Advanced Studies in Pure Mathematics 28, 2000
Combinatorial Methods in Representation Theory
pp. 133–142

Minor Summation Formulas of Pfaffians, Survey and A New Identity

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Abstract.

In this paper we treat the minor summation formulas of pfaffians presented in [IW1] and derive several basic formulas concerning pfaffians from it. We also present a pfaffian version of the Plücker relation and give a new pfaffian identity as its application.

Chapter I. Introduction

In this short note we treat the minor summation formulas of pfaffians presented in [IW1] and derive several basic formulas concerning pfaffians. We also present a pfaffian version of the Plücker relations and give a new pfaffian identity as its application in Chapter III.

The minor summation formula we call here is an identity which involves pfaffians for a weighted sum of minors of a given matrix. The first appearance of this kind of minor sum is when one tries to count the number of the totally symmetric plane partitions (see [O1]). Once we establish the minor summation formula full in general, one gets various applications (see, e.g., [IOW], [KO], [O2]). Indeed, for example, using the minor summation formula we obtained quite a number of generalizations of the Littlewood formulas concerning various generating functions of the Schur polynomials (see [IW2,3,4]).

Though the notion of pfaffians is less familiar than that of determinants it is also known by a square root of the determinant of a skew

Received March 1, 1999.
2000 *Mathematics Subjects Classification.* Primary 05A15, 15A15; Secondary 22E46, 33C45.
Key words and phrases. Pfaffian, generating function, Schur's polynomial, partition, Plučker's relation, Lewis-Carroll's formula, Frobenius notation.
[1]Partially supported by Grant-in-Aid for Scientific Research (C) No.09640037, the Ministry of Education, Science, Sports and Culture of Japan.
[2]Partially supported by Grant-in-Aid for Scientific Research (B) No.09440022, the Ministry of Education, Science, Sports and Culture of Japan.

symmetric matrix. We recall now a more combinatorial definition of pfaffians. Let $\mathfrak{S}_n$ be the symmetric group on the set of the letters $1, 2, \ldots, n$ and, for each permutation $\sigma \in \mathfrak{S}_n$, let $\operatorname{sgn}\sigma$ stand for $(-1)^{\ell(\sigma)}$, the sign of σ, where $\ell(\sigma)$ is the number of inversions of σ.

Let $n = 2r$ be even. Let H be the subgroup of $\mathfrak{S}_n$ generated by the elements $(2i-1, 2i)$ for $1 \le i \le r$ and $(2i-1, 2i+1)(2i, 2i+2)$ for $1 \le i < r$. We set a subset $\mathfrak{F}_n$ of $\mathfrak{S}_n$ to be

$$\mathfrak{F}_n = \left\{ \sigma = (\sigma_1, \ldots, \sigma_n) \in \mathfrak{S}_n \;\middle|\; \begin{array}{l} \sigma_{2i-1} < \sigma_{2i} \ (1 \le i \le r) \\ \sigma_{2i-1} < \sigma_{2i+1} \ (1 \le i \le r-1) \end{array} \right\}.$$

An element of $\mathfrak{F}_n$ is called a *perfect* matching or a 1-*factor*. For each $\pi \in \mathfrak{S}_n$, $H\pi \cap \mathfrak{F}_n$ has a unique element σ. Let $B = (b_{ij})_{1 \le i,j \le n}$ be an n by n skew-symmetric matrix with entries b_{ij} in a commutative ring. The *pfaffian* of B is then defined as follows:

$$\operatorname{pf}(B) = \sum_{\sigma \in \mathfrak{F}_n} \operatorname{sgn}\sigma\, b_{\sigma(1)\sigma(2)} \cdots b_{\sigma(n-1)\sigma(n)}. \tag{1.1}$$

Chapter II. Pfaffian Identities

Let us denote by $\mathbb{N}$ the set of nonnegative integers, and by $\mathbb{Z}$ the set of integers. Let $[n]$ denote the subset $\{1, 2, \ldots, n\}$ of $\mathbb{N}$ for a positive integer n.

Let n, M and N be positive integers such that $n \le M, N$ and let T be any M by N matrix. For n-element subsets $I = \{i_1 < \cdots < i_n\} \subseteq [M]$ and $J = \{j_1 < \cdots < j_n\} \subseteq [N]$ of row and column indices, let $T_J^I = T_{j_1 \ldots j_n}^{i_1 \ldots i_n}$ denote the sub-matrix of T obtained by picking up the rows and columns indexed by I and J. In the case that $n = M$ and I contains all row indices, we omit $I = [M]$ from the above expression and simply write $T_J = T_J^I$. Similarly we write T^I for T_J^I if $n = N$ and $J = [N]$.

