Advanced Studies in Pure Mathematics 24, 1996
Progress in Algebraic Combinatorics
pp. 285–308

Spin Models and Almost Bipartite 2-Homogeneous Graphs

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Abstract.

A connected graph of diameter d is said to be almost bipartite if it contains no cycle of length $2\ell+1$ for all $\ell < d$. An almost bipartite distance-regular graph $\Gamma = (X, E)$ is 2-homogeneous if and only if there are constants $\gamma_1, \ldots, \gamma_d$ such that $|\Gamma_{i-1}(u) \cap \Gamma_1(x) \cap \Gamma_1(y)| = \gamma_i$ holds for all $u \in X$ and for all $x, y \in \Gamma_i(u)$ with $\partial(x, y) = 2$ $(i = 1, \ldots, d)$.

In this paper, almost bipartite 2-homogeneous distance-regular graphs are classified. This determines triangle-free connected graphs affording spin models (for link invariants) with certain weights.

§1. Introduction

A spin model is one of the statistical mechanical models which were introduced by Vaughan Jones to construct invariants of knots and links [12]. A spin model is defined as a complex-valued symmetric function w on $X \times X$, where X is a finite set of "spins", satisfying several axioms. Each spin model S gives a corresponding link invariant through its partition function. Three examples of spin models are mentioned in Jones' paper [12]; Potts models, cyclic models and square models. It must be remarked that the Jones polynomial can be obtained from the partition function of the Potts models.

A connection between spin models and distance-regular graphs was found by François Jaeger [9] by constructing a new spin model on the Higman-Sims graph, a distance-regular graph of diameter $d = 2$ with $n = 100$ vertices, which was discovered by D. Higman and C. Sims [8], where we say that a spin model $S = (X, w)$ is constructed on a connected graph $\Gamma = (X, E)$ if $w(x, y)$ depends only on the distance $\partial(x, y)$ in the graph Γ. Jaeger [9] proved that the corresponding link invariant

Received April 10, 1995.
Revised June 7, 1995.

of the Higman-Sims model becomes a specialization of the Kauffman polynomial [14]. After Jaeger's discovery, a new infinite family of spin models were constructed on Hadamard graphs by the author [14]. The corresponding link invariants of the Hadamard models were determined by Jaeger [10,11], and then Jones [13] gave a pair of two links which can be detected by this invariant but not by Jones polynomial.

These examples of spin models can be constructed on almost bipartite distance-regular graphs. Moreover these graphs have extra regularity which we call 2-homogeneity; an almost bipartite distance-regular graph $\Gamma = (X, E)$ is 2-homogeneous if and only if $|\Gamma_{i-1}(u) \cap \Gamma_1(x) \cap \Gamma_1(y)|$ is a constant for all u, x, $y \in X$ with $\partial(u,x) = \partial(u,y) = i$, $\partial(x,y) = 2$ ($i = 1, \ldots, d$, where d denotes the diameter of Γ). In fact it was shown [21] that if a triangle-free connected graph affords a spin model with certain weights then the graph must be distance-regular and almost bipartite.

This paper contains two main results. At first we show that if a spin model is constructed on an almost bipartite distance-regular graph then the graph must be 2-homogeneous (under some conditions, see Theorem 4.3). Next we classify almost bipartite 2-homogeneous distance-regular graphs (Theorem 5.1). The proofs of these results are given in Section 4 and Section 5. In Section 2, some preliminaries on spin models and distance-regular graphs are given. In Section 3, two necessary and sufficient conditions (H1), (H2) for 2-homogeneity of almost bipartite distance-regular graphs are given. Then we slightly generalize of Yamazaki's sufficient condition for 2-homogeneity [22].

There are two generalizations of Jones' spin models by Kawagoe-Munemasa-Watatani [15] and Bannai-Bannai [1] (see also [2, 3, 16]). In this paper we restrict our interest to the original spin models defined in [12].

§2. Spin models and distance-regular graphs

2.1. Definition and examples of spin models

A *spin model* is a pair $S = (X, w)$ of a finite set of size $|X| = n > 0$ and a complex-valued function w on $X \times X$ such that (for all a, b, c in X)

$$\text{(S1)} \qquad w(a,b) = w(b,a) \neq 0,$$

$$\text{(S2)} \qquad \sum_{x \in X} w(a,x)w(b,x)^{-1} = n\delta_{a,b},$$

$$\text{(S3)} \quad \sum_{x \in X} w(a,x)w(b,x)w(c,x)^{-1} = \sqrt{n}\, w(a,b)w(a,c)^{-1}w(b,c)^{-1}.$$

The equation (S3) is called the "star-triangle" relation. The elements of X is called the *spins*, and the function w is called the (*Boltzmann*) *weight*. Putting $a = c$ in (S3), we have

$$\sum_{x \in X} w(b,x) = \sqrt{n}\, w(a,a)^{-1},$$

so that $w(a,a) = \alpha$ is a constant, called the *modulus* of S, which is independent of the choice of a in X.

The *weight matrix* of a spin model $S = (X, w)$, $|X| = n$, is a $n \times n$ matrix W, indexed by $X \times X$, whose (x,y)-entry is $W_{x,y} = w(x,y)$. For b, c in X, we consider a vector $\mathbf{u}_{bc}$ in the n-space $V = \mathbf{C}^n$, where the entries of the vectors are indexed by X, whose x-entry is given by

$$(\mathbf{u}_{bc})_x = \frac{w(b,x)}{w(c,x)}, \qquad (x \in X).$$

Then the condition (S3) can be written as

$$W\mathbf{u}_{bc} = \sqrt{n}\, w(b,c)^{-1}\mathbf{u}_{bc}.$$

This means the vector $\mathbf{u}_{bc}$ is an eigenvector of W for the eigenvalue $\sqrt{n}\, w(b,c)^{-1}$. It can be easily shown from (S2) that, for a fixed $b \in X$, the vectors $\mathbf{u}_{bc}$, $c \in X$ are linearly independent and hence form a basis of V. Therefore the values $\sqrt{n}\, w(b,c)^{-1}$, $c \in X$ give all the eigenvalues of W, where multiplicities are counted. This means that the multiplicity of an eigenvalue $\sqrt{n}\, \lambda^{-1}$ agrees with the number of $x \in X$ such that $w(b,x) = \lambda$ (thus this number does not depend on the choice of b). The vector $\mathbf{u}_{bb}$ becomes the all one vector $\mathbf{j}$, and it is an eigenvector of W corresponding the eigenvalue $\sqrt{n}\, \alpha^{-1}$ (α is the modulus). From condition (S2), the other vectors $\mathbf{u}_{bc}$, $b \neq c$ are orthogonal to $\mathbf{j}$.

Now we give three basic examples of spin models.

Potts model. Let X be a finite set with $n > 1$ elements. Let β be a solution of $\beta^2 + \beta^{-2} + \sqrt{n} = 0$ and put $\alpha = -\beta^{-3}$. Define a function w on $X \times X$ by

$$w(x,y) = \begin{cases} \alpha & x = y, \\ \beta & \text{otherwise.} \end{cases}$$

Then (X, w) is a spin model called the *Potts model* [12]. Potts model with $n = 2$ is also called the *Ising model.*

Cyclic model. Let $X = \{0, 1, \ldots, n-1\}$, and let θ be a primitive n-root of unity when n is odd, or a primitive $2n$-root of unity when n is even. Define a function w on $X \times X$ by

$$w(x,y) = \alpha\theta^{(x-y)^2},$$

where

$$\alpha^2 = \frac{\sqrt{n}}{\sum_{i=0}^{n-1} \theta^{i^2}}.$$

Then (X, w) becomes a spin model, called the *cyclic model* [2,6,12].

Square model. Let $X = \{1, 2, 3, 4\}$ and let α be an arbitrary non-zero complex number. Let us consider the following matrix:

$$W = \begin{pmatrix} \alpha & \alpha^{-1} & -\alpha & \alpha^{-1} \\ \alpha^{-1} & \alpha & \alpha^{-1} & -\alpha \\ -\alpha & \alpha^{-1} & \alpha & \alpha^{-1} \\ \alpha^{-1} & -\alpha & \alpha^{-1} & \alpha \end{pmatrix},$$

and define a function w on $X \times X$ by $w(x, y) = W_{x,y}$. Then (X, w) becomes a spin model, called the *square model* [7,12].

