Z from the above equalities and $F(X,Y,Z)=0$.

Let $\phi:\tilde{C}\to C$ be a normalization of C and let t be the (local) coordinate of $\tilde{C}$. Let $(x(t),y(t))$ be the affine parameterization of C. Then the tangent line is given by $y-y(t)=\frac{y'(t)}{x'(t)}(x-x(t))$. Thus the dual curve is parameterized in homogeneous coordinates as follows:

$$(1)\qquad U(t)=y'(t),\quad V(t)=-x'(t),\quad W(t)=x'(t)y(t)-x(t)y'(t)$$

Applying (1) to C^* again, we see easily that the dual curve operation enjoys the reciprocity law $C^{**} = C$ and thus C and C^* are birationally equivalent.

2.1. Action of the automorphism

The group G:= $PSL(3, \mathbf{C})$ acts on $\mathbf{P}^2$ from the right side as: $\mathbf{P}^2 \times G \to \mathbf{P}^2$, $((X, Y, Z), A) \mapsto (X, Y, Z)A$. Let $A \in G$ and we denote by φ_A the automorphism induced by the right multiplication. Then the image $\varphi_A(C)$ of the curve is defined by the polynomial $\varphi_{A^{-1}}^* F(X, Y, Z) = F((X, Y, Z)A^{-1})$. Put $C^A := \varphi_A(C)$. The following is easy to be proved.

Lemma 2. *We have $(C^A)^* = (C^*)^{{}^tA^{-1}}$. Thus if C^* is defined by $G(U, V, W) = 0$, $(C^A)^*$ is defined by $\varphi_{{}^tA}^* G(U, V, W) = G((U, V, W)^t A)$. In particular, if C^* is a torus curve, $(C^A)^*$ is a torus curve for any $A \in PSL(3, \mathbf{C})$.*

2.2. Class formula

Assume that C is an irreducible curve of degree n with k singularities P_i for $i = 1, \ldots, k$. Let m_i be the multiplicity, let μ_i be the Milnor number and let r_i be the number of irreducible components of C at P_i respectively and let g be the genus of the normalization $\tilde{C}$. The degree n^* of the dual curve is called *the class number* of C and n^* is given by the formula:

$$(3) \quad n^* = 2(g - 1 + n) - \sum_{i=1}^{k} (m_i - r_i) = n(n-1) - \sum_{i=1}^{k} (\mu_i + m_i - 1)$$

The second equality follows from the (modified) Plücker formula:
$2 - 2g = 3n - n^2 + \sum_{i=1}^{k} (\mu_i + r_i - 1)$.

2.3. Defining polynomial of the dual curve.

Let $F(X, Y, Z)$, $f(x, y)$ and C be as before. Let $G(U, V, W)$ be the defining homogeneous polynomial of C^* and let $g(u, v)$ be the affine equation, given by $g(u, v) = G(u, v, 1)$. G is given by eliminating X, Y, Z from $F(X, Y, Z)$ and $F_X - U$, $F_Y - V$, $F_Z - W$. However this elimination involves a tremendous computation. We prefer the following simple formula.

Lemma 4. *Assume that the line $Z = 0$ cuts C transversely. Let $P_i = (\alpha_i, \beta_i)$, $i = 1, \ldots, k$ be the singular points of C and let μ_i be the Milnor number and let m_i be the multiplicity of C at P_i. Put $f_1(x_1, p, y_1) := f(x_1 - py_1, y_1)$ and let $h(x_1, p) := \Delta_{y_1}(f_1)$ be the discriminant polynomial of f_1 with respect to y_1. Then $h(x_1, p)$ is a polynomial of degree $n(n-1)$. Put $\tilde{g}(u, v) = h(-1/u, v/u)u^{n(n-1)}$. Then*

$\tilde{g}(u,v)$ can be written as $\tilde{g}(u,v) = g(u,v)L(u,v)$ where L is given by $L(u,v) = \prod_{i=1}^{k}(\alpha_i u + \beta_i v + 1)^{\mu_i+m_i-1}$ and the polynomial $g(u,v)$ is a defining polynomial of the dual curve in the affine coordinates $u = U/W, v = V/W$.

Remark 1. *This formula also holds without the genericity assumption of the line at infinity with a slight modification $\tilde{g}(u,v) = h(-1/u, v/u)u^{\deg(h)}$. The defining polynomial $g(u,v)$ is obtained by throwing away all the multiple factors from $\tilde{g}(u,v)$. Therefore for the determination of g, we only need an elimination of one variable. Thus the computation is very easy.*

Proof. Let $f^*(u,v)$ be the defining polynomial of the dual curve. Consider p as a fixed constant. (We consider p as a variable later.) First observe that $h(a,p) = 0$ with $a \neq -(\alpha_i + p\beta_i)$, $i = 1, \dots, k$, if and only if $x + py - a = 0$ is tangent to C. Thus $(-1/a, -p/a) \in C^*$ when $h(a,p) = 0$. Thus $g(u,v) = 0$ defines C^* as a set. By a standard argument of discriminant, $\deg_{x_1} h(x_1,p) = n(n-1)$ and the solutions of $h(x_1,p) = 0$ in x_1 are all simple except $x_1 = \alpha_i + \beta_i p$, while the contribution from the singular point P_i is given by $(x_1 - (\alpha_i + \beta_i p))^{\nu_i}$ where ν_i is the intersection multiplicity of C and $\frac{\partial f_1}{\partial y_1} = 0$ at P_i, considering p as a constant (see for example, [O5]). Furthermore we have the equality: $\nu_i = \mu_i + m_i - 1$ by [Le] for a generic p. We need to show $\deg\ h(x_1,p) = n(n-1)$ as a polynomial of two variables x_1, p.

Step 1. Assume that C is a smooth curve. Then it is well-known that $f^*(u,v)$ is an irreducible polynomial of degree $n(n-1)$. Let $h^*(x_1,p) := f^*(-1/x_1, -p/x_1)x_1^{n(n-1)}$. Then h^* is also an irreducible polynomial of degree $n(n-1)$ and by the above consideration, $h^*(x_1,p)$ divides $h(x_1,p)$. So we conclude that $h(x_1,p) = h^*(x_1,p)$ up to a multiplication of a constant.

Step 2. Our case. Let $f_t(x,y) = f(x,y) - t$. Then for $t \neq 0$, sufficiently small, $C_t := \{f_t(x,y) = 0\}$ is a smooth curve of degree n. Let $h_t(x_1,p)$ be the discriminant polynomial of $f_t(x_1 - py, y)$ in y. Then $h_t(x_1,p)$ has degree $n(n-1)$ as a polynomial of x_1, p. Thus as $h_0 = h$, $\deg\ h(x_1,p) \leq n(n-1)$. As we have already seen that $\deg_{x_1} h(x_1,p) = n(n-1)$, we conclude that $\deg\ h(x_1,p) = n(n-1)$. Q.E.D.

2.4. Flex points

Let C be an irreducible plane curve of degree n defined by a homogeneous polynomial $F(X,Y,Z) = 0$ and put $f(x,y) = F(x,y,1)$ as before. A regular point $P \in C$ is called a *flex of order r* if the intersection multiplicity $I(C, T_PC; P)$ of C and the tangent line T_PC at P is $r+2$. We simply say a flex in the sense of a flex of order 1. It is well-known that flex points are defined by $H(X,Y,Z)=0$ on C where $H(X,Y,Z)$ is the Hessian of F which is a homogeneous polynomial of degree $3(n-2)$. Using the Euler equality $nF = XF_X + YF_Y + ZF_Z$, we can easily obtain $Z^2H \equiv -(n-1)^2(F_{X,X}F_Y^2 - 2F_{X,Y}F_YF_X + F_{Y,Y}F_X^2)$ modulo (F). We consider the polynomial $\mathcal{F}(f) := f_{x,x}f_y^2 - 2f_{x,y}f_yf_x + f_{y,y}f_x^2$ (see also §5, [O5]) and let J be the plane curve defined by $\mathcal{F}(f)(x,y)=0$. Note that $\deg \mathcal{F}(f) = 3n-4$. We define *the flex defect at P* of C by the intersection multiplicity $I(C,H;P)$ of C and H at P and we denote this integer by $\delta(P;f)$ or $\delta(P;C)$. By the above equality, the flex defect $\delta(P;f)$ is equal to the intersection number $I(C,J;P)$ for $P \in C \cap \mathbf{C}^2$. Let $P_1,\dots,P_k$ be the singular points of C. Thus we have

Proposition 5. *The number of flexes $i(C)$, counted with the multiplicity, is given by* $3(n-2)n - \sum_{i=1}^k \delta(P_i;C)$.

Remark 2. *The multiplicity of a flex point is counted by the flex defect, which turns out to be equal with the order by Corollary 9.*

2.5. Flex defect formula and flex stratification

Let σ be an equivalence class of an isolated plane curve singularity germ. Here two germs (C,O) and (C',O) at the origin are equivalent if they are joined by an equisingular family (i.e., μ-constant family). We define *the generic flex defect of* σ by $\min\{\delta(f;O); f\in\sigma\}$ and we denote the generic flex defect by $\bar{\delta}(\sigma)$. Let $f(x,y)$ be a polynomial and let $C(f)$ be the plane curve $\{f(x,y)=0\}$. We say that $f(x,y)$ or $C(f)$ is *generic* (at O) in σ if $(C(f),O)\in\sigma$ and $\delta(O;f)=\bar{\delta}(\sigma)$.

Let $\mathcal{P} = \{(m_1,n_1),\dots,(m_\ell,n_\ell)\}$ be a given set of Puiseux pairs and let $\sigma(\mathcal{P})$ be the equivalent class of the irreducible curve singularity having $\mathcal{P}$ as Puiseux pairs. Assume that $(C,O)\in\sigma(\mathcal{P})$ is defined by $f(x,y)=0$ and $y=0$ is the tangential direction. Then y can be expanded in a Puiseux series as $y=\varphi(x^{1/N})$,

$$(6)\qquad \varphi(x^{1/N}) = \textstyle\sum_{i=s}^{k_0} a_i x^i + h_1(x^{1/N_1}) + \cdots + h_\ell(x^{1/N_\ell}),$$
$$N_j := n_1\cdots n_j,\ N = N_\ell$$

where $h_j(x^{1/N_j}) = c_j x^{m_j/N_j} + \sum_{m_j<k<m_{j+1}/n_{j+1}} c_{j,k}x^{k/N_j}$ and c_1, c_2, $\dots$, $c_\ell \neq 0$, $k_0 := [m_1/n_1]$, $\gcd(n_i,m_i)=1$ and $m_i > m_{i-1}n_i$. Note that $a_1 = 0$ by the assumption on the tangent direction. Let $\mathcal{S} = \{j; a_i \neq 0, j\geq 2\}$. We call the order s of $\varphi(x)$ in x *the Puiseux order* of f and we denote it by Puiseux ord(f). Note that $sN = I(C, y=0; O)$ where $y=0$ is the tangential direction. Thus the Puiseux order does not depend on the choice of linear coordinates.

Let s = Puiseux order(f). By the definition, $s = \min\{j \in \mathcal{S}\}$ if $\mathcal{S} \neq \emptyset$ and $s = m_1/n_1$ if $\mathcal{S} = \emptyset$. As a function of x, φ is well-defined in the region, say $-\pi \leq \arg(x) < \pi$, when a branch $x^{1/N}$ is fixed. We fix a branch of $x^{1/N}$ hereafter. We consider the canonical stratification of $\sigma(\mathcal{P})$ given by $\{\sigma(\mathcal{P};2)\dots,\sigma(\mathcal{P};[m_1/n_1]),\sigma(\mathcal{P};m_1/n_1)\}$ where

$$\sigma(\mathcal{P};s) = \{(C(f),O) \in \sigma(\mathcal{P}); \text{Puiseux order}(f) = s\}.$$

We call this stratification *the flex stratification* of $\sigma(\mathcal{P})$.

Theorem 7. *Assume that $f(x,y) \in \sigma(\mathcal{P};s)$. Then we have*

$$\delta(O;f) = (s-2)n_1\cdots n_\ell + \textstyle\sum_{j=1}^{\ell} 3(n_j-1)m_j(n_{j+1}\cdots n_\ell)^2 \tag{8}$$

and f is generic if and only if $s \leq 2$, namely if either $s = 2$ or $m_1/n_1 \leq 2$ and $s = m_1/n_1$.

The formula (8) seems to be equivalent to Satz 2, p. 780, [B-K].

Proof. Put $\omega = \exp(2\pi\sqrt{-1}/N)$ and consider functions of $x^{1/N}$ defined by $\varphi_j(x^{1/N}) := \varphi(x^{1/N}\omega^j)$ for $j = 0,\dots,N-1$. Note $\varphi_0 = \varphi$. The defining function $f(x,y)$ is given by the product $f(x,y) = Ug$ where U is a unit and $g(x,y) = (y-\varphi_0(x^{1/N}))\cdots(y-\varphi_{N-1}(x^{1/N}))$. The intersection number $I(C,J;O)$ is given as $\mathrm{val}_t\mathcal{F}(f)(x(t),y(t))$, using the parameterization $x^{1/N} = t$ (so $x(t) = t^N$) and $y(t) = \varphi_0(t)$. First it is easy to show:

Assertion 1. *$\mathcal{F}(f)(x(t),y(t)) = U^3\mathcal{F}(g)(x(t),y(t))$ and $\mathrm{val}_t(\mathcal{F}(f)(x(t),y(t)))$ is equal to $\mathrm{val}_t(\mathcal{F}(g)(x(t),y(t)))$.*

Composing the parameterization mapping $t \mapsto \psi(t) := (x(t),y(t))$, we have:

$$\begin{aligned}
g_{x,x}g_y^2(\psi(t)) &= \Big(2\textstyle\sum_{j=1}^{N-1}\frac{\partial\varphi_0}{\partial x}\frac{\partial\varphi_j}{\partial x}\prod_{k\neq 0,j}(\varphi_0-\varphi_k)\\
&\quad -\frac{\partial^2\varphi_0}{\partial x^2}\textstyle\prod_{j=1}^{N-1}(\varphi_0-\varphi_j)\Big)\times\Big(\prod_{j=1}^{N-1}(\varphi_0-\varphi_j)\Big)^2(\psi(t))\\
-2g_{x,y}g_xg_y(\psi(t)) &= 2\Big(\textstyle\sum_{j=1}^{N-1}\frac{\partial\varphi_0}{\partial x}\prod_{k\neq 0,j}(\varphi_0-\varphi_k)\\
&\quad +\textstyle\sum_{j=1}^{N-1}\frac{\partial\varphi_j}{\partial x}\prod_{k\neq 0,j}(\varphi_0-\varphi_k)\Big)\times\big(-\frac{\partial\varphi_0}{\partial x}\big)\Big(\prod_{j=1}^{N-1}(\varphi_0-\varphi_j)\Big)^2(\psi(t))\\
g_{y,y}g_x^2 &= \Big(2\textstyle\sum_{j=1}^{N-1}\prod_{k\neq 0,j}(\varphi_0-\varphi_k)\Big)\Big(\frac{\partial\varphi_0}{\partial x}\Big)^2\Big(\prod_{j=1}^{N-1}(\varphi_0-\varphi_j)\Big)^2(\psi(t))
\end{aligned}$$

Thus by an easy computation we get $\mathcal{F}(f)(x(t),y(t)) = -U^3(\psi(t))\frac{\partial^2\varphi_0}{\partial x^2}(x(t))\prod_{i=1}^{N-1}(\varphi_0(x(t))-\varphi_i(x(t))^3$. As the number of $\{0<k<N; k \equiv 0(N_{j-1})\}$ is $n_j\cdots n_\ell - 1$, the assertion follows from the equalities $\mathrm{val}_t\frac{\partial^2\varphi_0}{\partial x^2}$

$(x(t)) = (s-2)N$ and $\mathrm{val}_t(\varphi_0 - \varphi_k)(x(t)) = m_j N/N_j$, if $k \equiv 0$ (N_{j-1}) and $k \not\equiv 0$ (N_j). Q.E.D.