Let B be an arbitrary N by N skew symmetric matrix; that is, $B = (b_{ij})$ satisfies $b_{ij} = -b_{ji}$. In Theorem 1 of the paper [IW1], we obtained a formula concerning a certain summation of minors which we call the minor summation formula of pfaffians:

Theorem 2.1. *Let $n \le N$ and assume n is even. Let $T = (t_{ij})_{1 \le i \le n, 1 \le j \le N}$ be any n by N matrix, and let $B = (b_{ij})_{1 \le i,j \le N}$ be any N by N skew symmetric matrix. Then*

$$\sum_{\substack{I \subset [N] \\ \sharp I = n}} \operatorname{pf}(B_I^I) \det(T_I) = \operatorname{pf}(Q), \tag{2.1}$$

where Q is the n by n skew-symmetric matrix defined by $Q = TB\,{}^tT$, i.e.

$$Q_{ij} = \sum_{1 \le k < l \le N} b_{kl} \det(T^{ij}_{kl}), \qquad (1 \le i, j \le n). \tag{2.2}$$

We note that another proof of this minor summation formula and some other extensions using the so-called lattice path methods will be given in the forthcoming paper [IW5].

We now add on one useful formula which relates to the skew symmetric part of a general square matrix. Actually the following type of pfaffians may arise naturally when we consider the imaginary part of a Hermitian form.

Corollary 2.1. *Fix positive integers m, n such that $m \le 2n$. Let A and B be arbitrary $n \times m$ matrices, and X be an $n \times n$ symmetric matrix. (i.e. ${}^tX = X$). Let P be the skew symmetric matrix defined by $P = {}^tAXB - {}^tBXA$. Then we have*

$$\operatorname{pf}(P) = \sum_{\substack{K \subseteq [2n] \\ \sharp K = m}} \operatorname{pf}\left(\begin{pmatrix} O_n & X \\ -X & O_n \end{pmatrix}^K_K \right) \det\left(\begin{pmatrix} A \\ B \end{pmatrix}^K \right).$$

In particular, when $m = 2n$ we have

$$\operatorname{pf}(P) = \det(X) \det\left(\begin{pmatrix} A \\ B \end{pmatrix} \right).$$

Proof. Apply the above theorem to the $2n \times 2n$ skew symmetric matrix $\begin{pmatrix} O_n & X \\ -X & O_n \end{pmatrix}$ and the $2n \times m$ matrix $\begin{pmatrix} A \\ B \end{pmatrix}$. Then the elementary identity

$${}^t\begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} O_n & X \\ -X & O_n \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = {}^tAXB - {}^tBXA$$

immediately asserts the corollary.

As a corollary of the theorem above we have the following expansion formula (cf. [Ste], [IW1]):

Corollary 2.2. *Let A and B be m by m skew symmetric matrices. Put $n = [\frac{m}{2}]$, the integer part of $\frac{m}{2}$. Then*

$$\operatorname{pf}(A + B) = \sum_{r=0}^{n} \sum_{\substack{I \subseteq [m] \\ \sharp I = 2r}} (-1)^{|I| - r} \operatorname{pf}(A^I_I) \operatorname{pf}(B^{\overline{I}}_{\overline{I}}), \tag{2.3}$$

where we denote by $\overline{I}$ the complement of I in $[m]$ and $|I|$ is the sum of the elements of I (i.e. $|I| = \sum_{i \in I} i$).

In particular, we have the expansion formula of pfaffian with respect to any column (row): For any i, j we have

$$\delta_{ij}\,\mathrm{pf}(A) = \sum_{k=1}^{m} a_{ki}\gamma(k,j), \tag{2.4}$$

$$\delta_{ij}\,\mathrm{pf}(A) = \sum_{k=1}^{m} a_{ik}\gamma(j,k), \tag{2.5}$$

where

$$\gamma(i,j) = \begin{cases} (-1)^{i+j-1}\,\mathrm{pf}(A^{[ij]}) & \text{if } i < j, \\ 0 & \text{if } i = j, \\ (-1)^{i+j}\,\mathrm{pf}(A^{[ij]}) & \text{if } j < i. \end{cases} \tag{2.6}$$

and $A^{[ij]}$ stands for the $(m-2)$ by $(m-2)$ skew symmetric matrix which is obtained from A by removing both the i, j-th rows and i, j-th columns for $1 \leq i \neq j \leq m$.

