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Mapping Properties of Functions of Schrödinger Operators between L^p-Spaces and Besov Spaces

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Abstract.

Sufficient conditions are given for the boundedness of $f(H)$, $H = -\triangle + V$, in $L^p(\mathbf{R}^d)$, $1 \leq p \leq \infty$. Optimal results with respect to the decay of f are obtained for L^p-boundedness of $e^{-itH}f(H)$ and the nearly-optimal norm-estimate $\left\|e^{-itH}f(H)\right\|_{\mathcal{B}(L^p)} \leq C(1+|t|)^\gamma$, $t \in \mathbf{R}$, $\gamma > d|1/2 - 1/p|$ is proved. Results are also obtained on the mapping properties of e^{-itH} between certain Besov spaces.

§1. Introduction

In this paper we consider mapping properties of functions $f(H)$ of a Schrödinger operator $H = -\triangle + V$ between L^p-spaces. Let f be a bounded Borel function on $\mathbf{R}$. Then $f(H)$ is defined using the functional calculus and is a bounded operator on $L^2(\mathbf{R}^d)$. For $1 \leq p < \infty$ the operator $f(H)$ is densely defined on $L^p(\mathbf{R}^d)$ and one may ask whether it can be extended to a bounded operator on $L^p(\mathbf{R}^d)$. Results for $p = \infty$ are obtained from those for $p = 1$ via duality. If $H = H_0 = -\triangle$, then $f(H_0)$ is a Fourier multiplier, and conditions for L^p-boundedness are well-known. One of the goals of this paper is to extend to $f(H)$ results from the theory of Fourier multipliers.

The results in this paper extend and complement the results obtained in [JN]. The main new ingredient here is a scaling result. We also

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obtain several results on mapping properties between Besov spaces. To state the results, we need some definitions. Our main assumption on the potential V is the following:

Assumption A. V is real-valued function on $\mathbf{R}^d$, and it is decomposed as $V(x) = V_+(x) - V_-(x)$ such that $V_\pm \geq 0$, $V_+ \in K_d^{\mathrm{loc}}$ and $V_- \in K_d$, where K_d is the Kato class of potentials.

For the sake of completeness, we recall the definitions of K_d and K_d^{loc} (cf. [S, Section A2] for details, discussion and examples):

Definition 1.1. $V \in K_d$, if:

$$\text{For} \quad d \geq 3, \quad \lim_{r\to 0} \sup_{x\in\mathbf{R}^d} \int_{|x-y|\leq r} \frac{|V(y)|}{|x-y|^{d-2}} dy = 0;$$

$$\text{For} \quad d = 2, \quad \lim_{r\to 0} \sup_{x\in\mathbf{R}^d} \int_{|x-y|\leq r} \log\{|x-y|^{-1}\}|V(y)|dy = 0;$$

$$\text{For} \quad d = 1, \quad \sup_{x\in\mathbf{R}^d} \int_{|x-y|\leq 1} |V(y)|dy < \infty.$$

$V \in K_d^{\mathrm{loc}}$ if $\chi_{\{|x|<R\}}(x)V(x) \in K_d$ for any $R > 0$, where χ_Ω denotes the characteristic function of Ω.

Let V satisfy Assumption A. Then $H = -\Delta + V$ is defined on $L^2(\mathbf{R}^d)$ using the quadratic form technique, see [S] for the details.

We consider functions in the following symbol class, which may be denoted by $S^\alpha = S(\langle\lambda\rangle^\alpha, d\lambda^2/\langle\lambda\rangle^2)$ in the notation of Hörmander's $S(m,g)$-class of pseudodifferential operators. Here $\langle\lambda\rangle = (1+\lambda^2)^{1/2}$ as usual.

Definition 1.2. Let $\alpha \in \mathbf{R}$. $f \in S^\alpha$ if and only if $f \in C^\infty(\mathbf{R})$ and for any $k \geq 0$,

$$\left|\partial_\lambda^k f(\lambda)\right| \leq C_k \langle\lambda\rangle^{\alpha-k}, \qquad \lambda \in \mathbf{R}.$$

We now describe our main results and the contents of the paper. In §2 we prove three main theorems. The following result is a variant of one of the results in [JN].

Theorem 1.3. *Let $\varepsilon > 0$. If $f \in S^{-\varepsilon}$, then $f(H)$ is extended to a bounded operator in $L^p(\mathbf{R}^d)$, $1 \leq p \leq \infty$.*

The results in [JN] on the t-dependence of the norm of $e^{-itH}f(H)$ are extended in the following result:

Theorem 1.4. *Let $1 \le p \le \infty$ and let $\beta > d\left|\frac{1}{2} - \frac{1}{p}\right|$, $\gamma > d\left|\frac{1}{2} - \frac{1}{p}\right|$. If $f \in S^{-\beta}$, then $e^{-itH} f(H)$ is bounded in $L^p(\mathbf{R}^d)$ and*

$$\left\|e^{-itH} f(H)\right\| \le C \langle t \rangle^{\gamma}, \qquad t \in \mathbf{R}.$$

This result is optimal with respect to the decay of f in the sense that for $H = H_0 = -\Delta$ the L^p-boundedness of $e^{-itH_0}(H_0+1)^{-\gamma}$ implies $\gamma \ge d|\frac{1}{2} - \frac{1}{p}|$, see [Sj]. For results with optimal t-estimates, see [JN] and the comments in §2.

We prove the following resolvent estimate:

Theorem 1.5. *Let $1 \le p \le \infty$ and let $\beta = d\left|\dfrac{1}{2} - \dfrac{1}{p}\right|$. Then there exists $C > 0$ such that*

$$\left\|(H-z)^{-1}\right\|_{\mathcal{B}(L^p)} \le C \frac{\langle z \rangle^{\beta}}{|\operatorname{Im} z|^{\beta+1}}, \qquad z \notin \mathbf{R}.$$

This estimate was proved by Pang [P] with $\beta = d$. Computing the L^1-norm of the explicit integral kernel of the free resolvent one finds that this estimate holds with $\beta = (d-1)/2$ $(p = 1)$. Thus we have no reason to believe that our estimate is optimal.

An alternate method for obtaining L^p-boundedness of $f(H)$ can be based on resolvent estimates as Theorem 1.5 and the representation formula (cf. [HS])

$$f(H) = \frac{1}{2\pi i} \int_{\mathbf{C}} (\partial_{\bar{z}} \tilde{f}(z))(H-z)^{-1} dz\, d\bar{z},$$

where $\tilde{f}$ is an almost analytic continuation of f. We discuss this approach and give some results in §3 and in the Appendix.

In §§4–5 we obtain results on mapping properties of e^{-itH} between Besov spaces. We first introduce a class of generalized Besov spaces and then show that under certain regularity assumptions on V these spaces can be identified with ordinary Besov spaces. Generalized Besov spaces have previously been considered in [Pe] in a different context. For one particular case this approach was also used in [JP]. The advantage of using the Besov spaces is that one obtains results for e^{-itH} directly, avoiding the localization $f(H)$. The main result is stated as Theorem 5.2.

§2. Scaling and L^p-estimates

In this section we show that estimates in [JN] are uniform with respect to the scaling: $H \to \theta H$, $0 < \theta \le 1$, and apply it to improve L^p-estimates for $f(H)$ and $e^{-itH}f(H)$. Throughout this section, we suppose V satisfies Assumption A and assume $\sigma(H) \subseteq [0,\infty)$ without loss of generality.

Theorem 2.1. *Let* $1 \le p \le \infty$, $\beta > d\left|\frac{1}{2} - \frac{1}{p}\right|$, *and let* $g \in C_0^\infty(\mathbf{R})$. *Then there exists* $C > 0$ *such that*

$$\left\| g(\theta H)e^{-it\theta H} \right\|_{\mathcal{B}(L^p)} \le C \langle t \rangle^\beta, \qquad t \in \mathbf{R}, \tag{2.1}$$

uniformly in $0 < \theta \le 1$. *In addition, the estimate is uniform with respect to* g, *if* g *runs in a bounded set* G *in* C_0^∞, *i.e., if there is* $R > 0$ *such that* $\operatorname{supp} g \subset [-R, R]$ *and* $|\partial_\lambda^\alpha g| \le C_\alpha$ *for any* α *and any* $g \in G$.

