Advanced Studies in Pure Mathematics 23, 1994
Spectral and Scattering Theory and Applications
pp. 283–294

Trudinger's Inequality and Related Nonlinear Elliptic Equations in Two-Dimension

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§1. Introduction and results

We are concerned with the following nonlinear elliptic equations:

$$\begin{cases} -\Delta u = \lambda u e^{u^2}, & x \in B, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $B = B_1(0) \subset \mathbb{R}^2$ is a unit disk in $\mathbb{R}^2$ and λ is a positive parameter. We consider a family of solutions of (1) satisfying

$$\|u\|_{L^\infty} \to \infty \quad \text{as } \lambda \to 0. \tag{2}$$

The nonlinearity of the equation (1) is the Sobolev critical exponent in two-dimension. For any domain $\Omega \in \mathbb{R}^2$, It is well known that the Sobolev space $H_0^1(\Omega)$ is continuously imbedded in $L^p(\Omega)$ for any $p < \infty$ but is false in the case $p = \infty$. Trudinger [18] showed that for any $u \in H_0^1(\Omega)$ with $\|\nabla u\|_2 = 1$, there are two constants $\alpha > 0$ and $C > 0$ such that

$$\int_\Omega \exp\{\alpha u^2\} dx \le C|\Omega|. \tag{3}$$

Later, Moser [7] simplified the proof and improved that (3) is also valid for $\alpha \le 4\pi$. Here 4π is the constant of the isoperimetric inequality. The inequality (3) is also valid for any unbounded domain (Ogawa [9]). That is when Ω is any domain in $\mathbb{R}^2$, we have for all $u \in H_0^1(\Omega)$,

$$\int_\Omega \{\exp(u^2) - 1\} dx \le C\|u\|_2^2, \quad \|\nabla u\|_2 = 1. \tag{4}$$

(See also Ogawa-Ozawa [10] and Ozawa [12] for further extensions).

Received December 28, 1992.

These inequalities (3)–(4) indicates that the order of local singularities of H^1 functions are allowed as far as $\exp(u^2)$ is integrable. In other words e^{u^2} is the critical order of integrability for H^1-functions.

Concerning our problem (1), there are two different approaches. One is the variational method. When we consider the maximizing problem of the functional

$$(5) \qquad \int_\Omega \exp\{\alpha u^2\}dx \quad \text{for } u \in H_0^1(\Omega), \quad \|\nabla u\|_2 = 1$$

on a bounded domain. Then the extremal function (if it is achieved) becomes a solution of (1). Shaw [14] showed the existence of a positive solution of (1) for each parameter $\lambda > 0$ (see also Adimurti [1]). When the domain is a ball in $\mathbb{R}^n$, the maximum can be attained by some function even when $n = 2$ and $\alpha = 4\pi$ (Carleson-Chang [4]).

When the domain is a unit disk, all the positive smooth solution must be radially symmetric by Gidas-Ni-Nirenberg's result [5]. Therefore the Dirichlet problem may be written as the nonlinear ordinary differential equation:

$$(6) \qquad \begin{cases} -u_{rr} - \dfrac{1}{r}u_r = \lambda u e^{u^2}, & x \in [0,1), \\ u(1) = 0, \quad u'(0) = 0. \end{cases}$$

By solving (6), we can obtain the details of the properties of the positive solution of (1), which is the second method. Atkinson-Peletier [2], [3] applied the shooting method to (6) and proved that the existence of radially symmetric solution of (1) satisfying

$$\|u\|_{L^\infty} \to \infty \text{ as } \lambda \to 0.$$

Our aim of this paper is to specify more precise behavior of the family of solutions $\{(u, \lambda)\}$ as $\lambda \to 0$. We have two results. First one states a *global behavior* of the solutions.

