Advanced Studies in Pure Mathematics 29, 2000
Singularities - Sapporo 1998
pp. 135–161

The quotients of log-canonical singularities by finite groups

Shihoko Ishii[1]

Abstract.

In this paper we study the quotient of an isolated strictly log-canonical singularity by a finite group. As a result, we obtain the boundedness of indices of these singularities of dimension 3 and determine all possible indices. We also determine the ramification indices of the quotient map of a 2-dimensional strictly log-canonical singularities by a finite group.

§1. Introduction

A log-canonical, non-log-terminal singularity is called strictly log-canonical. Let (X, x) be an isolated strictly log-canonical singularity over $\mathbb{C}$. If its dimension is 2, then the index is 1, 2, 3, 4 or 6. This is observed by checking the list of the weighted dual graphs of all strictly log-canonical singularities. This is also proved by Shokurov [21] by means of complements and by Okuma [18] by means of plurigenera. In the 3-dimensional case, the author heard that the boundedness of indices of such singularities is proved by Shokurov in [22]. In this paper, we study the quotient of isolated strictly log-canonical singularities by finite group actions. First, in case the group acts freely in codimension 1, we obtain a formula for the indices of the quotient singularity (Lemma 3.3). By this formula, we obtain a different proof of the above fact on indices for dimension 2. We then prove that the index of 3-dimensional strictly log-canonical singularity is less than or equal to 66. More precisely, a positive integer r can be the index of such a singularity if and only if $\varphi(r) \leq 20$ and $r \neq 60$, where φ is the Euler function. This is related to the finite automorphisms on $K3$-surfaces, Abelian surfaces and elliptic

---

Received March 8, 1999
Revised September 24, 1999
[1]Partially supported by the Grant-in-Aid for Scientific Research (No.09640016), the Ministry of Education, Japan.

curves. Next we study finite groups which act non-freely in codimension 1. For the 2-dimensional case, we determine the quotients by these groups with the branch divisors. Thus it follows that the ramification index of each ramification divisor is 2, 3, 4 or 6.

The author would like to express her gratitude to Professor Viyacheslav Shokurov for asking her the question on indices, which gave the motivation for this work. She is also grateful to Professors Viyacheslav Nikulin, Shigeyuki Kondo and Keiji Oguiso for providing her with useful information.

§2. Isolated strictly log-canonical singularities.

2.1. Isolated strictly log-canonical singularities are studied in [6]. In this section we summarize those results and add some basic facts on these singularities.

Definition 2.2. Let (X, x) be a germ of normal singularity. If there is an integer r such that $\omega_X^{[r]}$ is invertible, the singularity is called a $\mathbb{Q}$-*Gorenstein singularity*. We call the minimum positive such number r the *index* of (X, x) and denote by $\mathrm{Ind}(X, x)$.

Definition 2.3. A $\mathbb{Q}$-Gorenstein singularity (X, x) is called a *log-canonical singularity* (resp. *log-terminal singularity*) if for a good resolution $f : Y \to X$ the canonical divisor on Y has an expression in $\mathrm{Div}(Y) \otimes \mathbb{Q}$:

$$K_Y = f^*K_X + \sum_i m_i E_i$$

with $m_i \geq -1$ (resp. $m_i > -1$) for every irreducible exceptional divisor E_i with $x \in f(E_i)$. Here a good resolution means a resolution whose exceptional set is a normally crossing divisor with the non-singular irreducible components. We call m_i the *discrepancy* over X at E_i or the *discrepancy* for f at E_i for each irreducible component E_i.

2.4. In the case of index 1, a strictly log-canonical singularity is equivalent to a purely elliptic singularity ([6]). In this case we define the essential divisor in the exceptional divisor of a good resolution. It actually plays an essential role in the exceptional divisor (cf. Lemma 3.7 [6]).

Definition 2.5. Let (X, x) be an isolated strictly log-canonical singularity of index 1 and $f : Y \to X$ a good resolution. Then one has a representation

$$K_Y = f^*K_X + \sum_{i\in I} m_iE_i - \sum_{j\in J} E_j,$$

with $m_i \geq 0$, $I \cap J = \emptyset$ and $J \neq \emptyset$. The divisor $E_J := \sum_{j\in J} E_j$ is called the *essential divisor* for a good resolution f.

2.6. Let (X, x) be an n-dimensional isolated strictly log-canonical singularity of index 1 and $f : Y \to X$ a good resolution with the essential divisor E_J. Since E_J is a complete variety with normal crossings,

$$H^{n-1}(E_J, \mathcal{O}_{E_J}) \simeq Gr_F^0 H^{n-1}(E_J, \mathbb{C}) = \bigoplus_{i=0}^{n-1} H^{0,i}_{n-1}(E_J),$$

where F is the Hodge filtration and $H^{i,j}_m(*)$ is the (i, j)-Hodge-component of $H^m(*, \mathbb{C})$. As the left hand side is a 1-dimensional $\mathbb{C}$-vector space (Lemma 3.7 [6]), it must coincide with one of $H^{0,i}_{n-1}(E_J)$ $(i = 0, 1, 2, \dots, n-1)$.

Definition 2.7. An n-dimensional isolated strictly log-canonical singularity (X, x) of index 1 is said to be of type $(0, i)$, if $H^{n-1}(E_J, \mathcal{O}_{E_J}) = H^{0,i}_{n-1}(E_J)$.

2.8. The type is independent of the choice of a good resolution (Proposition 4.2 in [6]).

Example 2.9. A 2-dimensional srictly log-canonical singularity (X, x) of index 1 is of type $(0, 1)$ if and only if (X, x) is a simple elliptic singularity and of type $(0, 0)$ if and only if it is a cusp singularity.

Proposition 2.10. *Let (X, x) be a 3-dimensional isolated strictly log-canonical singularity of index 1 and of type $(0, 2)$ and $f : Y \to X$ the canonical model, i.e. Y has at worst canonical singularities and K_Y is f-ample. Let D be the exceptional divisor of f with the reduced structure. Then Y has at worst terminal singularities and D is isomorphic to either a normal $K3$-surface or an Abelian surface. Here a normal $K3$-surface is a normal surface whose minimal resolution is a $K3$-surface.*

Proof. First note that E_J is irreducible by Lemma 6, [8]. Since the discrepancy for f at each exceptional component is negative (the proof of Lemma 3.7 [8]), D is irreducible. Let $g : Y' \to Y$ be a proper birational morphism whose composite $f \circ g : Y' \to X$ is a good resolution. One sees that Y has at worst terminal singularities. Indeed, if not, there exists an exceptional divisor E_0 which is crepant for g. Then the discrepancy at E_0 for $f \circ g$ is less than 0, so E_0 becomes another component of the essential divisor, which is a contradiction. Now one can prove that Y is non-singular away from finite points. If D has 1-dimensional singular locus, then by the blowing-up at a 1-dimensional irreducible component of the singular locus one obtains a component E_1 whose discrepancy for $f \circ g$ is $-m + 1 < 0$, where m is the multiplicity of D at a general point on the curve. It implies that E_1 is another component of the essential divisor, which is a contradiction. Therefore D is non-singular away from finite points. On the other hand, since $\omega_Y \simeq \mathcal{O}_Y(-D)$ is Cohen-Macaulay, so is D. Hence by Serre's criterion D is normal. The condition $\omega_Y \simeq \mathcal{O}_Y(-D)$ yields that $\omega_D \simeq \mathcal{O}_D$. A normal surface with this condition and $H^2(E_J, \mathcal{O}_{E_J}) = \mathbb{C}$, where E_J is a resolution of D, is either a normal $K3$-surface or an Abelian surface ([23]). Q.E.D.

§3. Finite groups which act freely in codimension 1.

Definition 3.1. Let G be a group and (X, x) a germ of a singularity. We say that G acts on (X, x) if G acts on a neighbourhood of x and fixes the point x. We say that G acts on (X, x) freely in codimension 1, if there exists a closed subset S of codimension greater than or equal to 2 on a neighbourhood X such that G acts freely on $X \setminus S$.

3.2. We denote the set of non-singular points of X by X_{reg}. Let (X, x) be a $\mathbb{Q}$-Gorenstein singularity of index m and a group G act on (X, x). We denote the germ $(X/G, x')$ by $(X, x)/G$, where $x' \in X/G$ is the image of x. Denote the maximal ideal of x by $\mathfrak{m}_x$. Then it induces a canonical representation

$$\rho : G \to GL(\omega_X^{[m]}/\mathfrak{m}_x\omega_X^{[m]}) \simeq \mathbb{C}^*.$$

because G fixes the point x.

Lemma 3.3. *Let (X, x) be a $\mathbb{Q}$-Gorenstein normal singularity of index m. Let G be a finite group which acts on (X, x) freely in codimension 1 and $\rho : G \to GL(\omega_X^{[m]}/\mathfrak{m}_x\omega_X^{[m]}) \simeq \mathbb{C}^*$ the canonical representation. Then*

$$\mathrm{Ind}((X, x)/G) = m|\operatorname{Im}\rho|.$$

In particular,

$$\mathrm{Ind}((X, x)/G) \leq m|G|.$$

Proof. Denote the order of G by d, $|\operatorname{Im}\rho|$ by r and $\operatorname{Ind}((X,x)/G)$ by I. Let g be a generator of $\operatorname{Im}\rho$ and ϵ the primitive r-th root of 1 which corresponds to g. Let ω be a generator of $\omega_X^{[m]}$.

By the pull-back of a generator of $\omega_{X/G}^{[I]}$, one has a G-invariant I-ple n-form θ which is holomorphic and does not vanish on X_{reg}. Therefore $I = mm'$ for some $m' \in \mathbb{N}$ and $\theta = h\omega^{\otimes m'}$, where h is a nowhere vanishing holomorphic function on X. Since $\theta^g = \theta$ as an element of $\omega_X^{[I]}/\mathfrak{m}_x\omega_X^{[I]}$, one obtains that $\epsilon^{m'}h(x)\omega^{\otimes m'} = h(x)\omega^{\otimes m'}$. Hence $\epsilon^{m'} = 1$. This shows $I \geq mr$. Next, to prove $I \leq mr$, we construct a G-invariant mr-ple n-form which is holomorphic and does not vanish on X_{reg}. Denote an element of G which corresponds to $g \in \operatorname{Im}\rho$ by the same symbol g. Let θ be an mr-ple n-form $\omega \otimes \omega^g \ldots \otimes \omega^{g^{r-1}}$ and $\tilde{\theta}$ be $(1/d)\sum_{\sigma\in G}\theta^\sigma$. Then $\tilde{\theta}$ is an invariant mr-ple n-form. Let $\rho(\sigma) = g^i$ for $\sigma \in G$. Then in $\omega_X^{[mr]}/\mathfrak{m}_x\omega_X^{[mr]}$, $\theta^\sigma = \epsilon^{ri+(1+2+\ldots+r-1)}\omega^{\otimes r}$ which is $\omega^{\otimes r}$ if r is odd and $-\omega^{\otimes r}$ if r is even. Therefore $\tilde{\theta} = \pm\omega^{\otimes r} + \lambda$, where $\lambda \in \mathfrak{m}_x\omega_X^{[mr]}$. Since $\tilde{\theta} \notin \mathfrak{m}_x\omega_X^{[mr]}$, $\tilde{\theta}$ does not vanish on X_{reg}, which shows that $\tilde{\theta}$ is a required form. Q.E.D.

