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Submanifolds of Symmetric Spaces and Gauss Maps

Hiroo Naitoh

Dedicated to Professor Tadashi Nagano on his sixtieth birthday

Abstract.

We study Gauss maps for submanifolds of riemannian symmetric spaces and show that they have the same properties as the Gauss maps for submanifolds of euclidean spaces.

Let (M, g) be a simply connected riemannian symmetric space without Euclidean factor and denote by R the curvature tensor. A linear subspace V of a tangent space T_pM is called *strongly curvature invariant* if it satisfies that

$$(0.1) \qquad R_p(V, V)V \subset V \qquad \text{and} \qquad R_p(V^\perp, V^\perp)V^\perp \subset V^\perp,$$

where $V^\perp$ denotes the orthogonal complement of V in T_pM. Strongly curvature invariant subspaces V of T_pM and W of T_qM are said to be *equivalent* to each other if there exists an isometry φ of (M, g) such that $\varphi(p) = q$, $\varphi_{*p}(V) = W$. Denote by $[V]$ the equivalence class of V and by $\mathcal{S}(M, g)$ the set of all the equivalence classes. For $\mathcal{V} \in \mathcal{S}(M, g)$ a connected submanifold S of M is called a *$\mathcal{V}$-submanifold* if it holds that $[T_pS] = \mathcal{V}$ for any $p \in S$. For each $\mathcal{V}$ there exists a unique complete totally geodesic $\mathcal{V}$-submanifold except the congruence by isometries, and for any $\mathcal{V}$-submanifold we can construct "Gauss map" (Naitoh [5]).

In this paper we first show that the target space of this Gauss map is a connected component of the space of all the complete totally geodesic $\mathcal{V}^\perp$-submanifolds. Here $\mathcal{V}^\perp$ is the equivalence class of the orthogonal complement of a subspace representing $\mathcal{V}$. We next show that the following two properties hold for our Gauss map. These properties seem to be fundamental for "Gauss map". One is that a $\mathcal{V}$-submanifold has

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the parallel mean curvature vectors if and only if the Gauss map is harmonic, and another is that a $\mathcal{V}$-submanifold has the parallel second fundamental form if and only if the Gauss map is totally geodesic. Last we concretely give the target spaces of the Gauss maps associated with $\mathcal{V}$-submanifolds of the rank one symmetric spaces.

§1. The space of the totally geodesic $\mathcal{V}^{\perp}$-submanifolds

Fix an equivalence class $\mathcal{V}$ in $\mathcal{S}(M,g)$. Denote by $\mathcal{T}_{\mathcal{V}^{\perp}}$ the set of all complete totally geodesic $\mathcal{V}^{\perp}$- submanifolds of M and by $C_{\mathcal{V}}$ the set of the strongly curvature invariant subspaces representing $\mathcal{V}$. We first define a relation on the set $C_{\mathcal{V}}$ in the following: Two subspaces in $C_{\mathcal{V}}$ are *related* to each other if they are normal spaces of a complete totally geodesic $\mathcal{V}^{\perp}$-submanifold. This relation is an equivalence relation since a strongly curvature invariant subspace representing $\mathcal{V}^{\perp}$ determines a unique complete totally geodesic $\mathcal{V}^{\perp}$-submanifold such that the subspace is a tangent space of it ([2]). Denote by $\langle V\rangle$ the equivalence class of V in $C_{\mathcal{V}}$ and by $\mathcal{C}_{\mathcal{V}}$ the set of all the equivalence classes.

Lemma 1.1. *For $S \in \mathcal{T}_{\mathcal{V}^{\perp}}$ the normal spaces N_pS,$p \in S$, of S are related to each other in $C_{\mathcal{V}}$ and the correspondence:*

$$\mathcal{T}_{\mathcal{V}^{\perp}} \ni S \longrightarrow \langle N_pS\rangle \in \mathcal{C}_{\mathcal{V}}$$

is bijective.

Proof. This follows again since a strongly curvature invariant subspace representing $\mathcal{V}^{\perp}$ determines a unique complete totally geodesic $\mathcal{V}^{\perp}$-submanifold such that the subspace is a tangent space of it.

Q.E.D.

Now denote by r the dimension of the subspaces representing $\mathcal{V}$. Let $\Lambda^r(p)$ be the Grassmannian manifold of all the r-dimensional subspaces of T_pM and $\Lambda^r(M)$ the fibre bundle over M with the fibres $\Lambda^r(p), p \in M$. Then, since the isometry group $I(M,g)$ of (M,g) is a Lie transformation group of M, it is also a Lie transformation group of $\Lambda^r(M)$ in the following action: $\varphi\cdot V = \varphi_*(V)$ for $\varphi \in I(M,g)$, $V \in \Lambda^r(M)$. The set $C_{\mathcal{V}}$ is a closed topological subspace of $\Lambda^r(M)$ by (0.1), and it is preserved by this action. Hence the restriction to $C_{\mathcal{V}}$ of this action makes $I(M,g)$ a topological transformation group of $C_{\mathcal{V}}$. Consider the quatient topology on $\mathcal{C}_{\mathcal{V}}$ induced from $C_{\mathcal{V}}$. Then, since the action on $C_{\mathcal{V}}$ preserves the above relation , it also makes $I(M,g)$ a topological transformation group of $\mathcal{C}_{\mathcal{V}}$. Since $I(M,g)$ acts transitively on $C_{\mathcal{V}}$ and $\mathcal{C}_{\mathcal{V}}$, these spaces have unique differentiable structures so that $I(M,g)$ is Lie transformation groups, respectively. Moreover the identity component G of $I(M,g)$ acts transitively on each connected component of $C_\mathcal{V}$ (resp. $\mathcal{C}_\mathcal{V}$) and all the connected components of $C_\mathcal{V}$ (resp. $\mathcal{C}_\mathcal{V}$) are quatient manifolds of G diffeomorphic to each other.