Proof. The scaling operator $U_p(\theta)$ on $L^p(\mathbf{R}^d)$ is given by

$$U_p(\theta)\varphi(x) = \theta^{d/p}\varphi(\theta x), \qquad 0 < \theta \le 1, \quad x \in \mathbf{R}^d,$$

and $U_p(\theta)$ is an isometry in $L^p(\mathbf{R}^d)$. Then we have

$$\theta H = U_p(\sqrt{\theta})^{-1}(H_\theta)U_p(\sqrt{\theta})$$

where $H_\theta = H_0 + V_\theta$ and $V_\theta(x) = \theta V(\sqrt{\theta}x)$. In particular, this holds for $p = 2$, and by the functional calculus we learn

$$f(\theta H) = U_p(\sqrt{\theta})^{-1} f(H_\theta) U_p(\sqrt{\theta}),$$

in $L^2(\mathbf{R}^d)$, which in turn holds in any $L^p(\mathbf{R}^d)$ by a density argument. Thus it suffices to show

$$\left\| g(H_\theta)e^{-itH_\theta} \right\|_{\mathcal{B}(L^p)} \le C \langle t \rangle^\beta, \qquad t \in \mathbf{R} \tag{2.2}$$

uniformly in $0 < \theta \le 1$.

The idea of the proof is now to check all the computations in [JN] in order to conclude that the proof of (2.2) with $\theta = 1$ can be carried out with constants uniform in $0 < \theta \le 1$. It seems that two points in the argument require some comments. We discuss only these two points and omit other details.

At first, the proof of [JN, Theorem 2.1] uses the Gaussian kernel estimate for e^{-tH}. We note that if T on $L^p(\mathbf{R}^d)$ has an integral kernel

$T(x,y)$, then the scaled operator $T(\theta) = U_p(\sqrt{\theta})TU_p(\sqrt{\theta})^{-1}$ has the integral kernel given by $\theta^{d/2}T(\sqrt{\theta}x, \sqrt{\theta}y)$. Thus

$$e^{-tH_\theta} = U_p(\sqrt{\theta})e^{-t\theta H}U_p(\sqrt{\theta})^{-1}$$

has the integral kernel

$$e^{-tH_\theta}(x,y) = \theta^{d/2}\left(e^{-t\theta H}\right)(\sqrt{\theta}x, \sqrt{\theta}y). \tag{2.3}$$

On the other hand, under Assumption A, the integral kernel of e^{-tH} satisfies the bound

$$\left|e^{-tH}(x,y)\right| \le C_\varepsilon t^{-d/2}e^{Lt}\exp\left(-\frac{|x-y|^2}{4(1+\varepsilon)t}\right), \qquad t>0, \quad x,y\in\mathbf{R}^d,$$

for some $L>0$ and any $\varepsilon>0$ (see, e.g., [S, Theorem B.6.7] or [D]). Hence $e^{-t\theta H}$ satisfies

$$\left|e^{-t\theta H}(x,y)\right| \le C_\varepsilon \theta^{-d/2}t^{-d/2}e^{L\theta t}\exp\left(-\frac{|x-y|^2}{4(1+\varepsilon)\theta t}\right). \tag{2.4}$$

Combining (2.3) and (2.4), we derive

$$\begin{aligned}\left|e^{-tH_\theta}(x,y)\right| &\le C_\varepsilon t^{-d/2}e^{L\theta t}\exp\left(-\frac{|x-y|^2}{4(1+\varepsilon)t}\right)\\ &\le C_\varepsilon t^{-d/2}e^{Lt}\exp\left(-\frac{|x-y|^2}{4(1+\varepsilon)t}\right),\end{aligned}$$

which is uniform in $0<\theta\le 1$.

The second part is concerned with the commutator estimates in [JN, §3], where we need to have estimates for the operator norms on $L^2(\mathbf{R}^d)$ for $\|(H_\theta+M)^{-1/2}\|$ and $\|\partial_x(H_\theta+M)^{-1/2}\|$ on $L^2(\mathbf{R}^d)$ ($M>0$ is a sufficiently large constant). The former one is clear because it is bounded by $M^{-1/2}$. The latter follows once more from the scaling argument:

$$\begin{aligned}\left\|\partial_x(H_\theta+M)^{-1/2}\right\| &= \left\|\partial_x U_2(\sqrt{\theta})(\theta H+M)^{-1/2}U_2(\sqrt{\theta})^{-1}\right\|\\ &= \left\|\left(U_2(\sqrt{\theta})^{-1}\partial_x U_2(\sqrt{\theta})\right)(\theta H+M)^{-1/2}\right\|\\ &= \left\|\partial_x(H+\theta^{-1}M)^{-1/2}\right\| \le \left\|\partial_x(H+M)^{-1/2}\right\|.\end{aligned}$$

Remark 2.2. Under additional assumptions, e.g., if $d\le 3$, we know that (2.1) holds with $\theta=1$, $\beta=d\,|1/2-1/p|$ (see [JN, Theorems 1.4, 5.2]). In these cases, the estimate also holds with $\beta = d\,|1/2 - 1/p|$ uniformly in $0 < \theta \le 1$. The modifications needed are essentially the same as above, so we omit the details.

Proof of Theorem 1.3. Without loss of generality, we may suppose $\operatorname{supp} f \subset [-1, \infty)$. We choose $\varphi \in C_0^\infty(1/2, 2)$ so that

$$\sum_{n=-\infty}^{\infty} \varphi(2^n \lambda) = 1, \qquad \lambda > 0.$$

We let

$$\varphi_k(\lambda) = \varphi(2^{-k}\lambda), \qquad \lambda \in \mathbf{R},\ k = 1, 2, \ldots,$$

and let $\varphi_0(\lambda) \in C_0^\infty(\mathbf{R})$ such that

$$\varphi_0(\lambda) + \sum_{k=1}^{\infty} \varphi_k(\lambda) = 1, \qquad \lambda \ge -1.$$

We decompose f using $\{\varphi_k(\lambda)\}$ as follows:

$$f(\lambda) = \sum_{k=0}^{\infty} f(\lambda)\varphi_k(\lambda) = \sum_{k=0}^{\infty} f_k(2^{-k}\lambda),$$

where $f_k(\mu) = \varphi(\mu) f(2^k \mu)$ for $k \ge 1$. Then it is easy to see that $\operatorname{supp} f_k \subset (1/2, 2)$ for $k \ge 1$, and

$$\left|\partial_\mu^\alpha f_k(\mu)\right| \le C_\alpha 2^{k\alpha} \left\langle 2^k \mu \right\rangle^{-\varepsilon - \alpha} \le C_\alpha 2^{-\varepsilon k}, \qquad \mu \in \mathbf{R}, \quad k \ge 0.$$

Hence $\left\{2^{\varepsilon k} f_k(\mu)\right\}_{k=0}^{\infty}$ is a bounded set in $C_0^\infty(\mathbf{R})$. By Theorem 2.1, we learn

$$\left\|f_k(2^{-k}H)\right\|_{\mathcal{B}(L^p)} \le C\ 2^{-\varepsilon k}, \qquad k \ge 0. \tag{2.5}$$

Thus we conclude

$$\|f(H)\|_{\mathcal{B}(L^p)} \le \sum_{k=0}^{\infty} \left\|f_k(2^{-k}H)\right\|_{\mathcal{B}(L^p)} \le C \sum_{k=0}^{\infty} 2^{-\varepsilon k} < \infty.$$