2.2. Preliminaries for distance-regular graphs

Let $\Gamma = (X, E)$ be a connected (undirected simple) graph of diameter d with the vertex set X and the edge set E with the usual metric ∂ on X. For vertices u, v and for integers r, s, define

$$\Gamma_r(u) = \{x \in X \mid \partial(u,x) = r\},$$

$$\mathrm{D}_s^r(u,v) = \Gamma_r(u) \cap \Gamma_s(v).$$

Γ is said to be *distance-regular* if there are integers b_r, c_r such that for any two vertices u, x at distance $r = \partial(u, x)$, there are precisely c_r neighbours of x in $\Gamma_{r-1}(u)$ and b_r neighbours of x in $\Gamma_{r+1}(u)$. In particular Γ is regular of valency $k = b_0$, and there are $a_r = k - c_r - b_r$ neighbours of x in $\Gamma_r(u)$. The parameters c_r, b_r, a_r $(r = 0, \ldots, d)$ satisfy (see [5], Proposition 4.1.6)

$$1 = c_1 \leq c_2 \leq \cdots \leq c_{d-1} \leq c_d,$$

$$k = b_0 \geq b_1 \geq \cdots \geq b_{d-1} \geq b_d = 0.$$

The array

$$\left\{ \begin{array}{cccccc} 0 & c_1 & c_2 & \cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\ k & b_1 & b_2 & \cdots & b_{d-1} & 0 \end{array} \right\}$$

is called the *intersection array* of Γ.

It is known (see [5], Section 4.1) that the parameters

$$p^t_{r,s} = |\mathrm{D}^r_s(u,v)|, \qquad (t = \partial(u,v))$$

are well-defined, i.e., these parameters depends only on r, s and $t = \partial(u,v)$, rather than on the individual vertices u, v with $t = \partial(u,v)$. The parameters $p^t_{r,s}$ are called the *intersection numbers* of Γ. Clearly $c_r = p^r_{r-1,1}$, $a_r = p^r_{r,1}$ and $b_r = p^r_{r+1,1}$ hold.

Let A_i $(i = 0, 1, \ldots, d)$ denote the i-th adjacency matrix of Γ, i.e., A_i is the $n \times n$ matrix, indexed by $X \times X$, whose (x,y)-entry is

$$(A_i)_{x,y} = \begin{cases} 1 & \partial(x,y) = i, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $A_0 = I$ the identity matrix of size n and $A_1 = A$ the usual adjacency matrix of Γ. The matrices $A_0, A_1, \ldots, A_d$ satisfy

$$A_i A_j = A_j A_i = \sum_{\ell=0}^{d} p^\ell_{ij} A_\ell.$$

In particular,

$$AA_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}$$

holds. Using this relation recursively, A_i can be written as a polynomial in A, i.e., there are polynomials $v_i(x)$ of degree i such that $A_i = v_i(A)$ holds for $i = 0, 1, \ldots, d$.

It is known that the adjacency matrix A has distinct eigenvalues $\theta_0 = k, \theta_1, \ldots, \theta_d$, and the corresponding eigenspaces $V_0, V_1, \ldots, V_d$ in $V = \mathbf{C}^n$ $(n = |X|)$ are mutually orthogonal (see [5], Section 4.1):

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_d \quad \text{(orthogonal sum).}$$

Remark that V_0 is the 1-dimensional subspace spanned by $\mathbf{j}$.

More precise descriptions of distance-regular graphs can be found in [4,5].

2.3. Spin models on distance-regular graphs

Let $\Gamma = (X, E)$ be a connected graph of diameter d with the usual metric ∂ on X. Let R_i $(i = 0, 1, \ldots, d)$ be the set of pairs (x, y) in $X \times X$ such that $\partial(x, y) = i$. Then $X \times X$ is partitioned into $d+1$ relations:

$$X \times X = R_0 \cup R_1 \cup \cdots \cup R_d.$$

We consider spin models $S = (X, w)$ such that w takes a constant value t_i on R_i $(i = 0, 1, \ldots, d)$, i.e., $w(x, y) = t_i$ holds for all x, y in X at distance $\partial(x, y) = i$. In this case we say that the spin model $S = (X, w)$ is constructed on the graph $\Gamma = (X, E)$. We are particularly interested in spin models which are constructed on distance-regular graphs.

For three vertices x, y, z and for integers i, j, ℓ, define

$$P_{i,j,\ell}(x, y, z) = |\Gamma_i(x) \cap \Gamma_j(y) \cap \Gamma_\ell(z)|.$$

Lemma 2.1. *Let $\Gamma = (X, E)$ be a distance-regular graph of diameter d with the intersection numbers $p^\ell_{i,j}$, and let $t_0, \ldots, t_d$ be non-zero complex numbers. Define a function w on $X \times X$ by $w(x, y) = t_{\partial(x,y)}$. Then $S = (X, w)$ is a spin model if and only if the following conditions hold:*

(S2') *For $\ell = 1, \ldots, d$,*

$$\sum_{i=0}^{d} \sum_{j=0}^{d} p^\ell_{i,j} t_i t_j^{-1} = 0,$$

(S3') *For all x, y, z in X,*

$$\sum_{i=0}^{d} \sum_{j=0}^{d} \sum_{\ell=0}^{d} P_{i,j,\ell}(x, y, z) t_i t_j t_\ell^{-1} = \sqrt{n}\, t_{\partial(x,y)} t^{-1}_{\partial(x,z)} t^{-1}_{\partial(y,z)}.$$

Proof. It is not difficult to show that (S2), (S3) are equivalent to (S2'), (S3') respectively. Remark that (S1) holds for a spin model constructed on a connected graph. Q.E.D.

Now we give two examples which are constructed on distance-regular graphs.

Jaeger's Higman-Sims model. The Higman-Sims graph, which was discovered by D. Higman and C. Sims [8], is the unique distance-regular graph $\Gamma = (X, E)$ of diameter $d = 2$ with the following intersection array:

$$\left\{ \begin{array}{ccc} 0 & 1 & 6 \\ 0 & 0 & 16 \\ 22 & 21 & 0 \end{array} \right\}.$$

Γ has $|X| = 100$ vertices.

A spin model was constructed on the Higman-Sims graph by F. Jaeger [9] (see also [7]). Let $\tau = (1 + \sqrt{5})/2$ and put

$$t_0 = (5\tau + 3)\sqrt{-1}, \quad t_1 = \tau\sqrt{-1}, \quad t_2 = (-\tau + 1)\sqrt{-1}.$$

Define a function w on $X \times X$ by $w(x, y) = t_{\partial(x,y)}$ for x, $y \in X$. Then $S = (X, w)$ becomes a spin model. The corresponding link invariant becomes a specialization of the Kauffman polynomial [7].

Hadamard model. Hadamard graphs are distance-regular graphs of diameter $d = 4$ with the following intersection array:

$$\left\{ \begin{array}{ccccc} 0 & 1 & 2m & 4m-1 & 4m \\ 0 & 0 & 0 & 0 & 0 \\ 4m & 4m-1 & 2m & 1 & 0 \end{array} \right\},$$

where m is a positive integer. There is a natural correspondence between Hadamard graphs of valency $4m$ and Hadamard matrices of size $4m$ (see [5], Theorem 1.8.1). Let s, t_0, t_1 be complex numbers such that

$$s^2 + 2(2m-1)s + 1 = 0, \qquad t_0^2 = \frac{2\sqrt{m}}{(4m-1)s+1}, \qquad t_1^4 = 1,$$

and put

$$t_2 = st_0, \quad t_3 = -t_1, \quad t_4 = t_1.$$

Define a function w on $X \times X$ by $w(x, y) = t_{\partial(x,y)}$ for x, $y \in X$. Then $S = (X, w)$ is a spin model [17]. The corresponding link invariants of these models were determined by Jaeger [10,11].

§3. 2-Homogeneous distance-regular graphs

3.1. Definition of 2-homogeneity

Let $\Gamma = (X, E)$ be a distance-regular graph of diameter d. For a vertex x in X and for a subset A of X, let $e(x, A)$ denote the number of edges from x into A; $e(x, A) = |\Gamma_1(x) \cap A|$. Γ is said to be *t-homogeneous* (where t is an non-negative integer) if the following condition holds for all integers r, s, i, j and for all vertices u, v, u', v' with $\partial(u,v) = \partial(u',v') = t$:

$$x \in \mathrm{D}^r_s(u,v),\ x' \in \mathrm{D}^r_s(u',v') \Longrightarrow e(x, \mathrm{D}^i_j(u,v)) = e(x', \mathrm{D}^i_j(u',v')).$$

This means that, for two vertices u, v at distance t and for x in $\mathrm{D}^r_s(u,v)$, the number of edges from x into $\mathrm{D}^i_j(u,v)$ depends only on r, s, i, j rather than on the individual vertices u, v, x with $\partial(u,v) = t$ and $x \in \mathrm{D}^r_s(u,v)$.