Let M^* be a connected component of $\mathcal{C}_\mathcal{V}$ and fix a point p_* of M^*. Take a subspace V of T_pM such that $V \in C_\mathcal{V}$ and $\langle V \rangle = p_*$. Denote by K, K_* the isotropy subgroups of p, p_* in G, respectively. Denote by s_p the geodesic symmetry at p of (M,g) and by t_p the isometry of (M,g) satisfying that $t_p(p) = p$ and $(t_p)_{*p}x = -x$ or x according as $x \in V$ or $x \in V^\perp$. Such t_p uniquely exists from the condition (0.1) and the simple connectedness of M. The isometries induce involutive automorphisms σ, τ of G in the following way: $\sigma(h) = s_p \circ h \circ s_p$, $\tau(h) = t_p \circ h \circ t_p$ for $h \in G$. Then the followings hold ([2] and [5]):

$$(\mathrm{Fix}\ \sigma)_0 \subset K \subset \mathrm{Fix}\ \sigma, \qquad \text{and} \qquad (\mathrm{Fix}\ \tau)_0 \subset K_* \subset \mathrm{Fix}\ \tau,$$

where Fix $*$ denotes the Lie subgroup of the points fixed by $*$ and $(\mathrm{Fix}\ *)_0$ the identity component of Fix $*$. Hence (G,K) and (G,K_*) are symmetric pairs. Let $\mathfrak{g}$ be the Lie algebra of G and denote by the same notations σ, τ the differentials of σ, τ. Since s_p and t_p commute, the involutive automorphisms σ, τ also commute. Decompose the Lie algebra $\mathfrak{g}$ into the (± 1)-eigenspaces $\mathfrak{g}_{\pm 1}$ of σ, and moreover decompose $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ into the (± 1)-eigenspaces $\mathfrak{g}_{1\pm 1}$ and $\mathfrak{g}_{-1\pm 1}$ of τ, respectively. Then the Lie algebras of K, K_* are given by $\mathfrak{g}_1$, $\mathfrak{g}_{11} \oplus \mathfrak{g}_{-11}$ and the following identifications hold:

$$T_pM = \mathfrak{g}_{-1} = \mathfrak{g}_{-11} \oplus \mathfrak{g}_{-1-1}, V = \mathfrak{g}_{-1-1}, V^\perp = \mathfrak{g}_{-11},$$

and

$$T_{p*}M^* = \mathfrak{g}_{1-1} \oplus \mathfrak{g}_{-1-1}.$$

These identifications are given by corresponding $X \in \mathfrak{g}$ to the values at p, p_* of vector fields on M, M^* generated by the one parameter subgroup $\exp tX$ of G, respectively.

We define a riemannian metric g_* on M^* as follows. Under the identification $T_pM = \mathfrak{g}_{-1}$ regard the metric g_p on T_pM as an inner product on $\mathfrak{g}_{-1}$. Then the inner product is uniquely extended to a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ such that $\langle \mathfrak{g}_1, \mathfrak{g}_{-1} \rangle = \{0\}$ and that $\mathrm{ad}(X), X \in \mathfrak{g}$, are skew symmetric. Note that $\langle\ ,\ \rangle$ is τ-invariant and so nondegenerate on $\mathfrak{g}_{1-1} \oplus \mathfrak{g}_{-1-1}$. Hence the bi-invariant indefinite metric on G induced by $\langle\ ,\ \rangle$ induces a pseudo-riemannian metric g_* on M^*. This metric is determined independently of the fixed point p of M.

Theorem 1.2 (Naitoh[5]). *The space* (M^*, g_*) *is a pseudo-riemannian symmetric space. The geodesic symmetry at* p_* *is induced by the automorphism* τ *of* G. *Moreover if* (M, g) *is compact, the space* (M^*, g_*) *is a compact riemannian symmetric space.*

§2. Gauss maps for $\mathcal{V}$-submanifolds

Fix an equivalence class $\mathcal{V}$ in $\mathcal{S}(M, g)$ and let S be a $\mathcal{V}$-submanifold of M. Let M^* be the connected component of $\mathcal{C}_{\mathcal{V}}$ which contains the equivalence class $p_* = \langle T_pS \rangle$ for a point p of S. Since S is connected, the space M^* is determined independently of the base point p. On M^* we consider the pseudo-riemannian metric g_* defined in §1. In the following contents we retain the notations in §1.

The *Gauss map* κ is a smooth mapping of S to M^* defined in the following way: $\kappa(p) = \langle T_pS \rangle$ for $p \in S$. We first study the differential κ_* of κ. Fix a point p of S. Let Ω_p be the holonomy algebra at p of (M, g). Since (M, g) is a riemannian symmetric space, it holds that

$$\Omega_p = \{R(x, y) \in \mathrm{End}\ (T_pM); x, y \in T_pM\}_{\mathbb{R}} \tag{2.1}$$

where $\{*\}_{\mathbb{R}}$ denotes the linear subspace of $\mathrm{End}(T_pM)$ spanned by $\{*\}$ over $\mathbb{R}$. Decompose T_pM into the sum of the tangent space T_pS and the normal space N_pS of S and put $E_p^+ = (T_pS)^* \otimes T_pS \oplus (N_pS)^* \otimes N_pS$, $E_p^- = (T_pS)^* \otimes N_pS \oplus (N_pS)^* \otimes T_pS$. Here V^* denotes the dual space of a vector space V. Then they hold that $\mathrm{End}(T_pM) = E_p^+ \oplus E_p^-$ and moreover by the properties (0.1), (2.1) that