Proof of Theorem 1.4. Let $f \in S^{-\beta}$ with $\beta > d|1/2 - 1/p|$ and fix γ so that $d\,|1/2 - 1/p| < \gamma < \beta$. Let φ_k and f_k be chosen as in the proof of Theorem 1.3. Then by the above argument and Theorem 2.1, we learn

$$\left\| e^{-it\theta H} f_k(\theta H) \right\|_{\mathcal{B}(L^p)} \leq C 2^{-\beta k} \langle t \rangle^{\gamma}, \quad t \in \mathbf{R},\ k \geq 0,\ 0 < \theta \leq 1.$$

Setting $\theta = 2^{-k}$, $t = 2^k s$, we have

$$\left\| e^{-isH} f_k(2^{-k}H) \right\|_{\mathcal{B}(L^p)} \leq C\, 2^{-\beta k} \left\langle 2^k s \right\rangle^{\gamma} \leq C\, 2^{-(\beta-\gamma)k} \langle s \rangle^{\gamma}.$$

Summing over k we obtain

$$\begin{aligned} \left\| e^{-isH} f(H) \right\|_{\mathcal{B}(L^p)} &\leq \sum_{k=0}^{\infty} \left\| e^{-isH} f_k(2^{-k}H) \right\|_{\mathcal{B}(L^p)} \\ &\leq C \langle s \rangle^{\gamma} \sum_{k=0}^{\infty} 2^{-(\beta-\gamma)k} \leq C \langle s \rangle^{\gamma}. \end{aligned}$$

Lemma 2.3. *Let $m > d/2$ be an integer. Then there exists $C > 0$ such that for $z \in \{z \in \mathbf{C} \setminus \mathbf{R} \,|\, |z| \leq 2\}$,*

$$\left\| (H-z)^{-1}(H+1)^{-m} \right\|_{\mathcal{B}(L^1)} \leq C |\mathrm{Im}\, z|^{-1-d/2}. \tag{2.6}$$

Moreover, the estimate holds uniformly in $\theta \in (0,1]$, if we replace H by θH.

Proof. The idea is to mimic the proof of [JN, Theorems 1.1, 1.3], so we give only a sketch. For the notation and the details, we refer to [JN].

By commutator computations as in the proof of [JN, Lemma 3.2], we have

$$\sup_{n \in \mathbf{Z}^d} \left\| \langle \cdot - n \rangle^{l} (H-z)^{-1} \langle \cdot - n \rangle^{-l} \right\| \leq C_l |\mathrm{Im}\, z|^{-l-1}$$

for $z \in \{z \in \mathbf{R} \setminus \mathbf{R} \,|\, |z| \leq 2\}$ with any $l \in \mathbf{N}$. This implies

$$\begin{aligned} \left\| (H-z)^{-1} \right\|_{l} &\equiv \left\| (H-z)^{-1} \right\| + \sup_{n \in \mathbf{Z}^d} \left\| \langle \cdot - n \rangle^{l} (H-z)^{-1} \chi_{C(n)} \right\| \\ &\leq C |\mathrm{Im}\, z|^{-l-1}. \end{aligned}$$

We let $l > d/2$ and apply [JN, Theorem 2.4] to obtain

$$\begin{aligned}\left\|(H-z)^{-1}\right\|_{\mathcal{B}(l^1(L^2))} &\leq C \left\|\left|(H-z)^{-1}\right\|\right|_l^{d/2l} \left\|(H-z)^{-1}\right\|_{\mathcal{B}(L^2)}^{1-d/2l} \\ &\leq C \left|\operatorname{Im} z\right|^{-(l+1)d/2l} \left|\operatorname{Im} z\right|^{-(1-d/2l)} \\ &= C \left|\operatorname{Im} z\right|^{-1-d/2}.\end{aligned}$$

On the other hand, $(H+1)^{-m}$ is bounded from $L^1(\mathbf{R}^d)$ to $l^1(L^2)$ ([JN, Theorem 2.1]), hence

$$\begin{aligned}&\left\|(H-z)^{-1}(H+1)^{-m}\right\|_{\mathcal{B}(L^1)} \\ &\quad\leq \left\|(H-z)^{-1}\right\|_{\mathcal{B}(l^1(L^2),L^1)} \left\|(H+1)^{-m}\right\|_{\mathcal{B}(L^1,l^1(L^2))} \\ &\quad\leq C \left\|(H-z)^{-1}\right\|_{\mathcal{B}(l^1(L^2))} \leq C \left|\operatorname{Im} z\right|^{-1-d/2}.\end{aligned}$$

The proof of the last statement is analogous to the proof of Theorem 2.1, so we omit the details.

Proof of Theorem 1.5. It suffices to consider the case $p = 1$. Other cases follow by complex interpolation. We let $\beta = d/2$ and let $m > d/2$ be an integer. We first consider the case $|z| \leq 2$. We write $z = x + iy$, and suppose $0 < y < 2$. By iterations of the first resolvent equation (recall that we assume $\sigma(H) \subseteq [0, \infty)$), we have

$$(2.7)\quad (H-z)^{-1} = \sum_{k=1}^{m} (z+1)^{k-1}(H+1)^{-k} + (z+1)^m (H-z)^{-1}(H+1)^{-m}.$$

The first term is uniformly bounded, and we estimate the second term by Lemma 2.3 to obtain

$$(2.8)\qquad \left\|(H-z)^{-1}\right\|_{\mathcal{B}(L^1)} \leq C|\operatorname{Im} z|^{-\beta-1}, \qquad |z| \leq 2.$$

Now we use the scaling argument again. By the last statement in Lemma 2.3 we may replace H by θH in (2.8) :

$$\left\|(\theta H - z)^{-1}\right\|_{\mathcal{B}(L^1)} \leq C|\operatorname{Im} z|^{-\beta-1}, \qquad |z| \leq 2,\ 0 < \theta \leq 1.$$

For $|z| > 1$, we let $z = |z| \cdot \hat{z}$, $|\hat{z}| = 1$, and let $\theta = |z|^{-1}$. Then we obtain

$$\begin{aligned}\left\|(H-z)^{-1}\right\|_{\mathcal{B}(L^1)} &= \left\|\left(|z|(|z|^{-1}H - \hat{z})\right)^{-1}\right\|_{\mathcal{B}(L^1)} \\ &= |z|^{-1} \left\|(\theta H - \hat{z})^{-1}\right\|_{\mathcal{B}(L^1)} \\ &\leq C|z|^{-1} \left|\operatorname{Im} \hat{z}\right|^{-\beta-1} = C|z|^{\beta}|\operatorname{Im} z|^{-\beta-1}.\end{aligned}$$

This completes the proof.

Remark 2.4. We could have used Theorem 1.4 instead of Lemma 2.3 to estimate the second term in the right hand side of (2.7). This gives, however, a slightly weaker result, namely, the estimate with $\beta > d|1/2 - 1/p|$.

§3. The almost analytic continuation and L^p-boundedness

In this section we discuss an alternative approach to the proof of the L^p-boundedness of functions of Schrödinger operators. The idea is to combine the almost analytic continuation method with resolvent estimates.

We introduce the following definition concerning the almost analytic continuation. A construction is discussed in the Appendix, and it is used in the proof of Theorem 3.3.

Definition 3.1. Let $f \in S^\alpha$ for some $\alpha \in \mathbf{R}$. A function $\tilde{f}$ on $\mathbf{C}$ is called an almost analytic continuation of f, if it satisfies

(1) $\tilde{f}$ is a smooth function on $\mathbf{C}$ and $\tilde{f}(x) = f(x)$ for $x \in \mathbf{R}$.
(2) For any $N \geq 0$,

$$\left|\partial_{\bar{z}}\tilde{f}(z)\right| \leq C_N \langle z\rangle^{\alpha-1-N} |\operatorname{Im} z|^N, \qquad z \in \mathbf{C}, \tag{3.1}$$

where $\partial_{\bar{z}}\tilde{f}(x+iy) = (\partial_x + i\partial_y)\tilde{f}(x+iy)$.