Corollary 3.4. *Let (X,x) be an isolated strictly log-canonical singularity of index* 1 *on which a finite group G acts. Let $f : \tilde{X} \to X$ be a G-equivariant resolution of the singularities and $\rho : G \to GL(\omega_X/f_*\omega_X) \simeq \mathbb{C}$ the induced representation. Then* $\operatorname{Ind}((X,x)/G) = |\operatorname{Im}\rho|$.

Proof. For an isolated strictly log-canonical singularity of index 1, it follows that $\mathfrak{m}_x\omega_X = f_*\omega_{\tilde{X}}$. Q.E.D.

Corollary 3.5. *Let (X,x) be an n-dimensional isolated strictly log-canonical singularity of index* 1 *on which a finite group G acts. Assume that there exists the canonical model $\varphi : X' \to X$ and let E be the reduced exceptional divisor. Then the action induces a representation $\rho : G \to GL(H^{n-1}(E, \mathcal{O}_E))$ and* $\operatorname{Ind}(X,x)/G = |\operatorname{Im}\rho|$.

Proof. Take a G-equivariant resolution $f : \tilde{X} \to X$. Then $\bigoplus_{m\geq 0} f_*\omega_{\tilde{X}}^{\otimes m}$ admits the action of G. So the canonical model admits the equivariant action of G, therefore the exceptional divisor E also does. Since $\omega_{X'} \simeq \mathcal{O}_{X'}(-E)$ (proof of Lemma 7 of [8]) and X' is Gorenstein in codimension 2, E is Cohen-Macaulay and $\omega_E \simeq \mathcal{O}_E$. These yield that $H^{n-1}(E, \mathcal{O}_E) = \mathbb{C}$. As $R^{n-1}\varphi_*\mathcal{O}_{X'} \simeq R^{n-1}f_*\mathcal{O}_{\tilde{X}} \simeq \mathbb{C}$, the surjection $R^{n-1}\varphi_*\mathcal{O}_{X'} \to H^{n-1}(E, \mathcal{O}_E)$ is an isomorphism. On the other hand $R^{n-1}f_*\mathcal{O}_{\tilde{X}}$ is dual to $\omega_X/f_*\omega_{\tilde{X}}$, on which one can apply Corollary 3.4. Q.E.D.

Corollary 3.6. *Let (X,x) be an n-dimensional isolated strictly log-canonical singularity of index* 1 *on which a finite group G acts. Let $f : Y \to X$ be a G-equivariant good resolution and E_J the essential divisor. Then the action induces a representation $\rho : G \to GL(H^{n-1}(E_J, \mathcal{O}_{E_J}))$ and* $\mathrm{Ind}(X,x)/G = |\operatorname{Im}\rho|$.

Proof. It is clear that G acts on E_J. Since E_J is the essential divisor, $R^{n-1}f_*\mathcal{O}_{X'} \simeq H^{n-1}(E_J, \mathcal{O}_{E_J})$ by Lemma 3.7 [6]. On the other hand $R^{n-1}f_*\mathcal{O}_{\tilde{X}}$ is dual to $\omega_X/f_*\omega_{\tilde{X}}$, on which one can apply Corollary 3.4. Q.E.D.

§4. Index of isolated strictly log-canonical singularities

4.1. In this section, one proves that the indices of isolated strictly log-canonical singularities of dimension 2 and 3 are determined. Here one should note that the boundedness of indices does not hold for log-terminal singularities and non-log-canonical singularities even for 2-dimensional case.

Example 4.2. (1) Let (Z_m, z_m) be the cyclic quotient singularity $\mathbb{C}^2/G$, where G is generated by

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}.$$

Here ϵ is a primitive m-th root of unity. Then the exceptional curve on the minimal resolution is $\mathbb{P}^1$ and its self-intersection number is $-m$. Therefore the index of (Z_m, z_m) is m if m is odd and $m/2$ if m is even. This shows that the indices of log-terminal singularities are not bounded.

(2) Let $(X,x) \subset (\mathbb{C}^3, 0)$ be a hypersurface singularity defined by $x^4+y^4+z^4 = 0$ and (Z_m, z_m) is its quotient by the cyclic group generated by

$$\begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix},$$

where ϵ is a primitive m-th root of unity. Then the index of (Z_m, z_m) is m. This shows that the indices of non-log-canonical singularities are not bounded.

4.3. Let $\pi : (X,x) \to (Z,z)$ be a finite morphism étale in codimension 1. Then (X,x) is strictly log-canonical if and only if (Z,z) is (see for example Proposition 1.7, [7]). Hence by the canonical cover, an arbitrary strictly log-canonical singularity is regarded as the quotient of such a singularity of index 1 by a finite group which acts on the singularity freely in codimension 1.

Definition 4.4. An isolated strictly log-canonical singularity is called of type $(0,i)$, if its canonical cover is of type $(0,i)$.

Theorem 4.5. *An arbitrary dimensional isolated strictly log-canonical singularity of type* $(0,0)$ *has index either* 1 *or* 2.

Proof. This is proved in Theorem 3.10, [7]. One can also prove it by using 3.6. Let $\pi : (X,x) \to (Z,z)$ be the canonical cover of an n-dimensional isolated strictly log-canonical singularity (Z,z) and $G = \langle g \rangle$ the associated cyclic group. Let $f : \tilde{X} \to X$ be a G-equivariant good resolution of (X,x) such that $\pi \circ f$ factors through a good resolution $g : \tilde{Z} \to Z$ of (Z,z). Denote the essential divisor for f by E_J and its dual complex by Γ. Then g induces an automorphism g^* on $H^{n-1}(\Gamma, \mathbb{Z})$. Since (X,x) is of type $(0,0)$, $\mathbb{C} \simeq H^{0,0}_{n-1}(E_J)$ and this is isomorphic to $H^{n-1}(\Gamma, \mathbb{C})$ by 2.5, [12]. Therefore $H^{n-1}(\Gamma, \mathbb{Z})$ is of rank 1. Let λ be a free generator of $H^{n-1}(\Gamma, \mathbb{Z})$ Then $g^*(\lambda) = \pm\lambda + (torsion)$ in $H^{n-1}(\Gamma, \mathbb{Z})$. Therefore $g^*(\lambda) = \pm\lambda$ in $H^{n-1}(\Gamma, \mathbb{C})$. Hence the order of the action of G on $H^{n-1}(E_J, \mathcal{O}_{E_J})$ is 1 or 2. Now apply 3.6. Q.E.D.

4.6. A non-singular projective variety X is called a Calabi-Yau variety, if it satisfies that $\omega_X \simeq \mathcal{O}_X$. It is well known that a 1-dimensional Calabi-Yau variety is an elliptic curve and 2-dimensional one is either a $K3$-surface or an Abelian surface. An automorphism g on X induces a linear automorphism g^* on $\Gamma(X, \omega_X) = \mathbb{C}$ which is dual to $H^n(X, \mathcal{O}_X)$, where $n = \dim X$. Now let us introduce a conjecture on finite automorphisms on Calabi-Yau varieties, which is essential to our problem.

Conjecture 4.7. For $n \in \mathbb{N}$, there is a number B_n such that n-dimensional Calabi-Yau variety X and a finite automorphism g on X, the order of the induced automorphism g^* on $H^n(X, \mathcal{O}_X) = \mathbb{C}$ is bounded by B_n.

For $n = 1, 2$, the conjecture holds true.

Proposition 4.8. *For an arbitrary elliptic curve X, denote the order $|\operatorname{Im}\rho|$ by r, where $\rho : \operatorname{Aut}(X) \to GL(H^1(X, \mathcal{O}_X)) = \mathbb{C}^*$ is the induced representation. Then $\varphi(r) \leq 2$, which means $r =$* 1, 2, 3, 4 *or* 6.

Proof. This is a classical result and proved in various ways. For example, note that an automorphism of X is the composite of a group homomorphism and a translation. Since the translation has no effect on $H^1(X, \mathcal{O}_X) = \mathbb{C}$, $\operatorname{Im}\rho$ is $\rho(\operatorname{Aut}(X,0))$, where $\operatorname{Aut}(X,0)$ is the group of automorphisms. Since $\mathrm{Aut}(X,0)$ fixes the zero element of the group, it is a finite group of order 1, 2, 4 or 6 (see, for example, IV, 4.7, [5]). Q.E.D.

Proposition 4.9. (i) (10.1.2, [16]) *For an arbitrary $K3$-surface X, denote the order $|\mathrm{Im}\,\rho|$ by r, where $\rho : \mathrm{Aut}(X) \to GL(H^2(X, \mathcal{O}_X)) = \mathbb{C}^*$ is the induced representation. Then $\varphi(r) \leq 20$, in particular $r \leq 66$. Here φ is the Euler function.*

(ii) (3.2, [4]) *For an arbitrary Abelian surface X, the order r of a finite automorphism on X satisfies $\varphi(r) \leq 4$, which means that $r = 1$, 2, 3, 4, 5, 6, 8, 10, 12.*

Now one obtains a new proof of the following result.

Theorem 4.10. *A 2-dimensional strictly log-canonical singularity has index $1, 2, 3, 4$ or 6.*

Proof. Let $\pi : (X, x) \to (Z, z)$ be the canonical cover of the strictly log-canonical singularity (Z, z) and G be the associated cyclic group. By 4.5, it is sufficient to prove the case that (X, x) is of type $(0, 1)$. Let $f : Y \to X$ be the minimal resolution and E the exceptional curve. Then f is a G-equivariant good resolution with the essential divisor E which is an elliptic curve. By 4.8, $|\mathrm{Im}\,\rho| = 1$, 2, 3, 4 or 6, where $\rho : G \to GL(H^1(E, \mathcal{O}_E)) = \mathbb{C}^*$ is the induced representation. Now apply 3.6. Q.E.D.