Theorem A. *Let u be a positive solution of* (1) *with the blow up condition* (2). *That is*

$$\|u\|_{L^\infty(B)} = u(0) \to \infty \text{ as } \lambda \to 0.$$

Then we have

$$u(x) \to 0 \text{ as } \lambda \to 0$$

for all $x \in B \setminus \{0\}$. Moreover we have

$$\lim_{\lambda \to 0} \lambda \int_B u e^{u^2} dx = 0, \tag{7}$$

$$\lim_{\lambda \to 0} \lambda \int_B (e^{u^2} - 1) dx = 0, \tag{8}$$

$$\varliminf_{\lambda \to 0} \int_B |\nabla u|^2 dx \geq 4\pi. \tag{9}$$

This theorem says that the solution satisfying (2) must blow-up only at the origin. The inequality (9) shows the solution concentrates to the origin with its energy density $|\nabla u|^2$. The lower bound in (9) arise from the sharp exponent of the Trudinger inequality (3).

The second result is a *microscopic behavior* near the origin. When we rescale the solution by some sequence, then the solution has a limit function.

Theorem B. *There is a subsequence $\{(u_m, \lambda_m)\}$ of a family of solutions of* (1) *with* (2) *and a scaling sequance $\{\gamma_m\}$ such that $\gamma_m \to 0$ as $\lambda_m \to 0$ which satisfy*

$$u^2(\gamma_m x) - u^2(\gamma_m) \to 2\log(\frac{2}{1+|x|^2}) \quad \text{as } \lambda_m \to 0 \tag{10}$$

locally uniformly on $B \setminus \{0\}$.

The limit function of (10) is an exact solution of $-\Delta v = 2e^v$. Remark that since the nonlinearity of our problem is nonhomogeneous, the usual scaling $u \to \gamma^\mu u(\gamma x)$ does not work well. (For other nonlinearity or the higher dimensional case, see Nagasaki-Suzuki [8] and Itoh [6].)

The property (10) was firstly observed by Carleson-Chang in an implicit way. Later Struwe [15] obtained the similar result for the non-compact maximizing sequence for the variational problem (5) for the case $\alpha = 4\pi$. Our result Theorem B is, however, different from theirs, because in our case, the each factor of the sequence $\{u_m, \lambda_m)\}$ satisfies the equation (1). Moreover even the energy integral might blow up as $\lambda \to 0$ and therefore we can not obtain a priori estimate of $\{u_m\}$ from the Dirichlet integral. This is the crucial difference from the variational setting.

§2. Proof of Theorem A

We begin with the following lemma.

Lemma 1. *Let u be a positive, radially symmetric smooth solution of* (1). *We put $r = |x|$. Then we have*

$$r^2 u_r(r)^2 + 2\lambda r^2 (e^{u^2(r)} - 1) = \frac{\lambda}{2\pi} \int_{B_r} (e^{u^2} - 1) dx, \tag{11}$$

$$r u_r(r) = -\frac{\lambda}{2\pi} \int_{B_r} u e^{u^2} dx, \tag{12}$$

where $B_r = \{y \in \mathbb{R}^2, |y| < r\}$.

The first relation (11) is nothing else but the Pohozaev identity ([13]) associated to the equation (1).

Proof. Let u be a radially symmetric smooth solution of (1). Then u satisfies (6). Mutiplying (6) by $r u_r(r)$ and integrating on B_{r_0}, we have

$$-\int_0^{r_0} r^2 u_r u_{rr} dr - \int_0^{r_0} r u_r^2 dr = \lambda \int_0^{r_0} u e^{u^2} r^2 u_r dr.$$

Integrating by parts, we obtain

$$-\frac{1}{2} r_0^2 u_r(r_0)^2 = \frac{\lambda}{2} r_0^2 e^{u^2(r_0)} - \frac{\lambda}{2\pi} \int_{B_{r_0}} e^{u^2} dx,$$

which implies (11). The second relation (12) is a direct consequence of integration of the equation (6) on B_r. Q.E.D.