If $f \in S^{-\varepsilon}$, $\varepsilon > 0$, and if A is a selfadjoint operator in a Hilbert space, then it is known that $f(A)$ can be represented by the almost analytic continuation of f and the resolvent of A:

$$f(A) = \frac{1}{2\pi i}\int_{\mathbf{C}} \left(\partial_{\bar{z}}\tilde{f}(z)\right)(A-z)^{-1} dz d\bar{z} \tag{3.2}$$

(see [HS] and [G, Appendix]).

In order to apply this formula to Schrödinger operators on $L^p(\mathbf{R}^d)$, we need *a priori* estimates for the resolvent. Since the discussion of this section is methodological in its nature, we start from the following hypothesis, which includes the result of Theorem 1.5 as a special case.

Hypothesis (RE(β)). Let H be a Schrödinger operator on an $L^p(\mathbf{R}^d)$-space. We say that H satisfies RE(β), if

$$\left\|(H-z)^{-1}\right\|_{\mathcal{B}(L^p)} \leq C\frac{\langle z\rangle^\beta}{|\operatorname{Im} z|^{\beta+1}}, \qquad z \in \mathbf{C}\setminus\mathbf{R}.$$

Theorem 3.2. *Suppose H satisfies Hypothesis RE(β) on $L^p(\mathbf{R}^d)$ with $\beta \geq 0$, and suppose $f \in S^{-\varepsilon}$ with $\varepsilon > 0$. Then $f(H)$ is extended to a bounded operator in $L^p(\mathbf{R}^d)$.*

Proof. We take $N > \beta$, construct the almost analytic continuation, and then apply (3.1) and (3.2) to obtain

$$\begin{aligned}\|f(H)\|_{\mathcal{B}(L^p)} &\leq \frac{1}{2\pi}\int_{\mathbf{C}} C\frac{\langle z\rangle^{\beta}}{|\mathrm{Im}\, z|^{\beta+1}}\langle z\rangle^{-\varepsilon-1-N}|\mathrm{Im}\, z|^{N} dz d\bar{z}\\ &\leq C\int_{\mathbf{C}}\frac{\langle z\rangle^{-1-\varepsilon-(N-\beta)}}{|\mathrm{Im}\, z|^{1-(N-\beta)}} dz d\bar{z} < \infty.\end{aligned}$$

Theorem 1.3 follows easily from Theorem 3.2 and Theorem 1.5 (the proof of which is independent of Theorem 1.3) or a result by Pang [P]. We can also prove an analogue of Theorem 1.4 using the same idea. To simplify the argument, we consider only the case $f \in C_0^\infty(\mathbf{R})$.

Theorem 3.3. *Suppose H satisfies RE(β) on $L^p(\mathbf{R}^d)$ with $\beta \geq 0$ and let $f \in C_0^\infty(\mathbf{R})$. Then for any $\gamma > \beta + 1$,*

$$\left\|e^{-itH}f(H)\right\|_{\mathcal{B}(L^p)} \leq C\langle t\rangle^{\gamma}, \qquad t \in \mathbf{R}. \tag{3.3}$$

Proof. Let $f_t(x) = e^{-itx}f(x)$. Then it is easy to see that for any $s > 0$, $\|f_t\|_{H^s} \leq C_s\langle t\rangle^{s}$, $t \in \mathbf{R}$. Hence, by Lemma A.2 we learn

$$\int_{\mathbf{C}}|\mathrm{Im}\, z|^{-\gamma+\varepsilon}\left|\partial_{\bar{z}}\tilde{f}_t(z)\right| dz d\bar{z} \leq C_\varepsilon\langle t\rangle^{\gamma}, \qquad \varepsilon > 0,\ t \in \mathbf{R},$$

where $\tilde{f}_t(z)$ is the almost analytic continuation of f_t as constructed in the Appendix. Now letting $\varepsilon = \gamma - \beta - 1 > 0$, we obtain from RE($\beta$)

$$\begin{aligned}\left\|e^{-itH}f(H)\right\|_{\mathcal{B}(L^p)} &\leq \frac{1}{2\pi}\int\left|\partial_{\bar{z}}\tilde{f}_t(z)\right|\left\|(H-z)^{-1}\right\|_{\mathcal{B}(L^p)} dz d\bar{z}\\ &\leq C\int\frac{\langle z\rangle^{\beta}}{|\mathrm{Im}\, z|^{\beta+1}}\left|\partial_{\bar{z}}\tilde{f}_t(z)\right| dz d\bar{z}\\ &\leq C\int|\mathrm{Im}\, z|^{-\gamma+\varepsilon}\left|\partial_{\bar{z}}\tilde{f}_t(z)\right| dz d\bar{z}\\ &\leq C\langle t\rangle^{\gamma},\end{aligned}$$

since $\tilde{f}_t$ is compactly supported.

Combining this result with Theorem 1.5, we obtain (3.3) with $\gamma > 1 + d/2$ for $p = 1$. Thus this direct approach does not give the optimal result. Even if we had RE($(d-1)/2$) (free case), we would only get (3.3) with $\gamma > (d+1)/2$. We have lost at least order $O(\langle t\rangle^{1/2})$ in this procedure. There is another possibility, however. In the proof of Theorem 2.1 (or Theorem 1 in [JN]), the representation

$$f(H) = \int_{-\infty}^{\infty} e^{-isR} g(s)ds, \qquad R = (H+M)^{-1},$$

is used to obtain estimates for $\|\!|f(H)|\!\|_\beta$ and $\left|\!\left|\!\left| e^{-itH} f(H) \right|\!\right|\!\right|_\beta$ (see the proof of Lemma 2.3 for the definition of $\|\!|\cdot|\!\|_\beta$). An alternative is to use the representation (3.2) instead, and then we obtain optimal estimates for the t-dependence.

Remark 3.4. The almost analytic continuation technique was introduced by L. Hörmander in a series of lectures on Fourier integral operators held in 1969, see also [H1] and [H2, Chapter 3]. It was used extensively by A. Melin and J. Sjöstrand in their work on Fourier integral operators with complex phase functions. The representation formula (3.2) first appeared in [HS], and has recently been used extensively in the study of many-body Schrödinger operators.

Remark 3.5. An axiomatic approach to the functional calculus based on (3.2) and RE(β) has been given by Davies [D2].

§4. Generalized Besov spaces

Throughout this section we consider a fixed selfadjoint operator H on the Hilbert space $L^2(\mathbf{R}^d)$. Our goal is to associate with H a family of spaces in such a manner that this family becomes the usual Besov spaces for $H = -\triangle$. We define the spaces for an arbitrary selfadjoint operator on $L^2(\mathbf{R}^d)$, under certain assumptions on this operator, which are verified for $H = -\triangle + V$ by Theorem 2.1.

Assumption 4.1. For any $\varphi \in C_0^\infty(\mathbf{R})$ let $\varphi(H)$ denote the bounded operator on $L^2(\mathbf{R}^d)$ obtained via the functional calculus. Assume that $\varphi(H)$ extends to a bounded operator on $L^p(\mathbf{R}^d)$, $1 \le p \le \infty$.

REMARK. As mentioned in §1 the operator $\varphi(H)$ on $L^\infty(\mathbf{R}^d)$ is obtained as the adjoint of the corresponding operator on $L^1(\mathbf{R}^d)$, hence is uniquely determined.

Assumption 4.2. Let H satisfy Assumption 4.1. Let $\varphi \in C_0^\infty(\mathbf{R})$. Assume that for any p, $1 \le p \le \infty$, $\|\varphi(\theta H)\|_{\mathcal{B}(L^p)} \le c$ for all $\theta \in [0, 1]$, with c independent of θ.