Theorem 4.11. *An isolated 3-dimensional strictly log-canonical singularity of type $(0, 2)$ has index r, where $\varphi(r) \leq 20$.*

Proof. Let $\pi : (X, x) \to (Z, z)$ be the canonical cover of a 3-dimensional strictly log-canonical singularity (Z, z) and G the associated cyclic group. Let E be the exceptional divisor on the canonical model of X. Then by 2.10 E is either a normal $K3$-surface or an Abelian surface. Note that the action of G on E is lifted onto the minimal resolution $\tilde{E}$ of E. Since the singularities on E are at worst rational double, one obtains that $\Gamma(E, \omega_E) = \Gamma(\tilde{E}, \omega_{\tilde{E}})$. By the Serre duality, the action of G on $H^2(E, \mathcal{O}_E)$ is the same as the one on $H^2(\tilde{E}, \mathcal{O}_{\tilde{E}})$. Therefore by 3.5 and 4.9 $r = \mathrm{Ind}(Z, z)$ satisfies $\varphi(r) \leq 20$. Q.E.D.

Theorem 4.12. *An isolated 3-dimensional strictly log-canonical singularity of type $(0, 1)$ has index $1, 2, 3, 4$ or 6.*

4.13. For the proof of Theorem 4.12 one needs the discussion on the following divisor: Let E_J be a simple normal crossing divisor on a non-singular 3-fold. Assume $E_J = E_1 + E_2 + \ldots + E_s$ is a cycle of elliptic ruled surfaces E_i and every intersection curve is a section on the ruled surfaces. Decompose E_J into two connected chains $E^{(i)}$ $(i = 1, 2)$

with no common components. Let C_1 and C_2 be the irreducible curves of $E^{(1)} \cap E^{(2)}$. Let $p : E^{(1)} \to C$ and $q : E^{(2)} \to C$ be the rulings and $p_i : C_i \to C$ be the restriction of p on C_i. Then one obtains the Mayer-Vietoris exact sequence:

$$\begin{aligned} H^1(E^{(1)}, \mathbb{C}) \oplus H^1(E^{(2)}, \mathbb{C}) &\to H^1(C_1, \mathbb{C}) \oplus H^1(C_2, \mathbb{C}) \\ &\to H^2(E_J, \mathbb{C}) \to 0, \end{aligned}$$

which is an exact sequence of mixed Hodge structure. By taking Gr_F^0, where F is the Hodge filtration, one obtains the following:

$$\begin{aligned} H^1(E^{(1)}, \mathcal{O}) \oplus H^1(E^{(2)}, \mathcal{O}) &\xrightarrow{\Phi} H^1(C_1, \mathcal{O}) \oplus H^1(C_2, \mathcal{O}) \\ &\xrightarrow{\Psi} H^2(E_J, \mathcal{O}) \to 0. \end{aligned}$$

Lemma 4.14. *Assume that $H^2(E_J, \mathcal{O}) = \mathbb{C}$. Let $\Phi|_{H^1(E^{(i)}, \mathcal{O})} = \varphi_i$ and $\Psi|_{H^1(C_i, \mathcal{O})} = \psi_i$. Then the following hold:*

(i) $\operatorname{Im} \varphi_1 = \operatorname{Im} \varphi_2 = \operatorname{Im} \Phi$;

(ii) *ψ_i is an isomorphism for $i = 1, 2$ and* $\operatorname{Ker} \Psi \circ (p_1^* \oplus p_2^*) = \Delta$, *where Δ is the diagonal subspace of $H^1(C, \mathcal{O}) \oplus H^1(C, \mathcal{O})$;*

(iii) *fix C_1, then the isomorphism ψ_1 is independent of the choice of the decomposition of E_J as in* 4.13.

Proof. If (i) does not hold, then $\operatorname{Im} \Phi \neq \operatorname{Im} \varphi_1$, where $\operatorname{Im} \varphi_1$ is of dimension 1, because φ_1 is a non-zero map from 1-dimensional vector space. Therefore Φ becomes surjective, a contradiction to $H^2(E_J, \mathcal{O}_{E_J}) \neq 0$. For (ii), consider the composite:

$$\begin{aligned} H^1(E^{(i)}, \mathcal{O}_{E^{(i)}}) &\xrightarrow{\varphi_i} H^1(C_1, \mathcal{O}_{C_1}) \oplus H^1(C_2, \mathcal{O}_{C_2}) \\ &\xrightarrow{p_1^{*-1} \oplus p_2^{*-1}} H^1(C, \mathcal{O}_C) \oplus H^1(C, \mathcal{O}_C). \end{aligned}$$

One obtains that $\operatorname{Im}((p_1^{*-1} \oplus p_2^{*-1}) \circ \varphi_i) = \Delta$. Therefore ψ_i is not a zero map. For (iii), take another C_2' and $E^{(i)'}$ $(i = 1, 2)$ such that $E^{(1)'} \cap E^{(2)'} = C_1 \amalg C_2'$. One may assume that $C_2' \subset E^{(1)}$ and $E^{(1)'} \subset E^{(1)}$ and $E^{(2)} \subset E^{(2)'}$. Let $E^{(3)}$ be a subchain of E_J such that $E^{(1)} \cap E^{(2)'} = C_1 \amalg E^{(3)}$. Then C_2, $C_2' \subset E^{(3)}$. By these inclusions, we obtain the commutative diagram:

$$\begin{array}{ccccccc} H^1(E^{(1)}) \oplus H^1(E^{(2)}) & \to & H^1(C_1) \oplus H^1(C_2) & \xrightarrow{\Psi} & H^2(E_J) & \to & 0 \\ \| \quad \uparrow\wr & & \| \quad \uparrow\wr & & \| & & \\ H^1(E^{(1)}) \oplus H^1(E^{(2)'}) & \to & H^1(C_1) \oplus H^1(E^{(3)}) & \to & H^2(E_J) & \to & 0 \\ \downarrow\wr \quad \| & & \| \quad \downarrow\wr & & \| & & \\ H^1(E^{(1)'}) \oplus H^1(E^{(2)'}) & \to & H^1(C_1) \oplus H^1(C_2') & \xrightarrow{\Psi'} & H^2(E_J) & \to & 0. \end{array}$$

So the restrictions of Ψ and Ψ' on $H^1(C_1, \mathcal{O})$ are the same. Q.E.D.

Proof of Theorem 4.12. Let (Z, z) be an isolated strictly log-canonical singularity of type (0, 1), $\pi : (X, x) \to (Z, z)$ the canonical cover and G the associated cyclic group. Let $f : Y \to X$ be a G-equivariant good resolution and E_J the essential divisor. Then E_J is either as in (i) or (ii) of Theorem 6.8 in Appendix.

Case 1. The case that E_J is as in (ii) of Theorem 6.8.

Let $E_J = E^{(-)} + E^{(0)} + E^{(+)}$ be the decomposition as in (ii). Then there is a ruling $p : E^{(0)} \to C$ over an elliptic curve C. Since each fiber of p is mapped to a fiber of p by the action of G, C admits the action of G and p becomes a G-equivariant morphism. Now by the Mayer-Vietoris exact sequence:

$$\begin{aligned} &H^1(E^{(-)} + E^{(0)}, \mathcal{O}) \oplus H^1(E^{(0)} + E^{(+)}, \mathcal{O}) \to H^1(E^{(0)}, \mathcal{O}) \\ &\qquad \to H^2(E_J, \mathcal{O}) \to H^2(E^{(-)} + E^{(0)}, \mathcal{O}) \oplus H^2(E^{(0)} + E^{(+)}, \mathcal{O}) = 0, \end{aligned}$$

one obtains a G-equivariant isomorphism $H^1(E^{(0)}, \mathcal{O}) \simeq H^2(E_J, \mathcal{O})$. On the other hand there is a G-equivariant isomorphism $p^* : H^1(C, \mathcal{O}) \to H^1(E^{(0)}, \mathcal{O})$. Since the action of G on $H^1(C, \mathcal{O})$ is induced from that on C, the order of the action on G on $H^1(C, \mathcal{O})$ is 1, 2, 3, 4, 6 by Proposition 4.8.

Case 2. The case that E_J is as in (i) of Theorem 6.8.

If the intersection curves are all fixed under the action of G, the generater g of G induces an automorphism of each intersection curve. Take C_i and $E^{(i)}$ $(i = 1, 2)$ as in 4.13. Then one obtains the commutative diagram of isomorphisms:

$$\begin{array}{ccc} H^1(C_1) & \xrightarrow{\psi_1} & H^2(E_J) \\ g|_{C_1}^* \downarrow & & \downarrow g^* \\ H^1(C_1) & \xrightarrow{\psi_1} & H^2(E_J). \end{array}$$

Since $g|_{C_1}^*$ is of order 1, 2, 3, 4, 6 by Proposition 4.8, so is g^*.

If $g(C_1) = C_2$ for $C_1 \neq C_2$, then under the notation in 4.13 let $h : C \to C$ be an automorphism $p_2 \circ g|_{C_1} \circ p_1^{-1}$. By the definition of h, we obtain the commutative diagram of isomorphisms:

$$\begin{array}{ccccc} H^1(C) & \xrightarrow{p_2^*} & H^1(C_2) & \xrightarrow{\psi_2'} & H^2(E_J) \\ \downarrow h^* & & g|_{C_1}^* \downarrow & & \downarrow g^* \\ H^1(C) & \xrightarrow{p_1^*} & H^1(C_1) & \xrightarrow{\psi_1} & H^2(E_J), \end{array}$$

where ψ_2' is induced from ψ_1 through g. Here, note that $H^2(E_J, \mathcal{O}) = \mathbb{C}$ by the assumption of the singularity. So one can apply Lemma 4.14, (iii), obtaining that $\psi_2' = \psi_2$. On the other hand, as $\operatorname{Ker} \Psi \circ (p_1^* \oplus p_2^*) = \Delta$ by Lemma 4.14, (ii), it follows that $\psi_1 \circ p_1^* = -\psi_2 \circ p_2^*$. Hence, by the diagram above, the order of g^* is 1, 2, 3, 4, 6 since that of h^* is 1, 2, 3, 4, 6 by 4.8. Q.E.D.