It was shown [18] that, for a distance-regular graph Γ of diameter d in which $\mathrm{D}^1_1(u,v)$ is a (non-empty) clique for every edge uv, Γ is 1-homogeneous if and only if Γ is isomorphic to a regular near $2d$-gon (see [5], Section 6.4 for the definition).

Now we restrict our interest to the case $t = 2$. Let us consider the following conditions for a distance-regular graph Γ of diameter d:

(H1) There are integers $\delta_2, \ldots, \delta_d$ such that, for every pair of vertices u, v at distance $\partial(u,v) = 2$, and for every x in $\Gamma_r(u) \cap \Gamma_r(v)$, there are precisely δ_r neighbours of x in $\Gamma_{r-1}(u) \cap \Gamma_{r-1}(v)$ $(r = 2, \ldots, d)$.
(H2) There are integers $\gamma_1, \ldots, \gamma_d$ such that, for every vertex x and for every u, v in $\Gamma_r(x)$ with $\partial(u,v) = 2$, there are precisely γ_r common neighbours of u and v in $\Gamma_{r-1}(x)$ $(r = 1, \ldots, d)$.

Lemma 3.1. *Let $\Gamma = (X, E)$ be a distance-regular graph of diameter d. Then* (H1) *is equivalent to* (H2).

Proof. First assume Γ satisfies (H1). We must show that the size

$$|\Gamma_{r-1}(x) \cap \Gamma_1(u) \cap \Gamma_1(v)|$$

does not depend on the choice of x in X and u, v in $\Gamma_r(x)$ with $\partial(u,v) = 2$. Clearly this holds for $r = 1$. Assume $r > 1$. Fix a vertex x and fix two vertices u, v in $\Gamma_r(x)$ with $\partial(u,v) = 2$, and put

$$\mathrm{D}^i_j = \mathrm{D}^i_j(u,v) = \Gamma_i(u) \cap \Gamma_j(v)$$

for all integers i, j. We count the number N of paths of length $r-1$ from x to D^1_1. Let $x = x_r, x_{r-1}, \ldots, x_2, x_1$ be a path of length $r-1$ such that $x_1 \in \mathrm{D}^1_1$. Then we have $x_i \in \mathrm{D}^i_i$ for $i = 1, \ldots, r$. By (H1), there are precisely δ_i edges from x_i to D^{i-1}_{i-1} $(i = 2, \ldots, r)$. Hence we have

$$N = \delta_r \delta_{r-1} \cdots \delta_2.$$

On the other hand, for a fixed vertex y in $\Gamma_{r-1}(x) \cap \mathrm{D}^1_1$, there are precisely $c_{r-1}c_{r-2} \cdots c_2 c_1$ paths of length $r-1$ connecting x and y, since we have $\partial(x,y) = r-1$. Hence we have

$$N = |\Gamma_{r-1}(x) \cap \mathrm{D}^1_1| c_{r-1}c_{r-2} \cdots c_2 c_1.$$

So we obtain

$$|\Gamma_{r-1}(x) \cap \mathrm{D}_1^1| = \frac{\delta_r \delta_{r-1} \cdots \delta_2}{c_{r-1} c_{r-2} \cdots c_2 c_1}.$$

This means the number of common neighbours of u and v in $\Gamma_{r-1}(x)$ does not depend on the choice of x in X and u, v in $\Gamma_r(x)$ with $\partial(u, v) = 2$. Thus Γ satisfies (H2).

Next assume Γ satisfies (H2). We show by induction on r that the number of edges $e(x, \mathrm{D}_{r-1}^{r-1}(u, v))$ does not depend on the choice of u, v with $\partial(u, v) = 2$ and x in $\mathrm{D}_r^r(u, v)$ ($r = 2, \ldots, d$). This holds when $r = 2$, since for $x \in \mathrm{D}_2^2(u, v)$ we have u, $v \in \Gamma_2(x)$ and so

$$e(x, \mathrm{D}_1^1(u, v)) = |\Gamma_{r-1}(x) \cap \Gamma_1(u) \cap \Gamma_1(v)| = \gamma_2.$$

Assume $r > 2$ and assume that there are constants $\delta_2, \ldots, \delta_{r-1}$ such that $e(x, \mathrm{D}_{i-1}^{i-1}(u, v)) = \delta_r$ holds for every $x \in \mathrm{D}_i^i(u, v)$ ($i = 2, \ldots, r-1$). Fix two vertices u, $v \in X$ at distance $\partial(u, v) = 2$ and put $\mathrm{D}_j^i = \mathrm{D}_j^i(u, v)$. Pick a vertex $x \in \mathrm{D}_r^r$ and put

$$\delta(x) = e(x, \mathrm{D}_{r-1}^{r-1}).$$

We count the number N of paths $x = x_r, x_{r-1}, \ldots, x_1$ of length $r-1$ with $x_1 \in \mathrm{D}_1^1$. Since $x_i \in \mathrm{D}_i^i$ ($i = 1, \ldots, r$) holds for every path $x = x_r, \ldots, x_1$ with $x_1 \in \mathrm{D}_1^1$,

$$N = \delta(x) \delta_{r-1} \delta_{r-2} \cdots \delta_2.$$

On the other hand, since there are precisely γ_r common neighbours y of u, v in $\Gamma_{r-1}(x)$ by (H2),

$$|\mathrm{D}_1^1 \cap \Gamma_{r-1}(x)| = \gamma_r.$$

Since for each vertex y in $\mathrm{D}_1^1 \cap \Gamma_{r-1}(x)$ there are precisely $c_{r-1} c_{r-2} \cdots c_1$ paths of length $r-1$ connecting y and x, the number of paths is given by

$$N = |\mathrm{D}_1^1 \cap \Gamma_{r-1}(x)|\, c_{r-1} c_{r-2} \cdots c_2 c_1 = \gamma_r c_{r-1} c_{r-2} \cdots c_2 c_1.$$

Therefore we obtain

$$\delta(x) = \frac{\gamma_r c_{r-1} c_{r-2} \cdots c_2 c_1}{\delta_{r-1} \delta_{r-2} \cdots \delta_2}.$$

Thus Γ satisfies (H1). Q.E.D.

A connected graph Γ is said to be *bipartite* if there is no cycle of odd length, and *almost bipartite* if there is no cycle of odd length ℓ with $\ell < 2d+1$ (where d is the diameter of Γ). Let Γ be a distance-regular graph of diameter d with intersection numbers c_r, a_r, b_r ($r = 0, \ldots, d$). Clearly Γ is bipartite if and only if $a_r = 0$ for $r = 0, \ldots, d$, and Γ is almost bipartite if and only if $a_r = 0$ for $r = 0, \ldots, d-1$.

Lemma 3.2. *Let Γ be an almost bipartite distance-regular graph of diameter d. Then Γ is 2-homogeneous if and only if Γ satisfies* (H1).

Proof. The condition (H1) says that $e(x, \mathrm{D}^{r-1}_{r-1}(u,v)) = \delta_r$ holds for every u, v, x with $\partial(u,v) = 2$ and $x \in \mathrm{D}^r_r(u,v)$. Hence (H1) holds if Γ is 2-homogeneous.

Fix two vertices u, v at distance $\partial(u,v) = 2$ and let us denote $\mathrm{D}^i_j = \mathrm{D}^i_j(u,v)$ for all i, j. Remark that D^i_j is empty for all i, j with $|i-j| > 2$ since $\partial(u,v) = 2$. Also remark that D^i_j is empty for all i, j with $i+j \equiv 1 \pmod 2$ and $i+j < 2d-1$ since there is no cycle of odd length $\ell < 2d+1$. Therefore the vertex set of Γ is partitioned into the following subsets:

$$\begin{array}{cccccccc}
\mathrm{D}^0_2 & \mathrm{D}^1_3 & \mathrm{D}^2_4 & \cdots & \mathrm{D}^{d-3}_{d-1} & \mathrm{D}^{d-2}_d & & \\
 & & & & & & \mathrm{D}^{d-1}_d & \\
\mathrm{D}^1_1 & \mathrm{D}^2_2 & \mathrm{D}^3_3 & \cdots & \mathrm{D}^{d-2}_{d-2} & \mathrm{D}^{d-1}_{d-1} & & \mathrm{D}^d_d \\
 & & & & & & \mathrm{D}^d_{d-1} & \\
\mathrm{D}^2_0 & \mathrm{D}^3_1 & \mathrm{D}^4_2 & \cdots & \mathrm{D}^{d-1}_{d-3} & \mathrm{D}^d_{d-2} & &
\end{array}$$

Remark that there is no edge connecting D^i_j and $\mathrm{D}^{i'}_{j'}$ if $|i-i'| > 1$ or $|j-j'| > 1$. Remark also that there is no edge inside D^i_j for all i, j with $i < d$ or $j < d$ since $a_1 = \cdots = a_{d-1} = 0$.