Corollary 9. *For flex point P of order k, we have $\ell = 0$ and $s = k+2$. Thus $\delta(P; f) = k$.*

We can generalize Theorem 7 to reducible singularities. To avoid the complexity of notations, we do this only for the class of singularity which is equivalent to the Brieskorn singularity $B_{p,q} : y^p - x^q = 0$ with $2 \le p \le q$ at the origin. We denote this equivalence class by $\beta_{p,q}$. Put $r = \gcd(p,q)$ and write $p = rn_1$ and $q = rm_1$. Then each irreducible component has the unique Puiseux pair $\{(m_1, n_1)\}$. Take a function germ $f(x,y)$ which defines such a singularity at the origin. As the resolution complexity of a Brieskorn singularity is one ([L-O]), after a linear change of coordinates, we may assume that $f(x,y) = f_1(x,y) \cdots f_r(x,y)$ where $f_j(x,y) = (y - \sum_{2 \le i < [q/p]} a_i x^i)^{n_1} + c_{j,m_1} x^{m_1} +$ (higher terms) where a_i, $2 \le i < [q/p]$, are independent of j and $c_{1,m_1}, \dots, c_{r,m_1}$ are mutually distinct non-zero complex numbers. In particular, the Puiseux orders of $f_j, j = 1, \dots, r$, are the same. Let $\sigma(\beta_{p,q}; s)$ be the set of $f \in \beta_{p,q}$ whose irreducible components have the Puiseux order s.

Theorem 10. *Assume that $p < q$ and $f \in \sigma(\beta_{p,q}; s)$. Then the flex defect and the generic flex defect are given as follows.*

$$\delta(O; f) = 3pq - 3q + (s-2)p, \quad \bar{\delta}(\beta_{p,q}) = \begin{cases} 3pq - 3q, & q > 2p \\ 3pq - 2(p+q), & q \le 2p \end{cases}$$

Proof. Let $y = \varphi_j(x^{1/n_1})$ be the Puiseux expansion of y in x for $f_j(x,y) = 0$. By the assumption, it is written as $\varphi_j(x^{1/n_1}) = \sum_{s \le i < [q/p]} a_i x^i + \sum_{k=m_1}^{\infty} c_{j,k} x^{k/n_1}$ where $(c_{1,m_1})^{n_1}, \dots, (c_{r,m_1})^{n_1}$ are mutually distinct complex numbers. Let us consider $\varphi_{j,k}(x^{1/n_1}) = \varphi_j$ $(x^{1/n_1}\omega^k)$ with $\omega = \exp(2\pi\sqrt{-1}/n_1)$. Then $f_j(x,y)$ is given by the product $(y - \varphi_{j,0}) \cdots (y - \varphi_{j,n_1-1})$. Denote the i-th branch $f_i(x,y) = 0$ by C_i. To compute the intersection number $I(C_1, J; O)$, we consider the parameterization $x(t) = t^{n_1}$ and $y(t) = \varphi_1(t)$. Then by the same computation as in the proof of Theorem 7, we obtain $\mathcal{F}(f)(x(t), y(t)) = -\frac{\partial^2 \varphi_{1,0}}{\partial x^2}(x(t), y(t)) \prod_{(i,k) \ne (1,0)} (\varphi_{1,0} - \varphi_{i,k})(x(t))^3$. Therefore we obtain the formula

$$(11) \quad I(C_1, J; O) = \mathrm{val}_t(\mathcal{F}(f)(x(t), y(t))) = 3rn_1m_1 - 3m_1 + (s-2)n_1$$

The other intersection numbers $I(C_j, J; O), j = 2, \ldots, r$, are the same. Thus as $\delta(O; C)$ is the sum $I(C_1, J; O) + \cdots + I(C_r, J; O)$, the assertion follows immediately. Q.E.D.

Now we consider the case $p = q$. Then $r = p$ and we may assume that $f_j(x, y) = y - \sum_{k=1}^{\infty} c_{j,k} x^k$ where $\{c_{1,1}, \ldots, c_{p,1}\}$ are mutually distinct complex numbers. Put $\mathcal{S}_i = \{j; j \geq 2, c_{i,j} \neq 0\}$. We assume that $\mathcal{S}_i \neq \emptyset$ for each $i = 1, \ldots, p$. (Otherwise, C contains a line and it is contained in J.) Put s_i be the minimum of $\mathcal{S}_i$. Unlike the previous cases, $\delta(O; f)$ is not bounded.

Corollary 12. *Assume that $(C, O) \in \beta_{p,p+1}$, i.e., a cusp singularity of type $(p, p+1)$ at the origin. Then $\delta(O; C) = \bar{\delta}(\beta_{p,p+1}) = 3p^2 - p - 2$. For $A_{2p-1} = \beta_{2,2p}$, we have $\bar{\delta}(A_{2p-1}) = 6p$ for $p = 1, 2$ and $8p - 4$ for $p \geq 3$.*

By a similar computation, we have

Theorem 13. *The flex defect of the singularity $(C(f), O) \in \beta_{p,p}$ is given by $\delta(O; f) = 3p^2 - 3p + \sum_{i=1}^{p}(s_i - 2)$ and $\bar{\delta}(\beta_{p,p}) = 3p^2 - 3p$.*

Let σ_i, $i = 1, \ldots, k$, be equivalence classes of plane curve singularity and let $\Sigma = \{\sigma_1, \ldots, \sigma_k\}$. Consider the set of plane curves $\mathcal{M}(n; \Sigma)$ of degree n with k singularities which are equivalent to σ_i, $i = 1, \ldots, k$. Take a curve $C \in \mathcal{M}(n; \Sigma)$ and let $P_1, \ldots, P_k$ be the singular points of C. C is called *generic* in $\mathcal{M}(n; \Sigma)$ if the following conditions (1),(2),(3) are satisfied.
(1) (C, P_i) is a generic in σ_i and the tangent lines at P_i intersect C transversely except at P_i. (2) The flexes are of order one. (3) The multi-tangent lines are ordinary bi-tangent lines.

A Plücker curve is a generic curve in the case that Σ contains only nodes or cusps. The set of generic curves is an open subset of $\mathcal{M}(n; \Sigma)$ but it might be empty. See Example 17.

2.6. Dual singularity

Let $P \in C$ and P^* be the corresponding point of C^*. As is well-known, P is a $(k-1, k)$-cusp if and only if P^* is a flex of order $k - 2$. If P is a generic node, P^* consists of two tangent points with a bi-tangent line. We study the correspondence for other singularities. Take a point $O \in C$ and let $C_1, \ldots, C_k$ be the local irreducible components at O and let $\ell_1, \ldots, \ell_r$ be the corresponding tangent line at O. Then the dual image C_i^* of C_i passes through $\ell_i \in \mathbf{P}^{*2}$ for $i = 1, \ldots, k$. In the case of C being irreducible at O, we simply denote ℓ_1 by O^*. We call the germ (C_i^*, ℓ_i) *the dual singularity* of the germ of (C_i, O).
(1) **Irreducible case.** Let $\mathcal{P} = \{(m_1, n_1), \ldots, (m_\ell, n_\ell)\}$ and let $N_j = n_1 \cdots n_j$ ($N = N_\ell$) and assume that $(C, O) \in \sigma(\mathcal{P}; s)$ is an irreducible

germ at O defined by $f(x,y) = 0$ whose Puiseux series is given by $\varphi(x^{1/N}) = \sum_{i\geq 2}^{k_0} c_{0,i}x^i + h_1(x^{1/N_1}) + \cdots + h_\ell(x^{1/N_\ell})$ where $k_0 < m_1/n_1$ and $h_j(x^{1/N_j}) = \sum_{m_j \leq k < m_{j+1}/n_{j+1}} c_{j,k}x^{k/N_j}$, $c_{1,m_1}, c_{2,m_2}, \ldots, c_{\ell,m_\ell} \neq 0$. Let s be the Puiseux order. Let $\mathcal{S} = \{j; c_{i,0} \neq 0, j \geq 2\}$. The dual singularity is described by the following. The case $\ell = 0$ with $s \geq 3$ (a flex of order $s-2$) is also contained in the argument.

Theorem 14. (**Local Duality**) *Let $\sigma(\mathcal{P}; s)^* := \{(C^*, O^*); (C, O) \in \sigma(\mathcal{P}; s)\}$. Then the dual operation gives a well-defined mapping on the set of the strata of the flex stratification. More precisely,*

(1) *Assume that $\mathcal{S} \neq \emptyset$. Then $\sigma(\mathcal{P}; 2)^* = \sigma(\mathcal{P}, 2)$ and $\sigma(\mathcal{P}; s)^* = \sigma(\mathcal{P}^+; \frac{s}{s-1})$ if $s > 2$ where $\mathcal{P}^+ := \{(s, s-1), (m_1, n_1), \ldots, (m_\ell, n_\ell)\}$. The first equality says that the dual map $*$ gives an involution on $\sigma(\mathcal{P}; 2)$.*

(2) *Assume that $\mathcal{S} = \emptyset$. Then $s = m_1/n_1$ and $\sigma(\mathcal{P}; \frac{m_1}{n_1})^* = \sigma(\mathcal{P}^*; \frac{m_1}{m_1 - n_1})$, if $m_1 - n_1 > 1$ and $\sigma(\mathcal{P}; \frac{m_1}{n_1})^* = \sigma(\mathcal{P}^-; m_1)$, if $m_1 = n_1 + 1$, where $\mathcal{P}^* := \{(m_1, m_1 - n_1), (m_2, n_2), \ldots, (m_\ell, n_\ell)\}$ and $\mathcal{P}^- := \{(m_2, n_2), \ldots, (m_\ell, n_\ell)\}$.*

There is a related result by Wall [W2]. The cases $\ell = 0,\ s \geq 3$ or $\ell = 1$ and $m_1 = n_1 + 1$ are special cases of (1) and (2) respectively. It follows from (2) that a cusp of type $(k, k+1)$ and a flex of order $k-1$ corresponds each other by the dual operation.

Proof. Put $N_j = n_1 \cdots n_j$, $N^{(j)} = n_j \cdots n_\ell$ and $N = N_\ell$. Putting $x^{1/N} = t$, we can parameterize C as $x(t) = t^N$ and $y(t) = \varphi(t) = \sum_j b_j t^j$ where the coefficients are given by $b_k = c_{j,k/N^{(j+1)}}$, if $m_j \leq k/N^{(j+1)} < m_{j+1}/n_{j+1}$ and $k/N^{(j+1)} \in \mathbf{Z}$. Otherwise $b_k = 0$. By (1), we can parameterize C^* as $u(t) = -\sum_j \frac{jb_j}{N} t^{j-N}$, $\quad w(t) = \sum_j (\frac{j}{N} - 1) b_j t^j$ where (u, w) is the affine coordinates defined by $u = U/V$, $w = W/V$. Note that $\mathrm{val}_t u(t) = (s-1)N$. We take a change of parameter τ so that $u(\tau) = \tau^{(s-1)N}$. Write $t = \tau \sum_{k=0}^{\infty} \lambda(k)\tau^k$. The coefficients $\lambda(0), \lambda(1), \lambda(2), \ldots$ are inductively determined from the equality $u(t(\tau)) = \tau^{(s-1)N}$ after fixing $\lambda(0)$ which satisfies $\lambda(0)^{(s-1)N} = -1/sb_{sN}$.

Assertion 2. *For $p < m_k N^{(k+1)} - sN$, $\lambda(p) = 0$ if $p \not\equiv 0$ modulo $N^{(k)}$. The first non-trivial coefficient $\lambda(p)$ with $p \not\equiv 0$ $(N^{(k)})$ is $\lambda(m_k N^{(k+1)} - sN)$ and it is given by*

$$(15)\quad \lambda(m_k N^{(k+1)} - sN) = -\frac{m_k N^{(k+1)}}{s(s-1)N^2} \frac{b_{m_k N^{(k+1)}}}{b_{sN}} \lambda(0)^{m_k N^{(k+1)} - sN + 1}$$

Proof. Assume that we have shown $\lambda(p) = 0$ for $p \not\equiv 0$ modulo $N^{(k)}$ and $p < p'$ for some $p' \leq m_k N^{(k+1)} - sN$. Consider the equality:

$(PC): \tau^{(s-1)N} = -\sum_{j\geq sN} \frac{jb_j}{N}\tau^{j-N}(\sum_q \lambda(q)\tau^q)^{j-N}$. We compare the coefficients of $\tau^{p'+sN-N}$. Assume first that $p' \not\equiv 0\ (N^{(k)})$ and $p' < m_kN^{(k+1)} - sN$. Then the term $\tau^{p'+sN-N}$ in the right side comes only from the first term ($j = sN$) of the summation and the coefficient is $-s(s-1)Nb_{sN}\lambda(0)^{sN-N-1}\lambda(p')$. Thus $\lambda(p') = 0$. By an induction, we get $\lambda(p) = 0$ for $p < m_kN^{(k+1)} - sN$ with $p \not\equiv 0$ modulo $N^{(k)}$.

Now we consider the case $p'=m_kN^{(k+1)}-sN$. The term $\tau^{m_kN^{(k+1)}-N}$ in the right side summation comes from $j = sN$ and $j = m_kN^{(k+1)}$. Comparing the coefficient of $\tau^{m_kN^{(k+1)}-N}$ in (PC), we have

$$-s(s-1)Nb_{sN}\lambda(0)^{sN-N-1}\lambda(m_kN^{(k+1)} - sN)$$
$$-\frac{m_kN^{(k+1)}}{N}b_{m_kN^{(k+1)}}\lambda(0)^{m_kN^{(k+1)}-N} = 0$$

and the assertion follows from this equality. Q.E.D.