$$\Omega_p = \Omega_p^+ \oplus \Omega_p^- \tag{2.2}$$

where $\Omega_p^\pm = \Omega_p \cap E_p^\pm$. Under the identifications: $T_pS = \mathfrak{g}_{-1-1}$, $N_pS = \mathfrak{g}_{-11}$ the space Ω_p is identified with the adjoint representation $\mathrm{ad}_{\mathfrak{g}_{-1}}(\mathfrak{g}_1)$ of $\mathfrak{g}_1$ on $\mathfrak{g}_{-1}$ ([2]) and the subspaces $\Omega_p^\pm$ are identified with the adjoint representations $\mathrm{ad}_{\mathfrak{g}_{-1}}(\mathfrak{g}_{1\pm1})$ of $\mathfrak{g}_{1\pm1}$ on $\mathfrak{g}_{-1}$ since $[\mathfrak{g}_{11}, \mathfrak{g}_{-1\pm1}] \subset \mathfrak{g}_{-1\pm1}$, $[\mathfrak{g}_{1-1}, \mathfrak{g}_{-1\pm1}] \subset \mathfrak{g}_{-1\mp1}$. Moreover $\Omega_p^\pm$ are identified with $\mathfrak{g}_{1\pm1}$ since $\mathrm{ad}_{\mathfrak{g}_{-1}}(\mathfrak{g}_1)$ is faithful. Particularly the dimensions of $\Omega_p^\pm$ are constant independently of the base point p of S since the isometries s_q, t_q for other points q of S are conjugate to s_p, t_p in G. Put $\Omega = \cup_{p\in S}\Omega_p$ and $\Omega^\pm = \cup_{p\in S}\Omega_p^\pm$. Then Ω is the vector bundle over S induced by the holonomy bundle of (M, g) and $\Omega^\pm$ are vector subbundles of Ω. Now let $\kappa^{-1}TM^*$ be the pull bak of the tangent bundle TM^* by κ. Then it holds that

$$\kappa^{-1}TM^* = \Omega^- \oplus TS. \tag{2.3}$$

This identification is obvious by the following identifications: $T_{p_*}M^* = \mathfrak{g}_{1-1} \oplus \mathfrak{g}_{-1-1}$, $T_pS = \mathfrak{g}_{-1-1}$, and $\Omega_p^- = \mathrm{ad}_{\mathfrak{g}_{-1}}(\mathfrak{g}_{1-1}) = \mathfrak{g}_{1-1}$.

By the virture of (2.3) we regard the differential κ_* of κ as a bundle map of TS to $\Omega^- \oplus TS$. Denote by α the second fundamental form of the submanifold S of M and by B_ξ the shape operator for a normal vector ξ. For $x \in T_pS$ define an endomorphism T_x of T_pM in the following way: $T_x(y) = \alpha(x, y)$ for $y \in T_pS$ and $T_x(\xi) = -B_\xi(x)$ for $\xi \in N_pS$. It obviously follows that $T_x \in E_p^-$ and moreover the followings hold:

Proposition 2.1. *$T_x \in \Omega_p^-$ and*

$$\kappa_{*p}(x) = T_x + x$$

for $x \in T_pS$.

Proof. Fix a vector x of T_pS and let $\gamma(t)$ be a curve in S such that $\gamma(0) = p$ and $(d\gamma/dt)(0) = x$. Since S is a connected $\mathcal{V}$-submanifold, we can take a curve $u(t)$ in G such that $u(0) = e$, $u(t)(p) = \gamma(t)$, and $u(t)_*(T_pS) = T_{\gamma(t)}S$, where e denotes the identity map in G. Let Y be the Killing vector field on M generated by $u(t)$, i.e., $Y_q = (d/dt)\ |_{t=0}\ u(t)(q)$, $q \in M$. Identify Y with an element of $\mathfrak{g}$ and decompose Y into the sum of Y_{11}, Y_{1-1}, Y_{-1} where $Y_{1\pm1} \in \mathfrak{g}_{1\pm1}$ and $Y_{-1} \in \mathfrak{g}_{-1}$. Put $v(t) = u(t)\cdot\exp(-tY_{11})$. Then, since the one parameter subgroup $\exp(-tY_{11})$ of K satisfies that $(\exp\ -tY_{11})(p) = p$, $(\exp - tY_{11})_*T_pS = T_pS$ for all t, the curve $v(t)$ in G also satisfies that $v(0) = e$, $v(t)(p) = \gamma(t)$, and $v(t)_*(T_pS) = T_{\gamma(t)}S$. Let X be the Killing vector field on M generated by $v(t)$ and decompose X into the sum of X_1, X_{-1} where $X_{\pm1} \in \mathfrak{g}_{\pm1}$. Then it holds that $X_1 \in \mathfrak{g}_{1-1}$ and $X_{-1} \in \mathfrak{g}_{-1-1}$. In fact, it follows since

$$X_q = \frac{d}{dt}\ |_{t=0}\ (u(t)\exp(-tY_{11}))(q) = Y_q - (Y_{11})_q = (Y_{1-1})_q + (Y_{-1})_q$$

for $q \in M$, and

$$X_p = (Y_{-1})_p = x.$$

We first show that $\kappa_{*p}(x) = X$ under the identification: $T_{p_*}M^* = \mathfrak{g}_{1-1} \oplus \mathfrak{g}_{-1-1}$. In fact,regard X as a Killing vector field on M^*. Then it follows that

$$\begin{aligned}\kappa_{*p}(x) &= \frac{d}{dt}\ |_{t=0}\ \kappa(\gamma(t)) = \frac{d}{dt}\ |_{t=0}\ \langle T_{\gamma(t)}S\rangle \\ &= \frac{d}{dt}\ |_{t=0}\ \langle v(t)_*T_pS\rangle = \frac{d}{dt}\ |_{t=0}\ v(t)(p_*) = X_{p_*}.\end{aligned}$$

Hence it holds that $\kappa_{*p}(x) = X$ in $\mathfrak{g}_{1-1} \oplus \mathfrak{g}_{-1-1}$.