Theorem 4.15. *For a positive integer r the following are equivalent:*

(i) *r is the index of a 3-dimensional strictly log-canonical singularity;*
(ii) *$\varphi(r) \leq 20$ and $r \neq 60$, where φ is the Euler function.*

Proof. First assume (i), then by theorems 4.5, 4.11 and 4.12, it follows that $\varphi(r) \leq 20$. If there exists a 3-dimensional strictly log-canonical singularity (Z, z) of index 60, then by 4.5 and 4.12, (Z, z) must be of type $(0, 2)$. Let E be the exceptional divisor on the canonical model of the canonical cover (X, x), then E is a normal $K3$-surface. Let G be the corresponding group of the canonical cover, then G acts on E whose induced action on $H^2(E, \mathcal{O}_E)$ is of order 60. Since this action is lifted to the minimal resolution $\tilde{E}$ of E, one obtains a $K3$-surface $\tilde{E}$ which admits an automorphism whose action on $H^2(\tilde{E}, \mathcal{O}_{\tilde{E}})$ is of order 60. However, it is proved by Machida-Oguiso [13] that there is no $K3$-surface with such an automorphism.

Next assume (ii), then by [11] and [17], there is a $K3$-surface E with an automorphism $g : E \to E$ whose order and the order of induced automorphism on $H^2(E, \mathcal{O}_E)$ are both r. Let $G = \langle g \rangle$, $\pi : E \to E' = E/G$ the quotient map and $\mathcal{L}$ an ample invertible sheaf on E'. Let Y' and Y be the line bundles $\operatorname{Spec} \bigoplus_{m \geq 0} \mathcal{L}^{\otimes m}$ and $\operatorname{Spec} \bigoplus_{m \geq 0} \pi^* \mathcal{L}^{\otimes m}$ on E' and on E, respectively. Then $Y \to E$ has the zero section E_0 whose normal bundle is $\pi^* \mathcal{L}^{-1}$, so there is a contraction $f : (Y, E_0) \to (X, x)$ of E_0. Since the exceptional divisor E_0 is a $K3$-surface, the singularity (X, x) is strictly log-canonical of index 1 and of type $(0, 2)$ by [8]. One defines an action of G on (X, x) in the following way: Let σ be the action of G on E. On the other hand there is also an action τ of G on Y' which is trivial on E', because Y' admits a canonical action of $\mathbb{C}^*$ and G is considered as a subgroup of $\mathbb{C}^*$. Since Y is the fiber product $E \times_{E'} Y'$, one obtains the action of G on Y which is compatible with σ and τ. It is clear that this action is free on $Y \setminus E_0$ and E_0 is G-invariant. Therefore one can introduce the action of G on (X, x). The quotient $(Z, z) = (X, x)/G$ is strictly log-canonical of index r by Corollary 3.6. Q.E.D.

4.16. The boundedness of indices of higher dimensional strictly log-canonical singularities is also expected to follow from Conjecture 4.7. On the contrary, if indices of n-dimensional strictly log-canonical singularities are bounded, then Conjecture 4.7 holds for $(n-1)$-dimensional

Calabi-Yau varieties. Indeed, as in the proof of Theorem 4.15, for every Calabi-Yau $(n-1)$-fold E and a finite order automorphism g, one can construct a strictly log-canonical singularity of index r, where r is the order of the induced automorphism g^* on $H^{n-1}(E, \mathcal{O}_E)$. Hence the boundedness of indices implies Conjecture 4.7.

§5. Finite groups which act non-freely in codimension one.

5.1. Terminologies in [10] are used in this section. Here one considers a finite group action on a 2-dimensional strictly log-canonical singularity. If the action is not free in codimension 1, the index of the quotient is not bounded.

Example 5.2. Let $\pi : C \to \mathbb{P}^1$ be a double covering from an elliptic curve C. Then π is the quotient map by a group $G = \mathbb{Z}/(2)$. Let $\tilde{Z}_m$ and $\tilde{X}_m$ be $\operatorname{Spec} \bigoplus_{i \geq 0} \mathcal{O}_{\mathbb{P}^1}(mi)$ and $\operatorname{Spec} \bigoplus_{i \geq 0} \pi^* \mathcal{O}_{\mathbb{P}^1}(mi)$ respectively, then $\tilde{X}_m$ admits the canonical action of G and the induced morphism $\tilde{\pi} : \tilde{X}_m \to \tilde{Z}_m$ is the quotient map. Since the zero sections of $\tilde{X}_m$ and $\tilde{Z}_m$ are G-invariant, one obtains the quotient map $\pi' : X_m \to Z_m$, where X_m and Z_m are the contracted space of zero sections in $\tilde{X}_m$ and $\tilde{Z}_m$, respectively. Here the singularity of X_m is strictly log-canonical of index 1 and the singularity of Z_m has the index m if m is odd and $m/2$ if m is even as one sees in Example 4.2, which shows that the indices of the quotients $\{Z_m\}_{m \in N}$ are not bounded.

5.3. Let (X, x) be an n-dimensional normal singularity and G a finite group which acts on (X, x) non-freely in codimension 1. Let $\pi : (X, x) \to (Z, z) = (X, x)/G$ be the quotient map, then π ramifies at divisors on X. Let B_i $(i = 1, \ldots, s)$ be the branch divisors of π and R_{ij} $(j = 1, \ldots n_i)$ the ramification divisors over B_i. Then the ramification index of R_{ij} depends only on i, denote it by e_i, because the generic points of R_{ij}'s $(j = 1, \ldots, n_i)$ are mapped to each other transitively by the action of G. As for a Weil divisor D on Z the pull-back $\pi^*(D)$ by finite morphism π is defined (see for example 1.8 in [2]), one obtains the formula of $\mathbb{Q}$-divisors:

$$K_X = \pi^* \left(K_Z + \sum_{i=1}^{s} \frac{e_i - 1}{e_i} D_i \right).$$

Lemma 5.4. *Under the notation of* 5.3, (X, x) *is strictly log-canonical, if and only if the pair* $(Z, \sum_{i=1}^{s}(1 - 1/e_i)D_i)$ *is log-canonical, non-klt around* z.

Proof. By 3.16 of [10] $(X, \emptyset)$ is log-canonical, non-klt around x, if and only if $(Z, \sum_{i=1}^{s}(1 - 1/e_i)D_i)$ is log-canonical, non-klt around z. Here note that $(X, \emptyset)$ is klt around x, if and only if (X, x) is log-terminal. Q.E.D.

Lemma 5.5. *Let Z be a normal surface and D an effective $\mathbb{Q}$-divisor on Z such that* $\mathrm{Supp}(D)$ *contains a point $z \in Z$. If (Z, D) is log-canonical, then (Z, z) is a quotient singularity.*

Proof. Let $f : \tilde{Z} \to Z$ be a resolution of singularities on Z. First one will prove that $\omega_Z = f_*\omega_{\tilde{Z}}$ around z. Take a positive integer m such that mD is an integral divisor and $\omega_Z^{[m]}(mD)$ is trivial around z. Represent $mD = \sum_{i=1}^{u} r_i D_i$, where D_i's are the irreducible components. Let ω be a generator of $\omega_Z^{[m]}(mD)$, then $\nu_{D_i}(\omega) = -r_i < 0$ for every i. Since (Z, D) is log-canonical, one obtains

$$K_{\tilde{Z}} = f^*(K_Z + D) + \sum_{j=1}^{v} m_j E_j - D',$$

with $m_j \geq -1$ for every j, where D' is the proper transform of D and E_j's are the irreducible exceptional curves. Therefore

$$\omega_{\tilde{Z}}^{m}\left(-\sum m m_j E_j + mD'\right) = f^*(\omega_Z^{[m]}(mD)).$$

Hence $\nu_{E_j}(\omega) = m m_j \geq -m$ for every j. If an element $\theta \in \omega_Z$ satisfies $\nu_{E_j}(\theta) < 0$ for some E_j with $f(E_j) = \{z\}$, then $\nu_{E_j}(\theta^m) \leq -m$. Since $\theta^m \in \omega_Z^{[m]} \subset \omega_Z^{[m]}(mD)$, it follows that $\theta^m = h\omega$ with $h \in \mathcal{O}_Z$. Then $-m \leq \nu_{E_j}(\omega) \leq \nu_{E_j}(\theta^m) \leq -m$, and therefore $\nu_{E_j}(h) = 0$. Hence h does not vanish at z, from which one may assume that h does not vanish on Z by deleting Z sufficiently. But this yields a contradiction $\nu_{D_i}(\theta^m) = \nu_{D_i}(\omega) = -r_i < 0$. Now one obtains that $\omega_Z = f_*\omega_{\tilde{Z}}$ around z. Since Z is a normal surface, this equality implies that (Z, z) is a rational singularity, hence a $\mathbb{Q}$-Gorenstein singularity. So one can represent

$$K_{\tilde{Z}} = f^*K_Z + \sum n_j E_j,$$

with $n_j = m_j + m'_j$, where $f^*D = D' + \sum m'_j E_j$. By $z \in \mathrm{Supp}(D)$, it follows that $m'_j > 0$ for every E_j with $f(E_j) = \{z\}$, which yields that $n_j > -1$ for these j. A 2-dimensional log-terminal singularity is a quotient singularity. Q.E.D.

Theorem 5.6. *Let (X, x) be a 2-dimensional strictly log-canonical singularity and a finite group G act on (X, x) non-freely in codimension 1. Then the number of the branch divisors is at most 4 and the combination of the ramification indices of the quotient map $\pi : (X, x) \to (X, x)/G$ are* (6), $(4,4)$, $(3,3)$, $(3,3,3)$, $(2,2)$, $(2,2,2)$, $(2,2,2,2)$, $(6,2)$, $(4,2)$, $(3,2)$, $(6,3,2)$, $(4,4,2)$, $(4,2,2)$, $(3,3,2)$, $(3,2,2)$.

Proof. Use the notation of 5.3. By Lemma 5.4 $(Z, \sum(1-1/e_i)D_i)$ is log-canonical, not klt and by Lemma 5.5 (Z, z) is a quotient singularity. Let $\rho : \mathbb{C}^2 \to Z$ be the quotient map. Since ρ is étale in codimension 1, $K_{\mathbb{C}^2} = \rho^* K_Z$. Then by Lemma 5.4 $(\mathbb{C}^2, \sum(1 - 1/e_i)\rho^* D_i)$ is log-canonical, non-klt. In the following classification theorem of such pairs, one can see that the number of the branch divisors is at most 4 and the combination of the values of e_i's are (6), (4, 4), (3, 3), (3, 3, 3), (2, 2), (2, 2, 2), (2, 2, 2, 2), (6, 2), (4, 2), (3, 2), (6, 3, 2), (4, 4, 2), (4, 2, 2), (3, 3, 2), (3, 2, 2). Q.E.D.