Proof of Theorem A. Combining (11) and (12) in Lemma 1 with choosing $r = 1$, we get

$$\frac{1}{4\pi} \big(\lambda \int_B u e^{u^2} dx\big)^2 = \lambda \int_B (e^{u^2} - 1) dx. \tag{13}$$

For any $k > 0$, we put

$$C_k = \max_{u \geq k} \frac{1 - e^{-u^2}}{u}.$$

Then we see $C_k \leq 1/k \to 0$ as $k \to \infty$. From (13)

$$\begin{aligned} \frac{1}{4\pi} \big(\lambda \int_B u e^{u^2} dx\big)^2 &= \lambda \int_{u \geq k} (e^{u^2} - 1) dx + \lambda \int_{u < k} (e^{u^2} - 1) dx \\ &\leq \lambda C_k \int_B u e^{u^2} dx + \lambda |B| \{e^{k^2} - 1\}. \end{aligned}$$

Accordingly we have

$$\overline{\lim_{\lambda\to 0}}\,(\lambda \int_B ue^{u^2}dx) \le 4\pi C_k.$$

Since k is arbitrary, we can take k so large to obtain

$$\lim_{\lambda\to 0}(\lambda \int_B ue^{u^2}dx) = 0, \tag{14}$$

which shows (7) and therefore (8) by (13). Using (12) again, we have

$$ru_r \to 0 \quad \text{as } \lambda \to 0 \quad \text{uniformly on } B. \tag{15}$$

This proves that u vanishes except the origin, since

$$\begin{aligned} u(x) &= -\int_{|x|}^1 u_r dr \\ &\le \frac{1}{\varepsilon}\int_\varepsilon^1 ru_r(r)dr \to 0. \end{aligned}$$

Finally, if

$$\underline{\lim_{\lambda\to 0}} \int_B |\nabla u|^2 dx < 4\pi,$$

then there is a subsequence $\{(u_m, \lambda_m)\}$ such that $\lim_{m\to\infty} \|\nabla u_m\|_2^2 = 4\pi - \delta$ for some $\delta > 0$. By virtue of the sharp version of Trudinger's inequality (3), we see

$$\int_\Omega \exp\{\alpha u_m^2\}dx \le C|\Omega|.$$

with $\alpha = 1+\varepsilon$. Since $u \in L^p(B)$ for any $2 \le p < \infty$, we have $\lambda_m u_m e^{u_m^2} \in L^{1+\varepsilon/2}$. By the standard elliptic regularity theorem, $\|\Delta u_m\|_{L^{1+\varepsilon/2}} \le C$ and

$$\|u_m\|_{L^\infty(B)} \le C \quad \text{(independent of } m\text{)},$$

which contradicts our assumption (2). Therefore we obtain (9). Q.E.D.

§3. Proof of Theorem B

By the transform $r = e^{-t/2}$ and $u(r) = w(t)$, we rewrite the equation (6) into the following:

$$(16) \qquad \begin{cases} -w''(t) = \dfrac{\lambda}{4} w(t) e^{w(t)^2 - t} & \text{on } [0, \infty), \\ w(0) = 0, \\ w'(t)e^{t/2} \to 0 & (t \to 0). \end{cases}$$

For some scaling parameter τ such that $\tau \to \infty$, we define the rescaling function $v(t)$ as

$$v(t) \equiv w^2(t + \tau) - w^2(\tau).$$

Putting $w_\tau(t) \equiv w(t + \tau)$, we see that v satisfies

$$(17.a) \qquad -v''(t) = k(w_\tau(t)) e^{v(t) - t} - \rho(w_\tau),$$

$$(17.b) \qquad v(0) = 0,$$

$$(17.c) \qquad \lim_{t \to \infty} \left(\frac{v'(t) e^{(t+\tau)/2}}{w_\tau(t)} \right) = 0,$$

where we have put

$$k(w_\tau) = \frac{\lambda}{2} w_\tau(t)^2 e^{w(\tau)^2 - \tau},$$
$$\rho(w_\tau(t)) = 2w'_\tau(t)^2.$$

We first show that;

Lemma 2. *Let $\tau > 0$ satisfies $w(t + \tau) \geq 1$ as $\lambda \to 0$ for all $t \in [-\delta, \infty)$ where $0 < \delta < \tau$. Then we have*

$$(18) \qquad \rho(w_\tau(t)) \to 0 \text{ uniformly on } [-\tau, \infty),$$

$$(19) \qquad \frac{w_\tau(t)^2}{w(\tau)^2} \to 1 \quad \text{locally uniformly on } [-\delta, \infty)$$

as $\lambda \to 0$.