We next show that $X_1 = \mathrm{ad}_{\mathfrak{g}_{-1}}(X_1) = T_x$ under the identification: $\mathfrak{g}_{1-1} = \mathrm{ad}_{\mathfrak{g}_{-1}}(\mathfrak{g}_{1-1}) = \Omega_p^-$, while it is obvious that $X_{-1} = x$ under the identification: $\mathfrak{g}_{-1-1} = T_pS$. Denote by D, ∇ the riemannian connections of (M, g), (S, g), respectively. For the Killing vector field X of (M, g) define an endomorphism A_X of T_pM in the following way: $A_X(y) = -D_yX$ for $y \in T_pM$. Then we have the identification: $A_X = -\mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)$ ([3]) since (M, g) is a symmetric space. For a vector y of T_pS define a vector field Y_t tangent to S along γ in the following way: $Y_t = v(t)_*y$ and moreover extend it to a local vector field Y on M around p. Then, since X is a vector field on M generated by $v(t)$, it holds that $[X, Y]_p = 0$ ([3]). Hence it follows that

$$\begin{aligned}\mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)(y) &= -A_X(y) = D_yX = (D_YX)_p \\ &= (D_XY)_p = D_xY = \nabla_xY + \alpha(x, y)\end{aligned}$$

and, since $\mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)y \in N_pS$, it moreover follows that $\mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)(y) = \alpha(x, y)$ and $\nabla_xY = 0$.

Let ξ be a vector of N_pS. Then, since $\mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)(\xi) \in T_pS$, it follows that, for $z \in T_pS$,

$$\begin{aligned}\langle \mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)\xi, z\rangle &= -\langle \xi, \mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)z\rangle \\ &= -g(\xi, \alpha(x, z)) = -g(B_\xi(x), z).\end{aligned}$$

Hence it holds that $\mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)\xi = -B_\xi(x)$. Q.E.D.

Corollary 2.2. *The Gauss map κ is an immersion.*

Denote by ∇^* the Levi-Civita connection of (M^*, g_*). Then ∇^* induces the covariant differentiation ∇^* in the pull back $\kappa^{-1}TM^*$. We study the operation of ∇^* under the identification: $\kappa^{-1}TM^* = \Omega^- \oplus TS$

Proposition 2.3. *For a vector $x \in T_pS$ and a smooth vector field Z on S the covariant derivative ∇_x^*Z is contained in T_pS and it holds that $\nabla_x^*Z = \nabla_xZ$.*

Proof. Fix a vector x of T_pS and let $\gamma(t)$, $v(t)$ be the curves in S, G given in Proposition 2.1, respectively. Moreover for a vector y of T_pS let Y_t be the vector field along γ given in the proposition. Then, in the proof of the proposition, it holds that $\nabla_xY_t = 0$. If it moreover holds that $\nabla_x^*Y_t = 0$, our claim is proved as follows. Let $e_1, \cdots, e_r$ be a basis of T_pS and $(E_1)_t, \cdots, (E_r)_t$ be the base fields along γ constructed from $e_1, \cdots, e_r$ as Y_t is done from y. For a vector field Z on S put $Z_{\gamma(t)} =$

$\sum_{i=1}^{r} f^i(t)(E_i)_t$. Then it follows that $\nabla_x Z_{\gamma(t)} = \sum_{i=1}^{r}(df^i/dt)(0)e_i = \nabla^*_x Z_{\gamma(t)}$. Hence it holds that $\nabla^*_x Z = \nabla_x Z \in T_pS$.

We show that $\nabla^*_x Y_t = 0$. Note that the tangent spaces $T_{\gamma(t)}S$ are identified with the subspaces $\mathrm{Ad}(v(t))(\mathfrak{g}_{-1-1})$ in $\mathfrak{g}$ and moreover $\mathfrak{g}$ is identified with the Lie algebra of the Killing vector fields on M^*. Under these identifications let Y_0^* be the Killing vector field on M^* corresponding to the vector y of T_pS. Then the vectors Y_t of $T_{\gamma(t)}S$ correspond to the Killing vector fields $v(t)_*Y_0^*$ on M^*. Hence under the identification (2.1) the vector field Y_t is identified with the TM^*-valued vector field $v(t)_*((Y_0^*)_{p_*})$ along $\kappa\circ\gamma$. Extend this vector field to a local vector field Y^* on M^* around p_*. Next take the element X of $\mathfrak{g}$ defined in Proposition 2.1 and identify it with a Killing vector field X^* on M^*. Then X^* is generated by $v(t)$ and thus it holds that $[X^*, Y^*]_{p_*} = 0$. Let $A^*_{X^*}$ be the endomorphism of $T_{p_*}M^*$ defined as the endomorphism A_X of T_pM. Since $X \in \mathfrak{g}_{1-1} \oplus \mathfrak{g}_{-1-1}$, it holds that $A^*_{X^*} = 0$ ([3]). Then it follows that

$$\begin{aligned}\nabla^*_x Y &= (\nabla^*_{X^*}Y^*)_{p_*} = [X^*, Y^*]_{p_*} + (\nabla^*_{Y^*}X^*)_{p_*}\\ &= -A^*_{X^*}(Y^*_{p_*}) = 0\end{aligned}$$

Q.E.D.

Denote by $D^\perp$ the normal connection of the submanifold S of M. We define a covariant defferentiation D^* in the vector bundle $E^- = \cup_{p\in S}E_p^-$ over S. For a vector x of T_pS and a section K of E^- the covariant derivative $D^*_x K$ in E_p^- is given in the following way: For $y \in T_pS$ and $\xi \in N_pS$ extend them to a tangent local vector field Y on S and a normal local vector field N on S, respectively. Then,

$$(D^*_x K)(y) = D^\perp_x(K(Y)) - K(\nabla_x Y)$$

and

$$(D^*_x K)(\xi) = \nabla_x(K(N)) - K(D^\perp_x N).$$

We here note that $D^*_x K$ is skew symmetric if K is skew symmetric.