Theorem 5.7. *The pair $(\mathbb{C}^2, \sum(1-1/e_i)D_i)$ is log-canonical, non-klt around 0 if and only if (e_i) and (D_i) are as follows up to analytic isomorphism around 0:*

(1.1) $e_1 = 6$, $D_1 = (x^2 + g = 0)$, *where* $g = \sum_{a+b\geq 3} \alpha_{ab} x^a y^b$ $(\alpha_{03} \neq 0)$;

(1.2) $(e_1, e_2) = (4, 4)$, $D_1 = (x = 0)$, $D_2 = (x + y^2 + g = 0)$, *where* $g = \sum_{2a+b\geq 3} \alpha_{ab} x^a y^b$;

(1.3) $(e_1, e_2) = (3, 3)$, $D_1 = (x = 0)$, $D_2 = (x + y^3 + g = 0)$, *where* $g = \sum_{3a+b\geq 4} \alpha_{ab} x^a y^b$;

(1.4) $(e_1, e_2, e_3) = (3, 3, 3)$, $D_1 = (x = 0)$, $D_2 = (y = 0)$, $D_3 = (x + y = 0)$;

(1.5) $(e_1, e_2) = (2, 2)$, $D_1 = (x^2 + g = 0)$, $D_2 = (y^2 + h = 0)$, *where* $g = \sum_{na+b\geq 2n+1} \alpha_{ab} x^a y^b$ $(\alpha_{02n+1} \neq 0,\ n \geq 1)$, $h = \sum_{a+mb\geq 2m+1} \beta_{ab} x^a y^b$ $(\beta_{2m+1,0} \neq 0,\ m \geq 1)$;

(1.6) $(e_1, e_2, e_3) = (2, 2, 2)$, $D_1 = (x = 0)$, $D_2 = (x + y^2 + g = 0)$, $D_3 = (x + \beta y^n + h = 0)$, *where* $g = \sum_{2a+b\geq 3} \alpha_{ab} x^a y^b$, $h = \sum_{na+b\geq n+1} \beta_{ab} x^a y^b$ $(n \geq 2)$, $\beta \neq 0$ *and if* $n = 2$, $\beta \neq 1$;

(1.7) $(e_1, e_2, e_3) = (2, 2, 2)$, $D_1 = (x = 0)$, $D_2 = (x + y^n + g = 0)$, $D_3 = (y^2 + h = 0)$, *where* $g = \sum_{na+b\geq n+1} \alpha_{ab} x^a y^b$ $(n \geq 1)$, $h = \sum_{a+mb\geq 2m+1} \beta_{ab} x^a y^b$ $(\beta_{2m+1,0} \neq 0,\ m \geq 1)$;

(1.8) $(e_1, e_2, e_3, e_4) = (2,2,2,2)$, $D_i = (x + \alpha_i y + h_i = 0)$ *for* $i = 1, \dots, 4$, *where* $\deg h_i \geq 2$ *and* $\alpha_i \neq \alpha_j$ $(i \neq j)$;

(1.9) $(e_1, e_2, e_3, e_4) = (2,2,2,2)$, $D_1 = (x = 0)$, $D_2 = (y = 0)$, $D_3 = (x+y = 0)$, $D_4 = (x+\alpha y^n + g = 0)$, *where* $g = \sum_{na+b \geq n+1} \alpha_{ab} x^a y^b$ $(n \geq 2)$ *and* $\alpha \neq 0$;

(1.10) $(e_1, e_2, e_3, e_4) = (2,2,2,2)$, $D_1 = (x = 0)$, $D_2 = (x+y^n+g = 0)$, $D_3 = (y = 0)$, $D_4 = (y+x^m+h = 0)$, *where* $g = \sum_{na+b \geq n+1} \alpha_{ab} x^a y^b$ $(n \geq 2)$, $h = \sum_{a+mb \geq m+1} \beta_{ab} x^a y^b$ $(m \geq 2)$;

(2.1) $(e_1, e_2) = (6,2)$, $D_1 = (x = 0)$, $D_2 = (x + y^3 + g = 0)$, *where* $g = \sum_{3a+b \geq 4} \alpha_{ab} x^a y^b$;

(2.2) $(e_1, e_2) = (4,2)$, $D_1 = (x = 0)$, $D_2 = (x + y^4 + g = 0)$, *where* $g = \sum_{4a+b \geq 5} \alpha_{ab} x^a y^b$;

(2.3) $(e_1, e_2) = (3,2)$, $D_1 = (x = 0)$, $D_2 = (x + y^6 + g = 0)$, *where* $g = \sum_{6a+b \geq 7} \alpha_{ab} x^a y^b$;

(2.4) $(e_1, e_2) = (3,2)$, $D_1 = (x = 0)$, $D_2 = (x^2 + g = 0)$, *where* $g = \sum_{a+b \geq 3} \alpha_{ab} x^a y^b$ $(\alpha_{03} \neq 0)$;

(2.5) $(e_1, e_2) = (2,3)$, $D_1 = (x = 0)$, $D_2 = (y^2 + g = 0)$, *where* $g = \sum_{a+b \geq 3} \alpha_{ab} x^a y^b$ $(\alpha_{30} \neq 0)$;

(2.6) $(e_1, e_2, e_3) = (6,3,2)$, $D_1 = (x = 0)$, $D_2 = (y = 0)$, $D_3 = (x + y = 0)$;

(2.7) $(e_1, e_2, e_3) = (4,4,2)$, $D_1 = (x = 0)$, $D_2 = (y = 0)$, $D_3 = (x + y = 0)$;

(2.8) $(e_1, e_2, e_3) = (4,2,2)$, $D_1 = (x = 0)$, $D_2 = (y = 0)$, $D_3 = (x + y^2 + g = 0)$, *where* $g = \sum_{2a+b \geq 3} \alpha_{ab} x^a y^b$;

(2.9) $(e_1, e_2, e_3) = (3,3,2)$, $D_1 = (x = 0)$, $D_2 = (y = 0)$, $D_3 = (x + y^2 + g = 0)$, *where* $g = \sum_{2a+b \geq 3} \alpha_{ab} x^a y^b$;

(2.10) $(e_1, e_2, e_3) = (3,2,2)$, $D_1 = (x = 0)$, $D_2 = (x + y^3 + g)$, $D_3 = (y = 0)$, *where* $g = \sum_{3a+b \geq 4} \alpha_{ab} x^a y^b$.

Proof. Denote $\sum D_i$ by D. Since $(1 - 1/e_i) \geq 1/2$, $(\mathbb{C}^2, 1/2D)$ is log-canonical around 0. Therefore $1/2 \leq \operatorname{lcth}(\mathbb{C}^2, D, 0)$, where $\operatorname{lcth}(\mathbb{C}^2, D, 0)$ is the log-canonical threshold of $(\mathbb{C}^2, D)$ around 0. On the other hand $\operatorname{lcth}(\mathbb{C}^2, D, 0) \leq 2/\operatorname{mult}_0 D$ by 8.10 of [10]. Hence $\operatorname{mult}_0 D \leq 4$.

Case 1. $\#\{e_i\} = 1$.

In this case $e = e_i \leq 6$, because $(e-1)/e = \mathrm{lcth}(\mathbb{C}^2, D, 0)$ and the right hand side is shown to be $\leq 5/6$ by 8.16 of [10].

Subcase 1.1. $\mathrm{mult}_0 D = 2$.

First consider the case that D is analytically irreducible. Let n be the number of successive blowing-ups of $\mathbb{C}^2$ at the singular point of the proper transforms of D to get the resolution of D. Then by two more blowing-ups at the suitable centers, one obtains a log-resolution of $(\mathbb{C}^2, D)$. Let E_i $(i = 1, \ldots, n+2)$ be the exceptional curve of the i-th blowing-up and m_i the log-discrepancy of $(\mathbb{C}^2, (1-1/e)D)$ at E_i, which means:

$$K_{\tilde{\mathbb{C}^2}} + \frac{e-1}{e}\tilde{D} = f^*\left(K_{\mathbb{C}^2} + \frac{e-1}{e}D\right) + \sum_{i=1}^{n+2} m_i E_i,$$

where $f : \tilde{\mathbb{C}^2} \to \mathbb{C}^2$ is the log-resolution and $\tilde{D}$ is the proper transform of D. It follows that $m_i = i - (1-1/e)2i$ for $i = 1, \ldots, n$, $m_{n+1} = n+1-(1-1/e)(2n+1)$ and $m_{n+2} = 2n+2-(1-1/e)(4n+2)$. Therefore if $e = 2$, $(\mathbb{C}^2, (1-1/e)D)$ is klt for every n. If $e = 3$, it is klt for $n = 1, 2$ and non-log-canoninal for $n \geq 3$. If $e = 4$ and $e = 5$, it is klt for $n = 1$ and non-log-canonical for $n \geq 2$. If $e = 6$, it is non-log-canonical for $n \geq 2$ and log-canonical, non-klt for $n = 1$. Now one obtains that $(\mathbb{C}^2, (1-1/e)D)$ is log-canonical, non-klt, if and only if $e = 6$ and D has a double cusp at 0 which can be resolved by the blowing-up at 0. By Lemma 5.8 below one obtains the defining equation of D and this case turns out to be (1.1).

Lemma 5.8. *Let $(D, 0) \subset (\mathbb{C}^2, 0)$ be a double cusp defined by an equation $f = 0$. Let n be the number of successive blowing-ups of $\mathbb{C}^2$ at the singular point of the proper transforms of D to get the resolution of D. Then $f = x^2 + g$, where $g = \sum_{na+b \geq 2n+1} \alpha_{ab} x^a y^b$, $\alpha_{0\,2n+1} \neq 0$ by a suitable coordinate transformation.*

Next consider the remaining case that D is the union of two non-singular curves. Let n be as above, then the successive n-blowing-ups give a log-resolution. Define E_i and m_i $(i = 1, \ldots, n)$ in the same way as above. Then $m_i = i - (1-1/e)2i$ for $i = 1, \ldots, n$. Therefore $(\mathbb{C}^2, (1-1/e)D)$ is log-canonical, non-klt, if and only if $e = 4$ and $n = 2$ or $e = 3$ and $n = 3$. By Lemma 5.9 below, the former is (1.2) and the latter is (1.3).