We close this chapter by noting the fact that one may give a proof of the fundamental relation; $\mathrm{pf}(A)^2 = \det(A)$, for a skew symmetric matrix A without any use of a process of the "diagonalization" by employing the expansion formula above and the Lewis-Carroll formula for determinants discussing below.

Chapter III. The Lewis-Carroll formula, etc.

In this chapter we provide a Pfaffian version of Lewis-Carroll's formula and Plücker's relations. The latter relations are also treated in [DW], and in [Kn] it is called the (generalized) basic identity. First of all we recall the so-called Lewis-Carroll formula, or known as the Jacobi formula among minor determinants.

Proposition 3.1. *Let A be an n by n matrix and $\widetilde{A}$ be the matrix of its cofactors. Let $r \leq n$ and $I, J \subseteq [n]$, $\sharp I = \sharp J = r$. Then*

$$\det \widetilde{A}_I^J = (-1)^{r(|I|+|J|)}(\det A)^{r-1}\ \det A_{\overline{J}}^{\overline{I}}, \tag{3.1}$$

where $\overline{I}, \overline{J} \subseteq [n]$ stand for the complementary of I, J, respectively.

Example 1. We give here a few examples of Lewis-Carroll's formula for matrices of small degree.

$$\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{3.2}$$

We give one more;

$$\begin{aligned} &\begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} \\ &+ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} = a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}. \end{aligned} \tag{3.3}$$

Hereafter we write A_I for A_I^I for short. We hope that it doesn't cause the reader any confusion since we only treat square matrices. Let m be an even integer and A be an m by m skew symmetric matrix. Assume that $\mathrm{pf}(A)$ is nonzero, that is, A is non-singular.

Let $\Delta(i,j) = (-1)^{i+j} \det A^{ij}$ denote the (i,j)-cofactor of A. If we multiply the both sides of (2.6) by $\mathrm{pf}(A)$ and use the fundamental relation between determinants and pfaffians: $\det A = [\mathrm{pf}(A)]^2$, we obtain

$$\sum_{i=1}^{m} a_{ij}\gamma(i,k)\ \mathrm{pf}(A) = \delta_{jk}\ [\mathrm{pf}(A)]^2 = \delta_{jk}\ \det A. \tag{3.4}$$

Comparing this with the cofactor expansion of $\det A$, we obtain the following relation between $\Delta(i,j)$ and $\gamma(i,j)$:

$$\Delta(i,j) = \gamma(i,j)\ \mathrm{pf}(A). \tag{3.5}$$

The following relation is considered as a pfaffian version of the Lewis-Carroll formula.

Theorem 3.1. *Let m be an even integer and A be an m by m skew symmetric matrix. Let $\widehat{A} = (\gamma(j,i))$. Then, for any $I \subseteq [m]$ such that $\sharp I = 2r$, we have*

$$\mathrm{pf}\left[(\widehat{A})_I\right] = (-1)^{|I|}\,[\mathrm{pf}(A)]^{r-1}\,\mathrm{pf}(A_{\overline{I}}). \tag{3.6}$$

Example 2. Taking $m = 6$, $t = 1$ and $I = \{1,2,3,4\}$ in the above theorem, we see

$$\gamma(1,2)\gamma(3,4) - \gamma(1,3)\gamma(2,4) + \gamma(1,4)\gamma(2,3) = \mathrm{pf}(A)\,\mathrm{pf}(A_{\{5,6\}}).$$

Hence by definition, we see that this turns out to be

$$\begin{aligned}\mathrm{pf}(A_{\{3,4,5,6\}})\,\mathrm{pf}(A_{\{1,2,5,6\}}) &- \mathrm{pf}(A_{\{2,4,5,6\}})\,\mathrm{pf}(A_{\{1,3,5,6\}}) \\ &+ \mathrm{pf}(A_{\{2,3,5,6\}})\,\mathrm{pf}(A_{\{1,4,5,6\}}) = \mathrm{pf}(A)\,\mathrm{pf}(A_{\{5,6\}}),\end{aligned} \tag{3.7}$$

