Advanced Studies in Pure Mathematics 18-I, 1990
Recent Topics in Differential and Analytic Geometry
pp. 327–337

On Rotationally Symmetric Hamilton's Equation for Kähler-Einstein Metrics

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§0. Introduction

R.S. Hamilton [4] proved that any riemannian metric g_0 with positive Ricci curvature on a compact 3–dimensional manifold is deformed to an Einstein metric along the equation

$$\text{(H)} \qquad \frac{d}{dt}g_t = -r_t + \frac{1}{n}\overline{s}_t \cdot g_t \qquad (n = \text{dimension} = 3),$$

where r_t denotes the Ricci tensor of g_t and $\overline{s}_t$ the mean value of the scalar curvature. It is weakly generalized to higher dimensional cases (e.g. [8]).

Equation (H) has good properties: if the initial riemannian metric g_0 is invariant under a group action, then so is each g_t; if g_0 is a Kähler metric, then so is each g_t. In fact, H.D. Cao [2] proves that any Kähler metric on a compact Kähler manifold with vanishing or negative first Chern class is deformed to a Kähler-Einstein metric along equation (H).

This result suggests that, even on a compact Kähler manifold with positive first Chern class, the solution of equation (H) converges to a Kähler-Einstein metric if it exists. The first purpose of this paper is to show that it is true in some special cases given in Y. Sakane [9] and N. Koiso-Y. Sakane [6], [7], which contain rotationally symmetric metrics on the 2-dimensional sphere. On the other hand, if the manifold admits no Kähler-Einstein metrics, then the solution of equation (H) can not converge. But it is interesting to see the behaviour of the solution, which is the second purpose. These situations are unified as the following

Theorem. *Let* $(\widehat{L}, \tilde{g}_0)$ *be a Kähler manifold as in* Chapter VI *and* $\tilde{g}_t$ *be the solution of equation* (H). *Then there is a holomorphic vector*

Received February 13, 1989.
This research was supported by Max–Planck–Institut für Mathematik in Bonn.

field V on $\widehat{L}$ such that $\exp(tV)^*\tilde{g}_t$ *converges to a quasi-Einstein metric (see the definition below). Here,* $V=0$ *if and only if Futaki's obstruction* ([3]) *vanishes.*

§1. Hamilton's equation and quasi-Einstein metrics

At first, we need to modify the notion of Einstein metrics in order to analyze the behaviour of solutions of equation (H) which do not converge to Einstein metrics.

Definition 1.1. A riemannian metric g is called a *quasi-Einstein metric* if there is a vector field V such that $r-(\overline{s}/n)g = L_V g$.

We easily see

Proposition 1.2. *The solution of Hamilton's equation* (H) *whose initial riemannian metric g_0 is a quasi-Einstein metric is given by* $g_t = \gamma_t^{-1*} g_0$*, where* $\gamma_t = \exp tV$*. In particular, if g_0 is not Einstein, then g_t does not converge.*

From Proposition 1.2, even if a solution g_t of Hamilton's equation does not converge, $(\exp tV)^* g_t$ may converge for some vector field V on M. Remarking that the one parameter family $\tilde{g}_t = (\exp tV)^* g_t$ of riemannian metrics satisfies the equation

$$\frac{d}{dt}\tilde{g}_t = -\tilde{r}_t + \frac{1}{n}\overline{s}_t\cdot\tilde{g}_t + L_V\tilde{g}_t, \tag{H*}$$

where $\tilde{r}$ is the Ricci tensor of $\tilde{g}$, we treat equation (H*) as a modified Hamilton's equation.

On a compact complex manifold with positive first Chern class, we get the following

Proposition 1.3. *A Kähler metric g in the first Chern class is a quasi-Einstein metric if and only if* $r-g = L_V g$ *for some holomorphic vector field V. In particular, such a Kähler metric is an Einstein metric if and only if Futaki's obstruction vanishes.*

Proof. Put $r-g=\partial\overline{\partial}f$, where $\partial\overline{\partial}$ denotes the complex Hessian. Then by the definition of V,

$$D_\alpha V_{\overline{\beta}} + D_{\overline{\beta}}V_\alpha = D_\alpha D_{\overline{\beta}} f \text{ and } D_\alpha V_\beta + D_\beta V_\alpha = 0,$$

where D denotes the covariant derivative. These equations imply that $-D^\gamma(D_\gamma V_\alpha - D_\alpha V_\gamma)=0$, and so $D_\alpha V_\beta = 0$, i.e., V is holomorphic.

Assume that Futaki's obstruction vanishes. We know that there is a complex valued function η such that $V_\alpha = D_\alpha \eta$, because the first Chern class is positive and so there are no non-trivial harmonic 1-forms. Substituting it into the above equation, we get $D_\alpha D_{\overline{\beta}}(\eta + \overline{\eta} - f) = 0$, which means that there is a real valued function v such that $\eta = (1/2)f + \sqrt{-1}v$. Combining it with the definition of the obstruction ([3]) : $\int_M V[f]v_g = 0$, we see that $df = 0$. Q.E.D.