Proposition 2.4. *For a vector x of T_pS and a section K of Ω^- the covariant derivatives $\nabla^*_x K$, $D^*_x K$ are contained in Ω_p^- and it holds that $\nabla^*_x K = D^*_x K$.*

Proof. Fix a vector x of T_pS and let $\gamma(t)$, $v(t)$ be the curves in S, G given in Proposition 2.1, respectively . Moreover for $L_0 \in \Omega_p^-$ let L_t be the tensor field along γ given in the following way: $L_t = v(t)^*L_0$. Then the tensors L_t are contained in $\Omega^-(\gamma(t))$ since $v(t)$ are isometries of (M, g) satisfying that $v(t)_* T_p S = T_{\gamma(t)} S$ and $v(t)_* N_p S = N_{\gamma(t)} S$. If it holds that $\nabla^*_x L_t = D^*_x L_t = 0$, our claim can be proved in the same way as Proposition 2.3.

We first show that $\nabla^*_x L_t = 0$. Note that the spaces $\Omega^-(\gamma(t))$ are identified with the subspaces $\mathrm{Ad}(v(t))(\mathfrak{g}_{1-1})$ in $\mathfrak{g}$ and identify the tensors L_t with Killing vector fields L^*_t on M^*. Then, under the identification (2.1), the tensor field L_t is identified with the TM^*-valued vector field $(L^*_t)_{\kappa(\gamma(t))}$ along $\kappa \circ \gamma$ and it holds that $(L^*_t)_{\kappa(\gamma(t))} = v(t)_*((L^*_0)_{p_*})$ for all t. Extend $(L^*_t)_{\kappa(\gamma(t))}$ to a local vector field L^* on M^* around p_*. Then, in the same way as in Proposition 2.3, it follows that

$$\begin{aligned} \nabla^*_x L_t &= (\nabla^*_{X^*} L^*)_{p_*} = [X^*, L^*]_{p_*} + (\nabla^*_{L^*} X^*)_{p_*} \\ &= -A^*_{X^*}((L^*)_{p_*}) = 0 \end{aligned}$$

We next show that $D^*_x L_t = 0$. For $y \in T_p S$ put $Y_t = v(t)_* y$. Then, since $\nabla_x Y_t = 0$, it follows that $(D^*_x L_t)(y) = D^\perp_x(L_t(Y_t))$. Note that $L_t(Y_t) = v(t)_*(L_0(y))$ and extend $L_t(Y_t)$ to a local vector field Z on M around p. Then it follows that

$$\begin{aligned} D_x(L_t(Y_t)) &= (D_X Z)_p = [X, Z]_p + (D_Z X)_p \\ &= -A_X(Z_p) = -A_X(L_0(y)) = T_x(L_0(y)) \in T_p S. \end{aligned}$$

(See the proof of Proposition 2.1.) Hence it holds that $(D^*_x L_t)(y) = 0$. Also, since $L_t \in \Omega^-_{\gamma(t)} \subset \Omega_{\gamma(t)}$, the tensors L_t and thus $D^*_x L_t$ are skew symmetric. This, together with the above fact, implies that $(D^*_x L_t)(\xi) = 0$ for $\xi \in N_p S$. Q.E.D.

Now for a smooth mapping f of a riemannian manifold (S, g) to a pseudo-riemannian manifold (M^*, g_*), define a covariant differentiation $\bar{D} f_*$ of the differential f_* in the following way:

$$(\bar{D} f_*)(X, Y) = \nabla^*_X(f_* Y) - f_*(\nabla_X Y)$$

for vector fields X, Y on S. If it holds that $\bar{D} f_* = 0$, the mapping f is called *totally geodesic*. Define a TM^* -valued vector field T_f on S as follows. For $p \in S$,

$$(T_f)_p = (1/\dim S) \sum_{i=1}^{r} (\bar{D} f_*)(e_i, e_i)$$

where $\{e_i\}$ denotes an orthonormal basis of $T_p S$. If it holds that $T_f = 0$ on S, the mapping f is called *harmonic*. Next a submanifold S of a riemannian manifold (M, g) is called a *parallel submanifold* if it satisfies that

$$(\bar{D}\alpha)(X, Y, Z) = D_X^\perp(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z) = 0$$

for vector fields X, Y, Z on S.

Theorem 2.5. *Let $\mathcal{V} \in \mathcal{S}(M, g)$ and let S be a connected $\mathcal{V}$-submanifold of M. Then the followings hold.*

(1) *The submanifold S has the parallel mean curvature vectors if and only if the Gauss map κ is harmonic.*

(2) *The submanifold S is a parallel submanifold if and only if the Gauss map κ is totally geodesic.*

Proof. (1) Define a covariant derivative $\bar{D}H$ of the mean curvature vector field H as follows:

$$(\bar{D}H)(X) = D_X^\perp H \quad \text{and} \quad (\bar{D}H)(N) = -{}^t(D^\perp H)(N)$$

for a tangent vector field X and a normal vector field N on S, where ${}^t(F)$ denotes the transposed mapping of F. We show that

$$T_\kappa = \bar{D}H.$$

By this our claim (1) is obvious. Fix a point p of S and take an orthonormal local base field $E_1, \cdots, E_r$ on S around p satisfying that $(\nabla_{E_i} E_j)_p = 0$ for all i, j. Then it follows that

$$\begin{aligned}
(\dim S)(T_\kappa)_p &= \sum_{i=1}^{r} (\nabla^*_{E_i}(\kappa_*(E_i)))_p \\
&= \sum_{i=1}^{r} (\nabla^*_{E_i}(T_{E_i} + E_i))_p = \sum_{i=1}^{r} (D^*_{E_i} T_{E_i} + \nabla_{E_i} E_i)_p \\
&= \sum_{i=1}^{r} (D^*_{E_i} T_{E_i})_p
\end{aligned}$$

by Propositions 2.1, 2.3, and 2.4. Take a vector y of $T_p S$ and extend it to a local vector field Y on S satisfying that $(\nabla_{E_i} Y)_p = 0$ for all i.