Lemma 5.9. *Let $D \subset \mathbb{C}^2$ be the union of two non-singular curves D_1 and D_2 defined by equations $f_1 = 0$ and $f_2 = 0$. Let n be the number of successive blowing-ups of $\mathbb{C}^2$ at the singular point of the proper* *transforms of D to get the resolution of D. Then $f_1 = x$ and $f_2 = x + y^n + g$, where $g = \sum_{na+b \geq n+1} \alpha_{ab} x^a y^b$ by a suitable coordinate transformation.*

Subcase 1.2. $\mathrm{mult}_0 D = 3$.

In this case, $(\mathbb{C}^2, (1-1/e)D)$ is log-canonical, non-klt, if and only if (1.4) or (1.6) holds. It is proved in the same way as in Subcase 1.1, and the proof is omitted.

Subcase 1.3. $\mathrm{mult}_0 D = 4$.

In this case, $(\mathbb{C}^2, (1-1/e)D)$ is log-canonical, non-klt, if and only if (1.5), (1.7), (1.8), (1.9) or (1.10) holds. The proof is omitted.

Case 2. $\#\{e_i\} > 0$.

In this case $\mathrm{mult}_0 D \leq 3$. Indeed, if $\mathrm{mult}_0 D = 4$, then $\mathrm{lcth}(\mathbb{C}^2, D, 0) = 1/2$ by the inequalities in the beginning of the proof of the theorem. This is a contradiction to the fact that $(\mathbb{C}^2, \sum(1 - 1/e_i)D_i)$ is log-canonical around 0 with $\sum(1 - 1/e_i)D_i > 1/2D$.

Subcase 2.1. $\mathrm{mult}_0 D = 2$.

Since D is reducible, D is the union of two non-singular curves. Let n, E_i and m_i be as in Subcase 1.1. Then $m_i = i\{1 - (e_1 - 1)/e_1 - (e_2 - 1)/e_2\}$. Therefore $(\mathbb{C}^2, \sum(1 - 1/e_i)D)$ is log-canonical, non-klt, if and only if $(n, e_1, e_2) = (3, 6, 2)$, $(4, 4, 2)$ or $(6, 3, 2)$. These are the cases (2.1), (2.2) and (2.3), by Lemma 5.8 and Lemma 5.9.

Subcase 2.2. $\mathrm{mult}_0 D = 3$.

One can devide into two cases:

(1) $\mathrm{mult}_0 D_1 = 1$ and $\mathrm{mult}_0 D_2 = 2$ and

(2) $\mathrm{mult}_0 D_i = 1$ for $i = 1, 2, 3$.

Under the first case, $(\mathbb{C}^2, \sum(1 - 1/e_i)D)$ is log-canonical, non-klt, if and only if (2.4) or (2.5) holds, and under the second case, if and only if (2.6), (2.7), (2.8), (2.9) or (2.10) holds. The proof is in the same way as in Subcase 2.1. Q.E.D.

5.10. More generally, 2-dimensional log-canonical pairs are classified in [15] by the terminology of dual graphs of minimal good resolutions.

§6. Appendix : The essential divisors of type (0, 1)

In this section one studies the configurations of the essential divisors of strictly log-canonical singularities of index 1 and of type (0, 1). The configurations of such divisors were studied in [7]. But the proof skipped some steps and in ℓ.10, p.186, [7] it used a contraction criterion stated in p.61, §4, [20] which has a counter example (Proposition 3, Example, [3]). So in this appendix, we give a new proof including complete steps for the structure of the essential divisors. As a consequence we obtain a weaker result than stated in [7], but it is sufficient for our discussion in the preceding sections of this paper.

Definition 6.1. Let (X, x) be a normal singularity which admits an action of a group G. A birational proper morphism $g : Y \to X$ is called a G-equivariant $G\mathbb{Q}$-factorial terminal model of (X, x), if

(1) G acts on Y and g is G-equivariant,
(2) Y has at worst terminal singularities,
(3) every G-invariant divisor on Y is a $\mathbb{Q}$-Cartier divisor and
(4) K_Y is nef.

If (X, x) is of dimension 3, there exists a G-equivariant $G\mathbb{Q}$-factorial terminal model (relative version of 7.6 [1]).

Some parts of the following lemmas are proved in [7], but for the reader's convenience we give here the proofs.

Lemma 6.2. *Let (X, x) be a 3-dimensional isolated strictly log-canonical singularity of index 1 and of type $(0, 1)$, $f : \tilde{X} \to X$ a good resolution and E_J the essential divisor on $\tilde{X}$. Then*

(i) *E_J is not irreducible,*
(ii) *every intersection curve of E_J has positive genus and*
(iii) *there is no triple point on E_J.*

Proof. If E_J is irreducible, then $H^2(E_J, \mathcal{O}_{E_J}) = \mathbb{C}$ consists of $(0, 2)$-component, which is a contradiction. Take an irreducible component E_j of E_J and put $E_j^\vee = E_J - E_j$. Consider the exact sequence:

$$H^1(E_j, \mathcal{O}_{E_j}) \oplus H^1(E_j^\vee, \mathcal{O}_{E_j^\vee}) \to H^1(E_j \cap E_j^\vee, \mathcal{O}) \to H^2(E_J, \mathcal{O}_{E_J}) \to 0,$$

induced from the Mayer-Vietoris exact sequence and Proposition 3.8 of [6]. Since $H^2(E_J, \mathcal{O}_{E_J})$ consists of the $(0, 1)$-component, there is $(0, 1)$-component in $H^1(E_j \cap E_j^\vee, \mathcal{O})$. Therefore $E_j \cap E_j^\vee$ contains at least one curve of positive genus. Note that this holds for an arbitrary good resolution. Here, if ℓ is a rational intersection curve of E_J, take the blowing-up $\sigma : \tilde{X}' \to \tilde{X}$ with the center ℓ. Then the divisor $E_0 = \sigma^{-1}(\ell)$ is an essential component on $\tilde{X}'$ and the intersection curves of E'_J on E_0 are all rational, where E'_J is the essential divisor on $\tilde{X}'$. This is a contradiction to the fact proved above. If there is a triple point p on E_J, take the blowing-up at p. Then one also has an essential component with only rational double curves on it. Q.E.D.

Lemma 6.3. *Let $g : Y \to X$ be a G-equivariant $G\mathbb{Q}$-factorial terminal model of a 3-dimensional isolated strictly log-canonical singularity (X, x) of index 1 and D the reduced inverse image $g^{-1}(x)_{red}$. Then*

(i) $K_Y = -D$,

(ii) *the singularities of* D *are normal crossings except for finite points and*

(iii) D *is Cohen-Macaulay, therefore isolated singularities on* D *are normal.*

Proof. By the proof of Lemma 7 of [8], $K_Y = g^*K_X - \sum a_i D_i$ with $a_i > 0$ for all irreducible component D_i of D. Here by the assumption on the singularity, the negative discrepancy is -1, which yields (i). Let C be an irreducible component of 1-dimensional singular locus of D and m the multiplicity of D at a general point of C. Take the blowing-up $\sigma : Y' \to Y$ at the center C and denote the exceptional divisor over C by D_0. Then the discrepancy for $g \circ \sigma$ at D_0 is $1 - m$, because Y and C are both non-singular at a general point of C. Then by the assumption on the singularity (X, x), m must be 2. If the singularity of D is not ordinary at a general point of C, then, by successive blowing-ups of Y with suitable curves as centers, one obtains a partial resolution $g'' : Y'' \to X$ factored through g with $K_{Y''} = -D'_1 - D'_2 - D'_3 -$(other terms), where D'_1, D'_2 and D'_3 are components of ${g''}^{-1}(x)_{red}$ and intersect at a curve C'. By passing through the blowing-up of Y'' with center C', one obtains a good resolution $f : \tilde{X} \to X$, which has a discrepancy -2 at one component, a contradiction. (iii) follows from the fact that D is $\mathbb{Q}$-Cartier and the discussion as in 0.5 of [9]. Then by Serre's criterion, isolated singularities of D are normal. Q.E.D.

Definition 6.4. An irreducible component of 1-dimensional singular locus of D is called a double curve of D. If a double curve is the intersection of two irreducible components, it is called an intersection curve.

Proposition 6.5. *Let* (X, x) *be a* 3*-dimensional isolated strictly log-canonical singularity of index* 1 *and of type* $(0,1)$ *and* G *a finite group acting on* (X, x)*. Let* $g : Y \to X$ *be a* G*-equivariant* $G\mathbb{Q}$*-factorial terminal model of* (X, x) *and* D *the reduced inverse image* $g^{-1}(x)_{red}$*. Let* $\sigma : D' \to D$ *be the normalization. Then the structure of* D *is as follows:*

(i) *the case* D *is irreducible then* D *is one of the following:*

(i-1) *a normal elliptic ruled surface with two simple elliptic singularities or*

(i-2) *a normal rational surface with a simple elliptic singularity or*

(i-3) *a rational surface with a double curve* C *such that* $\sigma^{-1}(C)$ *is an elliptic curve or* (i-4) *an elliptic ruled surface with a simple elliptic singularity and a double curve* C *such that* $\sigma^{-1}(C)$ *is an elliptic curve or*

(i-5) *an elliptic ruled surface with two double curves* C_1 *and* C_2 *such that* $\sigma^{-1}(C_1)$, $\sigma^{-1}(C_2)$ *are disjoint elliptic curves or*

(i-6) *an elliptic ruled surface with a double curve* C *such that* $\sigma^{-1}(C)$ *consists of two disjoint elliptic curves;*

(ii) *the case* D *is not irreduible then* D *is one of the following:*

(ii-1) *a cycle of elliptic ruled surfaces with sections as double curves or*

(ii-2) *a chain of surfaces* $D = D_1 + \ldots + D_s (s \geq 2)$ *with elliptic intersection curves, where* $D_2, \ldots, D_{s-1}$ *are elliptic ruled surfaces and each of* D_1 *and* D_s *is as follows; rational surface or elliptic ruled surface with a simple elliptic singularity or elliptic ruled surface with a double curve* C *such that* $\sigma^{-1}(C)$ *is an elliptic curve.*

(iii) *the singularities of* D' *are at worst rational double points except for simple elliptic singularities appeared in* (i-1), (i-2), (i-4) *and* (ii-2). *Moreover,* D' *is non-singular along* $\sigma^{-1}(C)$, *where* C *is a double curve.*

Proof. First of all, note that the singularities on Y are isolated, because Y has at worst terminal singularities. By (i) of 6.3, the equality $\omega_D = \mathcal{O}_D$ holds away from finite points. Since D is Cohen-Macaulay by 6.3, this equality holds whole on D. Therefore

$$K_{D'} = -\sigma^{-1}(\text{double curves of } D).$$

Let $\varphi : \tilde{D} \to D'$ be the minimal resolution, then one obtains

$$K_{\tilde{D}} = \varphi^* K_{D'} - \Delta$$

with $\Delta \geq 0$, where $\varphi^* K_{D'}$ is the numerical pull-back defined in [19]. Now it follows that $-K_{\tilde{D}}$ is an effective divisor on each component of $\tilde{D}$. Denote an irreducible component of D by D_i and the corresponding component of D' and $\tilde{D}$ by D'_i and $\tilde{D}_i$, respectively. Then by [23], a pair $(\tilde{D}_i, \Gamma)$ $\Gamma \in |-K_{\tilde{D}_i}|$ is one of the following:

(1) $\tilde{D}_i$ is a rational surface and Γ is an elliptic curve;

(2) $\tilde{D}_i$ is a rational surface and Γ is a cycle of rational curves;

(3) $\tilde{D}_i$ is an elliptic ruled surface and Γ is two disjoint sections;

(4) $\tilde{D}_i$ is a ruled surface of genus ≥ 2 and $\Gamma = 2C_0 +$ rational curves, where C_0 is a section.