that is, in more familiar form we see

$$\begin{aligned}
&\mathrm{pf}\begin{pmatrix} 0 & a_{34} & a_{35} & a_{36} \\ -a_{34} & 0 & a_{45} & a_{46} \\ -a_{35} & -a_{45} & 0 & a_{56} \\ -a_{36} & -a_{46} & -a_{56} & 0 \end{pmatrix} \mathrm{pf}\begin{pmatrix} 0 & a_{12} & a_{15} & a_{16} \\ -a_{12} & 0 & a_{25} & a_{26} \\ -a_{15} & -a_{25} & 0 & a_{56} \\ -a_{16} & -a_{26} & -a_{56} & 0 \end{pmatrix} \\
&-\mathrm{pf}\begin{pmatrix} 0 & a_{24} & a_{25} & a_{26} \\ -a_{24} & 0 & a_{45} & a_{46} \\ -a_{25} & -a_{45} & 0 & a_{56} \\ -a_{26} & -a_{46} & -a_{56} & 0 \end{pmatrix} \mathrm{pf}\begin{pmatrix} 0 & a_{13} & a_{15} & a_{16} \\ -a_{13} & 0 & a_{35} & a_{36} \\ -a_{15} & -a_{35} & 0 & a_{56} \\ -a_{16} & -a_{36} & -a_{56} & 0 \end{pmatrix} \\
&+\mathrm{pf}\begin{pmatrix} 0 & a_{23} & a_{25} & a_{26} \\ -a_{23} & 0 & a_{35} & a_{36} \\ -a_{25} & -a_{35} & 0 & a_{56} \\ -a_{26} & -a_{36} & -a_{56} & 0 \end{pmatrix} \mathrm{pf}\begin{pmatrix} 0 & a_{14} & a_{15} & a_{16} \\ -a_{14} & 0 & a_{45} & a_{46} \\ -a_{15} & -a_{45} & 0 & a_{56} \\ -a_{16} & -a_{46} & -a_{56} & 0 \end{pmatrix} \\
&= \mathrm{pf}\begin{pmatrix} 0 & a_{56} \\ -a_{56} & 0 \end{pmatrix} \mathrm{pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 \end{pmatrix}.
\end{aligned}$$

We next state a pfaffian version of the Plücker relations (or known as the Grassmann-Plücker relations) for determinants which is a quadratic relations among several subpfaffians. This identity is also proved in the book [Hi] and a recent paper [DW] in the framework of an exterior algebra.

Theorem 3.2. *Suppose m, n are odd integers. Let A be an $(m+n)\times(m+n)$ skew symmetric matrices of odd degrees. Fix a sequence of integers $I = \{i_1 < i_2 < \cdots < i_m\} \subseteq [m+n]$ such that $\sharp I = m$. Denote the complement of I by $\overline{I} = \{k_1, k_2, \ldots, k_n\} \subseteq [m+n]$ which has the* *cardinality n. Then the following relation holds.*

$$\sum_{j=1}^{m}(-1)^{j-1}\operatorname{pf}(A_{I\setminus\{i_j\}})\operatorname{pf}(A_{\{i_j\}\cup\overline{I}})=\sum_{j=1}^{n}(-1)^{j-1}\operatorname{pf}(A_{I\cup\{k_j\}})\operatorname{pf}(A_{\overline{I}\setminus k_j}). \tag{3.8}$$

The following assertion, which is called by the basic identity in [Kn] is a special consequence of the formula above.

Corollary 3.1. *Let A be a skew symmetric matrix of degree N. Fix a subset $I=\{i_1,i_2,\dots,i_{2k}\}\subseteq[N]$ such that $\sharp I=2k$. Take an integer l which satisfies $2k+2l\le N$. Then*

$$\begin{aligned}&\operatorname{pf}(A_{1,2,\dots,2l})\operatorname{pf}(A_{i_1,i_2,\dots,i_{2k},1,\dots,2l})\\&=\sum_{j=1}^{2k-1}(-1)^{j-1}\operatorname{pf}(A_{i_1,1,2,\dots,2l,i_{j+1}})\operatorname{pf}(A_{i_2,\dots,\widehat{i_{j+1}},\dots,i_{2k},1,\dots,2l}).\end{aligned}\tag{3.9}$$

The theorem stated below is proved by induction using this basic identity. Its proof will be given in the forthcoming paper [IW5].