Then it follows that

$$\begin{aligned}(D^*_{E_i}T_{E_i})_p(y) &= (D^{\perp}_{E_i}(T_{E_i}(Y)))_p = (D^{\perp}_{E_i}(\alpha(E_i,Y)))_p\\ &= (\bar{D}\alpha)(E_i,E_i,Y)_p + \alpha(\nabla_{E_i}E_i,Y)_p + \alpha(E_i,\nabla_{E_i}Y)_p\\ &= (\bar{D}\alpha)_p(E_i,E_i,Y) = (\bar{D}\alpha)_p(Y,E_i,E_i)\\ &= D^{\perp}_Y(\alpha(E_i,E_i))_p - 2\alpha(\nabla_Y E_i,E_i)_p\\ &= D^{\perp}_y(\alpha(E_i,E_i))\end{aligned}$$

by the Codazzi equation and the condition (0.1). Hence it follows that $\sum_{i=1}^r (D^*_{E_i}T_{E_i})_p(y) = (\dim S)(D^{\perp}_y H)_p$. Since $D^*_{E_i}T_{E_i}$ are skew symmetric, it holds that $T_\kappa = \bar{D}H$.

(2) Define a covariant derivative $\bar{D}B$ of the shape operator B as follows:

$$(\bar{D}B)(X,Y,N) = \nabla_X(B_N(Y)) - B_{D^{\perp}_X N}Y - B_N(\nabla_X Y)$$

for tangent vector fields X, Y and a normal vector field N on S. Then it holds that

$$g(\bar{D}B(X,Y,N),Z) = g(\bar{D}\alpha(X,Y,Z),N) \tag{2.5}$$

for a tangent vector field Z on S. We show that $\bar{D}\kappa_* \in (TS)^*\otimes(TS)^*\otimes \Omega^-$ and the followings hold:

$$(\bar{D}\kappa_*)(X,Y)Z = (\bar{D}\alpha)(X,Y,Z)$$

and

$$(\bar{D}\kappa_*)(X,Y)N = -(\bar{D}B)(X,Y,N).$$

By these our claim(2) is obvious. It first follows that

$$\begin{aligned}(\bar{D}\kappa_*)(X,Y) &= \nabla^*_X(T_Y + Y) - (T_{\nabla_X Y} + \nabla_X Y)\\ &= D^*_X T_Y - T_{\nabla_X Y}\end{aligned}$$

by Propositions 2.1, 2.3, and 2.4. Hence it holds that $\bar{D}\kappa_* \in (TS)^*\otimes(TS)^*\otimes\Omega^-$. It next follows that

$$\begin{aligned}(\bar{D}\kappa_*)(X,Y)Z &= (D^*_X T_Y)(Z) - T_{\nabla_X Y}(Z)\\ &= D^{\perp}_X(\alpha(Y,Z)) - \alpha(Y,\nabla_X Z) - \alpha(\nabla_X Y, Z)\\ &= (\bar{D}\alpha)(X,Y,Z).\end{aligned}$$

Note that $(\bar{D}\kappa_*)(X,Y)$ is skew symmetric. Then by (2.5) it follows that $(\bar{D}\kappa_*)(X,Y)N = -(\bar{D}B)(X,Y,N)$. Q.E.D.

Remark. (a) A complete $\mathcal{V}$-submanifold S of (M,g) is parallel if and only if it is a symmetric submanifold. It has already been proved in [5] that the Gauss map of a symmetric $\mathcal{V}$-submanifold is totally geodesic. The proof is done by a concrete construction of the Gauss image of a geodesic in S. Refer [4], [6] for symmetric submanifolds.

(b) On the "classical" Gauss map for a submanifold of $\mathbb{R}^n$, a theorem of this type has been proved in Vilm [7].

§3. Examples

A *symmetric Lie algebra* $(\mathfrak{g},\sigma)$ is, by definition, a pair of a semisimple Lie algebra $\mathfrak{g}$ and an involutive automorphism σ of $\mathfrak{g}$ such that the adjoint representation $\mathrm{ad}_{\mathfrak{g}_{-1}}(\mathfrak{g}_1)$ is faithful, where $\mathfrak{g}_{\pm 1}$ denote the (± 1)-eigenspaces of σ. If $\mathfrak{g}$ is of compact type (resp. of noncompact type), the symmetric Lie algebra $(\mathfrak{g},\sigma)$ is also called *of compact type* (resp. *of noncompact type*). Let $(\mathfrak{g},\sigma)$ be a symmetric Lie algebra of compact type and take a σ-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ such that the endomorphisms $\mathrm{ad}(X)$, $X\in\mathfrak{g}$, of $\mathfrak{g}$ are skew symmetric. Let G be a compact simply connected Lie group with Lie algebra $\mathfrak{g}$ and K the connected closed subgroup of G with Lie algebra $\mathfrak{g}_1$. Put $M=G/K$ and let g be the riemannian metric on M induced from $\langle\ ,\ \rangle$. Then (M,g) is a compact simply connected riemannian symmetric space. Next put $\hat{\mathfrak{g}}=\mathfrak{g}_1\oplus\sqrt{-1}\mathfrak{g}_{-1}$ and let $\hat\sigma$ be the involutive automorphism of $\hat{\mathfrak{g}}$ induced by σ. Then $(\hat{\mathfrak{g}},\hat\sigma)$ is a symmetric Lie algebra of noncompact type. Let $\langle\ \hat{,}\ \rangle$ be the nondegenerate symmetric bilinear form on $\hat{\mathfrak{g}}$ induced by $-\langle\ ,\ \rangle$. Let $\hat G$ be a simply connected Lie group with Lie algebra $\hat{\mathfrak{g}}$ and $\hat K$ be the connected closed subgroup of $\hat G$ with Lie algebra $\mathfrak{g}_1$. Put $\hat M=\hat G/\hat K$ and let $\hat g$ be the riemannian metric on $\hat M$ induced from $\langle\ \hat{,}\ \rangle$. Then $(\hat M,\hat g)$ is a noncompact simply connected riemannian symmetric space. These spaces (M,g) and $(\hat M,\hat g)$ are called *dual* to each other.