Proof. Since from (15), we have for $\gamma = e^{-\tau/2}$,

$$(20) \qquad \rho(w_\tau(t)) = 2w'_\tau(t)^2 = \frac{1}{2}(\gamma r)^2 u_r(\gamma r)^2 \to 0$$

uniformly for $r \in [0, 1/\gamma]$ and therefore $t \in [-\tau, \infty)$. This shows (18).

To show (19), we use

$$w_\tau(t)^2 = w(\tau)^2 + 2\int_0^t w_\tau(s)w_\tau'(s)ds.$$

We only show the case when $t \geq 0$. The other case is similar. Since $w_\tau(t)$ is increasing in t,

$$\begin{aligned} 1 \leq \frac{w_\tau(t)^2}{w(\tau)^2} &= 1 + \frac{2}{w(\tau)^2}\int_0^t w_\tau(s)w_\tau'(s)ds \\ &\leq 1 + 2\int_0^t \frac{w_\tau(s)^2}{w(\tau)^2}\frac{w_\tau'(s)}{w_\tau(s)}ds. \end{aligned}$$

By (20), we can choose λ small so that $|w_\tau'(s)| < \varepsilon$. Then since $w_\tau(s) > 1$,

$$1 \leq X(t) \equiv \frac{w_\tau(t)^2}{w(\tau)^2} \leq 1 + 2\varepsilon\int_0^t X(s)ds.$$

This yields

$$1 \leq X(t) \leq e^{2\varepsilon t} \qquad \text{for } t \in [0, \infty).$$

In particular,

$$X(t) \to 1 \quad \text{uniformly for } t \in [0, T] \quad \text{as } \lambda \to 0$$

for some fixed T. Q.E.D.

Proof of Theorem B. In the following, we shall omit the subscriptions for each subsequences.

We split the proof into two cases.

Case 1.

$$\max_{t>0} \lambda w(t)^2 e^{w(t)^2 - t} \to \infty \qquad (\lambda \to 0).$$

Since $w(0) = 0$, we can choose the scaling sequence $\{\tau\}$ as

$$\lambda w(\tau)^2 e^{w(\tau)^2 - \tau} = 1 \tag{21}$$

for the family of solutions $\{(u, \lambda)\}$. It is easy to see

$$\begin{aligned} &\tau \to \infty, \\ &w(t) \to \infty \quad \text{as } \lambda \to 0. \end{aligned}$$

Therefore we may assume $w_\tau(t) \geq w(\tau) > 1$ for $t > -\delta$ and from Lemma 2,

$$w_\tau(t)^2 \to 0 \text{ uniformly on } [-\tau, \infty), \tag{22}$$

$$\frac{w_\tau(t)^2}{w(\tau)^2} \to 1 \quad \text{locally uniformly on } [-\delta, \infty). \tag{23}$$

Next we claim that for any fixed $T > 0$,

$$\|v\|_{L^\infty(0,T)} \leq C$$

and there is a limit function $v_0(t)$ such that

$$v(t) \to v_0(t) \quad \text{locally uniformly on } [0, \infty).$$

For that purpose, we set $q(r) = v(t)$ with $r = e^{-t/2}$. Then the equation (17) can be written as follows:

$$\begin{cases} -\Delta q = 4\tilde{k}(u(\gamma r))e^{q(r)} - \tilde{\rho}(u(\gamma r))r^{-2} & \text{on } B_{\gamma^{-1}}, \\ \quad q = 0 \text{ on } \partial B, \end{cases} \tag{24}$$

where $B_a = \{y \in \mathbb{R}^2, \quad |y| < a\}$ and

$$\tilde{k}(u(\gamma r)) = \frac{\lambda}{2}\gamma^2 u(\gamma r)^2 e^{u(\gamma)^2},$$
$$\tilde{\rho}(u(\gamma r)) = 2\gamma^2 r^2 u(\gamma r)^2.$$