The other coefficients $\lambda(j)$'s are complicated but they are not important. To determine the Puiseux pairs of the dual curve, we write $w(\tau) = \sum_j d(j)\tau^j$. Then by a similar argument,

Assertion 3. *(1) The coefficient $d(j)$ vanishes for any $j < sN$ and $d(sN) = (s-1)b_{sN}\lambda(0)^{sN}$.*
(2) The coefficient $d(j)$ for $j \not\equiv 0\ (N^{(k)})$ vanishes for $j < m_kN^{(k+1)}$ and the first non-vanishing coefficient $d(j)$ with $j \not\equiv 0\ (N^{(k)})$ is $d(m_kN^{(k+1)})$, which is given by $d(m_kN^{(k+1)}) = -b_{m_kN^{(k+1)}}\lambda(0)^{m_kN^{(k+1)}}$.

Proof. As $w(t) = \sum_j(\frac{j}{N} - 1)b_jt^j$, the first assertion of (2) follows immediately from Assertion 2. The second equality follows from

$$\begin{aligned} d(m_jN^{(j+1)}) &= (\frac{sN}{N} - 1)b_{sN}\lambda(0)^{sN-1}sN\lambda(m_jN^{(j+1)} - sN) \\ &\quad +(\frac{m_jN^{(j+1)}}{N} - 1)b_{m_jN^{(j+1)}}\lambda(0)^{m_jN^{(j+1)}} \\ &= -b_{m_jN^{(j+1)}}\lambda(0)^{m_jN^{(j+1)}} \quad \text{Q.E.D.} \end{aligned}$$

Assume that $\mathcal{S} \neq \emptyset$. Assume first that $s = 2$. Then $u = \tau^N$ and $(C^*, O^*) \in \sigma(\mathcal{P}; 2)$. If $s > 2$, $u(\tau) = \tau^{(s-1)N}$ and the assertion follows from $\frac{m_jN^{(j+1)}}{(s-1)N} = \frac{m_j}{(s-1)n_1\cdots n_j}$. Assume that $\mathcal{S} = \emptyset$ and $s = m_1/n_1$. Then $u(\tau) = \tau^{(m_1-n_1)N^{(2)}}$ and $\gcd(m_1 - n_1, n_1) = 1$. Thus the assertion follows. This completes the proof.

(2) **Reducible case**. A similar assertion can be proved for reducible curve germs. We do this for Brieskorn singularities. Let us consider a germ of a Brieskorn singularity $(C, O) \in \beta_{p,q}$ defined by a polynomial

$f(x,y)$ with the tangential direction $y=0$. Let $r=\gcd(p,q)$ and write $p=rn_1$ and $q=rm_1$. Let $f=f_1\cdots f_r$ be the factorization and let C_j be the irreducible component of C defined by $f_j(x,y)=0$. Recall that the Puiseux expansions of $f_j(x,y)$ in x for $i=1,\dots,r$ are the same up to the term x^{m_1/n_1}.

Theorem 16. **(Local Duality-bis)** *Assume that $p<q$ and $(C,O) \in \sigma(\beta_{p,q};s)$. Then $s=q/p$ and $(C^*,O^*)\in\sigma(\beta_{q-p,q};\frac{q}{q-p})$ if $q\le 2p$. If $2p<q$ and $s=2$, then $(C^*,O^*)\in\sigma(\beta_{p,q};2)$. If $2p<q$ and $s>2$, $(C^*,O^*)=\cup_{i=1}^r(C_i^*,O^*)$ and $(C_i^*,O^*)\in\sigma(\mathcal{P}^+;\frac{s}{s-1})$ with $\mathcal{P}^+=\{(s,s-1),(m_1,n_1)\}$. The Puiseux expansions of C_i^* in $u^{1/(s-1)n_1}, i=1,\dots,r$ coincide up to the term $u^{m_1/(s-1)n_1}$.*

Proof. Assume first that $m_1>2n_1$. Then C_j is defined by a polynomial $f_j(x,y)$ of the form $f_j(x,y)=(y-\sum_{i=s}^{k_0}a_ix^i)^{n_1}-c_j^{n_1}x^{m_1}+$ (higher terms) where $k_0=[m_1/n_1]$, $a_s\neq 0$ and $s\ge 2$. Here $a_s,\dots,a_{k_0}$ are independent of j. In the proof of Theorem 14, we have shown that $(C_j^*,O^*)\in\sigma(\mathcal{P}^+;\frac{s}{s-1})$ with $\mathcal{P}^+=\{(s,s-1),(m_1,n_1)\}$ and C_j^* is parameterized as $u(t)=\tau^{(s-1)n_1}$ and $w(t)=\sum_{s\le i<m_1/n_1}d(i)\tau^{in_1}+\sum_{i=m_1}^{\infty}d(j,i)\tau^i$. Thus the assertion follows from the observation $d(s)\neq 0$ and $d(s),\dots,d(k_0)$ are independent of j and $d(j,m_1)=c_j\times\lambda(0)^{m_1}$. In particular, this implies that if $s=2$, $(C_j^*,O^*)\in\sigma(\beta_{m_1,n_1};2)$ and C_j^* is defined by a polynomial of the type $g_j(u,w)=(w-\sum_{2\le i<m_1/n_1}d(i)u^i)^{n_1}$ $-d(j,m_1)^{n_1}\omega^Au^{m_1}+$ (higher terms) where $\omega=\exp(2\pi\sqrt{-1}/n_1)$ and $A=n_1(n_1-1)m_1/2$. Thus the assertion follows immediately. The case $m_1\le 2n_1$ can be treated similarly. Q.E.D.

Example 17. *Let us consider a rational curve $C=\{f(x,y)=0\}$ of degree 6 where $f(x,y)=(x^2+y^3)^2-4y^3x^3$. In the affine coordinate $(u,v)=(Z/X,Y/X)$, C is defined by $(u+v^3)^2-4v^3=0$. Thus C is a Jung transform of the rational curve $u^2-4v^3=0$ (See Example (6.6), [O4]). C has two singularities. One (2,3) cusp at $P:=(1,0,0)$ and one irreducible singularity of Puiseux pairs $\{(3,2),(9,2)\}$ at $Q:=(0,0,1)$. By Theorem 7, the flex defect at Q is 61 and the Milnor number is* 18. *Thus the dual curve should have three cusps and* 3 *nodes if C is generic in the moduli. The dual curve is given by $C^*=\{g(x,y)=0\}$ where $g(x,y)=16y^6+27y^3+540y^3x-216y^3x^2+729x+2187x^2+2187x^3+729x^4$. Thus C^* is a rational curve of degree 6 and it has three cusps and one singularity at $Q^*:=(1,0,0)$ of Milnor number* 8 *which is in the moduli $\sigma(\{(9,2)\};3)$ by Theorem 14. The discriminant of g in y is given by $cx^2(x+1)^6(8x-1)^9$, $c\neq 0$ and C^* has a $\beta_{3,3}$ singularity at $(-1,0,1)$. C is not generic as C^* does not have three nodes but a $\beta_{3,3}$. The reason* *is, C has a tri-tangent line $x = -1$. In fact, by computing the moduli space explicitly, we can show that there does not exist any generic curve in the moduli of C but every member has a tri-tangent line.*

§3. Moduli of certain sextics and their dual

In this section, we consider various moduli spaces of sextics. Unless otherwise stated, n, n^*, g are the degree, the class number and the genus of the curve in discussion respectively.

3.1. Moduli space $\mathcal{M} := \mathcal{M}(6; \Sigma)$.

Let $\Sigma = \{3\beta_{2,2}, 6\beta_{2,3}\}$ and consider the moduli space $\mathcal{M} := \mathcal{M}(6; \Sigma)$ of sextics with 6 cusps and 3 nodes. Let us denote the subset of $\mathcal{M}$ whose curves are generic (i.e., Plücker) by $\mathcal{M}'$. It is easy to see that $g(C) = 1$ for any $C \in \mathcal{M}$ by the modified Plücker formula. By the class formula (3), the dual curves C^* has degree 6. By Theorem 10 and Proposition 5, the dual curve C^* has also 6 cusps for $C \in \mathcal{M}'$. As $g(C^*) = 1$, they have 3 nodes. Thus we have the self-duality: $\mathcal{M}'^* = \mathcal{M}'$. However $\mathcal{M}^* \neq \mathcal{M}$. The reason is that there exists an interesting degeneration in this moduli as we will see below. First, the number of flexes on $C \in \mathcal{M}$ is 6 counting the multiplicity by Proposition 5. Thus the possible types of flexes are (0) 6 flexes of order 1, (i) 4 flexes of order 1 and one flex of order 2, (ii) 2 flexes of order 1 and 2 flexes of order 2, (iii) 3 flexes of order 2 and (iv) 3 flexes of order 1 and one flex of order 3. There do not exist other types as the dual curve has genus 1 and the sum of Milnor numbers of the singular points of C^* is less than or equal to 18 by the modified Plücker formula. The moduli space with these flex types are difficult to study directly. So we consider their dual moduli spaces.
(1) Let $\Sigma_1 = \{2\beta_{2,2}, 4\beta_{2,3}, \beta_{3,4}\}$ and let $\mathcal{N}_1 := \mathcal{M}(6; \Sigma_1)$. The genus of a curve in $\mathcal{N}_1$ is 1 and the class number is 6. Thus we have the inclusion: $\mathcal{N}_1'^* \subset \mathcal{M}$. Here we denote by $\mathcal{N}_1'$ the submoduli of $\mathcal{N}_1$ which consists of the generic curves. A curve $C \in \mathcal{N}_1'^*$ is not a Plücker curve but it has 4 flexes of order 1 and a flex of order 2. We put $\mathcal{M}_1 := \mathcal{N}_1'^*$. By reciprocity law, $C \in \mathcal{M}$ is in $\mathcal{M}_1$ if and only if C has 4 flexes of order 1, one flex of order 2 and two bi-tangents.
(2) Let $\Sigma_2 = \{\beta_{2,2}, 2\beta_{2,3}, 2\beta_{3,4}\}$ and $\mathcal{N}_2 := \mathcal{M}(6; \Sigma_2)$. For $C \in \mathcal{N}_2$, the genus $g(C) = 1$ and $n^* = 6$. The generic dual $\mathcal{N}_2'^*$ consists of curves C with 6 cusps and 3 nodes and 2 flexes of order 1 and 2 flexes of order 2. We denote this dual image $\mathcal{N}_2'^*$ by $\mathcal{M}_2$.
(3) Let $\Sigma_3 = \{3\beta_{3,4}\}$ and let $\mathcal{N}_3 := \mathcal{M}(6; \Sigma_3)$. We see that $g(C) = 1$ for any $C \in \mathcal{N}_3$ and the generic dual $\mathcal{N}_3'^*$ is again 6 cuspidal 3 nodal sextics with 3 flexes of order 2. The moduli of such curves is denoted by $\mathcal{M}_3$.

(4) Finally let $\Sigma_4 = \{\beta_{4,5}, 3\beta_{2,3}\}$ and let $\mathcal{N}_4 = \mathcal{M}(6;\Sigma_4)$. We see that $g = 1, n^* = 6$ and the generic dual $\mathcal{N}_3'^*$ is again 6 cuspidal 3 nodal sextics with 3 ordinary flexes and one flex of order 3. Put $\mathcal{M}_4 := \mathcal{N}_4'^*$.

Let $\mathcal{T}$ be the moduli space of (2,3)-torus curves of degree 6 and of type (2,3). The respective submoduli of torus type $\mathcal{M}\cap\mathcal{T}$, $\mathcal{M}_i\cap\mathcal{T}$ and $\mathcal{N}_i\cap\mathcal{T}$ are denoted simply by $\mathcal{M}_{torus}, \mathcal{M}_{i,torus}, \mathcal{N}_{i,torus}$ respectively. Non-torus moduli are denoted as $\mathcal{M}_{gen}, \mathcal{M}_{i,gen}, \mathcal{N}_{i,gen}$ respectively. The main result about the structure of the moduli spaces $\mathcal{M}$ is:

Theorem 18. *1. The union $\widehat{\mathcal{M}} := \mathcal{M}' \cup_{i=1}^4 \mathcal{M}_i \cup_{i=1}^4 \mathcal{N}_i'$ is invariant by the dual operation. Namely the dual operation $C \mapsto C^*$ gives an involution on $\widehat{\mathcal{M}}$. Furthermore the dual operation preserves curves of torus type and non-torus type. Namely ${\mathcal{M}_\alpha'}^* = \mathcal{M}_\alpha'$, ${\mathcal{N}_{i,\alpha}'}^* = \mathcal{M}_{i,\alpha}$ and ${\mathcal{M}_{i,\alpha}}^* = \mathcal{N}_{i,\alpha}'$ for $i = 1,\dots,4$ and α=torus or gen.*
2. (Stratification)*$\mathcal{M}_{torus}=\mathcal{M}_{torus}' \cup_{i=1}^3 \mathcal{M}_{i,torus}$ and $\mathcal{M}_{gen}=\mathcal{M}_{gen}' \cup_{i=1}^4 \mathcal{M}_{i,gen}$. Thus $\mathcal{M}_4 = \mathcal{M}_{4,gen}$ and $\mathcal{N}_4 = \mathcal{N}_{4,gen}$. The moduli spaces $\mathcal{M}_{torus}', \mathcal{M}_{i,torus}, \mathcal{N}_{i,torus}$, $i = 1,2,3$ and $\mathcal{N}_{3,gen}$ are irreducible. For the moduli of the curves of torus type, we have the adherence relation:*

$$\overline{\mathcal{M}_{torus}'} \supset \overline{\mathcal{M}_{1,torus}} \supset \overline{\mathcal{M}_{2,torus}} \supset \mathcal{M}_{3,torus},$$
$$\overline{\mathcal{M}_{torus}'} \supset \overline{\mathcal{N}_{1,torus}'} \supset \overline{\mathcal{N}_{2,torus}'} \supset \mathcal{N}_{3,torus}'$$

3. (Alexander polynomial) *For $C \in \widehat{\mathcal{M}}_{torus}$, the Alexander polynomial $\Delta_C(t)$ is given by $t^2 - t + 1$* ([Li1],[D]). *For non-torus curve $C \in \widehat{\mathcal{M}}_{gen}$, it is given by 1.*
4. (Fundamental groups) *$\pi_1(\mathbf{P}^2 - C) \cong \mathbf{Z}_2 * \mathbf{Z}_3$ or $\pi_1(\mathbf{P}^2 - C) \cong \mathbf{Z}_6$ according to $C \in \widehat{\mathcal{M}}_{torus}$ or $C \in \mathcal{M}_{3,gen}$ respectively.*

Remark 3. *We do not know if the other moduli spaces of non-torus type are irreducible. If this is the case, the adherence relations and the commutativity of the fundamental group holds for other non-torus type sextics $\mathcal{M}_{i,gen}, \mathcal{N}_{i,gen}$, $i = 1,2,3$. The moduli space $\mathcal{N}_4$ seems to be irreducible.*

3.2. Alexander polynomial

Let C be an irreducible plane curve of degree n and L_∞ be the line at infinity. We assume that L_∞ intersects C transversely. We consider the Alexander polynomial $\Delta_C(t)$ with respect to L_∞ and we call it *the generic Alexander polynomial.* It has integral coefficients. For the definition of the Alexander polynomial, we refer to [Li2]. We recall several basic properties of $\Delta_C(t)$.
(1) $\Delta_C(t)$ divides the Alexander polynomial at infinity $(t^n - 1)^{n-2}(t-1)$

and also the product of the local Alexander polynomials at singular points of C ([Li2] and [Li1]).