Put $p=K\in M$ and identify $\mathfrak{g}$ with the Lie algebra of the Killing vextor fields of (M,g). Then an isometry φ of (M,g) fixing p induces an automorphism $\varphi_\sharp$ of $\mathfrak{g}$ which commutes with σ and leaves $\langle\ ,\ \rangle$ invariant, in the following way: $\varphi_\sharp(X)=\varphi_*(X)\quad$ for $X\in\mathfrak{g}$. Conversely, such an automorphism of $\mathfrak{g}$ is induced by an isometry of (M,g) in this way. These facts also hold for $(\hat M,\hat g)$. The corresponding notations are denoted by attaching the hat to the notations for (M,g).

Now identify the tangent spaces T_pM, $T_{\hat p}\hat M$ with the subspaces $\mathfrak{g}_{-1}$, $\sqrt{-1}\mathfrak{g}_{-1}$, respectively. Then the curvature tensor R_p, (resp. $\hat R_{\hat p}$) is identified as follows: Let x, y, $z\in T_pM$ (resp. $\hat x$, $\hat y$, $\hat z\in T_{\hat p}\hat M$) and let X, Y, Z (resp. $\hat{X}$, $\hat{Y}$, $\hat{Z}$) be the Killing vector fields corresponding to x, y, z (resp. $\hat{x}$, $\hat{y}$, $\hat{z}$). Then it holds that $R_p(x,y)z = [[Y,X],Z]$ (resp. $\hat{R}_{\hat{p}}(\hat{x},\hat{y})\hat{z} = [[\hat{Y},\hat{X}],\hat{Z}]$). Hence, if a subspace V of T_pM is strongly curvature invariant, the subspace $\sqrt{-1}V$ of $T_{\hat{p}}\hat{M}$ is also strongly curvature invariant. Take an equivalence class $\mathcal{V}$ of $\mathcal{S}(M,g)$ and let V be a subspace in T_pM representing $\mathcal{V}$. Then we define an equivalence class $\hat{\mathcal{V}}$ of $\mathcal{S}(\hat{M},\hat{g})$ by putting $\hat{\mathcal{V}} = [\sqrt{-1}V]$.

Proposition 3.1. *The correspondence:* $\mathcal{S}(M,g) \ni \mathcal{V} \longmapsto \hat{\mathcal{V}} \in \mathcal{S}(\hat{M},\hat{g})$ *is a well-defined bijection.*

Proof. We first show that it is well defined. Let W be another subspace in T_pM representing $\mathcal{V}$. Then there exists an isometry φ of (M,g) such that $\varphi(p) = p$ and $\varphi_*(V) = W$. The isometry φ induces an automorphism $\varphi_\sharp$ of $\mathfrak{g}$. Since $\varphi_\sharp$ commutes with σ and leaves $\langle\ ,\ \rangle$ invariant, it moreover induces an automorphism $\hat{\varphi}_\sharp$ of $\hat{\mathfrak{g}}$ in the following way: $\hat{\varphi}_\sharp(X+\sqrt{-1}Y) = \varphi_\sharp(Y)+\sqrt{-1}\varphi_\sharp(X)$ for $X+\sqrt{-1}Y \in \hat{\mathfrak{g}}$. Then $\hat{\varphi}_\sharp$ commutes with $\hat{\sigma}$ and leaves $\widehat{\langle\ ,\ \rangle}$ invariant. Hence $\hat{\varphi}_\sharp$ induces the isometry $\hat{\varphi}$ of $(\hat{M},\hat{g})$ such that $\hat{\varphi}(\hat{p}) = \hat{p}$. It obviously follows that $\hat{\varphi}_*(\sqrt{-1}V) = \sqrt{-1}W$. This implies that $\sqrt{-1}V$ and $\sqrt{-1}W$ are equivalent. Hence the above correspondence is well defined.

The injectivity of the correspondence is proved in the same way as above, and the surjectivity is obvious. Q.E.D.

Now let (M,g) be a compact simply connected riemannian symmetric space and $(\mathfrak{g},\sigma)$ the corresponding symmetric Lie algebra. Let $\mathcal{V}$ be an equivalence class of $\mathcal{S}(M,g)$ and let V be a subspace of T_pM representing $\mathcal{V}$. Let τ be the involutive automorphism of $\mathfrak{g}$ induced by the isometry t_p associated with V, and moreover let $\hat{\tau}$ be the involutive automorphism of $\hat{\mathfrak{g}}$ induced by τ. Then, from the arguements in §1, the target spaces M^*, $\hat{M}^*$ assciated with $\mathcal{V}$, $\hat{\mathcal{V}}$ are locally determined by the symmetric Lie algebras $(\mathfrak{g},\tau)$, $(\hat{\mathfrak{g}},\hat{\tau})$, respectively. We concretely give the symmetric Lie algebras for the case that (M,g) is of rank one. An equivalence class is denoted by the unique complete totally geodesic submanifold which belongs to it, and a symmetric Lie algebra is denoted by the quotient of the Lie algebra by the subalgebra of the points fixed by the involution. Denote by S^n the n-dimensional sphere, by $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{Q}P^n$, $\mathbb{C}\mathrm{a}P^2$ the n-dimensional real, complex, quaternion projective spaces and the Cayley projective plane, and by $\mathbb{R}H^n$, $\mathbb{C}H^n$, $\mathbb{Q}H^n$, $\mathbb{C}\mathrm{a}H^2$ the n-dimensional real, complex, quaternion hyperbolic spaces and the Cayley hyperbolic plane, respectively.