First we show that the number of edges $e(x, \mathrm{D}^i_j)$ ($x \in \mathrm{D}^r_s$) is determined by the intersection numbers for all r, s with $r \neq s$. For x in D^{r-2}_r we have

$$e(x, \mathrm{D}^{r-3}_{r-1}) = e(x, \Gamma_{r-3}(u)) = c_{r-2},$$

$$e(x, \mathrm{D}^{r-1}_{r-1}) = e(x, \Gamma_{r-1}(v)) - e(x, \mathrm{D}^{r-3}_{r-1}) = c_r - c_{r-2}.$$

Moreover when $r < d$ we have

$$e(x, \mathrm{D}^{r-1}_{r+1}) = e(x, \Gamma_{r+1}(v)) = b_r,$$

and when $r = d$ we have

$$e(x, \mathrm{D}^{d-1}_d) = e(x, \Gamma_{d-1}(u)) - e(x, \mathrm{D}^{d-1}_{d-1}) = b_{d-2} - (c_d - c_{d-2}).$$

For x in D_d^{d-1} we have

$$e(x, \mathrm{D}_d^{d-2}) = e(x, \Gamma_{d-1}(u)) = c_{d-1},$$

$$e(x, \mathrm{D}_{d-1}^{d-1}) \leq e(x, \Gamma_{d-1}(u)) = a_{d-1} = 0,$$

$$e(x, \mathrm{D}_{d-1}^{d}) = e(x, \Gamma_{d-1}(v)) - e(x, \mathrm{D}_{d-1}^{d-1}) = c_d,$$

$$e(x, \mathrm{D}_d^{d}) = e(x, \Gamma_d(u)) - e(x, \mathrm{D}_{d-1}^{d}) = b_{d-1} - c_d.$$

Thus $e(x, \mathrm{D}_j^i)$ is determined by the intersection numbers for $x \in \mathrm{D}_s^r$ with $r \neq s$. Moreover for x in D_1^1 we have

$$e(x, \mathrm{D}_2^0) = e(x, \mathrm{D}_0^2) = c_1, \qquad e(x, \mathrm{D}_2^2) = b_1 - c_1.$$

Now we assume Γ satisfies (H1) and let $x \in \mathrm{D}_r^r$ $(2 \leq r \leq d)$. Then by (H1) we have

$$e(x, \mathrm{D}_{r-1}^{r-1}) = \delta_r.$$

When $r < d$ we have

$$e(x, \mathrm{D}_{r+1}^{r-1}) = e(x, \Gamma_{r-1}(u)) - e(x, \mathrm{D}_{r-1}^{r-1}) = c_r - \delta_r,$$

$$e(x, \mathrm{D}_{r-1}^{r+1}) = e(x, \Gamma_{r-1}(v)) - e(x, \mathrm{D}_{r-1}^{r-1}) = c_r - \delta_r,$$

$$e(x, \mathrm{D}_{r+1}^{r+1}) = e(x, \Gamma_{r+1}(u)) - e(x, \mathrm{D}_{r-1}^{r+1}) = b_r - (c_r - \delta_r),$$

here remark that there is no edge between D_{d-1}^{d-1} and D_{d-1}^{d}. For $x \in \mathrm{D}_d^d$ we have

$$e(x, \mathrm{D}_d^{d-1}) = e(x, \Gamma_{d-1}(u)) - e(x, \mathrm{D}_{d-1}^{d-1}) = c_d - \delta_d,$$

$$e(x, \mathrm{D}_{d-1}^{d}) = e(x, \Gamma_{d-1}(v)) - e(x, \mathrm{D}_{d-1}^{d-1}) = c_d - \delta_d,$$

$$e(x, \mathrm{D}_d^{d}) = e(x, \Gamma_d(u)) - e(x, \mathrm{D}_{d-1}^{d}) = a_d - (c_d - \delta_d).$$

Therefore Γ is 2-homogeneous. Q.E.D.

3.2. A sufficient condition for 2-homogeneity

Yamazaki [22] proved that every bipartite distance-regular graph with an eigenvalue of multiplicity k (k is the valency) satisfies condition (H1). Here we give a slight generalization.

Proposition 3.3. *Let Γ be an almost bipartite distance-regular graph of valency k. If the adjacency matrix A of Γ has an eigenvalue θ of multiplicity f with $1 < f \leq k$, then Γ is 2-homogeneous.*

In the following we prove the above proposition in a similar way as Yamazaki's proof [22].

Let $\Gamma = (X, E)$ be an almost bipartite distance-regular graph of diameter d and valency k. We may assume $d > 1$ and $k > 2$ since the graph is clearly 2-homogeneous if $d = 1$ or $k \leq 2$. Let c_i, b_i and a_i $(i = 0, 1, \ldots, d)$ be the usual intersection numbers of Γ. We have $a_1 = \cdots = a_{d-1} = 0$ since Γ is almost bipartite. In particular Γ has no triangle. Assume that the adjacency matrix A of Γ has an eigenvalue θ of multiplicity f with $1 < f \leq k$. By [5] Proposition 4.4.1, we have a mapping $^{-} : X \longrightarrow \mathbf{R}^f$ such that $\langle \overline{x}, \overline{y} \rangle = u_i$ holds for all x, y at distance $\partial(x, y) = i$, where $\langle \overline{x}, \overline{y} \rangle$ denote the ordinary inner product of the Euclidean space $\mathbf{R}^f$, and $(u_0, u_1, \ldots, u_d)$ is the standard sequence corresponding to θ, i.e., it is the sequence defined by the recurrence: $u_0 = 1$, $u_1 = \theta/k$, $c_i u_{i-1} + b_i u_{i+1} = \theta u_i$ $(i = 1, \ldots, d-1)$. It is known that an eigenvalue η of A has multiplicity 1 if and only if $\eta = \pm k$ [5] Proposition 4.4.8. So $\theta \neq \pm k$ by our assumption $f > 1$. Then we obtain $u_2 \neq u_0 = 1$ from the above recurrence. Hence $\overline{x} \neq \overline{y}$ holds for all vertices x, y with $\partial(x, y) = 2$.

Lemma 3.4. *Let $\sigma : Y \longrightarrow X$ be a mapping from a subset Y of X which preserves distances. Then for real numbers λ_y $(y \in Y)$, $\sum_{y \in Y} \lambda_y \overline{y} = 0$ if and only if $\sum_{y \in Y} \lambda_y \overline{\sigma(y)} = 0$.*

Proof. Use $\langle \overline{x}, \overline{y} \rangle = u_{\partial(x,y)}$ to show

$$\| \textstyle\sum_{y \in Y} \lambda_y \overline{\sigma(y)} \| = \| \sum_{y \in Y} \lambda_y \overline{y} \| = 0.$$

Q.E.D.

For a subset Y of X, we denote $\overline{Y} = \{ \overline{y} \mid y \in Y \}$, $\widetilde{Y} = \sum_{y \in Y} \overline{y}$.

Lemma 3.5. *For every $x \in X$, $\overline{\Gamma_1(x) \cup \{x\}}$ spans a k-dimensional subspace of $\mathbf{R}^f$. In particular $f = k$.*

Proof. Assume that the subspace U spanned by $\overline{\Gamma_1(x) \cup \{x\}}$ has dimension $m + 1 < k$. Choose m vertices $y_1, \ldots, y_m$ in $\Gamma_1(x)$ such that $\overline{x}, \overline{y_1}, \ldots, \overline{y_m}$ form a basis of U, and choose two distinct vertices y, $y' \in \Gamma_1(u)$ which are different from $y_1, \ldots, y_m$ (here remark that $m \leq k - 2$). Write $\overline{y} = \lambda \overline{x} + \sum_{i=1}^{m} \lambda_i \overline{y_i}$ $(\lambda, \lambda_i \in \mathbf{R})$. Applying Lemma 3.4 for $Y = \{x, y, y_1, \ldots, y_m\}$ and $\sigma : Y \longrightarrow X$ such that $\sigma(y) = y'$,

$\sigma(x) = x$, $\sigma(y_i) = y_i$ $(i = 1, \ldots, m)$, we obtain $\overline{y'} = \lambda\overline{x} + \sum_{i=1}^{m} \lambda_i \overline{y_i}$. Hence $\overline{y} = \overline{y'}$, contradicting $\partial(y, y') = 2$. Q.E.D.