If V satisfies Assumption A, then $H = -\Delta + V$ satisfies Assumption 4.2 by Theorem 2.1 (with $t = 0$). Fix $\varphi \in C_0^\infty(\mathbf{R})$ with $\mathrm{supp}(\varphi) \subseteq \{\lambda \,|\, 1/4 \le |\lambda| \le 4\}$ and

$$\sum_{j=-\infty}^{\infty} \varphi(4^{-j}\lambda) = 1, \qquad \lambda \ne 0.$$

Define

$$\psi_0(\lambda) = 1 - \sum_{j=1}^{\infty} \varphi(4^{-j}\lambda), \qquad \lambda \in \mathbf{R},$$

and

$$\psi_j(\lambda) = \varphi(4^{-j}\lambda), \qquad j = 1, 2, \dots, \quad \lambda \in \mathbf{R}.$$

Definition 4.3. Let H satisfy Assumption 4.2. Let p, q, s satisfy $1 \le p \le \infty$, $1 \le q < \infty$, and $s \ge 0$. For $v \in L^p(\mathbf{R}^d)$ define

$$\|v\|_{\boldsymbol{B}_p^{s,q}(H)} = \left(\sum_{j=0}^{\infty} \left(2^{sj} \left\| \psi_j(H) v \right\|_p \right)^q \right)^{1/q} \tag{4.1}$$

For $q = \infty$ the definition is modified in the obvious way. The generalized Besov space is defined by

$$\boldsymbol{B}_p^{s,q}(H) = \{ v \in L^p(\mathbf{R}^d) \,|\, \|v\|_{\boldsymbol{B}_p^{s,q}(H)} < \infty \}.$$

Lemma 4.4. *Let H satisfy Assumption 4.1. Let $u \in L^p(\mathbf{R}^d)$. Then*

$$\|u\|_p \le \sum_{j=0}^{\infty} \|\psi_j(H) u\|_p,$$

where the sum may equal $+\infty$.

Proof. Let $1 \le p < \infty$. If $u \in L^p(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$, then we have $u = \sum_{j=0}^{\infty} \psi_j(H) u$, and the assertion is clear. It follows for general

$u \in L^p(\mathbf{R}^d)$ by the density argument. The case $p = \infty$ follows by the duality argument.

Proposition 4.5. *For $s > 0$, $1 \le p, q \le \infty$ and $s = 0$, $q = 1$, $1 \le p \le \infty$, the space $\boldsymbol{B}_p^{s,q}(H)$ is a Banach space with the norm given by* (4.1). *It is a subspace of $L^p(\mathbf{R}^d)$.*

Proof. It is easy to see that (4.1) defines a norm on $\boldsymbol{B}_p^{s,q}(H)$. Let $(v^k)_{k\in\mathbf{N}}$ be a Cauchy sequence in $\boldsymbol{B}_p^{s,q}(H)$. Consider first the case $q = 1$. Then by Lemma 4.4 and $s \ge 0$

$$\|u\|_p \le \sum_{j=0}^{\infty} \|\psi_j(H)u\|_p \le \sum_{j=0}^{\infty} 2^{sj} \|\psi_j(H)u\|_p = \|u\|_{\boldsymbol{B}_p^{s,1}(H)}$$

Let $q > 1$. Let q' denote the exponent conjugate to q. Then $q' < \infty$ and for $s > 0$ we have

$$\|u\|_p \le \sum_{j=0}^{\infty} \|\psi_j(H)u\|_p \le \left(\sum_{j=o}^{\infty} 2^{-sjq'}\right)^{1/q'} \|u\|_{\boldsymbol{B}_p^{s,q}(H)}\,.$$

In either case we conclude that $\boldsymbol{B}_p^{s,q}(H)$ is a subspace of $L^p(\mathbf{R}^d)$ and that the given sequence $(v^k)_{k\in\mathbf{N}}$ is a Cauchy sequence in $L^p(\mathbf{R}^d)$, hence convergent in L^p to a limit $v \in L^p(\mathbf{R}^d)$. Define

$$\xi_j^k = 2^{sj} \left\|\psi_j(H)v^k\right\|_p$$
$$\xi_j = 2^{sj} \|\psi_j(H)v\|_p\,.$$

Then $\xi_j^k \to \xi_j$ as $k \to \infty$ for each $j = 0, 1, 2, \ldots$. Furthermore, since $\left\|v^k\right\|_{\boldsymbol{B}_p^{s,q}(H)} \le c$ for all k, we conclude that $(\xi_j)_{j\in\mathbf{N}} \in \ell^q(\mathbf{N})$.

We have now proved $v \in \boldsymbol{B}_p^{s,q}(H)$. It remains to prove convergence of the sequence $\boldsymbol{\xi}^k = (\xi_j^k)_{j\in\mathbf{N}}$ to $\boldsymbol{\xi} = (\xi_j)_{j\in\mathbf{N}}$ in $\ell^q(\mathbf{N})$. Since $(\boldsymbol{\xi}^k)_{k\in\mathbf{N}}$ is a Cauchy sequence in $\ell^q(\mathbf{N})$ and the components converge, this result is straightforward to prove. Details are omitted.

Now we prove a mapping property of e^{-itH} between abstract Besov spaces associated with H.

Theorem 4.6. *Let V satisfy Assumption* A *and let $H = -\triangle + V$. Assume $s \ge 0$, $1 \le p$, $q \le \infty$, and $\beta > d|\frac{1}{2} - \frac{1}{p}|$. Then*

$$e^{-itH} \in \mathcal{B}(\boldsymbol{B}_p^{s+2\beta,q}(H), \boldsymbol{B}_p^{s,q}(H)) \tag{4.2}$$

with norm bounded by $c\langle t\rangle^{\beta}$.

Remark 4.7. Note that the above result holds with $\beta = d\left|\frac{1}{2}-\frac{1}{p}\right|$ under restrictions on d (e.g. $d \le 3$) or under additional assumptions on V, see [JN, §5].

Proof. Fix $\chi \in C_0^\infty(\mathbf{R})$ such that $\varphi(\lambda) = \chi(\lambda)\varphi(\lambda)$ for all $\lambda \in \mathbf{R}$. For $j \ge 1$ and $u \in L^p(\mathbf{R}^d)$ we have from Theorem 2.1

$$\begin{aligned} 2^{sj}\left\|\varphi(4^{-j}H)e^{-itH}u\right\|_p &= 2^{sj}\left\|\chi(4^{-j}H)e^{-i(4^j t)4^{-j}H}\varphi(4^{-j}H)u\right\|_p \\ &\le c\,2^{j(s+2\beta)}\langle t\rangle^{\beta}\left\|\varphi(4^{-j}H)u\right\|_p. \end{aligned}$$

The estimate for $j = 0$ follows from Theorem 2.1. The result now follows from the definition of the norm (4.1) and the covering argument.

We note the following results, which are useful in the next section.

Proposition 4.8. *Let* V *satisfy Assumption* A. *Assume* $1 \le p, q, q_1 \le \infty$ *and* $s \ge s_1 > 0$. *If either* $s > s_1$ *or* $s = s_1$ *and* $q \le q_1$, *then* $\boldsymbol{B}_p^{s,q}(H)$ *is continuously embedded in* $\boldsymbol{B}_p^{s_1,q_1}(H)$.

Proof. The argument in the proof of [BTW, Theorem 2.2.1] carries over unchanged to our generalized Besov spaces.

Lemma 4.9. *Let* V *satisfy Assumption* A *and let* $s \ge 0$, $1 \le p, q \le \infty$. *Let* $M \in \mathbf{R}$. *Then* $\boldsymbol{B}_p^{s,q}(H+M) = \boldsymbol{B}_p^{s,q}(H)$ *with equivalent norms.*

Proof. A simple covering argument, which is omitted.

§5. Identification with ordinary Besov spaces

We have chosen the definition of $\boldsymbol{B}_p^{s,q}(H)$ in such a manner that for $H = -\Delta$ this space is identical with the usual Besov space, which we here denote $B_p^{s,q}$. In applications it is of interest to know conditions on V which imply $\boldsymbol{B}_p^{s,q}(H) = B_p^{s,q}$ (with equivalent norms).