Let $p : Y \to \mathbf{P}^2$ be the embedded resolution of the singularity of $C \cup L_\infty$. Let $q_m : X_m \to \mathbf{P}^2$ be the m-cyclic covering branched along $C \cup L_\infty$ and let $p_m : Z_m \to Y$ be the desingularization of the pull-back of q_m by p. Let $\Lambda := \mathbf{Q}[t, t^{-1}]$. Then $H_1(X_\infty; \mathbf{Q})$ is a Λ-module where t acts as the Deck transformation. Thus there are polynomials $\lambda_1(t), \dots, \lambda_k(t)$ with $\lambda_i | \lambda_{i+1}$, $i = 1, \dots, k-1$, such that $H_1(X_\infty; \mathbf{Q})$ is isomorphic to the direct sum $\sum_{i=1}^k \Lambda/(\lambda_i)$ and $\Delta_C(t) = \lambda_1(t) \cdots \lambda_k(t)$.
(2) The first Betti number $b_1(Z_m)$ of Z_m is equal to the sum $\sum_{i=1}^k \alpha_i$ where α_i is the the number of different m-th roots of unity in the roots of $\lambda_i(t) = 0$ ([Li2]).
(3) $\Delta_C(t)$ is a cyclotomic polynomial and $\Delta_C(1) = \pm 1$ (see for example, [R]).

Consider the case $m = n$ and we write $Z := Z_n$ for simplicity. Combining these properties, the determination of the Alexander polynomial is reduced to the calculation of the first Betti number of Z, or equivalently to the calculation of the irregularity of Z.

For the calculation of the irregularity $q(Z)$, the method by Esnault ([E]) and Artal ([A1]) is convenient. Let us recall it. Let $P_1, \dots, P_\nu$ be the singular points of C. Let $L^{(k)}$ be the divisor on Y introduced in [E]. Then $b_1(Z) = 2q(Z) = 2\sum_{k=0}^{n-1} \dim H^1(Y; \mathcal{O}(L^{(k)}))$ by [E] and $H^1(Y; \mathcal{O}(L^{(k)}))$ can be identified by the cokernel of the natural homomorphism $\sigma_{k-3,k} : H^0(\mathbf{P}^2; \mathcal{O}(k-3)) \to \sum_{P_i} \mathcal{O}_{\mathbf{P}^2, P_i}/\mathcal{I}_{P_i,k,n}$ where $\mathcal{I}_{P_i,k,n}$ is an ideal described as follows ([A1]). Let $E_{i,1}, \dots, E_{i,\ell_i}$ be the exceptional divisors over P_i and let $m_{i,j}$ be the multiplicity of p^*f along $E_{i,j}$. Let $K = -3L + \sum_{i,j} k_{i,j} E_{i,j}$ be a canonical divisor, where L is a generic line, not passing through any of $P_1, \dots, P_\nu$. Then the ideal $\mathcal{I}_{P_i,k,n}$ is generated by the function germs g such that the pull-back p^*g vanishes along $E_{i,j}$ at least with the multiplicity $-k_{i,j} + [k m_{i,j}/n]$.

Now we are ready to compute the Betti number of Z_6 for the sectics in $\widehat{\mathcal{M}}$. For the computation, we use canonical toric modifications at singular points ([O6]). Assume that the singularity P_i is non-degenerate and the restriction of $p : Y \to \mathbf{P}^2$ to a neighbourhood of P_i is a toric modification. Let Σ_i^* be a regualr fan subdividing the dual Newton diagram $\Gamma^*(f; P_i)$ at P_i which is used to construct the toric modification and let $P_{i,j} = {}^t(a_{i,j}, b_{i,j})$, $j = 1, \dots, \ell_i$ be the primitive covectors which generate 1-dimensional cones and let $\widehat{E}(P_{i,j})$ be the corresponding exceptional divisor. Then using the equality $\frac{dx}{x} \wedge \frac{dy}{y} = \frac{dx_\sigma}{x_\sigma} \wedge \frac{dy_\sigma}{y_\sigma}$, we have a simple formula: $K = -3L + \sum_{i,j}(a_{i,j} + b_{i,j} - 1)\widehat{E}(P_{i,j})$. Here (x_σ, y_σ)

are the toric coordinates of the coordinate chart $\mathbf{C}^2_\sigma$ and $\widehat{E}(P_{i,j})$ is the exceptional divisor corresponding to $P_{i,j}$. Refer to Chapter III, in [O6] for detail.

(a) For a cusp, $y^2 - x^3 +$ (higher terms) $= 0$, the exceptional divisors correspond to covectors $Q_1 = {}^t(1,1), Q_2 = {}^t(2,3),\ Q_3 = {}^t(1,2)$. We have $K = -3L + \widehat{E}(Q_1) + 4\widehat{E}(Q_2) + 2\widehat{E}(Q_3)$ and $(p^*f) = C' + 2\widehat{E}(Q_1) + 6\widehat{E}(Q_2) + 3\widehat{E}(Q_3)$ (locally at each P_i). Here C' is the strict transform of C. Recall the equivalence: *a curve $C \in \mathcal{M}'$ is of torus type if and only if six cusps are on a conic* (see [D]). Let $C \in \mathcal{M}$ and let $P_1, \dots, P_6$ be the cusps. The nodes have nothing to do with the Alexander polynomial. The non-trivial case is $H^1(Y; \mathcal{O}(L^{(5)}))$. The kernel of $\sigma_{2,5} : H^0(\mathbf{P}^2; \mathcal{O}(2)) \to \sum_{i=1}^6 \mathcal{O}_{\mathbf{P}^2,P_i}/\mathcal{I}_{P_i,5,6}$ consists of conics passing through $P_1, \dots, P_6$. Thus $\dim \operatorname{Ker}(\sigma_{2,5}) = 1$ or 0 and therefore $b_1(Z_6) = 2$ or 0 depending on whether C is of torus type or not. By (1), this also implies $\Delta_C(t) = (t^2 - t + 1)^\alpha,\ \alpha \geq 1$, or 1 respectively.

(b) Now we consider (3,4)-cusp, $y^3 - x^4 +$ (higher terms) $= 0$. We have four exceptional divisors, corresponding to $Q_1 = {}^t(1,1), Q_2 = {}^t(3,4),\ Q_3 = {}^t(2,3),\ Q_4 = {}^t(1,2)$. $K = -3L + \widehat{E}(Q_1) + 6\widehat{E}(Q_2) + 4\widehat{E}(Q_3) + 2\widehat{E}(Q_4)$ and $(p^*f) = C' + 3\widehat{E}(Q_1) + 12\widehat{E}(Q_2) + 8\widehat{E}(Q_3) + 4\widehat{E}(Q_4)$.

Let $C \in \mathcal{N}_3$ be a sextic with 3 (3,4)-cusps. The non-trivial case is again $\sigma_{2,5} : H^0(\mathbf{P}^2; \mathcal{O}(2)) \to \sum_{i=1}^3 \mathcal{O}_{\mathbf{P}^2,P_i}/\mathcal{I}_{P_i,5,6}$. Locally $\mathcal{I}_{P_i,5,6}$ is generated by function germs $g(x,y)$ such that either it has no linear term in a coordinate centered at P_i or the conic $g = 0$ is tangent to the tangent cone of C at P_i. Thus $\dim \mathcal{O}_{\mathbf{P}^2,P_i}/\mathcal{I}_{P_i,5,6} = 2$. q is in the kernel of $\sigma_{2,5}$ if and only if the conic $q = 0$ passes through P_1, P_2, P_3 and is tangent to (the tangent cones of) C at $P_i, i = 1,2,3$. Thus $b_1(Z_6) = 2$ ($\Delta_C(t) = (t^2 - t + 1)^\beta,\ \beta \geq 1$) if and only if C is of torus type (cf. Corollary 24). Otherwise $b_1(Z_6) = 0$. To show $\alpha = \beta = 1$, we need a little more discussion but in our case, this follows immediately from the assertion on the fundamental group (see §5) and the Fox calculus. The computation of $b_1(Z_6)$ for curves in $\mathcal{N}_1, \mathcal{N}_2$ are similar.

(c) We consider a (4,5)-cusp, $y^4 - x^5 +$ (higher terms) $= 0$. We need 5 exceptional divisors, corresponding to the covectors $Q_1 = {}^t(1,1), Q_2 = {}^t(4,5), Q_3 = {}^t(3,4),\ Q_4 = {}^t(2,3)$ and $Q_5 = {}^t(1,2)$. The canonical divisor is locally given by $K = \widehat{E}(Q_1) + 8\widehat{E}(Q_2) + 6\widehat{E}(Q_3) + 4\widehat{E}(Q_4) + 2\widehat{E}(Q_5)$ and $(p^*f) = C' + 4\widehat{E}(Q_1) + 20\widehat{E}(Q_2) + 15\widehat{E}(Q_3) + 10\widehat{E}(Q_4) + 5\widehat{E}(Q_5)$.

Now we compute the Alexander polynomial of $C \in \mathcal{N}_4$. Thus C has a (4,5)-cusp singularity at P_1 and 3 (2,3)-cusps at P_2, P_3, P_4. Observe first that any two of $P_i, i = 2,3,4$ can not be colinear with P_1 by the

Bezout theorem. Again we only need to compute $\mathrm{Ker}(\sigma_{2,5})$. We can see easily that $\mathcal{I}_{P_1,5,6}$ is generated by the functions without any linear term at P_1 and $\mathcal{I}_{P_i,5,6}$ is generated by functions vanishing at P_i for $i = 2, 3, 4$. Thus the dimension of the target is also 6. A conic $q = 0$ is in the kernel of $\sigma_{2,5}$ if $q = 0$ has multiplicity 2 at P_1 and passes through P_2, P_3, P_4. This is impossible. Thus $\Delta_C(t)$ is trivial. See also Proposition 27. We thank to Anatoly Libgober for communicating us that the computation can be also made using quasiadjunction formula as in [Li1].

3.3. Moduli space $\mathcal{M}$

We first compute the moduli space $\mathcal{M}_{torus} = \mathcal{M} \cap \mathcal{T}$ where $\mathcal{M} = \mathcal{M}(6; 6\beta_{2,3} + 3\beta_{2,2})$. We start from the expression $f(x,y) = f_2(x,y)^3 + f_3(x,y)^2$ where

$f_2(x,y) = y^2 + y(a_{1,0} + a_{1,1}x) + a_{0,0} + a_{0,1}x + a_{0,2}x^2$ and

$f_3(x,y) = b_{3,0}y^3 + y^2(b_{2,0} + b_{2,1}x) + y(b_{1,0} + b_{1,1}x + b_{1,2}x^2) + b_{0,0} + b_{0,1}x + b_{0,2}x^2 + b_{0,3}x^3$

First we may assume that the nodes are at $O = (0,0), A = (1,1), B = (1,-1)$ by the action of PSL(3, **C**). The submoduli of $\mathcal{M}_{torus}$ consisting of curves with three nodes at O, A, B is denoted by $\mathcal{M}^{\#}_{torus}$. As PSL(3, **C**) orbit of $\mathcal{M}^{\#}_{torus}$ is $\mathcal{M}$, it is enough to see the irreducibility of $\mathcal{M}^{\#}_{torus}$. Introducing the variables t_0, t_1, t_2 such that $f_2(O) = -t_0^2$, $f_2(A) = -t_1^2$ and $f_2(B) = -t_2^2$, we can explicitly solve the equations $f(Q) = \frac{\partial f}{\partial x}(Q) = \frac{\partial f}{\partial y}(Q) = 0$, $Q = O, A, B$ as they are linear conditions. We can solve these equations, one by one so that the moduli has 6 free parameters $a_{1,0}, a_{0,2}, b_{2,1}, t_0, t_1, t_2$ and the other coefficients are uniquely determined as follows.