Lemma 3.6. *There are constants λ_i, μ_i, ν_i $(i = 2, \ldots, d)$ such that $\overline{v} = \lambda_i\overline{x} + \nu_i\widetilde{C} + \mu_i\widetilde{B}$ holds for all v, x with $i = \partial(v, x)$, where $C = \Gamma_1(x) \cap \Gamma_{i-1}(v)$ and $B = \Gamma_1(x) \setminus C$.*

Proof. Remark that $B = \Gamma_1(x) \cap \Gamma_{i+1}(v)$ when $i < d$, and $B = \Gamma_1(x) \cap \Gamma_i(v)$ when $i = d$. From Lemma 3.5, $\overline{v}$ can be written as

$$\overline{v} = \lambda\overline{x} + \sum_{y \in C} \nu_y \overline{y} + \sum_{z \in B} \mu_z \overline{z}$$

for some λ, ν_y, $\mu_z \in \mathbf{R}$ $(y \in C,\ z \in B)$. We would like to show that $\nu_{y_1} = \nu_{y_2}$ holds for all y_1, $y_2 \in C$. Let y_1, $y_2 \in C$ with $y_1 \neq y_2$. We use Lemma 3.4 for $Y = \{v, x\} \cup B \cup C$ and $\sigma : Y \longrightarrow X$ which fixes all vertices in Y except $\sigma(y_1) = y_2$, $\sigma(y_2) = y_1$. Clearly σ preserves distances. Then the above equation implies

$$\overline{v} = \lambda\overline{x} + \nu_{y_1}\overline{y_2} + \nu_{y_2}\overline{y_1} + \sum_{y \in C \setminus \{y_1, y_2\}} \nu_y \overline{y} + \sum_{z \in B} \mu_z \overline{z}.$$

These two equations imply $\nu_{y_1}\overline{y_1} + \nu_{y_2}\overline{y_2} = \nu_{y_1}\overline{y_2} + \nu_{y_2}\overline{y_1}$, and this becomes $(\nu_{y_1} - \nu_{y_2})(\overline{y_1} - \overline{y_2}) = 0$. Here we have $\overline{y_1} \neq \overline{y_2}$ by $\partial(y_1, y_2) = 2$, so $\nu_{y_1} = \nu_{y_2}$. This means $\nu_y = \nu$ is a constant for $y \in C$. In the same way, $\mu_z = \mu$ is a constant for $z \in B$. Thus $\overline{v} = \lambda\overline{x} + \nu\widetilde{C} + \mu\widetilde{B}$. Use Lemma 3.4 again to show that λ, μ, ν do not depend on v and x with $\partial(v, x) = i$. Q.E.D.

Fix two vertices v, w with $\partial(v, w) = 2$ and put $\mathrm{D}_s^r = \mathrm{D}_s^r(v, w)$. We have

$$\|\overline{v} - \overline{w}\|^2 = \langle \overline{v}, \overline{v} \rangle + \langle \overline{w}, \overline{w} \rangle - 2\langle \overline{v}, \overline{w} \rangle = u_0 + u_0 - 2u_2 = 2(u_0 - u_2).$$

First take $x \in \mathrm{D}_i^i$ $(1 < i < d)$ and put $A = \Gamma_1(x) \cap \mathrm{D}_{i-1}^{i-1}$, $B = \Gamma_1(x) \cap \mathrm{D}_{i+1}^{i-1}$, $C = \Gamma_1(x) \cap \mathrm{D}_{i-1}^{i+1}$, $D = \Gamma_1(x) \cap \mathrm{D}_{i+1}^{i+1}$. Then we have a partition $\Gamma_1(x) = A \cup B \cup C \cup D$. Clearly we have $|A| + |B| = |A| + |C| = c_i$, so that $|B| = |C|$. By Lemma 3.6, we have

$$\overline{v} = \lambda_i\overline{x} + \nu_i(\widetilde{A} + \widetilde{B}) + \mu_i(\widetilde{C} + \widetilde{D}),$$

$$\overline{w} = \lambda_i\overline{x} + \nu_i(\widetilde{A} + \widetilde{C}) + \mu_i(\widetilde{B} + \widetilde{D}).$$

Hence

$$\|\overline{v}-\overline{w}\|^2 = \|(\nu_i-\mu_i)(\widetilde{B}-\widetilde{C})\|^2 = (\nu_i-\mu_i)^2\,(\|\widetilde{B}\|^2+\|\widetilde{C}\|^2-2\langle\widetilde{B},\,\widetilde{C}\rangle).$$

Here we have $\|\widetilde{B}\|^2 = \|\widetilde{C}\|^2 = |B|u_0 + |B|(|B|-1)u_2$ and $\langle\widetilde{B},\,\widetilde{C}\rangle = |B|^2u_2$. Hence $\|\overline{v}-\overline{w}\|^2 = 2(\nu_i-\mu_i)^2|B|(u_0-u_2)$. Therefore we obtain $(\nu_i-\mu_i)^2|B| = 1$ and hence $|A| = c_i - |B| = c_i - (\nu_i-\mu_i)^{-2}$. This means the size of $\Gamma_1(x)\cap \mathrm{D}_{i-1}^{i-1}$ depends only on i.

Next take $x\in \mathrm{D}_d^d$ and put $A=\Gamma_1(x)\cap \mathrm{D}_{d-1}^{d-1}$, $B=\Gamma_1(x)\cap \mathrm{D}_d^{d-1}$, $C=\Gamma_1(x)\cap \mathrm{D}_{d-1}^{d}$, $D=\Gamma_1(x)\cap \mathrm{D}_d^d$. Then we can show that $|A| = c_i-(\nu_i-\mu_i)^{-2}$ in the same way.

Thus Γ satisfies (H1) and hence Γ is 2-homogeneous by Lemma 3.2.

§4. Graphs with spin model structure

4.1. An observation

Here we observe that the examples of spin models given in Section 2 can be constructed on distance-regular graphs. Jaeger's Higman-Sims model and the Hadamard models are constructed on distance-regular graphs with the intersection arrays:

$$\left\{\begin{matrix} 0 & 1 & 6 \\ 0 & 0 & 16 \\ 22 & 21 & 0 \end{matrix}\right\},$$

and

$$\left\{\begin{matrix} 0 & 1 & 2m & 4m-1 & 4m \\ 0 & 0 & 0 & 0 & 0 \\ 4m & 4m-1 & 2m & 1 & 0 \end{matrix}\right\}.$$

The Potts models with n spins is constructed on a complete graph K_n, which is a distance-regular graph of diameter $d=1$ with the intersection array

$$\left\{\begin{matrix} 0 & 1 \\ 0 & k-1 \\ k & 0 \end{matrix}\right\}, \qquad k=n-1.$$

The weights are given by $t_0=\alpha$, $t_1=\beta$, where $\beta^2+\beta^{-2}+\sqrt{n}=0$ and $\alpha=-\beta^{-3}$.

The cyclic model with n spins is constructed on the n-cycle C_n which is a distance-regular graph of diameter d with the intersection array:

$$\left\{\begin{matrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 2 & 1 & 1 & \cdots & 1 & 0 \end{matrix}\right\} \quad \text{when } n=2d+1,$$

or

$$\left\{\begin{matrix} 0 & 1 & 1 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 1 & \cdots & 1 & 0 \end{matrix}\right\} \quad \text{when } n = 2d.$$

The weights are given by $t_i = \alpha\theta^{i^2}$ $(i = 0, \ldots, d)$, where θ is a primitive n-root of unity if $n = 2d+1$, a primitive $2n$-root of unity if $n = 2d$, and $\alpha = \sqrt{n}/(\sum_{i=0}^{n-1} \theta^{i^2})$.

The square model is constructed on the 4-cycle C_4 with $t_0 = \alpha$, $t_1 = \alpha^{-1}$, $t_2 = -\alpha$, where α is a non-zero complex number.

Observe that all the above distance-regular graphs are almost bipartite. Moreover, as easily observed, each successive three terms t_{i-1}, t_i, t_{i+1} are distinct $(0 < i < d)$ in each of the above spin models except the square model with $\alpha = \pm 1$.

Motivated by the above observation, the author obtained the following result [21].