Our result on this question is based on the real interpolation method and interpolation spaces defined via semigroups. We refer to [BL] for the results needed. We recall a few results from [BL, Section 6.7]. Let

$G(t)$, $t > 0$, be a strongly continuous bounded semigroup on a Banach space $\mathcal{X}$ with infinitesimal generator Λ. For $u \in \mathcal{X}$ define

$$\omega(t, u) = \sup_{s<t} \|G(s)u - u\|_{\mathcal{X}} .$$

The real interpolation method constructs a family of Banach spaces between the domain $\mathcal{D}(\Lambda)$ of Λ (with the graph norm) and $\mathcal{X}$, denoted $(\mathcal{X}, \mathcal{D}(\Lambda))_{\theta,q}$, $0 < \theta < 1$, $1 \leq q \leq \infty$. In [BL, Theorem 6.7.3] it is shown that the norm $\|u\|_{(\mathcal{X},\mathcal{D}(\Lambda))_{\theta,q}}$ is equivalent to the norm given by

$$\|u\|_{\mathcal{X}} + \left(\int_0^\infty t^{-\theta q - 1} \omega(t, u)^q dt \right)^{1/q} . \tag{5.1}$$

The usual L^p-type Sobolev space of order $m \in \mathbf{N}$ is denoted $W_p^m(\mathbf{R}^d)$.

Assumption $B(p, m)$. Let $1 \leq p \leq \infty$ and let $m \in \mathbf{N}$. Let V satisfy Assumption A, and let $H = -\Delta + V$. Assume there exists $M \geq 0$ such that $(H + M)^{-m}$ is a bounded map from $L^p(\mathbf{R}^d)$ to $W_p^m(\mathbf{R}^d)$ with a bounded inverse.

Theorem 5.1. *Let V satisfy Assumption $B(p, m)$ for some $m \in \mathbf{N}$ and $1 \leq p \leq \infty$. Then for $1 \leq q \leq \infty$, $0 < s < 2m$, $\boldsymbol{B}_p^{s,q}(H) = B_p^{s,q}$ (with equivalent norms).*

Proof. Let V satisfy Assumption $B(p, m)$. We first show that $-(H + M)^m - L$ generates a strongly continuous bounded semigroup with $M, L > 0$ and the domain of the generator is $W_p^m(\mathbf{R}^d)$. Without loss of generality we may assume $M = 0$ and $H > 1$. Then by Theorem 1.3, $U(t) = e^{-tH^m}$ is bounded in $L^p(\mathbf{R}^d)$. Moreover, by Theorem 2.1 $U(t) = e^{-(t^{1/m} H)^m}$ is uniformly bounded with respect to $t \in (0, 1]$. Hence there is $L \in \mathbf{R}$ such that $\|U(t)\|_{\mathcal{B}(L^p)} \leq Ce^{Lt}$ for any $t > 0$. Thus $-(H^m + L)$ generates a bounded C_0 semigroup. The strong continuity follows from the fact that it is strongly continuous in $L^2(\mathbf{R}^d)$. The expression of the resolvent by the semigroup:

$$(\Lambda + K)^{-1} = -\int_0^\infty e^{-Kt} U(t) dt, \qquad K > L,$$

where Λ is the generator of $U(t)$, implies $(H^m + K)^{-1} = (\Lambda + K)^{-1}$, and hence the domain of Λ is $W_p^m(\mathbf{R}^d)$. We assume $L = 0$ in the sequel in order to simplify the notation.

Now we let $\Lambda = -H^m$ and let $G(t)$, $t > 0$, denote the semigroup generated by Λ. Let $\mathcal{D} = \mathcal{D}(\Lambda) = W_p^m(\mathbf{R}^d)$. Note that the usual Sobolev norm and the graph norm of Λ are equivalent norms on $\mathcal{D}$, as can be seen using the closed graph theorem.

Fix q, $1 \le q < \infty$ (the case $q = \infty$ requires obvious modifications in the arguments below) and s, $0 < s < 2m$. Define $\theta = \dfrac{s}{2m}$. It follows from the real interpolation method (see [BL]) that

$$B_p^{s,q} = (L^p, \mathcal{D})_{\theta,q}.$$

Thus to prove the theorem it suffices to prove

$$\boldsymbol{B}_p^{s,q}(H) = (L^p, \mathcal{D})_{\theta,q}$$

with equivalent norms. We follow essentially the arguments in [BL, p. 160–1]. Let φ, ψ_j denote the functions from §4 used in our definition of the generalized Besov spaces.

Assume first $u \in (L^p, \mathcal{D})_{\theta,q}$. Let $\Phi(\lambda) = \varphi(\lambda)(\exp(-\lambda^m) - 1)^{-1}$, $\lambda \in \mathbf{R}$. Note $\Phi \in C_0^\infty(\mathbf{R})$. Using Theorem 2.1 we find $\|\Phi(4^{-j}H)\|_{\mathcal{B}(L^p)} \le c$ for $j = 0, 1, 2, \ldots$. Therefore

$$\|\psi_j(H)u\|_p = \left\|\Phi(4^{-j}H)(G(4^{-mj}) - 1)u\right\|_p \le c\,\omega(4^{-mj}, u).$$

Using (4.1) we conclude

$$\|u\|_{\boldsymbol{B}_p^{s,q}(H)} \le c\left(\|u\|_p + \left(\sum_{j=0}^{\infty}(2^{sj}\omega(2^{-2mj}, u))^q\right)^{1/q}\right).$$

Since $\omega(t, u)$ is an increasing function of t and we have

$$\int_{2^{-2mj}}^{2^{-2m(j-1)}} t^{-\theta q - 1}\,dt = c\,2^{2m\theta jq} = c\,2^{sjq},$$

we get

$$\begin{aligned}
\sum_{j=0}^{\infty} 2^{sjq}\omega(2^{-2mj}, u)^q &= c\sum_{j=0}^{\infty}\int_{2^{-2mj}}^{2^{-2m(j-1)}} t^{-\theta q-1}\omega(2^{-2mj}, u)^q dt \\
&\le c\sum_{j=0}^{\infty}\int_{2^{-2mj}}^{2^{-2m(j-1)}} t^{-\theta q-1}\omega(t, u)^q dt \\
&\le c\int_0^\infty t^{-\theta q-1}\omega(t, u)^q dt.
\end{aligned}$$

Using (5.1) we conclude

$$\|u\|_{B_p^{s,q}(H)} \leq c\,\|u\|_{(L^p,\mathcal{D})_{\theta,q}} \tag{5.2}$$

which proves the first half of the theorem. To prove the second half, assume $u \in B_p^{s,q}(H)$. Theorem 2.1 implies

$$\|\Lambda\psi_j(H)u\|_p \leq c\,4^{mj}\,\|\psi_j(H)u\|_p\,, \qquad j = 0,1,2,\ldots.$$

Using

$$\|G(s)u - u\|_p \leq \int_0^s \|G(\tau)\Lambda u\|_p\,d\tau$$

and

$$\|G(s)u - u\|_p \leq 2\,\|u\|_p$$

we get (see also Lemma 4.4)

$$\omega(t,u) \leq c\sum_{j=0}^{\infty} \min\{1, t\,4^{mj}\}\,\|\psi_j(H)u\|_p\,.$$

We estimate the integral term in (5.1). The integral is split as

$$\int_0^\infty \cdots\, dt = \int_1^\infty \cdots\, dt + \sum_{k=0}^{\infty}\int_{4^{-m(k+1)}}^{4^{-mk}} \cdots\, dt. \tag{5.3}$$