$$
\begin{aligned}
a_{0,0} &= -t_0^2,\\
a_{0,1} &= -1 - \tfrac{1}{2}t_1^2 - \tfrac{1}{2}t_2^2 + t_0^2 - a_{0,2},\\
a_{1,1} &= -a_{1,0} - \tfrac{1}{2}t_1^2 + \tfrac{1}{2}t_2^2,\\
b_{0,0} &= t_0^3\\
b_{0,1} &= -\tfrac{3}{2}t_0(-1 - \tfrac{1}{2}t_1 - \tfrac{1}{2}t_2 + t_0 - a_{0,2}),\\
b_{1,0} &= -\tfrac{3}{2}t_0a_{1,0},\\
b_{0,2} &= b_{2,1} + \tfrac{3}{2}t_2 - 3t_0 + \tfrac{3}{2}t_1 - \tfrac{3}{2}t_0a_{1,0} + \tfrac{15}{16}t_1^3 - 3t_0a_{0,2}\\
&\quad -\tfrac{9}{4}t_0t_1^2 + \tfrac{3}{4}t_1t_0^2 + \tfrac{3}{4}t_1a_{0,2} + \tfrac{3}{4}t_1a_{1,0} + \tfrac{3}{2}t_1(-a_{1,0} - \tfrac{1}{2}t_1^2 + \tfrac{1}{2}t_2^2)\\
&\quad -\tfrac{3}{2}t_0(-a_{1,0} - \tfrac{1}{2}t_1^2 + \tfrac{1}{2}t_2^2) + \tfrac{3}{16}t_2 - \tfrac{3}{16}t_1t_2\\
&\quad -\tfrac{3}{4}t_0t_2^2 + \tfrac{3}{4}t_2t_0^2 + \tfrac{3}{4}t_2a_{0,2} + \tfrac{3}{4}t_2a_{1,0} + \tfrac{9}{16}t_2t_1^2,
\end{aligned}
$$

$$\begin{aligned}
b_{0,3} &= \tfrac{1}{8}t_2^2 - \tfrac{3}{8}t_1t_2^2 + \tfrac{3}{4}t_0t_2^2 - \tfrac{3}{4}t_2t_0^2 - \tfrac{3}{4}t_2a_{0,2} - \tfrac{3}{4}t_2a_{1,0} \\
&\quad -\tfrac{3}{4}t_2 - \tfrac{3}{4}t_1t_0^2 - \tfrac{3}{4}t_1a_{0,2} + \tfrac{3}{4}t_1a_{1,0} + \tfrac{3}{2}t_0a_{0,2} \\
&\quad +\tfrac{1}{2}t_0^3 - \tfrac{3}{8}t_2t_1^2 + \tfrac{1}{8}t_1^3 - \tfrac{3}{4}t_1 - b_{2,1} + \tfrac{3}{4}t_0t_1^2 + \tfrac{3}{2}t_0, \\
b_{1,1} &= -\tfrac{3}{4}t_2 + \tfrac{3}{4}t_1 + \tfrac{3}{2}t_0a_{1,0} + \tfrac{3}{8}t_1^3 - \tfrac{3}{4}t_0t_1^2 + \tfrac{3}{4}t_1t_0^2 \\
&\quad +\tfrac{3}{4}t_1a_{0,2} - \tfrac{3}{4}t_1a_{1,0} - \tfrac{3}{2}t_0(-a_{1,0} - \tfrac{1}{2}t_1^2 + \tfrac{1}{2}t_2^2) - \tfrac{3}{8}t_2^3 + \tfrac{3}{8}t_1t_2^2 \\
&\quad +\tfrac{3}{4}t_0t_2^2 - \tfrac{3}{4}t_2t_0^2 - \tfrac{3}{4}t_2a_{0,2} - \tfrac{3}{4}t_2a_{1,0} - \tfrac{3}{8}t_2t_1^2, \\
b_{1,2} &= -\tfrac{9}{16}t_1^3 + \tfrac{3}{4}t_0t_1^2 - \tfrac{3}{4}t_1t_0^2 - \tfrac{3}{4}t_1a_{0,2} - \tfrac{3}{4}t_1a_{1,0} \\
&\quad -\tfrac{3}{2}t_1(-a_{1,0} - \tfrac{1}{2}t_1^2 + \tfrac{1}{2}t_2^2) + \tfrac{3}{2}t_0(-a_{1,0} - \tfrac{1}{2}t_1^2 + \tfrac{1}{2}t_2^2) - \tfrac{3}{16}t_2^3 \\
&\quad +\tfrac{9}{16}t_1t_2^2 - \tfrac{3}{4}t_0t_2^2 + \tfrac{3}{4}t_2t_0^2 + \tfrac{3}{4}t_2a_{0,2} + \tfrac{3}{4}t_2a_{1,0} + \tfrac{3}{16}t_2t_1^2, \\
b_{2,0} &= \tfrac{3}{16}t_2^3 - \tfrac{3}{16}t_1t_2^2 - \tfrac{3}{16}t_2t_1^2 - \tfrac{3}{4}t_2 + \tfrac{3}{16}t_1^3 - \tfrac{3}{4}t_1 - b_{2,1}, \\
b_{3,0} &= \tfrac{1}{16}t_2^3 - \tfrac{3}{16}t_1t_2^2 + \tfrac{3}{16}t_2t_1^2 + \tfrac{3}{4}t_2 - \tfrac{1}{16}t_1^3 - \tfrac{3}{4}t_1
\end{aligned}$$

Thus the moduli space $\mathcal{M}_{torus}^{\#}$ is a Zariski-open subset of $\mathbf{C}^6$ and this proves the irreducibility of the moduli $\mathcal{M}_{torus}^{\#}$ and $\mathcal{M}_{torus}$.

Remark 4. *Let $\mathcal{M}_{torus,col}$ be the submoduli space of $\mathcal{M}_{torus}$ for which three nodes are colinear. $\mathcal{M}_{torus,col}$ is a* codimention one sub-variety of *$\mathcal{M}_{torus}$ and $\mathcal{M}_{torus} - \mathcal{M}_{torus,col}$ is* Zariski dense in *$\mathcal{M}$. To see this, first we consider the submoduli $\mathcal{M}_{torus,col}^{\#}$ whose curves have theree nodes on O and $D := (1,0)$ and $E = (0,1)$. They are defined by $h(x,y) = 0$ where $h(x,y) = h_2(x,y)^3 - h_3(x,y)^2$ and $h_2(x,y) := y^2 + (A_{10} + A_{11}x)y + T_0^2 + 3T_0^2x^2$ and $h_3(x,y) := B_{30}y^3 + y^2(B_{20} + B_{21}x) + y(\frac{3}{2}T_0A_{10} - 3xT_0A_{11} - \frac{9}{2}T_0A_{10}x^2) + T_0^3 - 9T_0^3x^2$.*

For a given generic curve $C_0 \in \mathcal{M}_{torus,col}^{\#}$, we can explicitly find a family of curves $C_s := \{f(x,y,s) = 0\}$ in $\mathcal{M}$ such that three nodes of C_s are at D, E and $O_s := (0,s)$. We omit the explicit polynomial equation as it is long and the computation is boring. Instead we give an example. $f := f_2^3 - f_3^2$ where $f_2 := 1 + y^2 - s^2 + 3x^2 + \frac{2}{9}syx + s^2x^2$ and $f_3 = 2y^3 + 6y^2 + 3s^2 - 2yx^2s^2 + 1 - 9x^2 + y^2x - \frac{2}{3}syx - 3s^2x^2 + 9yx^2s - 9ys - 4y^2s + 2ys^2$.

3.4. Moduli spaces $\mathcal{N}_{i,torus}$ and the degeneration

We consider the moduli spaces $\mathcal{N}_{1,torus}$, $\mathcal{N}_{2,torus}$ and $\mathcal{N}_{3,torus}$. Let $O = (0,0), A = (1,1), B = (1,-1)$ be as above. We compute the sub-moduli spaces $\mathcal{N}_{1,torus}^{\#}, \mathcal{N}_{2,torus}^{\#}, \mathcal{N}_{3,torus}^{\#}$.

(1) Moduli space $\mathcal{N}_{1,torus}$. Consider first $\mathcal{N}_{1,torus}^{\#}$, the moduli of torus sextics $f_2(x,y)^3 + f_3(x,y)^2 = 0$ with a (3,4)-cusp singularity at O and 2 nodes at A, B and four ordinary cusps. As the sum of the intersection multiplicity of $f_2 = f_3 = 0$ is 6, it is necessary that $f_2(O) =$

0 and four other cusps are also on the conic $f_2(x,y)=0$. The condition for O to be a (3,4)-cusp is given by the following four linear equations: $f_2(O)=f_3(O)=\frac{\partial f_3}{\partial x}(O)=\frac{\partial f_3}{\partial y}(O)=0$.

Proposition 19. *The above (3,4)-cusp condition is the same as the limit of the node condition at O for $t_0 \to 0$: $f(O)=\frac{\partial f}{\partial x}(O)=\frac{\partial f}{\partial y}(O)=0$.*

Proof. In fact, using $f_2(O)=-t_0^2$ and $f_3(O)=t_0^3$, we have $\frac{\partial f}{\partial x}(O)=t_0^3(3t_0\frac{\partial f_2}{\partial x}(O)+2\frac{\partial f_3}{\partial x}(O))$. Thus the limit for $t_0\to 0$ gives $\frac{\partial f_3}{\partial x}(O)=0$. The same argument applies for $\frac{\partial f_3}{\partial y}(O)$. Q.E.D.

Therefore the moduli is given by substituting $t_0=0$ in $\mathcal{M}$ and it has 5 free parameters $a_{1,0},a_{0,2},b_{2,1},t_1,t_2$ where $f_2(A)=-t_1^2$ and $f_2(B)=-t_2^2$. We see that $\mathcal{N}_{1,torus}^{\#}$ and (thus $\mathcal{N}_{1,torus}$ also) is irreducible. Geometrically this implies the following. Let $f_t(x,y)$ be the family given by fixing generic $a_{1,0},a_{0,2},b_{2,1},t_1,t_2$ and $t_0=t$ in the moduli space $\mathcal{M}_{torus}$. Then the conic $f_{2,t}(x,y)=0$ approches to the node at O when $t\to 0$. Actually one can see by a direct computation that there are two cusps among six cusps on a conic which approach to O so that they produce a (3,4)-cusps on $C_0=\{f_0=0\}$.

(2) Moduli space $\mathcal{N}_{2,torus}$. Now we consider the moduli space $\mathcal{N}_{2,torus}^{\#}$. The curves in this moduli have 2 (3,4)-cusps at A and B (and 2 other cusps) on the conic $f_2(x,y)=0$ and a node at O. By Proposition 19, the conditions at A,B are replaced by $t_1=t_2=0$ in $\mathcal{M}$. Thus it has 4 free parameters $a_{1,0},a_{2,0},b_{2,1},t_0$ where $f_2(O)=-t_0^2$. and the moduli space coincides again to the one which is obtained by substituting $t_1=t_2=0$ in the moduli space $\mathcal{M}_{torus}^{\#}$. Thus we see that $\mathcal{N}_{2,torus}^{\#}$ and $\mathcal{N}_{2,torus}$ are irreducible.

(3) Moduli space $\mathcal{N}_{3,torus}$. Finally the moduli space $\mathcal{N}_{3,torus}^{\#}$ with three (3,4)-cusps are given by $\mathcal{M}\cap\{t_0=t_1=t_2=0\}$. The corresponding polynomials are given by $f=f_2^3+f_3^2$ where $f_2=y^2+y(a_{1,0}-a_{1,0}x)+(-1-a_{0,2})x+a_{0,2}x^2$ and $f_3=b_{2,1}(y^2-x^2)(x-1)$. This is equal to the subspace of $\mathcal{M}_{torus}^{\#}$ given by $\mathcal{M}_{torus}^{\#}\cap\{t_0=t_1=t_2=0\}$.

We have shown in the above argument that $\mathcal{N}_{i,torus}$ is on the boundary of $\mathcal{M}_{torus}$. By the same argument, we can see that $\overline{\mathcal{N}_{i,torus}}\supset\mathcal{N}_{i+1,torus}$ for $i=1,2$. This proves the stratification assertion in Theorem 18. The fact $\mathcal{N}_{4,torus}=\emptyset$ will be proved in 4.2.

3.5. Proof of $(\widehat{\mathcal{M}_{torus}})^*=\widehat{\mathcal{M}_{torus}}$.

A polynomial $f(x,y)$ is called *even* in y if $f(x,y)=f(x,-y)$ for any (x,y). To prove the assertion, it is enough to show that there is a $C_0 \in \mathcal{M}'_{torus}$ such that $C_0^* \in \mathcal{M}'_{torus}$. In fact, assuming this for a moment and taking $C \in \widehat{\mathcal{M}}_{torus}$, we can connect C and C_0 by a piecewise analytic path $C_{\tau(t)}$, $0 \le t \le 1$ such that $C_{\tau(0)} = C_0$, $C_{\tau(1)} = C$ and $C_{\tau(t)} \in \mathcal{M}'_{torus}$ for any $t < 1$. For $0 \le t < 1$, the topology of the complements $\mathbf{C}^2 - C_{\tau(t)}, t < 1$ and $\mathbf{C}^2 - C^*_{\tau(t)}$ is independent of t as they are locally μ-constant family at every singular point. Thus they have the same topology and therefore they have the same Alexander polynomial. In particular, they are torus curves. By Lemma 4, the polynomial $g_t(u,v)$ which defines the dual curves $C^*_{\tau(t)}$ can be assumed to be analytic in t at $t = 1$. Thus this implies that $g_1(u,v)$ is also a torus curve. By the reciprocity law, this implies that the dual of a non-torus sextic in $\widehat{\mathcal{M}}$ is again a non-torus curve. Now we prove the existence of C_0. In fact, we can take any torus curve C defined by an even polynomial $f(x,y) \in \mathcal{M}'_{torus}$. Even curves are given by putting $a_{1,0} = 0$ and $t_2 = t_1$ in the moduli parameters. It is easy to see that the dual curve C^* is also even. Thus it has six cusps which are symmetric with respect to the y-axis and generically these 6 cusps are not on the x-axis. Thus there exists a conic which passes through these 6 points. Now by [D], C^* is a torus curve. Or more directly, we can give C_0 as the following curve. Q.E.D.

Example 20. *For example, we take an even polynomial* $f = f_2^3 + f_3^2$ *where* $f_2(x,y) = y^2 - 1 - 2x + x^2$ *and* $f_3(x,y) = 1 + y^2(-\frac{5}{2} + x) + 3x - \frac{1}{2}x^2 - x^3$. *The dual curve is defined by Lemma 4 by the polynomial* $g(x,y) = 484x^6 + 720y^2x^4 + 357y^4x^2 + 59y^6 + 2068x^5 + 962y^2x^3 - 24y^4x - 761x^4 + 11516y^2x^2 - 1486y^4 - 14078x^3 + 14620y^2x - 24661x^2 + 12699y^2 - 21924x - 6728$. *Now the torus decomposition is obtained as follows:* $g(x,y) = 59g_2(x,y)^3 - \frac{1}{3481}g_3(x,y)^2$ *where* $g_2(x,y) = y^2 + \frac{241}{59} + \frac{86}{59}x + \frac{122}{59}x^2$ *an* $g_3(x,y) = -6117 - 7463x - 4639x^2 + 362x^3 + 2773y^2 + 177y^2x$.

§4. Moduli space of three cuspidal sextics of type (3,4)

In this section , we study the moduli space $\mathcal{N}_3$ of plane curves of degree 6 with 3 $(3,4)$ cusps which are not necessarily of torus type. To study the moduli of sextics with 3 (3,4)-cusps, we may assume hereafter that the cusps are on $O = (0,0)$, $A = (1,1)$ and $B = (1,-1)$.

Lemma 21. *Let* $\mathcal{Q}$ *be the set of smooth conics which pass through* O, A, B *and let* $\pi : \mathcal{Q} \to \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ *be the mapping defined by* $\pi(Q) = (T_O Q, T_A Q, T_B Q)$, $Q \in \mathcal{Q}$. *Here* $T_P Q$ *is the tangent line of* Q *at* P. *Then* π *is an embedding and the image* $\pi(\mathcal{Q})$ *is characterized as follows. Let* $\alpha, \beta, \gamma \in \mathbf{P}^1$ *be the respective tangent directions of* Q *at* O, A

and B. Then we can write $\beta = (b,1)$, $\gamma = (c,1)$ and $\alpha = (a_1, a_2)$ and they satisfy the equality: $(b+c)a - (2-b+c) = 0$ (respectively $b+c=0$) if $a_2 \neq 0$ with $a := a_1/a_2$ (resp. if $a_2 = 0$). The corresponding conic is defined by $q(x,y) = y^2 + y(c+b)(1-x) + (-2-c+b)x + (1+c-b)x^2$.