But in our situation, (2) and (4) do not occur. Indeed, assume D_i is a component such that $\tilde{D}_i$ and Γ are as in (4). Take a good resolution $f : \tilde{X} \to Y$ isomorphic on points which are non-singular on

D and on Y. Let E_k be the proper transform of D_k on $\tilde{X}$. Represent $K_{\tilde{X}} = -\sum_k E_k + \sum_{F_j:f-\text{exceptional}} m_j F_j$. Then

$$K_{E_i} = -\sum_{k\neq i} E_k|_{E_i} + \sum_{F_j:f-\text{exceptional}} m_j F_j|_{E_i}. \tag{6.5.1}$$

Here non-empty $F_j|_{E_i}$ is either corresponding to a double curve of D or a point on D, while $E_k|_{E_i}$ corresponds to a double curve of D. Note that $f|_{E_i}$ factors through $\tilde{D}_i$ and an irreducible component of Γ is either corresponding to a double curve or a point on D. Therefore

$$K_{E_i} = -2C_0' + \sum n_j e_j, \tag{6.5.2}$$

where C_0' is the proper transform of C_0 and e_j is either corresponding to a double curve or a point on D. By the uniqueness of the representation, (6.5.1) and (6.5.2) coincide, which shows that there is a component F_j with $m_j = -2$, a contradiction to the condition on the singularity (X, x). Next if D_i is a component such that $\tilde{D}_i$ and Γ are as in (2). Then in the same way as above one can prove that there exists an essential component F_j which intersects E_i at a rational curve, which is a contradiction to Lemma 6.2.

Now one has only to consider the case (1) or (3). Note that each component of Γ corresponds to either a double curve or a point on D.

First assume that D is irreducible. Consider the case that $\tilde{D}$ and Γ are as in (1). If Γ corresponds to a double curve, then one obtains (i-3). If Γ corresponds to a point, then one obtains (i-2). Next consider the case that $\tilde{D}$ and Γ are as in (3). If both components of Γ correspond to points, then one obtains (i-1). If both components of Γ correspond to double curves, then one obtains (i-5) and (i-6). If one component of Γ corresponds to a double curve and the other to a point, then one obtains (i-4).

Next assume that D is reducible. Then at least one component of Γ of $\tilde{D}_i$ corresponds to a double curve of D. Hence the structure of D is either (ii-1) or (ii-2).

For the statement (iii), take any point $p \in D'$ which is not the simple elliptic singularity stated in (i-1), (i-2), (i-4) and (ii-2). If p is not in the curve corresponding to a double curve of D, then p is rational double, because $K_{\tilde{D}} = \varphi^* K_{D'}$ around p. Assume p is on the curve $C' \subset D_i'$ corresponding to a double curve of D and φ is not isomorphic at p. As $K_{D_i'} = -C'$ around p, it follows that $K_{\tilde{D}_i} = -\tilde{C} - \Delta$, where $\tilde{C}$ is the proper transform of C' on $\tilde{D}_i$, $\Delta > 0$ and $\Delta \cap \tilde{C} \neq \emptyset$, which is a contradiction to the configuration of Γ. Therefore this point p is non-singular. Q.E.D.

In order to prove the structure theorem of the essential part of 3-dimensional isolated strictly log-canonical singularities of index 1 and of type 1), one need the following lemmas.

Lemma 6.6. *Let X_i $(i = 1, 2)$ be non-singular 3-folds, E an irreducible non-singular divisor with $K_{X_1} = -E$, C a non-singular curve on E. and $f : X_2 \to X_1$ a proper birational morphism isomorphic away from C. Denote the proper transform of E by E' and represent*

$$K_{X_2} = -E' + \sum_{F_j : f-\text{exceptional}} m_j F_j.$$

Then $m_j \geq 0$ for an irreducible component F_j with $f(F_j) = C$, and $m_j = 0$ for such F_j with moreover $f(F_j \cap E') = C$.

Proof. By replacing X_1 by a small analytic neighbourhood of a point on C, one obtains a smooth morphism $\pi : X_1 \to \Delta \subset \mathbb{C}$ such that $H_t \cap C$ is one point $\{p_t\}$, where $H_t = \pi^{-1}(t)$ for $t \in \Delta$. Denote $f^{-1}(H_t)$ by $\tilde{H}_t$, $\tilde{H}_t \cap E'$ by $\tilde{e}$ and $H_t \cap E$ by e. Then for a general $t \in \Delta$, $\tilde{H}_t$ is irreducible, non-singular and the intersection $\tilde{H}_t \cap F_j = e_j$ is a reduced curve for F_j with $f(F_j) = C$. Therefore by $K_{X_1}|_{H_t} = K_{H_t}$ and $K_{X_2}|_{\tilde{H}_t} = K_{\tilde{H}_t}$, it follows that

$$K_{H_t} = -e,$$

$$K_{\tilde{H}_t} = -\tilde{e} + \sum_{f(F_j)=C} m_j e_j.$$

Here $f|_{\tilde{H}_t} : \tilde{H}_t \to H_t$ is a proper birational morphism between non-singular surfaces, therefore the composite of blowing-ups at points. Hence $m_j \geq 0$ for all e_j and $m_j = 0$ for e_j with $e_j \cap \tilde{e} \neq \emptyset$. Q.E.D.

Lemma 6.7. *Let X_i $(i = 1, 2)$ be non-singular 3-folds, E_1 and E_2 irreducible non-singular divisors which cross normally at a curve C and $K_{X_1} = -E_1 - E_2$. Let $f : X_2 \to X_1$ be a proper birational morphism such that $E_1' \cap E_2' = \emptyset$ and $E_1' + E_2' + \sum F_j$ is of normal crossings, where E_i''s are the proper transforms of E_i's and F_j's are exceptional divisors. Represent*

$$K_{X_2} = -E_1' - E_2' + \sum m_j F_j.$$

Then there exist ruled surfaces $F_1, \ldots, F_r$ over C such that $E_1' + F_1 + \ldots + F_r + E_2'$ is a chain whose intersection curves are all sections of F_j's and $m_j = -1$ for $j = 1, \ldots, r$, $m_j \geq 0$ for $j \neq 1, \ldots, r$ and $f(F_j) = C$.

Proof. Take the same π as in the previous lemma and use the same notation H_t, $\tilde{H}_t$, p_t, e_j. Denote $H_t \cap E_i$ by e_i and $\tilde{H}_t \cap E'_i$ by e'_i. Then for general $t \in \Delta$,

$$K_{H_t} = -e_1 - e_2,$$
$$K_{\tilde{H}_t} = -e'_1 - e'_2 + \sum_{f(F_j)=C} m_j e_j.$$

Since $f|_{\tilde{H}_t}$ is a composite of blowing-ups at points, there exist $e_1, \dots, e_r$ such that $\{e'_1, e_1, \dots, e_r, e'_2\}$ forms a chain of rational curves in some order and $m_j = -1$ for $j = 1, \dots, r$ and $m_j \geq 0$ for $j \neq 1, \dots, r$ such that $f(F_j) = C$. For the assertion on F_j's ($j = 1, \dots, r$), note first that the general fiber of $F_j \to C$ is a disjoint union of non-singular rational curves, therefore F_j is a ruled surface. Next take F_1 such that $F_1 \cap E'_1 \neq \emptyset$. Then $f|_{F_1} : F_1 \to C$ is the projection of ruled surface, because F_1 intersects E'_1 at a curve isomorphic to C. Therefore e_1 is irreducible. Then take F_2 such that $F_1 \cap F_2 \neq \emptyset$. If $F_1 \cap F_2$ is not a section of $f|_{F_1}$, $e_1 \cap e_2$ consists of more than one point, which contradicts to that $\{e'_1, e_1, e_2, \dots\}$ forms a chain. So $f|_{F_2} : F_2 \to C$ has a section $F_1 \cap F_2$, which shows that it is a projection of a ruled surface over C and e_2 is irreducible. Inductively one obtains the assertion on F_j's for $j = 1, \dots, r$. Q.E.D.

Theorem 6.8. *Let (X, x) be a 3-dimensional isolated strictly log-canonical singularity of index 1 and of type 1) and a finite group G act on (X, x). Then either:*

(i) *there is a G-equivariant good resolution $f : \tilde{X} \to X$ such that the essential divisor E_J is a cycle $E_1 + E_2 + \dots + E_s$, ($s \geq 2$) of elliptic ruled surfaces, where E_i and E_{i+1} intersect at a section on each component for $i = 1, \dots, s$ ($E_{s+1} = E_1$) or*

(ii) *there is a G-equivariant good resolution $f : \tilde{X} \to X$ such that the essential divisor E_J contains a G-invariant chain $E^{(0)} = E_1 + \dots + E_s$ ($s \geq 1$) of elliptic ruled surfaces, where E_i and E_{i+1} intersect at a section on each component for $i = 1, \dots, s-1$. There are mutually disjoint subdivisors $E^{(-)}$ and $E^{(+)}$ of E_J such that $E_J = E^{(-)} + E^{(0)} + E^{(+)}$, where $E^{(-)} \cap E^{(0)}$ is a section of E_1 and $E^{(+)} \cap E^{(0)}$ is a section of E_s.*

Proof. Let $g : Y \to X$ be a G-equivariant $G\mathbb{Q}$-factorial terminal model of (X, x) and D the reduced inverse image $g^{-1}(x)_{red}$.