We introduce the notation $\alpha_j = \|\psi_j(H)u\|_p$. For $t \in (4^{-m(k+1)},\, 4^{-mk})$ we have $\min\{1, t\,4^{mj}\} = 1$, if $j \geq k+1$, and $\min\{1, t\,4^{mj}\} = t\,4^{mj}$, if $j \leq k$. This result is inserted in the sum in (5.3) to get

$$\begin{aligned}
&\sum_{k=0}^{\infty}\int_{4^{-m(k+1)}}^{4^{-mk}} t^{-\theta q-1}\left(\sum_{j=0}^{k} t\,4^{mj}\alpha_j + \sum_{j=k+1}^{\infty}\alpha_j\right)^q dt\\
&\qquad \leq c\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k} 4^{mj-(1-\theta)mk}\alpha_j\right]^q + c\sum_{k=0}^{\infty}\left[\sum_{j=k+1}^{\infty} 4^{\theta mk}\alpha_j\right]^q\\
&= c\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k} 4^{m(1-\theta)(j-k)}(4^{\theta jm}\alpha_j)\right]^q + c\sum_{k=0}^{\infty}\left[\sum_{j=k+1}^{\infty} 4^{\theta m(k-j)}(4^{\theta jm}\alpha_j)\right]^q.
\end{aligned}$$

Since $u \in B_p^{s,q}(H)$, $(4^{\theta jm}\alpha_j)_{j\in\mathbf{N}} \in \ell^q(\mathbf{N})$, and in both cases above we have convolution by a sequence in ℓ^1, so we use Young's inequality to conclude

$$\left(\int_0^1 t^{-\theta q-1}\omega(t,u)^q dt\right)^{1/q} \le c\left(\sum_{j=0}^{\infty}\left(4^{\theta mj}\left\|\psi_j(H)u\right\|_p\right)^q\right)^{1/q}.$$

The other term in (5.3) is estimated using Hölder's inequality:

$$\left(\int_1^\infty t^{-\theta q-1}\omega(t,u)^q dt\right)^{1/q} \le \left(\int_1^\infty t^{-\theta q-1}\left(\sum_{j=0}^{\infty}\alpha_j\right)^q dt\right)^{1/q}$$

$$= c\sum_{j=0}^{\infty}\alpha_j \le c\left(\sum_{j=0}^{\infty}\left(4^{\theta mj}\alpha_j\right)^q\right)^{1/q}.$$

Combining these estimates we get

$$\|u\|_{(L^p,\mathcal{D})_{\theta,q}} \le c\,\|u\|_{\boldsymbol{B}_p^{s,q}(H)} \tag{5.4}$$

which proves the second half of the theorem.

Theorem 5.1 combined with Theorem 4.6 implies the following mapping property of e^{-itH} between (usual) Besov spaces.

Theorem 5.2. *Let V satisfy Assumptions* A *and* $B(p,m)$, *and let* $H=-\Delta+V$. *Assume* $1\le p,\ q\le\infty$, $\beta>d|\frac{1}{2}-\frac{1}{p}|$, $\gamma>d|\frac{1}{2}-\frac{1}{p}|$, *and* $0\le s<2(m-\beta)$. *Then*

$$e^{-itH}\in\mathcal{B}(B_p^{s+2\beta,q},B_p^{s,q}) \tag{5.5}$$

with norm bounded by $c\,\langle t\rangle^\gamma$.

Concerning the Assumption $B(p,m)$ we note that for $m=1$ we can use standard perturbation results to show that if V is bounded relative to the Laplacian on $L^p(\mathbf{R}^d)$ with relative bound less than one, then the condition is satisfied. Several sufficient conditions for this to hold can be found in [Sc]. For $m>1$ some regularity is needed. If $V\in C^\infty(\mathbf{R}^d)$ with all derivatives bounded, then Assumption $B(p,m)$ holds for all $m\ge1$ and all p, $1\le p\le\infty$.

Remark 5.3. Note that the proof of Theorem 5.2 also yields

$$e^{-itH}\in\mathcal{B}(B_p^{2\beta,q},L^p(\mathbf{R}^d)) \tag{5.6}$$

under the same assumptions. In this form the result is a direct generalization of the results on the free Schrödinger equation in [BTW].

Remark 5.4. In the proof of Theorem 5.1 we have shown that $-(H+M)^m - L$ generates a bounded C_0 semigroup. This result has also been obtained by Davies [D2] in an abstract framework, cf. Remark 3.5.

Appendix. A construction of an almost analytic continuation

In this appendix we propose a construction of an almost analytic continuation, and discuss its properties. We start by constructing an almost analytic continuation of $f \in C_0^\infty(-2,2)$.

We fix $\chi \in C_0^\infty(\mathbf{R})$ such that $0 \le \chi(x) \le 1$,

$$\chi(x) = \begin{cases} 1, & \text{if } |x| \le 1, \\ 0, & \text{if } |x| \ge 2, \end{cases}$$

and let $\rho(x) = \int_0^x \chi(y)dy$. For $f \in C_0^\infty(-2,2)$, we define $\tilde{f}(z)$, $z \in \mathbf{C}$ by

$$\tilde{f}(x+iy) = (2\pi)^{-1/2}\chi(x/2)\chi(y)\int_{-\infty}^{\infty} e^{-\rho(y\xi)}e^{ix\xi}\hat{f}(\xi)d\xi, \tag{A.1}$$

where $\hat{f}(\xi)$ denotes the Fourier transform of $f(x)$.

Lemma A.1. *$\tilde{f}(z)$ is an almost analytic continuation of $f(x)$.*

Proof. It is easy to see that $\tilde{f}(z) \in C_0^\infty(\mathbf{C})$ because $\hat{f} \in \mathcal{S}$, and $e^{-\rho(y\xi)}$ is a smooth bounded function. It is also easy to see that $\tilde{f}(x) = f(x)$ for $x \in \mathbf{R}$ since $\rho(0) = 0$. It remains to show (3.1). By direct computation we have

$$\begin{aligned} \left(\partial_{\bar{z}}\tilde{f}\right)(x+iy) &= \left(\partial_x + i\partial_y\right)\tilde{f}(x+iy) \\ &= (2\pi)^{-1/2}\chi(x/2)\chi(y)\int i\xi(1-\rho'(y\xi))e^{-\rho(y\xi)}e^{ix\xi}\hat{f}(\xi)d\xi \\ &\quad + (2\pi)^{-1/2}2^{-1}\chi'(x/2)\chi(y)\int e^{-\rho(y\xi)}e^{ix\xi}\hat{f}(\xi)d\xi \\ &\quad + i(2\pi)^{-1/2}\chi(x/2)\chi'(y)\int e^{-\rho(y\xi)}e^{ix\xi}\hat{f}(\xi)d\xi \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{A.2}$$

To estimate the first term, we note that $\rho'(y\xi) = \chi(y\xi) = 1$ if $|y\xi| \le 1$, hence

$$|1-\rho'(y\xi)| = |1-\chi(y\xi)| \le \chi_{\{|y\xi|\ge 1\}}(y,\xi)$$

where χ_Ω denotes the characteristic function of Ω. Then we have

$$
\begin{aligned}
|\mathrm{I}| &\le C\int |\xi|\chi_{\{|y|\,|\xi|\ge 1\}}(y,\xi)|\hat{f}(\xi)|d\xi \\
&\le C\int |\xi|^{N+1}|y|^N|\hat{f}(\xi)|d\xi \le C|y|^N \qquad \text{(A.3)}
\end{aligned}
$$

since $\hat{f}\in\mathcal{S}$. To estimate the second term, we note that $f(x)\equiv 0$ on $\operatorname{supp}\chi'(x/2)$. Hence

$$
0=\chi'\Big(\frac{x}{2}\Big)\sum_{k=0}^{N}\frac{(iy)^k f^{(k)}(x)}{k!}=(2\pi)^{-1/2}\chi'\Big(\frac{x}{2}\Big)\int\sum_{k=0}^{N}\frac{(-y\xi)^k}{k!}e^{ix\xi}\hat{f}(\xi)d\xi.
$$

We subtract this from (II) to obtain

$$
\begin{aligned}
|\mathrm{II}| &\le C\int\left|\left(e^{-\rho(y\xi)}-\sum_{k=0}^{N}\frac{(-y\xi)^k}{k!}\right)\right||\hat{f}(\xi)|d\xi \\
&\le C\int |y|^{N+1}|\xi|^{N+1}|\hat{f}(\xi)|d\xi \\
&\le C|y|^{N+1}. \qquad \text{(A.4)}
\end{aligned}
$$

The estimate for (III) is easy since it is supported away from the real axis.