Theorem 4.1. *Let $\Gamma = (X, E)$ be a connected graph of diameter d which has no 3-cycle. Let $t_0, \ldots, t_d$ be non-zero complex numbers such that $t_1 \neq t_i$ and $t_{i-2} \neq t_i \neq t_{i-1}$ for $i = 2, \ldots, d$. Define a function w on $X \times X$ by $w(x, y) = t_{\partial(x,y)}$ for x, $y \in X$. If $S = (X, w)$ is a spin model, then Γ is an almost bipartite distance-regular graph.*

This was obtained by "localizing" the star-triangle relation (S3). This technique of localization was introduced in [19].

4.2. 2-homogeneity

Lemma 4.2. *Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $d > 1$ and valency k, and let $t_0, \ldots, t_d$ be non-zero complex numbers such that $t_i \neq t_1$ for $i = 2, \ldots, d$. Assume $S = (X, w)$ is a spin model, where w is a function on $X \times X$ defined by $w(x, y) = t_{\partial(x,y)}$ for x, $y \in X$. Then the adjacency matrix A of Γ has an eigenvalue θ of multiplicity f with $1 < f \leq k$.*

Proof. Let $\theta_0 = k, \theta_1, \ldots, \theta_d$ be the eigenvalues of the adjacency matrix A of Γ and let V_i be the eigenspace corresponding to $\theta_i, i = 0, \ldots, d$, where V_0 is the 1-dimensional subspace of $V = \mathbf{C}^n$ spanned by the all 1 vector $\mathbf{j}$. V splits into an orthogonal direct sum:

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_d \qquad \text{(orthogonal)}.$$

On the other hand, let $\mathbf{u}_{bc}$, b, $c \in X$ be the vector defined in Section 2.1, which is an eigenvector of the weight matrix W of S for the eigenvalue $\sqrt{n}\, w(b, c)^{-1}$.

Now fix a vertex $b \in X$. Then the vectors $\mathbf{u}_{bc}$, $c \in X$, form a basis of V. Let V_i' be the subspace of V spanned by the vectors $\mathbf{u}_{bc}$, $c \in \Gamma_i(b)$ $(i = 0, \dots, d)$. Remark that $V_0' = \langle \mathbf{j} \rangle = V_0$. So V splits into a direct sum:

$$V = V_0 \oplus V_1' \oplus \cdots \oplus V_d',$$

where we have $V_i' \subset V_0^{\perp}$ for $i = 1, \dots, d$. Since $\mathbf{u}_{bc}$ is an eigenvector of W for the eigenvalue $\sqrt{n}\, w(b,c)^{-1}$, V_i' is included in the eigenspace of W for the eigenvalue $\sqrt{n}\, t_i^{-1}$, $i = 0, \dots, d$. Since $t_1 \neq t_i$ for $i = 2, \dots, d$, the eigenspace of W for the eigenvalue $\sqrt{n}\, t_1^{-1}$ is included in $V_0 \oplus V_1'$. Now consider the action of W on

$$V_0^{\perp} = V_1' \oplus \cdots \oplus V_d'.$$

Then V_1' is the eigenspace of W in $V_0^{\perp}$ for the eigenvalue $\sqrt{n}\, t_1^{-1}$.

On the other hand, W is written as

$$W = \sum_{i=0}^{d} t_i A_i,$$

where A_i denotes the i-th adjacency matrix of the distance-regular graph Γ $(i = 0, \dots, d)$. Since A_i is a polynomial in A, $A_i = v_i(A)$, W is written as a polynomial in A:

$$W = \sum_{i=0}^{d} t_i v_i(A).$$

Hence for each eigenvector $\mathbf{x}$ of A for the eigenvalue θ_j of A, $j > 0$, we have

$$W\mathbf{x} = \sum_{i=0}^{d} t_i v_i(A)\mathbf{x} = \sum_{i=0}^{d} t_i v_i(\theta_j)\mathbf{x},$$

so $\mathbf{x}$ is an eigenvector of W for the eigenvalue $\sum_{i=0}^{d} t_i v_i(\theta_j)$. Since $\mathbf{x} \in V_0^{\perp}$, $\mathbf{x}$ must belong to some eigenspace (in $V_0^{\perp}$) of W.

Therefore we can conclude that V_1' is a sum of some eigenspaces of A, say:

$$V_1' = V_1 \oplus \cdots \oplus V_\ell,$$

so that

$$k = \dim V_1' = f_1 + \cdots + f_\ell,$$

where $f_i = \dim V_i$. This implies $f_i \leq k$ $(i = 1, \dots, \ell)$. We must show that $1 < f_i \leq k$ holds for some i $(1 \leq i \leq \ell)$. If $\ell = 1$ then we have $f_1 = k$ and $f_1 > 1$ since $k > 1$ by our assumption $d > 1$. So we may assume $\ell > 1$. If $f_i > 1$ holds for some i, then we have the conclusion. So we may assume $f_1 = \cdots = f_\ell = 1$. Now it is known that an eigenvalue θ of a distance-regular graph has multiplicity 1 if and only if $\theta = \pm k$ [5] Proposition 4.4.8. Hence $f_i = 1$ occurs at most one i, that is when $\theta_i = -k$ (remark that $\theta_i \neq k$ since $\theta_0 = k$). This implies $\ell = 1$, a contradiction. Q.E.D.

Theorem 4.3. *Let $\Gamma = (X, E)$ be an almost bipartite distance-regular graph of diameter d, and let $t_0, t_1, \ldots, t_d$ be non-zero complex numbers such that $t_1 \neq t_i$ for $i = 2, \ldots, d$. If $S = (X, w)$ is a spin model with the weight w defined by $w(x, y) = t_{\partial(x,y)}$, x, $y \in X$, then Γ is 2-homogeneous.*

Proof. It is obtained from Lemma 4.2 and Proposition 3.3. Q.E.D.

Corollary 4.4. *Let $\Gamma = (X, E)$ be a triangle-free connected graph of diameter d, and let $t_0, \ldots, t_d$ be non-zero complex numbers such that $t_1 \neq t_i$ and $t_{i-2} \neq t_i \neq t_{i-1}$ for $i = 2, \ldots, d$. If $S = (X, w)$ is a spin model with the weight w defined by $w(x, y) = t_{\partial(x,y)}$, x, $y \in X$, then Γ is an almost bipartite 2-homogeneous distance-regular graph.*

Proof. It is obtained from Theorem 4.1 and Theorem 4.3. Q.E.D.

Remark. The assumption 'triangle-free' in Corollary 4.4 is essential. Actually there exists a distance-regular graph Γ (with triangles) such that Γ affords a spin model structure with weights $t_0, \ldots, t_d$ satisfying the same conditions but Γ is not 2-homogeneous. Also remark that every connected graph can have a spin model structure with the weights $t_1 = \cdots = t_d$ (Potts model), and so we need some conditions on the weights $t_0, \ldots, t_d$ in Corollary 4.4.

§5. Classification of almost bipartite 2-homogeneous graphs

In this section we determine the intersection arrays of almost bipartite 2-homogeneous distance-regular graphs.

Theorem 5.1. *Let Γ be an almost bipartite 2-homogeneous distance-regular graph of diameter $d > 0$ and valency k. Then Γ has one of the following intersection arrays:*

$$(1)\quad \left\{\begin{matrix} 0 & 1 \\ 0 & k-1 \\ k & 0 \end{matrix}\right\}, \quad k > 0,$$

$$(2)\quad \begin{Bmatrix} 0 & 1 & k \\ 0 & 0 & 0 \\ k & k-1 & 0 \end{Bmatrix},\quad k > 1,$$

$$(3)\quad \begin{Bmatrix} 0 & 1 & c \\ 0 & 0 & k-c \\ k & k-1 & 0 \end{Bmatrix},\quad \begin{array}{l} k = \gamma(\gamma^2 + 3\gamma + 1), \\ c = \gamma(\gamma + 1),\ \gamma > 0, \end{array}$$

$$(4)\quad \begin{Bmatrix} 0 & 1 & k-1 & k \\ 0 & 0 & 0 & 0 \\ k & k-1 & 1 & 0 \end{Bmatrix},\quad k > 1,$$

$$(5)\quad \begin{Bmatrix} 0 & 1 & 2\gamma & 4\gamma-1 & 4\gamma \\ 0 & 0 & 0 & 0 & 0 \\ 4\gamma & 4\gamma-1 & 2\gamma & 1 & 0 \end{Bmatrix},\quad \gamma > 0,$$

$$(6)\quad \begin{Bmatrix} 0 & 1 & c & k-c & k-1 & k \\ 0 & 0 & 0 & 0 & 0 & 0 \\ k & k-1 & k-c & c & 1 & 0 \end{Bmatrix},\quad \begin{array}{l} k = \gamma(\gamma^2 + 3\gamma + 1), \\ c = \gamma(\gamma + 1),\ \gamma > 0, \end{array}$$

$$(7)\quad \begin{Bmatrix} 0 & 1 & \cdots & 1 & 2 \\ 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & \cdots & 1 & 0 \end{Bmatrix},\quad d > 1,$$

$$(8)\quad \begin{Bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ 2 & 1 & \cdots & 1 & 0 \end{Bmatrix},\quad d > 1,$$

$$(9)\quad \begin{Bmatrix} 0 & 1 & 2 & 3 & \cdots & k-1 & k \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ k & k-1 & k-2 & k-3 & \cdots & 1 & 0 \end{Bmatrix},\quad k = d,$$

$$(10)\quad \begin{Bmatrix} 0 & 1 & 2 & 3 & \cdots & d-1 & d \\ 0 & 0 & 0 & 0 & \cdots & 0 & d+1 \\ 2d+1 & 2d & 2d-1 & 2d-2 & \cdots & d+2 & 0 \end{Bmatrix}\quad d > 1.$$