Assume that D is as in (i-1) of Proposition 6.5. Then there are two simple elliptic singularities p_1, p_2 on D. Take a G-equivariant good resolution $f : \tilde{X} \to Y$ and denote the proper transform of D by E and f-exceptional divisors by F_j's. Represent $K_{\tilde{X}} = -E + \sum m_j F_j$. Since

$m_j = 0$ for non-$f|_E$-exceptional curve $F_j|_E$ by Lemma 6.6, it follows that

$$(6.8.1) \qquad K_E = \sum_{F_j : f|_E-\text{exceptional}} m_j F_j|_E.$$

On the other hand, recall that $K_{\tilde{D}} = -C_1 - C_2$, where C_i's are the fibers of the simple elliptic singularities and disjoint sections of the elliptic ruled surface. Hence denoting the proper transform of C_i by $\tilde{C}_i$ and the canonical morphism $E \to \tilde{D}$ by ψ, one obtains:

$$(6.8.2) \qquad K_E = -\tilde{C}_1 - \tilde{C}_2 + \sum_{e_j : \psi-\text{exceptional}} n_j e_j,$$

where $n_j \geq 0$ because $\psi : E \to \tilde{D}$ is a composite of blowing-ups at points.

Noting that an $f|_E$-exceptional divisor is either $\tilde{C}_i$ or ψ-exceptional, compare (6.8.1) and (6.8.2). Then one obtains that there are components F_1 and F_2 such that $F_i|_E = \tilde{C}_i$ with $m_1 = m_2 = -1$ and $m_j \geq 0$ for every F_j $(j \neq 1, 2)$. Let $E^{(-)}$ be the sum of the essential components in $f^{-1}(p_1)$, $E^{(+)}$ that in $f^{-1}(p_2)$. If one puts $E^{(0)} = E$, then these satisfy the condition in (ii) of the theorem.

Assume that D is as in (i-2) of Proposition 6.5. In the same way as above, one obtains that there exists only one essential component F which intersects E. Since the intersection curve $F \cap E$ is G-invariant elliptic curve, one obtains another G-equivariant good resolution with the properties in (ii) of the theorem by compositing the blowing-up at $F \cap E$. In this case, the exceptional divisor of the blowing-up becomes $E^{(0)}$.

Assume D is as in (i-3) of Proposition 6.5. Let $f : \tilde{X} \to Y$ be a G-equivariant good resolution passing through the blowing-up at the double curve C which is G-invariant. Denote the proper transform of D on $\tilde{X}$ by E and the elliptic curve on E corresponding to C by $\tilde{C}$. Then there exists an f-exceptional curve F_1 such that $F_1|_E = \tilde{C}$. Represent $K_{\tilde{X}} = -E + \sum m_j F_j$. Then by Lemma6.6, it follows that

$$(6.8.3) \qquad K_E = m_1 F_1|_E + \sum_{F_j : f|_E-\text{exceptional}} m_j F_j|_E.$$

Since $K_{\tilde{D}} = -C'$, where C' is an elliptic curve corresponding to the double curve C, it follows that

$$(6.8.4) \qquad K_E = -\tilde{C} + \sum_{e_j : \psi-\text{exceptional}} n_j e_j.$$

Here one obtains $n_j \geq 0$, because $\psi : E \to \tilde{D}$ is a composite of blowing-ups at points. Noting that an $f|_E$ -exceptional curve is ψ-exceptional, compare (6.8.3) and (6.8.4). Then it follows that $m_1 = -1$ and $m_j \geq 0$ for $j \neq 1$ such that $F_j|_E \neq \emptyset$. Therefore there exists only one essential component F_1 which intersects E and the intersection $F_1 \cap E$ is G-equivariant elliptic curve. By taking the blowing-up at $F_1 \cap E$, one obtains $E^{(0)}$ which satisfies the conditions in (ii) of the theorem

Assume that D is as in (i-4) or (i-5) of Proposition 6.5. In the same way as in (i-1), one obtains that the conditions in (ii) of the theorem hold by denoting the proper transform of D by $E^{(0)}$.

Asssume that D is as in (i-6) or (ii-1) of Proposition 6.5. Take a G-equivariant good resolution $f : \tilde{X} \to Y$, decompose D into the irreducible components $D_1 + \ldots + D_s$ $(s \geq 1)$ and denote the proper transform of D_i on $\tilde{X}$ by E_i. By Lemma 6.7, the essential divisor E_J on $\tilde{X}$ contains a subdivisor E'_J with the property in (i) of the theorem. Represent $E'_J = \sum_{i=1}^{s} E_i + \sum_{j=1}^{t} F_j$. Let F be an f-exceptional divisor not contained in E'_J. Suppose first $F|_{F_j} \neq \emptyset$ for some $j = 1, \ldots, t$. If $f(F|_{F_j})$ is a point, then it is contained in a fiber of the ruling of F_j, therefore it is rational. Then by (ii) of Lemma 6.2, F is not an essential component. If $f(F|_{F_j})$ is a curve, then by Lemma 6.7, F is not essential. Next suppose that $F|_{E_i} \neq \emptyset$ for some $i = 1, \ldots, s$. If $F|_{E_i}$ is $f|_{E_i}$-exceptional, then it is rational, because the singularities on the normalization D'_i of D_i are all rational by (iii) of Proposition 6.5. Therefore by (ii) of Lemma 6.2, F is not essential. If $F|_{E_i}$ is not $f|_{E_i}$-exceptional, then by Lemma 6.6, F is not essential. Now it follows that $E_J = E'_J$ by connectedness of the essential divisor.

Assume that D is as in (ii-2) of Proposition 6.5. Decompose D into irreducible components $D_1 + \ldots + D_s$. For the case $s = 2$, by taking the blowing-up at $D_1 \cap D_2$ one can reduce into the case $s = 3$. So one may assume that $s \geq 3$. Let $f : \tilde{X} \to Y$ be a G-equivariant good resolution and E_i the proper transform of D_i. Then by Lemma 6.7 in the essential divisor E_J on $\tilde{X}$ there exists a chain of elliptic ruled surfaces starting with E_2, including E_i $(2 < i < s-1)$ and finishing with E_{s-1} such that the intersection curves are all sections on ruled surfaces. Note that this chain is G-invariant, because $D_2 + \ldots + D_{s-1}$ is G-invariant. In the same way as in the case (ii-1), one obtains that there are only two essential components which intersect this chain, and the intersection is sections of E_2 and of E_{s-1}. Denote this chain by $E^{(0)}$ and the sum of the essential components in $f^{-1}(D_1)$ by $E^{(-)}$ and that in $f^{-1}(D_s)$ by $E^{(+)}$. Then these satisfy the conditions in (ii) of the theorem. Q.E.D.

References

[1] H. Clemens, J. Kollár, S. Mori, *Higher Dimensional Complex Geometry*, Astérisque **166**, (1988) Société Mathématique de France.

[2] M. Demazure, *Anneaux Gradués normaux*, Introduction a la théorie des singularités II; Methodes algebriques et géométriques, Travaux En Cours **37**, (1988) 35–68 Paris Herman.

[3] A. Fujiki, *On the blowing down of analytic spaces*, Publ. RIMS, Kyoto Univ. **10**, (1975) 473–507.

[4] A. Fujiki, *Finite automorphism groups of complex tori of dimension two*, Publ. RIMS, Kyoto Univ. **24**, (1988) 1–97.

[5] R. Hartshorne, *Algebraic Geometry*, Graduate Text of Mathematics **52**, (1977) Springer-Verlag, New York-Heidelberg-Berlin.

[6] S. Ishii, *On isolated Gorenstein singularities*, Math. Ann. **270**, (1985) 541–554.

[7] S. Ishii, *Isolated $\mathbb{Q}$-Gorenstein singularities of dimension three*, Advanced Studies in Pure Math. **8**, Complex Analytic Singularities (1986) 165–198.

[8] S. Ishii & K. Watanabe, *A geometric characterization of a simple $K3$-singularity*, Tohoku Math. J. **44**, (1992) 19–24.

[9] S. Ishii, *On Fano 3-folds with non-rational singularities and two-dimensional base*, Abh. Math. Sem. Univ. Hamburg **64**, (1994) 249–277.

[10] J. Kollár, *Singularities of pairs*, Proceeding of Symposia in Pure Math. **62.1**, (1997) 221–287.

[11] S. Kondo, *Automorphisms of algebraic $K3$-surfaces which act trivially on Picard groups*, J. Math. Soc. Japan **44**, (1992) 75–98.

[12] V. Kulikov, *Degenerations of $K3$-surfaces and Enriques surfaces*, Math. USSR Izv. **11**, (1977) 957–989.

[13] N. Machida & K. Oguiso, *On $K3$-surfaces admitting finite non-symplectic group actions*, J. Math. Sci. Univ. Tokyo **5**, (1998) 273–297.

[14] S. Mori, *Flip theorem and the existence of minimal models for 3-folds*, J. Amer. Math. Soc. **1**, (1988) 117–253.

[15] S. Nakamura, *A classification of dual graphs of log-canonical singularities and local uniformization of quasi-regular log-canonical singularities*, Master Thesis, Saitama University (1989) (in Japanese).

[16] V.V. Nikulin, *Factor groups of automorphisms of hyperbolic forms with respect to subgroups generated by 3-reflections. Algebrogeometric applications*, J. Soviet Math., **22**, (1983) 1401–1476.

[17] K. Oguiso, *A remark on the global indices of $\mathbb{Q}$-Calabi-Yau 3-folds*, Math. Proc. Camb. Phil. Soc. **114**, (1993) 427–429.

[18] T. Okuma, *The pluri-genera of surface singularities*, Tohoku Math. J. **50**, (1998) 119–132.

[19] F. Sakai, *Weil divisors on normal surfaces*, Duke Math. J. **51**, (1984) 877–887.

[20] N.I. Shepherd-Barron, *Degenerations with numerical effective canonical divisor*, The Birational Geometry of Degenerations. ed. by R. Friedman, D. R. Morrison, Progress in Mathematics, Birkhäuser (1983) 33–84.
[21] V.V. Shokurov, 3-*fold log flips.* Russian Acad. Sci. Izv. Math. **40**, (1993) 95–202.
[22] V.V. Shokurov, *Complement on surfaces*, to appear in J. of Math. Sci.
[23] Y. Umezu, *On normal projective surfaces with trivial dualizing sheaf*, Tokyo J. Math. **4**, (1981) 343–354.

Department of Mathematics
Tokyo Institute of Technology
Oh-Okayama, Meguro, Tokyo
Japan