Once an almost analytic extension is constructed for a C_0^∞-function, it is then standard procedure to extend it to $f\in S^\alpha$. We include the construction for the sake of completeness. Let $\varphi\in C_0^\infty(1/2,2)$ as in the proof of Theorem 1.3, and let $\varphi_j(x)\in C_0^\infty(\mathbf{R})$ defined by

$$
\begin{aligned}
&\varphi_{\pm k}(x)=\varphi(\pm 2^{-k}x),\quad k=1,2,\dots,\ x\in\mathbf{R},\\
&\varphi_0(x)=1-\sum_{k\ne 0}\varphi_k(x),\quad x\in\mathbf{R}.
\end{aligned}
$$

We decompose $f\in S^\alpha$ as

$$
f(x)=\sum_{j=-\infty}^{\infty}f(x)\varphi_j(x)=\sum_{j=-\infty}^{\infty}f_j(2^{-|j|}x),
$$

where $f_k(y)=\varphi(\operatorname{sign}(k)y)f(2^{|k|}y)$ for $k\ne 0$ and $f_0(y)=f(y)\varphi_0(y)$. Now we can apply the above construction to each $f_j(x)$ to obtain $\tilde{f}_j(z)$. Note that we can modify the construction such that $\tilde{f}_j(z)$ is supported in $\{z \mid \operatorname{Re} z \in [1/4, 4], |\operatorname{Im} z| \leq 2\}$ for $j > 0$ and in $\{z \mid \operatorname{Re} z \in [-4, -1/4], |\operatorname{Im} z| \leq 2\}$ for $j < 0$. Then $\tilde{f}(z) = \sum_{j=-\infty}^{\infty} \tilde{f}_j(2^{-j}z)$ defines an almost analytic continuation of f. Further details are omitted.

Compared with the other known constructions of an almost analytic continuation, our method seems to have the advantage of being straightforward, namely, we do not use asymptotic sums. On the other hand, we need no differentiability of f to define $\tilde{f}(z)$, and the proof of Lemma A.1 shows that (3.1) with $N = a \in \mathbf{R}_+$ follows from $f \in H_0^s$, $s > a + 3/2$. In fact, it is known that $f \in C_0^{1+a}(\mathbf{R})$ is sufficient to construct $\tilde{f}(z)$ satisfying (3.1) with $N = a$ (E. B. Davies, private communication, see also [D2]). Our construction may be not as precise as Davies', but the next lemma is sufficient for our application in §3.

Lemma A.2. *Let $R > 0$ be fixed, and let $f \in H_0^s([-R, R])$ with $s \geq 1$. Then for any $\varepsilon > 0$ there is $C = C(R, \varepsilon)$ such that*

$$\int_{\mathbf{C}} |\operatorname{Im} z|^{-s+\varepsilon} \left|\partial_{\bar{z}} \tilde{f}(z)\right| dz d\bar{z} \leq C \|f\|_{H_0^s}. \tag{A.5}$$

Proof. It suffices to consider the case $R = 1$, and we may assume $\tilde{f}(z)$ is defined by (A.1). As in the proof of Lemma A.1, we decompose $\partial_{\bar{z}} \tilde{f}$ as $\partial_{\bar{z}} \tilde{f} = \mathrm{I} + \mathrm{II} + \mathrm{III}$. We start by estimating (I). As in the computation to derive (A.3), for each y we have

$$\begin{aligned}\left(\int \left|\xi(1 - \rho'(y\xi))\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} &\leq |y|^{s-1} \left(\int |\xi|^{2s} |\hat{f}(\xi)|^2\right)^{1/2} \\ &\leq C|y|^{s-1} \|f\|_{H_0^s}.\end{aligned}$$

Hence by Plancherel's theorem, we have

$$\begin{aligned}\int_{\mathbf{C}} |\operatorname{Im} z|^{-s+\varepsilon} |\mathrm{I}(z)| dz d\bar{z} &= \int_{|x|,|y| \leq 2} |y|^{-s+\varepsilon} |\mathrm{I}(x+iy)| dx dy \\ &\leq C \int_{|y| \leq 2} \left(\int |\mathrm{I}(x+iy)|^2 dx\right)^{1/2} |y|^{-s+\varepsilon} dy \\ &\leq C \int_{|y| \leq 2} \left(|y|^{s-1} \|f\|_{H_0^s}\right) |y|^{-s+\varepsilon} dy \\ &= C \left(\int_{|y| \leq 2} |y|^{-1+\varepsilon} dy\right) \|f\|_{H_0^s} = C \|f\|_{H_0^s}.\end{aligned}$$

On the other hand, (A.4) implies

$$|\,\mathrm{II}(x+iy)| \le C|y|^{s-1}\,\|f\|_{H_0^s}\,,$$

and the estimate for (II) follows from this. The estimate for (III) is easy and we omit it.

References

[BL] J. Bergh, J. Löfström, "Interpolation spaces", Springer Verlag, Berlin, Heidelberg, New York, 1976.

[BTW] P. Brenner, V. Thomée, L. B. Wahlbin, "Besov Spaces and Applications to Difference Methods for Initial Value Problems", Springer Lecture Notes in Math. **434**, Springer Verlag, Berlin, Heidelberg, New York, 1975.

[D] E. B. Davies, "Heat Kernels and Spectral Theory", Cambridge University Press, Cambridge, 1989.

[D2] E. B. Davies, The functional calculus, Report no. 2, Mittag-Leffler Institute 1992..

[G] C. Gérard, Sharp propagation estimates for N-particle systems, Duke Math. J., **67** (1992), 483–515.

[H1] L. Hörmander, On the singularities of solutions of partial differential equations, in "Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969)", Univ. of Tokyo Press, Tokyo, 1970, pp. 31–40.

[H2] L. Hörmander, "The Analysis of Linear Partial Differential Operators, Vol. I", Springer Verlag, Berlin, Heidelberg, New York, 1983.

[HS] B. Helffer, J. Sjöstrand, Equation de Schrödinger avec champ magnétique et équation de Harper, in "Schrödinger Operators", Springer Lecture Notes in Physics vol. **345**, Springer Verlag, Berlin, Heidelberg, New York, 1989, pp. 118–197.

[JN] A. Jensen, S. Nakamura, L^p-mapping properties of functions of Schrödinger operators and their applications to scattering theory, Preprint, Aalborg University, March 1992.

[JP] A. Jensen, P. Perry, Commutator methods and Besov space estimates for Schrödinger operators, J. Operator Theory, **14** (1985), 181–188.

[P] M. M. H. Pang, Resolvent estimates for Schrödinger operators in $L^p(\mathbf{R}^N)$ and the theory of exponentially bounded C-semigroups, Semigroup Forum, **41** (1990), 97–114.

[Pe] J. Peetre, "New Thoughts on Besov Spaces", Duke University Mathematics Series I, Duke Univ. Math. Dept., 1976.

[Sc] M. Schechter, "Spectra of Partial Differential Operators", North-Holland, Amsterdam, London, 1971.

[S] B. Simon, Schrödinger semigroups, Bull. Amer. Math. Soc., **7** (1982), 447–526.

[Sj] S. Sjöstrand, On the Riesz mean of the solution of the Schrödinger equation, Ann. Scuola Norm. Sup. Pisa, **24** (1970), 332–348.

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