Lemma 22. *Assume that $C = \{(x,y) \in \mathbf{C}^2; r(x,y) = 0\}$ be a reduced plane curve of degree 3 which has singularities at O, A, B. Then C is the union of 3 lines $(x-1)(y^2-x^2) = 0$.*

The proofs of Lemma 21 and Lemma 22 are elementary and omitted.

Lemma 23. *Assume that $C_1 = \{(x,y) \in \mathbf{C}^2; f(x,y) = 0\}$ a germ of a smooth curve at the origin. Let $C_2 = \{(x,y) \in \mathbf{C}; g(x,y) = 0\}$ be another germ of a curve at the origin. Let d be the multiplicity of g at the origin and let $g_d(x,y)$ be the homogeneous part of g of degree d, which defines the tangent cone of C_2. Let p, q be positive integers such that $p < dq$. Consider the germ of a plane curve $C = \{(x,y) \in \mathbf{C}^2; f(x,y)^p - g(x,y)^q = 0\}$. Assume that each irreducible component of $g_d(x,y) = 0$ intersects C_1 transversely at the origin. Then $(C,O) \in \beta_{p,dq}$ and the tangential direction at the origin coincides with that of $f = 0$.*

Proof. Changing local coordinates if necessary, we may assume that $f(x,y) = y$ and $g_d(x,y) = \sum_{i=0}^{d} a_i y^i x^{d-i}$. The assumption implies that $a_0 \neq 0$. Thus $f^p(x,y) = y^p$ and $g^q(x,y) = g_d(x,y)^q + R$ where order$R \geq dq+1$. Thus we can write $f(x,y)^p - g(x,y)^q = y^p - a_0^q x^{dq} + R'(x,y)$ where the order of $R'(x,y)$ with respect to the weight $\mathrm{wt}(x) = p$ and $\mathrm{wt}(y) = dq$ is strictly larger than pqd. Thus the assertion follows. Q.E.D.

Corollary 24. *Let $C = \{(x,y) \in \mathbf{C}^2; f(x,y) = 0\}$ be a reduced sextic with 3 (3,4)-cusps at O, A, B. The following conditions are equivalent.*

(1) $f(x,y)$ is written as $c_1 x^2 (y^2 - x^2)^2 + c_2 q(x,y)^3$ for non-zero $c_1, c_2 \in \mathbf{C}^*$ and the conic $q(x,y) = 0$ is smooth and passes through O, A, B.

(2) There exists a conic $q(x,y) = 0$ which passes throuh O, A, B such that the respective tangent line of the conic is equal to that of C at O, A, B.

(3) C is a torus curve of type (2,3).

Proof. The implication (2) $\Longrightarrow$ (3) follows from Degtyarev [D] or Tokunaga [T]. Q.E.D.

4.1. Moduli space $\mathcal{N}_3$.

Now we compute the moduli space $\mathcal{N}_3^{\#}$ of sextics with 3 (3,4)-cusps at O, A, B. Assume that $C \in \mathcal{N}_3^{\#}$. By Bezout theorem, the tangent cone at O is not $y \pm x = 0$. The stabilizer $H^{\#}$ of $\mathcal{N}_3^{\#}$ in $\mathrm{PSL}(3,\mathbf{C})$ has dimensioin two. Thus under the action of $H^{\#}$, we may assume also that *the tangent cone of C at O is given by $x = 0$*. So we compute the submoduli $\mathcal{N}_3^{\#\#}$ of $\mathcal{N}_3^{\#}$ whose tangent cone at O is $x = 0$. Let $H^{\#\#}$ be the stabilizer of $\mathcal{N}_3^{\#\#}$. It has dimension one. We start from the expression $f(x,y) = \sum_{i+j\le 6} a_{i,j}y^i x^j$. We can normalize the coefficient $a_{6,0} = 1$ and we have 27 coefficients. The multiplicities of f at O, A, B are 3 by the assumption. Thus at each of these three points, the partial derivatives of order ≤ 2 must vanish. This gives $3 \times 6 = 18$ linear relations and we can eliminate 18 coefficients and we have still 9 coefficients left. For the other computation, we consider the projection $\pi : \mathcal{M} \to \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ which is defined by the tangent cone directions at O, A, B. We fix $(\alpha, \beta, \gamma) \in \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ and we study the fiber $\pi^{-1}(\alpha, \beta, \gamma)$. First we observe that $\beta, \gamma \ne (1,0)$, i.e., β and γ are transverse to the vertical line $x = 1$ by Bezout theorem. Thus we can put $\beta = (b,1)$, $\gamma = (c,1)$. By the assumption, $\alpha = (1,0)$. Let $h_3(f)(Q)(u,v)$ be the following homogeneous polynomial of degree 3: $\frac{1}{6}\frac{\partial^3 f}{\partial x^3}(Q)u^3 + \frac{1}{2}\frac{\partial^3 f}{\partial x^2 \partial y}(Q)u^2 v + \frac{1}{2}\frac{\partial^3 f}{\partial x \partial y^2}(Q)uv^2 + \frac{1}{6}\frac{\partial^3 f}{\partial y^3}(Q)v^3$.

The condition for O, A, B to be (3,4)-cusps with the above tangent cones is $h_3(f)(A) = c_A(v-bu)^3$, $h_3(f)(B) = c_B(v-cu)^3$ and $h_3(f)(O) = c_O u^3$ for some non-zero constants $c_A, c_B, c_O \in \mathbf{C}^*$. By an easy computation, we have $c_A = 8$, $c_B = -8$. Solving $h_3(f)(A) = 8(v-bx)^3$, $h_3(f)(B) = -8(v-cu)^3$ and $h_3(f)(O) = c_O u^3$, we can eliminate the remaining coefficients so that the moduli space $\mathcal{N}_3^{\#\#}$ is given by

$$\begin{aligned}\mathcal{N}_3^{\#\#} := \pi^{-1}(\{((1,0),(b,1),(c,1)) \in \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1;\\ (b+c)(b^2-3b-bc+3+3c+c^2)=0\})\end{aligned}$$

The other coefficients are given by

$$\begin{aligned}
a_{5,0} &= 3(b+c),\ a_{5,1} = -3(b+c),\\
a_{4,0} &= -1 + a_{0,6} - 6(b^2+c^2) - 4(b^3-c^3) + 3(b-c),\\
a_{4,1} &= -4 - 2a_{0,6} + 12(b^2+c^2) + 3(c-b) - 8(b^3-c^3),\\
a_{4,2} &= 2 + a_{0,6} - 6(b^2+c^2), +4(b^3-c^3),\\
a_{3,1} &= -12(b+c) + 6(b^2-c^2),\ a_{3,2} = 18(b+c) - 12(b^2-c^2),\\
a_{3,3} &= -6(b+c) + 6(b^2-c^2),\\
a_{2,2} &= 14 - 18(b-c) + 18(b^2+c^2) - 8(b^3-c^3) - 2a_{0,6},\\
a_{2,3} &= -16 + 4a_{0,6} - 36(b^2+c^2) + 30(b-c) + 16(b^3-c^3),\\
a_{2,4} &= 5 + 8(c^3-b^3) + 18(b^2+c^2) + 12(c-b) - 2a_{0,6},\\
a_{1,2} &= 12(b+c) + 12(c^2-b^2) + 4(b^3+c^3),\\
a_{1,3} &= -24(b+c) + 30(c^2-b^2) - 12(b^3+c^3),
\end{aligned}$$

$$\begin{aligned}
a_{1,4} &= 15(b+c) - 24(b^2 - c^2) + 12(b^3 + c^3), \\
a_{1,5} &= -3(b+c) + 6(b^2 - 6c^2) - 4(b^3 + c^3), \\
a_{0,3} &= -8 - 4(c^3 - b^3) - 12(b^2 + c^2) + 12(b - c), \\
a_{0,4} &= 11 + a_{0,6} + 24(b^2 + c^2) + 21(c - b) + 8(c^3 - b^3), \\
a_{0,5} &= -4 - 2a_{0,6} - 12(b^2 + c^2) - 9(c - b) - 4(c^3 - b^3)
\end{aligned}$$

where $a_{0,6}$ is a free parameter. The quotient of the moduli $\mathcal{N}_3^{\#\#}/H^{\#\#}$ has two irreducible components, given by the respective quotients of $\mathcal{N}_{3,1}^{\#\#} := \pi^{-1}(\{b+c=0\})$ and $\mathcal{N}_{3,2}^{\#\#} := \pi^{-1}(\{b^2 - 3b - bc + 3 + 3c + c^2 = 0\})$. Therefore the quotient of moduli space $\mathcal{N}_3/\mathrm{PSL}(3;\mathbf{C})$ has also 2 irreducible components $\mathcal{N}_{3,1}/\mathrm{PSL}(3;\mathbf{C})$ and $\mathcal{N}_{3,2}/\mathrm{PSL}(3;\mathbf{C})$.

Remark 5. *The moduli space $\mathcal{N}_{3,2}^{\#\#}$ consists of two irreducible components $L_\pm := \pi^{-1}(\{(a_{0,6}, b, c); c - (b-3)/2 \pm (b-1)\sqrt{3}I/2\})$. However taking a $\psi \in H^{\#\#}$ such that $\psi(O) = A, \psi(A) = O$ and $\psi(B) = B$, we can easily see that $\psi(L_+) = L_-$, $\psi(L_-) = L_+$ and thus $\mathcal{N}_{3,2}^{\#\#}/H^{\#\#}$ is irreducible.*

Lemma 25. *The component $\mathcal{N}_{3,1}^{\#}$ coincides with the submoduli of sextics of torus type $C \in \mathcal{N}_{3,torus}$ which has 3 (3,4)-cusps at O, A, B. $\mathcal{N}_{3,2}^{\#}$ coincides with $\mathcal{N}_{3,gen}^{\#}$ defined in the section 3.*

Proof. The assertion follows from Lemma 21 and Corollary 24. In fact, for f corresponding to the above parameters and $c = -b$, the torus decomposition is given by $f(x,y) = f_2(x,y)^3 + kf_3(x,y)^2$ where $f_2(x,y) = y^2 + (2b-2)x + (1-2b)x^2$, $f_3(x,y) = (y^2 - x^2)(x-1)$ and $k = 6b - 1 + 8b^3 - 12b^2 + a_{0,6}$.

4.2. Moduli space $\mathcal{N}_4$.

We consider the moduli space of sextics with one (4,5)-cusp at the origin and 3 (2,3)-cusps. First we will show that $\mathcal{N}_{4,torus} = \emptyset$. In fact, assume that there exists a sextic $f(x,y) = f_2(x,y)^3 + f_3(x,y)^2 = 0$ in $\mathcal{N}_4$. It can be easily observed that O must be on the conic $f_2(x,y) = 0$. As the multiplicity of f at O is 4, f_3 has multiplicity at least 2 at the origin and thus f_2 also has multiplicity 2 at O. Thus $f_2(x,y)^3$ has multiplicity 6 at O and O can not be a (4,5)-cusp.

By Bezout theorem, any two of 3 cusps and the origin can not be colinear. Therefore by the action of $\mathrm{PSL}(3,\mathbf{C})$, we can assume that the locus of 3 cusps are either $A = (1,1), B = (1,-1)$ and $C = (1,0)$ if they are colinear or $A = (1,1), C = (1,0)$ and $C' = (0,1)$. The moduli space $\mathcal{N}_4$ seems to be irreducible but we only give examples in this paper.

Example 26. *1. Let $C_0 = \{f(x,y) = 0\}$ where*
$f(x,y) := y^6 - 6y^5 + 6y^5x + 16y^4 - 22y^4x + 4y^4x^2 - 32y^3x + 68y^3x^2 - 36y^3x^3 + 24y^2x^2 - 58y^2x^3 + 35y^2x^4 - 8yx^3 + 18yx^4 - 10yx^5 + x^4 - 2x^5 + x^6$. *$C_0$ has a (4,5)-cusp singularity at the origin and 3 (2,3)-cusps at $A = (1,1), B = (1,-1)$ and $C = (1,0)$.*
2. Let $C_1 \in \mathcal{N}_4$ be defined by $f(x,y) = 0$ where
$f(x,y) = y^6 + y^4 - 2y^5 - 2x^5 + 6y^5x - 10y^4x - 5y^4x^2 + 4y^3x - 4y^3x^2 + 12y^3x^3 + 6y^2x^2 - 4y^2x^3 - 5y^2x^4 + 4yx^3 - 10yx^4 + 6yx^5 + 4Iy^5x - 4Iy^4x - 8Iy^4x^2 + 12Iy^3x^2 - 12Iy^2x^3 + 8Iy^2x^4 + 4Iyx^4 - 4Iyx^5 + x^6 + x^4$ *where $I = \sqrt{-1}$. Then C_1 has three cusps at A, C, C'.*

We can check that the dual curve has 6 cusps and 3 nodes in both examples. We assert that

Proposition 27. *For any C in the irreducible component of $\mathcal{N}_4$ containing C_1, $\pi_1(\mathbf{P}^2 - C) \cong \mathbf{Z}_6$.*

Proof. We show that $\pi_1(\mathbf{P}^2 - C_1 \cup \{x = 0\}) \cong \mathbf{Z}$, using a pencil lines through O where C_1 is in 2 of Example 26. Identifying $\mathbf{P}^2 - \{x = 0\}$ with $\mathbf{C}^2$, the generic pencil line intersect the affine curve $C_1 \cap \mathbf{C}$ 2 at two points and therefore $\pi_1(\mathbf{P}^2 - C_1 \cup \{x = 0\})$ is generated by two generators. Thus it is enought to show the existence of a pencil line which is tangent to C. This can be done by taking $y = 2/7x$ or $y = (-3 + 4i)/5\ x$ respectively. Now the surjectivity $\pi_1(\mathbf{P}^2 - C_1 \cup \{x = 0\} \to \pi_1(\mathbf{P}^2 - C_1)$ proves the commutativity of $\pi_1(\mathbf{P}^2 - C_1)$. Q.E.D.

We thank to Artal Bartolo for the suggestion of this choice of the pencil.

§5. Fundamental group of torus curves

In this section, we prove that

Theorem 28. *$\pi_1(\mathbf{P}^2 - C) \cong \mathbf{Z}_2 * \mathbf{Z}_3$ and $\pi_1(\mathbf{C}^2 - C) \cong B_3$ for a generic $C \in \mathcal{N}_{3,1}$.*

Here B_3 is the braid group of three strings. This theorem implies the next stronger assertion.