Remark. The intersection arrays in the above list are realized by the following graphs:

(1) complete graph K_{k+1},
(2) complete bipartite graph $K_{k,k}$,
(3) antipodal quotient of 5-dimensional hypercube when $\gamma = 1$, Higman-Sims graph when $\gamma = 2$, the existence of graphs is unknown when $\gamma > 2$,
(4) complement of $2 \times (k+1)$-grid,
(5) Hadamard graph of valency $k = 4\gamma$,
(6) antipodal double cover of (3),
(7) cycle C_{2d+1} of length $2d + 1$,
(8) cycle C_{2d} of length $2d$,
(9) d-dimensional hypercube,

(10) antipodal quotient of $(2d+1)$-dimensional hypercube.

Now we prove Theorem 5.1. Let $\Gamma = (X, E)$ be an almost bipartite 2-homogeneous distance-regular graph of diameter d and valency k with the intersection array:

$$\left\{ \begin{array}{cccccc} 0 & c_1 & c_2 & \cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \cdots & b_{d-1} & 0 \end{array} \right\}.$$

We have $a_i = 0$ $(i = 1, \ldots, d-1)$, $c_1 = 1$, $b_0 = k$, $b_1 = k-1$ and $a_d = k - c_d$. If $k \leq 2$ or $d \leq 1$, then Γ is isomorphic to a cycle or a complete graph and the intersection array of Γ becomes (1), (7) or (8). So in the following we assume $k > 2$ and $d > 1$. In particular we have $a_1 = 0$ and hence Γ has no 3-cycle.

By Lemma 3.1 and Lemma 3.2, Γ satisfies condition (H2), so that there are constants $\gamma_1, \ldots, \gamma_d$ such that

$$\gamma_i = |\Gamma_{i-1}(u) \cap \Gamma_1(x) \cap \Gamma_1(y)|$$

holds for all vertices u, x, $y \in X$ with $\partial(u,x) = \partial(u,y) = i$ and $\partial(x,y) = 2$ $(i = 1, \ldots, d)$.

Lemma 5.2. (i) $c_2 > 1$,
(ii) $(k-2)(\gamma_2 - 1) = (c_2 - 1)(c_2 - 2)$,
(iii) $\gamma_i(c_{i+1} - 1) = c_i(c_2 - 1)$, $(0 < i < d)$,
(iv) $(c_2 - 1)(\gamma_i - 1) = (c_i - 1)(\gamma_2 - 1)$, $(0 < i < d)$.

Proof. Fix a vertex u in X.

(i) We claim that $\gamma_i > 0$ if $c_i = 1$. Pick a vertex w in $\Gamma_{i-1}(u)$. Then w has at least two neighbours x, y in $\Gamma_i(u)$, since we have $b_{i-1} = k - c_{i-1} \geq k - c_i = k - 1 > 1$. So we have $\partial(x,y) = 2$ and $w \in \Gamma_{i-1}(u) \cap \Gamma_1(x) \cap \Gamma_1(y)$, and hence $\gamma_i > 0$.

First assume $c_d = 1$. We have $\gamma_d > 0$ as shown above. Each vertex v in $\Gamma_d(u)$ has at least two distinct neighbours x, y in $\Gamma_d(u)$ since $a_d = k - c_d = k - 1 \geq 2$. Then $\partial(x,y) = 2$ since Γ has no 3-cycle, and hence x and y has at least one common neighbour z in $\Gamma_{d-1}(u)$ by $\gamma_d > 0$. We have $\partial(v,z) = 2$ and x, y are common neighbours of v and z, so that $c_2 > 1$.

Next assume $c_d > 1$. Since $1 = c_1 \leq c_2 \leq \cdots \leq c_d$ and $c_d > 1$, there is an integer r such that $1 = c_1 = c_2 = \cdots = c_r < c_{r+1}$. Pick a vertex z in $\Gamma_{r+1}(u)$. Since $c_{r+1} > 1$, z has at least two distinct neighbours x', $y' \in \Gamma_r(u)$. Since $\partial(x', y') = 2$ and $\gamma_r > 0$ by the above claim, x' and y'

have a common neighbour v in $\Gamma_{r-1}(u)$. Then $z \in \Gamma_2(v)$, and z has two distinct neighbours x', y' in $\Gamma_1(v)$. This implies $c_2 \geq 2$.

(ii) Fix an edge vw with $v \in \Gamma_1(u)$ and $w \in \Gamma_2(u)$. We count the number N of edges xy with $x \in \Gamma_1(u) \cap \Gamma_1(w) \cap \Gamma_2(v)$ and $y \in \Gamma_1(v) \cap \Gamma_2(u) \cap \Gamma_2(w)$ in two different ways. Since $w \in \Gamma_2(u)$, there are precisely $c_2 - 1$ vertices $x \in \Gamma_1(u) \cap \Gamma_1(w)$ with $x \neq v$. Fix such a vertex x. Since $x \in \Gamma_2(v)$, there are precisely $c_2 - 2$ vertices $y \in \Gamma_1(v) \cap \Gamma_1(x)$ with $y \neq u$, $y \neq w$. So we have $N = (c_2 - 1)(c_2 - 2)$. On the other hand, there are precisely $k - 2$ vertices $y \in \Gamma_2(u) \cap \Gamma_1(v)$ with $y \neq w$. Fix such a vertex y. Since w, $y \in \Gamma_2(u)$ and $\partial(w, y) = 2$, w and y have precisely $\gamma_2 - 1$ common neighbours x in $\Gamma_1(u)$ with $x \neq v$. So we obtain $N = (k-2)(\gamma_2 - 1)$.

(iii) Fix an edge vw with $v \in \Gamma_i(u)$ and $w \in \Gamma_{i+1}(u)$. We count the number N of edges xy with $x \in \Gamma_{i-1}(u) \cap \Gamma_1(v)$ and $y \in \Gamma_i(u) \cap \Gamma_1(w) \cap \Gamma_2(v)$ in two different ways. Since $v \in \Gamma_i(u)$, v has precisely c_i neighbours x in $\Gamma_{i-1}(u)$. Fix such a vertex x. Since $w \in \Gamma_2(x)$, w has precisely $c_2 - 1$ neighbours y in $\Gamma_1(x)$ with $y \neq v$. Hence we have $N = c_i(c_2 - 1)$. On the other hand, since $w \in \Gamma_{i+1}(u)$, w has precisely $c_{i+1} - 1$ neighbours y in $\Gamma_i(u)$ with $y \neq v$. Fix such a vertex y. Since v, $y \in \Gamma_i(u)$ and $\partial(v, y) = 2$, v and y have precisely γ_i common neighbours x in $\Gamma_{i-1}(u)$. So we obtain $N = (c_{i+1} - 1)\gamma_i$.

(iv) Fix a path zvw with $z \in \Gamma_{i-1}(u)$, $v \in \Gamma_i(u)$, $w \in \Gamma_{i+1}(u)$, and count the number of edges xy with $x \in \Gamma_{i-1}(u) \cap \Gamma_1(v) \cap \Gamma_2(z)$ and $y \in \Gamma_i(u) \cap \Gamma_1(z) \cap \Gamma_1(w) \cap \Gamma_2(v)$ in two different ways. Since $v \in \Gamma_i(u)$, v has precisely $c_i - 1$ neighbours x in $\Gamma_{i-1}(u)$ with $x \neq z$. Fix such a vertex x. Since x, $z \in \Gamma_2(w)$ and $\partial(x, z) = 2$, x and z have precisely $\gamma_2 - 1$ common neighbours y in $\Gamma_1(w)$ with $y \neq v$. So we have $N = (c_i - 1)(\gamma_2 - 2)$. On the other hand, since $w \in \Gamma_2(z)$, w has precisely $c_2 - 1$ neighbours y in $\Gamma_1(z)$ with $y \neq v$. Fix such a vertex y. Since v, $y \in \Gamma_i(u)$ and $\partial(v, y) = 2$, v and y have precisely $\gamma_i - 1$ common neighbours x in $\Gamma_{i-1}(u)$ with $x \neq z$. So we obtain $N = (c_2 - 1)(\gamma_i - 1)$.
Q.E.D.