Example 1. Let $(M,g) = S^n$ and $(\hat{M},\hat{g}) = \mathbb{R}H^n$. Moreover let $\mathcal{V}$, $\hat{\mathcal{V}}$ be the totally geodesic sphere S^r and the totally geodesic real hyperbolic space $\mathbb{R}H^r$, respectively. Then it holds that

$$(\mathfrak{g},\tau) = \mathfrak{so}(n+1)/\mathfrak{so}(r) \oplus \mathfrak{so}(n+1-r)$$

and

$$(\hat{\mathfrak{g}},\hat{\tau}) = \mathfrak{so}(n,1)/\mathfrak{so}(n-r,1) \oplus \mathfrak{so}(r).$$

Example 2. Let $(M,g) = \mathbb{C}P^n$ and $(\hat{M},\hat{g}) = \mathbb{C}H^n$.

(1) Let $\mathcal{V}$, $\hat{\mathcal{V}}$ be the totally real totally geodesic submanifolds $\mathbb{R}P^n$, $\mathbb{R}H^n$, respectively. Then it holds that

$$(\mathfrak{g},\tau) = \mathfrak{su}(n+1)/\mathfrak{so}(n+1)$$

and

$$(\hat{\mathfrak{g}},\hat{\tau}) = \mathfrak{su}(1,n)/\mathfrak{so}(1,n).$$

(2) Let $\mathcal{V}$, $\hat{\mathcal{V}}$ be the kaehlerian totally geodesic submanifolds $\mathbb{C}P^r$, $\mathbb{C}H^r$, respectively. Then it holds that

$$(\mathfrak{g},\tau) = \mathfrak{su}(n+1)/\mathfrak{s}(\mathfrak{u}(r) \oplus \mathfrak{u}(n+1-r))$$

and

$$(\hat{\mathfrak{g}},\hat{\tau}) = \mathfrak{su}(n,1)/\mathfrak{su}(n-r,1) \oplus \mathfrak{su}(r) \oplus \mathbb{T}.$$

Example 3. Let $(M,g) = \mathbb{Q}P^n$ and $(\hat{M},\hat{g}) = \mathbb{Q}H^n$.

(1) Let $\mathcal{V}$, $\hat{\mathcal{V}}$ be the quaternionic totally geodesic submanifolds $\mathbb{Q}P^r$, $\mathbb{Q}H^r$, respectively. Then it holds that

$$(\mathfrak{g},\tau) = \mathfrak{sp}(n+1)/\mathfrak{sp}(r) \oplus \mathfrak{sp}(n+1-r)$$

and

$$(\hat{\mathfrak{g}},\hat{\tau}) = \mathfrak{sp}(n,1)/\mathfrak{sp}(n-r,1) \oplus \mathfrak{sp}(r).$$

(2) Let $\mathcal{V}$, $\hat{\mathcal{V}}$ be the totally complex totally geodesic submanifolds $\mathbb{C}P^n$, $\mathbb{C}H^n$, respectively. Then it holds that

$$(\mathfrak{g},\tau) = \mathfrak{sp}(n+1)/\mathfrak{u}(n+1)$$

and

$$(\hat{\mathfrak{g}},\hat{\tau}) = \mathfrak{sp}(1,n)/\mathfrak{su}(1,n) \oplus \mathbb{T}.$$

Example 4. Let $(M,g) = \mathbb{C}\mathrm{a}P^2$ and $(\hat{M},\hat{g}) = \mathbb{C}\mathrm{a}H^2$.

(1) Let $\mathcal{V}$, $\hat{\mathcal{V}}$ be the totally geodesic submanifolds $\mathbb{Q}P^2$, $\mathbb{Q}H^2$, respectively. These imbeddings are induced from the inclusion: $\mathbb{Q} \hookrightarrow \mathbb{C}\mathrm{a}$. Then it holds that

$$(\mathfrak{g}, \tau) = \mathfrak{F}_4/\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$$

and

$$(\hat{\mathfrak{g}}, \hat{\tau}) = \mathfrak{F}_4^2/\mathfrak{sp}(1,2) \oplus \mathfrak{su}(2).$$

(2) Let $\mathcal{V}$, $\hat{\mathcal{V}}$ be the totally geodesic submanifolds S^8, $\mathbb{R}H^8$, respectively. The space S^8 is a line in $\mathbb{C}\mathrm{a}P^2$. Then it holds that

$$(\mathfrak{g}, \tau) = \mathfrak{F}_4/\mathfrak{so}(9)$$

and

$$(\hat{\mathfrak{g}}, \hat{\tau}) = \mathfrak{F}_4^2/\mathfrak{so}(1,8).$$

Remark. (a) On the case of Example 1, if we regard a $\mathcal{V}$-submanifold of S^n as a submanifold in $\mathbb{R}^{n+1}$, our Gauss map is the "classical" Gauss map.

(b) On the case of Example 3 (1), $\mathcal{V}$-submanifolds of M and $\hat{\mathcal{V}}$-submanifolds of $\hat{M}$ are always totally geodesic ([1]).

(c) Refer [5] for the details of these examples and the target spaces M^*, $\hat{M}^*$ in the case that (M, g), $(\hat{M}, \hat{g})$ are other riemannian symmetric spaces.

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Department of Mathematics
Yamaguchi University
Yamaguchi 753
Japan