Theorem 29. *$\pi_1(\mathbf{P}^2 - C) \cong \mathbf{Z}_2 * \mathbf{Z}_3$ and $\pi_1(\mathbf{C}^2 - C) \cong B_3$ for any $C \in \mathcal{M}'_{torus}, \mathcal{N}'_{i,torus}$ and $\mathcal{M}_{i,torus}$ for $i = 1, 2, 3$.*

Proof. This can be proved by a direct computation. Here is another proof. Take $C \in \mathcal{M}'_{torus}$ for example. Then we can take a family C_t so that $C_0 = C$ and C_t is a 6 cuspidal sextic (without nodes) for $t \neq 0$. We can also find another family D_t such that $D_1 = C$ and

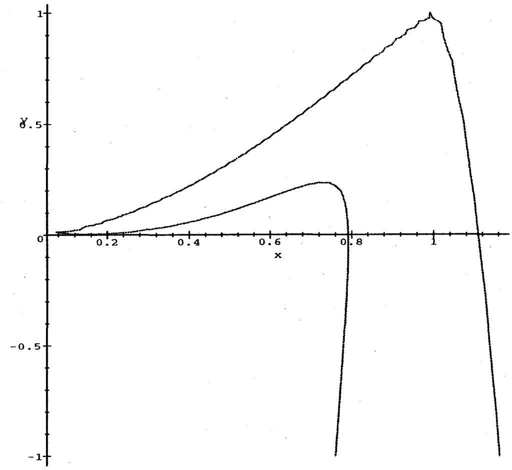

Fig. 1. Graph of $g = 0$

$D_0 \in \mathcal{N}_{3,1}$ and $D_t \in \mathcal{M}'_{torus}$ for $t \neq 0$. By a standard argument, we have surjective homomorphisms $\psi_1 : \pi_1(\mathbf{P}^2 - D_0) \to \pi_1(\mathbf{P}^2 - C)$ and $\psi_2 : \pi_1(\mathbf{P}^2 - C) \to \pi_1(\mathbf{P}^2 - C_1)$ which are isomorphisms on the first homology groups. Thus they induce surjections on the commutator groups. On the other hand, we know that $\pi_1(\mathbf{P}^2 - C_1) \cong \mathbf{Z}_2 * \mathbf{Z}_3$ and the commutator group $D(\mathbf{Z}_2 * \mathbf{Z}_3)$ is a free group $F(2)$ of rank two ([Z],[O1]). Thus we obtain a surjective homomorphism $\psi_2 \circ \psi_1 : F(2) = D(\pi_1(\mathbf{P}^2 - D_0)) \to F(2) = D(\pi_1(\mathbf{P}^2 - C_1))$. This implies that the kernel of $\psi_2 \circ \psi_1$ is trivial by Theorem 2.13, [M-K-S]. Thus ψ_1, ψ_2 are isomorphisms. Q.E.D.

Proof of Theorem 28. For the proof, we take the following sextic curve $C_1 := \{(x,y) \in \mathbf{C}^2; f(x.y) = 0\} \in \mathcal{N}_{3,1}$ where $f(x,y) = f_2(x,y)^3 + \frac{103}{2} f_3(x,y)^2$ and $f_2(x,y) = y^2 + x^2 - 2x$ and $f_3(x,y) = (x-1)(x^2 - y^2)$. Our curve C_1 is even in y. Let us consider a polynomial $g(x,y)$ defined by $g(x,y) := f(x, \sqrt{y})$. Then C_1 is obtained by the double covering branched along $y = 0$ of the curve $g(x,y) = 0$ and the singular fiber for $g(x,y) = 0$ with respect to the pencil $\{x = \eta; \eta \in \mathbf{C}\}$ is defined by the roots of $\Delta_y(g) = -42436x^3(130x - 103)(x-1)^8 = 0$. The graph of the real curve $C(g) := \{g(x,y) = 0\}$ is given in Figure 1. It has two compact components in its real graph. By Lemma 2.2 of [O5] and by the equality $g(x,0) = 1/2 \cdot x^3(-16 + 127x - 218x^2 + 105x^3)$, we get $\Delta_y(f)(x) = cx^9(-16 + 127x - 218x^2 + 105x^3)(130x - 103)^2(x-1)^{16}$, with some constant $c \in \mathbf{C}^*$. Let $p : \mathbf{C}^2 \to \mathbf{C}$ be the first projection and

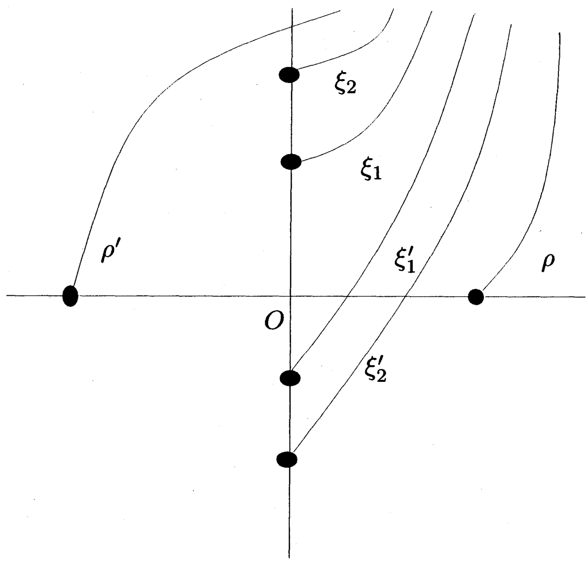

Fig. 2. Generators ($x = \beta_3 - \varepsilon$)

we consider the pencil given by $L_\eta = p^{-1}(\eta)$ as usual. We have chosen f so that the singular pencil lines are all real and given by $\beta_0 < \cdots < \beta_5$ where $\beta_0 = 0$, $\beta_1 = 0.173\cdots$, $\beta_2 = 0.792\cdots$, $\beta_3 = 103/130$, $\beta_4 = 1$, $\beta_5 = 1.110\cdots$. Here $\beta_i, i = 1,2,5$ are non-zero roots of $g(x,0) = 0$ and the corresponding line $x = \beta_i$ is simply tangent to C at $(\beta_i, 0)$ for $i = 1,2,5$. Hereafter ε is assumed to be a sufficiently small positive number. We use the notation $\{\sigma,\tau\} := \sigma\tau\sigma\tau^{-1}\sigma^{-1}\tau^{-1}$. Thus $\{\sigma,\tau\} = e$ is equivalent to $\sigma\tau\sigma = \tau\sigma\tau$ where e is the unit. We often use the equivalence: $\{\sigma,\tau\} = e \iff \{\sigma,\sigma\tau\sigma^{-1}\} = e \iff \{\sigma,\sigma^{-1}\tau\sigma\} = e$. We compute the fundamental group $\pi_1(\mathbf{C}^2 - C_1)$ by Zariski's pencil method. We first take generators $\rho,\xi_1,\xi_2,\rho',\xi_1',\xi_2'$ of $\pi_1(L_{\beta_3-\varepsilon} - L_{\beta_3-\varepsilon}\cap C_1)$ as in Figure 2. In the following figures, for simplicity of drawing pictures, we denote a small lasso oriented counterclockwise by a path ending by a bullet ———• as in [O5]. As the monodromy relation at $x = \beta_3$, we get tangent relations $\xi_1 = \xi_2$, $\xi_1' = \xi_2'$. At $x = \beta_2$, we also get a tangent relation $\xi_1 = \xi_1'$. Thus we can put $\xi := \xi_1 = \xi_2 = \xi_1' = \xi_2'$. The generators are reduced to ξ,ρ,ρ'. For further computation, we freely use the relations which have been obtained. Figure 3 shows the situation of our generators at $x = \beta_4 - \varepsilon$. We get the monodromy relations at $x = \beta_4$: $\xi = (\xi^2\rho)(\xi\rho\xi^{-1})(\xi^2\rho)^{-1}$ and $\xi\rho\xi^{-1} = (\xi^2\rho)\xi(\xi^2\rho)^{-1}$ at $(1,1)$ and $\xi = (\rho'\xi^2)(\xi^{-1}\rho'\xi)(\rho'\xi^2)^{-1}$ and $\xi^{-1}\rho'\xi = (\rho'\xi^2)\xi(\rho'\xi^2)^{-1}$ at $(1,-1)$

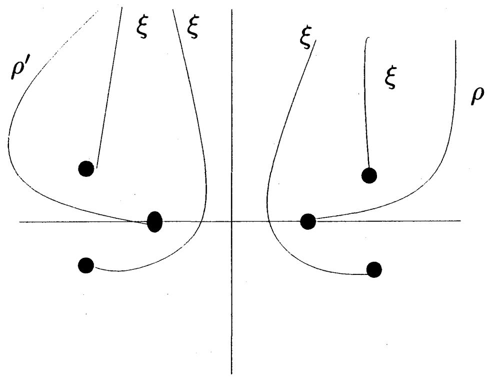

Fig. 3. $x = \beta_4 - \varepsilon$

which reduce to:

$$\{\xi, \rho\} = e, \qquad \{\xi, \rho'\} = e \tag{30}$$

At $x = \beta_5$ we get a tangent relation: $(\xi^2\rho)\xi(\xi^2\rho)^{-1} = (\rho'\xi^2)^{-1}\xi(\rho'\xi^2)$ which reduces to

$$\xi\rho\xi^{-1} = \xi^{-1}\rho'\xi \tag{31}$$

Put $\hat{\rho} = \xi\rho\xi^{-1}$. Then we can take $\xi, \hat{\rho}$ as new generators. The relation (30) gives the relation $\{\xi, \hat{\rho}\} = e$. We can see that the monodromy relation at $x = \beta_0$ is derived from the above relations. Thus we have shown that

$$\pi_1(\mathbf{C}^2 - C_1) = \langle \xi, \hat{\rho}; \xi\hat{\rho}\xi = \hat{\rho}\xi\hat{\rho} \rangle \cong B_3 \tag{32}$$

The fundamental group $\pi_1(\mathbf{P}^2 - C_1)$ is obtained by adding the relation $\rho'\xi^4\rho = e$ which is equivalent to $(\xi\hat{\rho}\xi)^2 = e$. Thus this group is isomorphic to $\mathbf{Z}_2 * \mathbf{Z}_3$. See [O3] for the proof.

§6. Non-torus sextic with three (3,4)-cusps

In this section, we will show that the fundamental groups $\pi_1(\mathbf{C}^2 - C)$ and $\pi_1(\mathbf{P}^2 - C)$ are isomorphic to cyclic groups $\mathbf{Z}$ and $\mathbf{Z}_6$ respectively for a generic member C of $\mathcal{N}_{3,2}$. The main difficulty is that, it seems, there does not exist a generic curve in $\mathcal{N}_{3,2}$ which is defined over real numbers for which the singular points are real and the singular fibers are all real. Thus we have to admit some singular points which are not

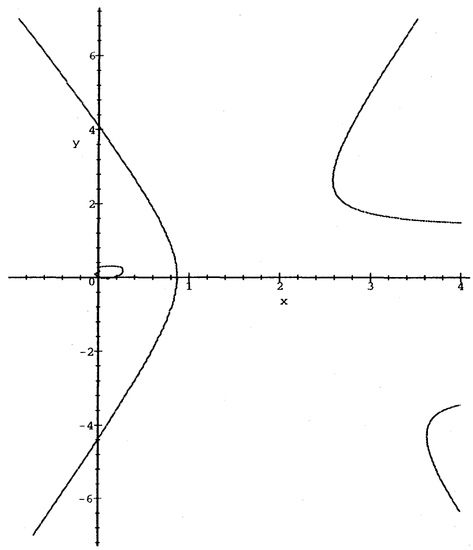

Fig. 4. Graph of C_2

real points or some non-real singular fibers. We take the following curve C_2 defined by

$$\begin{aligned} f(x,y) &= y^6 + y^4(18 - 30x + 9x^2) \\ &+y^3(3\sqrt{3}I - 9\sqrt{3}Ix + 9\sqrt{3}Ix^2 - 3\sqrt{3}Ix^3) \\ &+y^2(9x - 51x^2 + 63x^3 - 18x^4) \\ &+y(-3\sqrt{3}Ix^2 + 9\sqrt{3}Ix^3 - 9\sqrt{3}Ix^4 + 3\sqrt{3}Ix^5) - x^3 + 9x^4 - 9x^5 \end{aligned}$$

where $I = \sqrt{-1}$. We can easily see that $C_2 \in \mathcal{N}_{3,2}$. By the construction, C_2 has three (3,4)-cusps at O, A, B. Now we change the affine coordinates by $(x, y) \mapsto (x, yI)$, to make the defining polynomial to have real coefficients. Thus in the new coordinates, C_2 has three cusps at O, A', B' where $A' = (1, I), B' = (1, -I)$ and the defining polynomial $F(x, y)$ is a real polynomial given by $F(x, y) = f(x, yI)$. The discriminant of $F(x, y)$ in y, $\Delta_y(F)(x)$, which describes the singular fibers is given by $cx^8(9463x^6 + 135838x^5 - 1346423x^4 + 3270132x^3 - 2370951x^2 + 364014x + 22599)(x - 1)^{16}$ with some $c \neq 0$. The singular pencil lines are on the real line and correspond to $x = \eta_i$, $i = 1, \ldots, 8$, where $\eta_1 < \eta_2 < \eta_3 < \eta_4 < \eta_5 < \eta_6 < \eta_7 < \eta_8$ and $\eta_1 = -21.678\cdots$, $\eta_2 = -0.468\cdots$, $\eta_3 = 0$, $\eta_4 = 0.287\cdots$ $\eta_5 = 0.872\cdots$, $\eta_6 = 1$, $\eta_7 = 2.580\cdots$ and $\eta_8 = 3.629\cdots$. The real graph is given as in Figure 4.

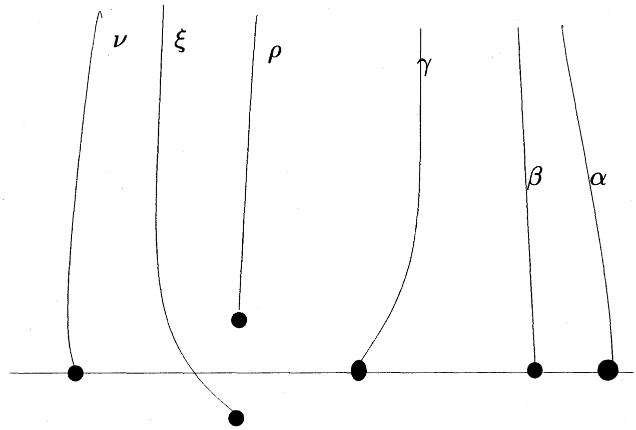

Fig. 5. Generators of $\pi_1(\mathbf{C}^2 - C_2)$

We observe that in the real graph of F, there is a small oval passing through the origin and 4 non-compact components. (One branch is far left outside of the figure.) The singular fibers $x = \eta_1, \eta_2, \eta_4, \eta_5, \eta_7, \eta_8$ are tangent to C_2 in the real graph. The lines $x = \eta_2, \eta_4$ are tangent to the oval. The singular fiber $x = \eta_3$ passes through a cusps at the origin and $x = \eta_6$ passes through two cusps at A', B'. By an easy computation, the principal part of the defining polynomial at three cusps O, A', B' (with respect to the coordinates centered at the singular points) are given by $(\sqrt{3}y - x)^3 + 16x^4 = 0$ at O, $-8(2x + yI)^3 + (54 - 6\sqrt{3}I)x^4 = 0$ at A' and $8(2x - yI)^3 + (54 + 6\sqrt{3}I)x^4 = 0$ at B'. First we take generators $\alpha, \beta, \gamma, \rho, \xi, \nu$ in the fiber $x = \eta_3 + \varepsilon = \varepsilon$ as in Figure 5.