Lemma 5.3. *If* $a_d > 0$,
(v) $c_d(c_2 - 1) = (k - c_d - 1)\gamma_d$,
(vi) $k \geq 2c_d$.

Proof. (v) Since $a_d > 0$, there is an edge vw in $\Gamma_d(u)$. We count the number N of edges xy with $x \in \Gamma_{d-1}(u) \cap \Gamma_1(v)$ and $y \in \Gamma_d(u) \cap \Gamma_1(w) \cap \Gamma_2(v)$ in two different ways. Since $v \in \Gamma_d(u)$, v has precisely c_d neighbours x in $\Gamma_{d-1}(u)$. Fix such a vertex x. Since $x \in \Gamma_2(w)$, x has precisely $c_2 - 1$ neighbours y in $\Gamma_1(w)$ with $y \neq v$, where we have

$y \in \Gamma_d(u)$ since there is no edge in $\Gamma_{d-1}(u)$. So we have $N = c_d(c_2 - 1)$. On the other hand, since $w \in \Gamma_d(u)$, w has precisely $a_d - 1$ neighbours y in $\Gamma_d(u)$ with $y \neq v$. Fix such a vertex y. Since v, $y \in \Gamma_d(u)$ and $\partial(v, y) = 2$, v and y have precisely γ_d common neighbours x in $\Gamma_{d-1}(u)$. So we obtain $N = (a_d - 1)\gamma_d = (k - c_d - 1)\gamma_d$.

(vi) Let vw be an edge in $\Gamma_d(u)$. If there is a vertex x in $\Gamma_1(u) \cap \Gamma_{d-1}(v) \cap \Gamma_{d-1}(w)$, then uv is an edge in $\Gamma_{d-1}(x)$, contradicting $a_{d-1} = 0$. Hence $\Gamma_1(u) \cap \Gamma_{d-1}(v)$ and $\Gamma_1(u) \cap \Gamma_{d-1}(w)$ are mutually disjoint, each of which has size c_d since $u \in \Gamma_d(v)$ and $u \in \Gamma_d(w)$. Hence $k = |\Gamma_1(u)| \geq 2c_d$. Q.E.D.

To simplify notations, we put

$$c = c_2, \quad \gamma = \gamma_2.$$

When $\gamma = 1$, we have $c > 1$ by Lemma 5.2 (i), and hence $c = 2$ by Lemma 5.2 (ii). Then $\gamma_i = 1$ $(i = 1, \ldots, d-1)$ by Lemma 5.2 (iv) and this implies $c_i = i$ $(i = 1, \ldots, d)$ by Lemma 5.2 (iii). If $a_d = 0$ then we have $k = c_d = d$, so that the intersection array becomes of type (9). If $a_d > 0$ then Lemma 5.3 (v) implies $d = (k - d - 1)\gamma_d$, here we have $k \geq 2c_d = 2d$ by Lemma 5.3 (vi). Hence we must have $\gamma_d = 1$ and $k = 2d + 1$ so that the intersection array becomes of type (10).

Now we assume $\gamma > 1$. By Lemma 5.2 (i), (ii), we have $c > 1$ and

$$k = \frac{(c-1)(c-2)}{\gamma - 1} + 2.$$

First we consider the case $a_d > 0$.

When $d = 2$, Lemma 5.3 (v) becomes

$$k = \frac{c(c-1)}{\gamma} + c + 1,$$

and hence we have

$$\frac{(c-1)(c-2)}{\gamma - 1} + 2 = \frac{c(c-1)}{\gamma} + c + 1.$$

This becomes

$$c = \gamma(\gamma + 1),$$

and hence

$$k = \frac{(c-1)(c-2)}{\gamma - 1} + 2 = \gamma(\gamma^2 + 3\gamma + 1),$$

so that the intersection array becomes of type (3) in the case $d = 2$.

Assume $d > 2$. We have $2c_3 \le k$ by Lemma 5.3 (vi) and by $c_3 \le c_d$. By Lemma 5.2 (iii), we have

$$c_3 = \frac{c(c-1)}{\gamma} + 1.$$

So $2c_3 \le k$ implies

$$2\left(\frac{c(c-1)}{\gamma} + 1\right) \le \frac{(c-1)(c-2)}{\gamma - 1} + 2,$$

and this becomes

$$2(\gamma-1)c(c-1) \le \gamma(c-1)(c-2).$$

By Lemma 5.2 (i), we have $c - 1 > 0$, so the above inequality implies

$$2(\gamma-1)c \le \gamma(c-2)$$

and hence

$$(\gamma-2)c + 2\gamma \le 0.$$

This is impossible by our assumption $\gamma \ge 2$. Thus the case $d > 2$ does not occur.

Next we consider the case $a_d = 0$. Since $b_0 = k$, $b_1 = k - 1$ and $b_2 = k - c$, we have

$$b_0 = \frac{(c-1)(c-2)}{\gamma-1} + 2, \qquad b_1 = \frac{(c-1)(c-2)}{\gamma-1} + 1,$$

$$b_2 = \frac{(c-1)(c-2)}{\gamma-1} + 2 - c = \frac{(c-\gamma)(c-2)}{\gamma-1}.$$

From Lemma 5.2 (iii) with $i = 2$, we obtain

$$c_3 = \frac{c(c-1)}{\gamma} + 1 = \frac{c^2 - c + \gamma}{\gamma},$$

and $b_3 = k - c_3$ implies

$$b_3 = \frac{(c-1)(c-2)}{\gamma-1} + 2 - \frac{c^2 - c + \gamma}{\gamma} = \frac{(c-\gamma)(c-\gamma-1)}{\gamma(\gamma-1)}.$$

When $d > 3$, Lemma 5.2 (iii), (iv) and $c_3 = (c^2 - c + \gamma)/\gamma$ imply

$$c_4 = \frac{c(c^2 - 2c + 2\gamma)}{\gamma + \gamma c - c},$$

and $b_4 = k - c_4$ implies

$$b_4 = \frac{(c-\gamma)(c-2\gamma)}{(\gamma-1)(c\gamma+\gamma-c)}.$$

When $d > 4$, Lemma 5.2 (iii), (iv) imply

$$c_5 = \frac{c^4 - 3c^3 + c^2 + 3\gamma c^2 - 2\gamma c + \gamma^2}{\gamma c^2 + \gamma^2 - c^2},$$

and $b_5 = k - c_5$ implies

$$b_5 = \frac{(c-\gamma)(c-\gamma-\gamma^2)}{(\gamma-1)(c^2\gamma+\gamma^2-c^2)}.$$

If $d > 5$, we have $b_5 \geq 1$, so the above equation implies (noting that the denominator is positive since $\gamma > 1$)

$$(c-\gamma)(c-\gamma-\gamma^2) \geq (\gamma-1)(c^2\gamma+\gamma^2-c^2),$$

and this becomes

$$\gamma(c-1)(2c-2\gamma-c\gamma) \geq 0.$$

This implies a contradiction since $c \geq 2$ and $\gamma \geq 2$. Hence we have $d \leq 5$.

When $d = 5$, we have $b_5 = 0$, and this implies $c = \gamma$ or $c = \gamma^2 + \gamma$. But $c = \gamma$ does not occur by Lemma 5.2 (ii) since $c_2 = k - b_2 < k$. So we have $c = \gamma^2 + \gamma$. Substituting this value of c in the above equations, we obtain $k = \gamma(\gamma^2 + 3\gamma + 1)$, $c_3 = k - c$, $c_4 = k - 1$. So the intersection array becomes of type (6).

When $d = 4$, we have $b_4 = 0$, and this implies $c = 2\gamma$ ($c = \gamma$ is impossible as above). So we obtain $k = 4\gamma$, $c_3 = k - 1$, so the intersection array becomes of type (5).

When $d = 3$, we have $b_3 = 0$, and this implies $c = \gamma + 1$. So we obtain $c = k - 1$, and the intersection array becomes of type (4).

This completes the proof of Theorem 5.1.

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