Since from (21), (22) and (23), we have for $r \in [\varepsilon, 1 + \delta]$,

$$|\tilde{\rho}(u)r^{-2}| \leq C\frac{\eta^2}{\varepsilon^2} \to 0 \tag{25}$$

and

$$\tilde{k}(u(\gamma r) = \frac{\lambda}{2}w_\tau(t)^2 e^{w(\tau)^2 - \tau} = \frac{w_\tau(t)^2}{2w(\tau)^2} \to \frac{1}{2} \tag{26}$$

as $\lambda \to 0$. Therefore by the standard elliptic estimate, we have for fixed $\varepsilon > 0$,

$$|q_r(1)| \leq C, \tag{27}$$

$$\|q\|_{L^\infty(B_{1+\delta} \setminus B)} \leq C. \tag{28}$$

According to (24), (25) and (27),

(29)

$$\begin{aligned}\|\tilde{k}(u)e^q\|_{L^1(B\setminus B_\varepsilon)} &= \int_{B\setminus B_\varepsilon} \tilde{k}(u)e^q dx \\ &= \int_{B\setminus B_\varepsilon} -\Delta q dx + \int_{B\setminus B_\varepsilon} \rho(u)r^{-2}dx \\ &= 2\pi \int_\varepsilon^1 -(rq_{rr} + q_r)dr + 4\pi \int_\varepsilon^1 (\gamma r)^2 u_r^2(\gamma r)r^{-1}dr \\ &\leq -2\pi q_r(1) + C\eta^2 \int_\varepsilon^1 r^{-1}dr \\ &\leq C - C\eta^2 \log\varepsilon \leq C.\end{aligned}$$

Hence by (24), (25), (26) with (29), q satisfies

$$-\Delta q = 4\tilde{k}(u)e^q - \tilde{\rho}r^{-2} \leq 3e^q$$

with

$$\|3e^q\|_{L^1(B\setminus B_\varepsilon)} \leq C \quad \text{independent of } \lambda.$$

Then the nonlinear Harnack principle (Suzuki [16], [17]) implies the blow-up points of q in $B \setminus B_\varepsilon$ is finite. However q is radially symmetric, the blow-up points of q must be empty set. That is

$$\overline{\lim_{\lambda\to 0}} \|q\|_{L^\infty(B\setminus B_\varepsilon)} < \infty.$$

This proves

$$\|v\|_{L^\infty(0,T)} \leq C \quad \text{for small } \lambda.$$

By this a priori estimate with the equation (17) and Lemma 2, we obtain by Ascori-Arzela theorem, that there is a smooth function v_0 such that

$$v(t) \to v_0(t) \quad \text{locally uniformly on } [0,\infty)$$

with

$$-v_0''(t) = \frac{1}{2}e^{v_0(t)-t}. \tag{30}$$

We may solve (30) and conclude that

$$v(t) = u(\gamma x)^2 - u(\gamma)^2 \to v_0(t) = 2\log(\frac{2}{1+e^{-t}}) = 2\log(\frac{2}{1+|x|^2}).$$

This proves the theorem in the case 1.

Case 2.

$$\max_{t>0} \lambda w(t)^2 e^{w(t)^2-t} < \infty \qquad (\lambda \to 0). \tag{31}$$

This case is rather simple. We choose $\{\tau\}$ as

$$\lim_{t\to\infty} w(t)^2 - w(\tau)^2 = 2\log 2. \tag{32}$$

This choice of τ assures us that

$$\begin{gathered} \tau \to \infty, \\ w(\tau)^2 \to \infty \end{gathered}$$

and a priori estimate

$$0 \le v(t) \le 2\log 2. \tag{33}$$

By the assumption (31), we can choose a subsequence such that

$$\lambda w(\tau)^2 e^{w(\tau)^2-\tau} \to 2\mu \quad \text{as } \lambda \to 0 \tag{34}$$

for some constant $\mu > 0$. Lemma 2 with (33) and (34) implies that

$$v(t) \to v_0(t) \quad \text{locally uniformly on } [0,\infty)$$

with

$$\begin{cases} -v_0''(t) = \dfrac{\mu}{2} e^{v_0(t)-t}, \\ \quad v_0(0) = 0. \end{cases}$$

In fact, by the boundary condition at $t \to \infty$, we find that $\mu = 1$ and

$$v_0(t) = 2\log\left(\frac{2}{1+e^{-t}}\right).$$

This proves our conclusion of Theorem B. Q.E.D.

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