The monodromy relations at $x = \eta_2, \eta_4$ are tangential relations and they are given by
$(R1)$: $\beta = \xi$, $\beta = \gamma$. Eliminating the generators γ, ξ using (R1), the monodromy relation at $x = \eta_3$ is given by $\beta(\beta\rho\beta) = (\beta\rho\beta)\rho$, $\rho(\beta\rho\beta) = (\beta\rho\beta)\xi$ which reduces to the cusp relation: $(R2)$: $\beta\rho\beta = \rho\beta\rho$. To read the monodromy relations at $x = \eta_5$ and η_6, we need to know how the six roots $y_1(x), \cdots, y_6(x)$ of $F(x, y) = 0$ in y move when x moves on the real axis from $x = \eta_4 + \varepsilon \to \eta_5 - \varepsilon$ and then on the circle $|x - \eta_5| = \varepsilon$ clockwise to $x = \eta_5 + \varepsilon$ and then on the real line from $x = \eta_5 + \varepsilon$ to $x = \eta_6 - \varepsilon$. Here we have chosen $y_i(x)$ to be continuous on x so that

1. the imaginary parts $\Im(y_1(x)), \Im(y_2(x))$ are positive and $y_3(x) = \overline{y_1}(x)$ and $y_4(x) = \overline{y_2}(x)$ on $\eta_4 + \varepsilon \le x \le \eta_5 - \varepsilon$ and $\eta_5 + \varepsilon \le x \le \eta_6 - \varepsilon$. We assume that $\Im(y_1(\eta_4 + \varepsilon)) < \Im(y_2(\eta_4 + \varepsilon))$.

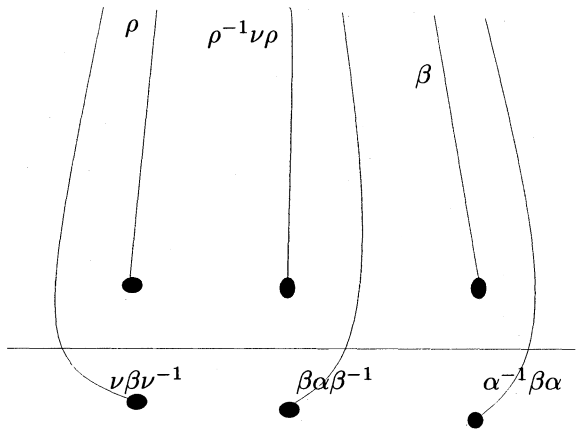

Fig. 6. Generators in $x = \eta_5 + \varepsilon$

2. $y_5(x)$ and $y_6(x)$ are real and $y_5(x) < y_6(x)$ for $\eta_4 + \varepsilon \le x \le \eta_5 - \varepsilon$ and
3. $\Im(y_5(x)) > 0$ and $y_6(x) = \overline{y_5}(x)$ for $\eta_5 + \varepsilon \le x \le \eta_6 - \varepsilon$.

The most delicate part of the argument is the determination of the braid of these six roots $y_j(x), j = 1, \dots, 6$ over $\eta_4 + \varepsilon \le x \le \eta_5 - \varepsilon$ and over $\eta_5 + \varepsilon \le x \le \eta_6 - \varepsilon$. We claim that

Assertion 4. *The ordering by the real part on non-real solutions is preserved on $\eta_4 + \varepsilon \le x \le \eta_5 - \varepsilon$ and $\eta_5 + \varepsilon \le x \le \eta_6 - \varepsilon$. Namely we have*

$$\Re(y_1(x)) < \Re(y_2(x)), \quad \eta_4 + \varepsilon \le x \le \eta_5 - \varepsilon \tag{33}$$

$$\Re(y_1(x)) < \Re(y_5(x)) < \Re(y_2(x)), \quad \eta_5 + \varepsilon \le x \le \eta_6 - \varepsilon \tag{34}$$

We assume this for a while. Then braids over the intervals $(\eta_4 + \varepsilon, \eta_5 - \varepsilon)$ and $(\eta_5 + \varepsilon, \eta_6 - \varepsilon)$ are uniquely determined. Thus in the fiber of $x = \eta_5 + \varepsilon$, the generators are deformed as in Figure 6. Then the monodromy relation at $x = \eta_5$ is given by
$(R3): \ \rho^{-1}\nu\rho = \beta\alpha\beta^{-1}$. Now we have to read the monodromy relations at $x = \eta_6 (= 1)$. Thus we start from the fiber $x = \eta_5 + \varepsilon$ as in Figure 6. The local equation of our curve at A', B' are given by the equations $-8(2x + yI)^3 + (54 - 6\sqrt{3}I)x^4$ and $-8(2x - yI)^3 + (54 + 6\sqrt{3}I)x^4$. Thus the topological behaviors of three roots y_1, y_2, y_5 or y_3, y_4, y_6 over the circle $|x - \eta_6| = \varepsilon$ look like satellites going arround the earth $(= \pm 2xI)$. The generators are deformed as in Figure 7 on the fiber $x = \eta_6 - \varepsilon$ and the monodromy relations are given by $\theta(\rho\beta\theta) = (\rho\beta\theta)\beta, \quad \beta(\rho\beta\theta) =$

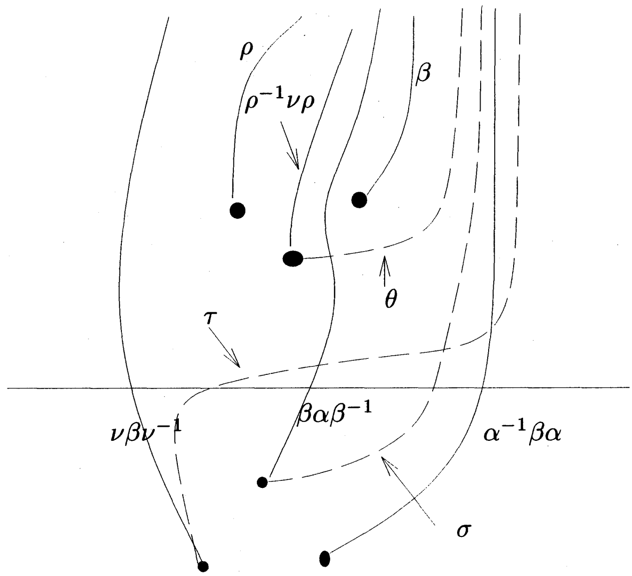

Fig. 7. Generators at $x = \eta_6 - \varepsilon$

$(\rho\beta\theta)\rho$ at A' and $(\alpha^{-1}\beta\alpha)(\tau\sigma\alpha^{-1}\beta\alpha) = (\tau\sigma\alpha^{-1}\beta\alpha)\sigma, \quad \sigma(\tau\sigma\alpha^{-1}\beta\alpha) = (\tau\sigma\alpha^{-1}\beta\alpha)\tau$, at B'. As $\theta = \beta^{-1}\rho^{-1}\nu\rho\beta = \alpha$ by (R3), $\sigma = \alpha$ and $\tau = (\nu\rho\beta)^{-1}\nu\beta\nu^{-1}(\nu\rho\beta) = \beta^{-1}\rho^{-1}\beta\rho\beta = \rho$ by (R2), the above relations reduces to:

$$\alpha(\rho\beta\alpha) = (\rho\beta\alpha)\beta, \quad \beta(\rho\beta\alpha) = (\rho\beta\alpha)\rho \tag{35}$$

$$(\alpha^{-1}\beta\alpha)(\rho\beta\alpha) = (\rho\beta\alpha)\alpha, \quad \alpha(\rho\beta\alpha) = (\rho\beta\alpha)\rho \tag{36}$$

The second relation of (35) reduces to $\rho\alpha = \alpha\rho$ by (R2). By the last relation, the first relation of (35) reduces to the braid type relation: $\alpha\beta\alpha = \beta\alpha\beta$. As $\alpha(\rho\beta\alpha) = \rho\alpha\beta\alpha = \rho\beta\alpha\beta$, we get from (36) that $\beta = \rho$. Thus $\beta\alpha = \alpha\beta$ by (35). Combining the last braid relation, we get $\alpha = \beta$. By (R3), we obtain the relation $\nu = \alpha$. Therefore $\pi_1(\mathbf{C}^2 - C)$ is generated by a single generator α and thus $\pi_1(\mathbf{C}^2 - C) \cong \mathbf{Z}$ and therefore $\pi_1(\mathbf{P}^2 - C) \cong \mathbf{Z}_6$. Q.E.D.

Appendix. Outline of the proof of Assertion 4. The following proof is essentially due to Maple. We consider the polynomial $h(x,u,v) := F(x, u+vI)$ for x,u,v real and let $F_e(x,u,v)$ and $F_o(x,u,v)$ be the real and the imaginary part of $h(x,u,v)$ respectively. They are given by

$$\begin{aligned} F_e(x,u,v) &:= v^6 + b_4v^4 + b_2v^2 + b_0, \\ F_o(x,u,v) &:= d_5v^5 + d_3v^3 + d_1v \end{aligned} \tag{37}$$

where the coefficients are polynomials of x, u. We omit their explicit forms.

Suppose that there exists an $x_0 \in (\eta_4+\varepsilon, \eta_5-\varepsilon) \cup (\eta_5+\varepsilon, \eta_6-\varepsilon)$ so that either $\Re(y_1(x_0)) = \Re(y_2(x_0))$ or $\Re(y_2(x_0)) = \Re(y_3(x_0))$. We may assume $\Re(y_1(x_0)) = \Re(y_2(x_0))$ for example and put $u_0 = \Re(y_1(x_0)) \in \mathbf{R}$. This implies that the equation $h(x_0, u_0, v)$ for v has four real solutions $\pm\Im(y_1(x_0))$, $\pm\Im(y_2(x_0))$. Therefore the equation $F_e(x_0, u_0, v) = F_o(x_0, u_0, v) = 0$ has four real solutions. As $\Delta_y(F)(x) = 0$ has no solutions on the intervals $(\eta_4+\varepsilon, \eta_5-\varepsilon) \cup (\eta_5+\varepsilon, \eta_6-\varepsilon)$, v can not be 0. Thus putting $F'_o(x, u, v) = F_o(x, u, v)/v$, $F_e(x_0, u_0, v) = F'_o(x_0, u_0, v) = 0$ has four real solutions $\pm\Im(y_1(x_0)), \pm\Im(y_2(x_0))$. As $F'_o(x_0, u_0, v)$ has degree 4 in v, this implies that $F'_o(x_0, u_0, v)$ divides $F_e(x_0, u_0, v)$. Thus the remainder $R(x, u, v)$ of F_e by F'_o as a polynomial of v must be identically zero for $x = x_0, u = u_0$. Put $R = c_2v^2 + c_0$. c_2 and c_0 are polynomials of x, u. Thus (x_0, u_0) is a common real solution of $c_2 = c_0 = 0$. Let $S(x)$ be the resultant of c_2, c_0 as polynomials of u. We do not give the explicit forms of $c_0(x, u), c_2(x, u), S(x)$ here but $S(x)$ is a polynomial of degree 48 and $(x-1)$ has the multiplicity 27. Note that $S(x_0) = 0$ is a necessary condition to have a real partner u_0 so that $c_2(x_0, u_0) = c_0(x_0, u_0) = 0$ but it is not a sufficient condition as the possible partner u_0 might be not real. Similarly even if we have a real solution $(x_0, u_0) \in \mathbf{R}^2$ of $c_2 = c_0 = 0$, the four roots of $F'_o(x_0, u_0, v) = 0$ might not be real numbers. Anyway Maple gives the unique real solution on the interval $(0, 1)$: $x_0 = .29572934753\cdots$. We check the solutions of $F(x_0, y) = 0$. We see that this does satisfy our requirement. Q.E.D.

§7. Application

In our previuos paper, we have constructed a Zariski's triple for plane curves of degree 12 with 27 cusps. In this section, we construct a new example of Zariski's triple $\{F_1, F_2, F_3\}$. They have degree 12 and 12 (3,4)-cusps.

(1) Let F_1 be a torus curve of type (3,4) defined by $f_3(x, y)^4 + f_4(x, y)^3 = 0$ where f_3 and f_4 are generic polynomials of degree 3 and 4 respectively. The Alexander polynomial $\Delta_{F_1}(t)$ is given by $(t^2 - t + 1)(t^4 - t^2 + 1)$. The fundamental groups are given by

$$\begin{aligned}\pi_1(\mathbf{C}^2 - F_1) \cong \langle \rho_1, \rho_2, \rho_3; \rho_1(\rho_3\rho_2\rho_1) &= (\rho_3\rho_2\rho_1)\rho_2, \\ \rho_2(\rho_3\rho_2\rho_1) &= (\rho_3\rho_2\rho_1)\rho_3\rangle\end{aligned}$$

and $\pi_1(\mathbf{P}^2 - F_1) \cong \mathbf{Z}_3 * \mathbf{Z}_4$ by [O1].

(2) Let F_2 be a generic cyclic (2,2)-covering $C_{2,2}(C_1)$ where C_1 is a torus sextic of type (2,3) with three (3,4)-cusps which is, for example, defined by $f(x,y)$ used in the proof of Theorem 28. Then F_2 is defined by $f((x-a)^2+a,(y-b)^2+b)$ for generic a,b. The Alexander polynomial $\Delta_{F_2}(t)$ is given by t^2-t+1 by Theorem 3.4 of [O4]. The fundamental group $\pi_1(\mathbf{C}^2-F_2)$ is isomorphic to the braid group B_3 and $\pi_1(\mathbf{P}^2-F_2)$ is a central extention of $\mathbf{Z}_2 * \mathbf{Z}_3$ by $\mathbf{Z}_2$ (Theorem 3.4 , [O4]).

(3) Let F_3 be a generic cyclic (2,2)-covering of non-torus three (3,4)-cuspidal sextic C_2, constructed in Section 4. The fundamental groups $\pi_1(\mathbf{C}^2-F_3)$ and $\pi_1(\mathbf{P}^2-F_3)$ are isomorphic to cyclic groups $\mathbf{Z}$, $\mathbf{Z}_{12}$ respectively.

Thus there are at least three connected components in the moduli of 12 (3,4)-cuspidal plane curves of degree 12.

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