Theorem 3.3.

$$\begin{aligned}&\operatorname{pf}\left(\frac{y_i-y_j}{a+b(x_i+x_j)+cx_ix_j}\right)_{1\le i,j\le 2n}\times\prod_{1\le i<j\le 2n}\{a+b(x_i+x_j)+cx_ix_j\}\\&=(ac-b^2)^{\frac{n(n-1)}{2}}\sum_{\substack{I\subseteq[2n]\\ \sharp I=n}}(-1)^{|I|-\frac{n(n+1)}{2}}y_I\Delta_I(x)\Delta_{\overline{I}}(x)J_I(x)J_{\overline{I}}(x),\end{aligned}$$

where the sum runs over all n-element subset $I=\{i_1<\cdots<i_n\}$ of $[2n]$ and $\overline{I}=\{j_1<\cdots<j_n\}$ is the complementary subset of I in $[2n]$. Further we write

$$\Delta_I(x)=\prod_{\substack{i,j\in I\\ i<j}}(x_i-x_j),$$

$$J_I(x)=\prod_{\substack{i,j\in I\\ i<j}}\{a+b(x_i+x_j)+cx_ix_j\},$$

$$y_I=\prod_{i\in I}y_i.$$

As a corollary of this theorem we obtain the following identity in [Su2]. Indeed, if we put $a = c = 1, b = 0$ in the theorem, then we have the

Corollary 3.2.

$$\mathrm{pf}\left(\frac{y_i - y_j}{1 + x_i x_j}\right)_{1\le i,j\le 2n} \times \prod_{1\le i<j\le 2n} (1 + x_i x_j) = \sum_{\lambda,\mu} a_{\lambda+\delta_n,\mu+\delta_n}(x, y),$$

where the sums runs over pairs of partitions

$$\lambda = (\alpha_1 - 1, \cdots, \alpha_p - 1|\alpha_1, \cdots, \alpha_p), \mu = (\beta_1 - 1, \cdots, \beta_p - 1|\beta_1, \cdots, \beta_p)$$

in Frobenius notation with $\alpha_1, \beta_1 < n - 1$. Also, for α and β partitions (compositions, in general) of length n, we put

$$a_{\alpha,\beta}(x, y) = \sum_{\sigma\in\mathfrak{S}_{2n}} \epsilon(\sigma)\sigma(x_1^{\alpha_1} y_1 \cdots x_n^{\alpha_n} y_n x_{n+1}^{\beta_1} \cdots x_{2n}^{\beta_n}),$$

where $\sigma \in \mathfrak{S}_{2n}$ acts on each of two sets of variables $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$ by permuting indices, and $\delta_n = (n - 1, n - 2, \cdots, 0)$.

Proof. Recall that

$$\sum_{\lambda=(\alpha_1-1,\cdots,\alpha_p-1|\alpha_1,\cdots,\alpha_p)} s_\lambda(x_1, \cdots, x_n) = \prod_{1\le i<j\le n} (1 + x_i x_j), \tag{3.10}$$

where $s_\lambda = s_\lambda(x_1, \cdots, x_n) = a_{\lambda+\delta_n}/a_{\delta_n}$ and $a_\alpha = \det(x_i^{\alpha_j})_{1\le i,j\le n}$ for a composition α. We write $a_\alpha(I) = a_\alpha(x_{i_1}, \cdots, x_{i_n})$ for $I = \{i_1 < \cdots < i_n\} \subseteq [2n]$. By the theorem and (3.10) we see

$$\begin{aligned}
&\mathrm{pf}\left(\frac{y_i - y_j}{1 + x_i x_j}\right)_{1\le i,j\le 2n} \times \prod_{1\le i<j\le 2n} (1 + x_i x_j) \\
&= \sum_{\substack{I\subseteq[2n] \\ \sharp I=n}} \sum_{\lambda,\mu} (-1)^{|I|-\frac{n(n+1)}{2}} y_I a_{\lambda+\delta_n}(I) a_{\mu+\delta_n}(\overline{I}) \\
&= \sum_{\lambda,\mu} \sum_{i_1<\cdots<i_n} \sum_{\sigma,\tau\in\mathfrak{S}_n} (-1)^{|I|-\frac{n(n+1)}{2}} \epsilon(\sigma)\epsilon(\tau) \\
&\quad \times \sigma(x_{i_1}^{\lambda_1+n-1} y_{i_1} \cdots x_{i_n}^{\lambda_n} y_{i_n}) \tau(x_{j_1}^{\mu_1+n-1} \cdots x_{j_n}^{\mu_n}),
\end{aligned}$$

where $\overline{I} = \{j_1, \cdots, j_n\}$. Thus, the last sum is turned to be

$$= \sum_{\lambda,\mu} \sum_{\sigma,\tau \in \mathfrak{S}_{2n}} \epsilon(\sigma)\sigma(x_1^{\lambda_1+n-1} y_1 \cdots x_n^{\lambda_n} y_n x_{n+1}^{\mu_1+n-1} \cdots x_{2nn}^{\mu_n})$$

$$= \sum_{\lambda,\mu} a_{\lambda+\delta_n,\mu+\delta_n}(x,y).$$

This completes the proof of the corollary.

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