If we take a Kähler metric g_0 in the first Chern class, then equation (H*) becomes

$$\text{(H*K)} \qquad \frac{d}{dt}\tilde{g} = -\tilde{r} + \tilde{g} + L_V \tilde{g}.$$

Cao's result [2] for the case of negative first Chern class can be easily modified as follows (for detail, readers can refer to [5]).

Lemma 1.4. *The solution $\tilde{g}_t$ of equation* (H*K) *exists for all time. If $\tilde{g}_t$ converges uniformly to a riemannian metric $\tilde{g}_\infty$ and is bounded in C^1-topology, then $\tilde{g}_t$ converges to $\tilde{g}_\infty$ in C^∞-topology.*

§2. Rotationally symmetric equations

From now on we treat only Kähler manifolds $(\widehat{L}, \tilde{g})$ of form (6.1.1) in Chapter VI, and assume hypothesis A_1) and A_2) in Chapter VI. We will use the same notation as VI, unless otherwise stated. In particular, t denotes a space variable in VI while we will reserve t as the time variable. We will use the results in VI up to Lemma 6.5 and the discussion in Proof of Theorem 6.7. We summarize them as follows.

Lemma 2.1. *Let $\tilde{g}$ be a Kähler metric of form* VI (6.1.1). *Then there are a function $U\colon \widehat{L} \to [-D_{\min}, D_{\max}]$ and a function $\varphi\colon [-D_{\min}, D_{\max}] \to \mathbb{R}$ such that $\tilde{g}(H,H) = \varphi \circ U$ and that $H[f \circ U] = (\varphi \cdot f') \circ U$ for all functions f on $[-D_{\min}, D_{\max}]$. The function φ satisfies the following properties.*

(1) *φ is C^∞-ly extended over the boundary.*
(2) *φ is positive on $(-D_{\min}, D_{\max})$ and vanishes at the boundary.*
(3) *$\varphi'(-D_{\min}) = 2$, $\varphi'(D_{\max}) = -2$.*

Conversely, for a function φ with properties above, we can construct a Kähler metric $\tilde{g}$ of form VI (6.1.1), *uniquely up to holomorphic $\mathbb{R}^+$-action.*

Let $\tilde{g}$, φ, U be as above. Denoting by x the variable on $[-D_{\min}, D_{\max}]$, we set for a function φ of x

$$P(\varphi)(x) = \varphi'(x) + \frac{\varphi(x)}{Q(x)}Q'(x) + 2x,$$

where $'$ denotes the derivation d/dx. If we choose a function f of x as VI (6.5.1), i.e., $\tilde{r} - \tilde{g} = \partial\overline{\partial}(f \circ U)$, then by VI Lemma 6.5 we see that

$$P(\varphi) \circ U = -H[f \circ U].$$

From VI (6.1.3), VI Lemmas 6.2 and 6.3, we easily see that the formulae:

$$\text{(2.1.1)} \qquad \begin{aligned} (\tilde{r} - \tilde{g})_{0\overline{0}} &= -H[P(\varphi) \circ U], \ (\tilde{r} - \tilde{g})_{\alpha\overline{\beta}} = (1/2)P(\varphi) \circ U \cdot B_{\alpha\overline{\beta}}, \\ (L_H\tilde{g})_{0\overline{0}} &= 2H[\varphi \circ U], \ (L_H\tilde{g})_{\alpha\overline{\beta}} = -\varphi \circ U \cdot B_{\alpha\overline{\beta}} \end{aligned}$$

hold under assumption VI (6.3.1).

Now we define a real number E by

$$\int_{-D_{\min}}^{D_{\max}} xe^{-Ex}Q(x)dx = 0.$$

Since $Q(x) > 0$ on $(-D_{\min}, D_{\max})$, such a real number E uniquely exists, and $E = 0$ if and only if Futaki's obstruction vanishes. If we set $V = -(E/2)H$, then from (2.1.1), we get the following formulae.

$$\text{(2.1.2)} \qquad \begin{aligned} (\tilde{r} - \tilde{g} - L_V\tilde{g})_{0\overline{0}} &= -H[P(\varphi) \circ U - E\varphi \circ U], \\ (\tilde{r} - \tilde{g} - L_V\tilde{g})_{\alpha\overline{\beta}} &= (1/2)\big(P(\varphi) \circ U - E\varphi \circ U\big) \cdot B_{\alpha\overline{\beta}}. \end{aligned}$$

Using the real number E we define a function $\varphi^\circ(x)$ by

$$\varphi^\circ(x) = -2(e^{Ex}/Q(x)) \int_{-D_{\min}}^{x} xe^{-Ex}Q(x)dx,$$

and choose a function U° so that $H[f \circ U^\circ] = (\varphi f') \circ U$ for any function f. Since the function φ° satisfies the properties (1), (2) and (3) in Lemma 2.1, the pair (U°, φ°) defines a Kähler metric $\tilde{g}^\circ$ on $\widehat{L}$. Moreover, from the definition of φ°, we see that

$$\text{(2.1.3)} \qquad P(\varphi^\circ) = E\varphi^\circ.$$

Combining it with (2.1.2), we get

Lemma 2.2. *The pair (U°, φ°) defines a quasi-Einstein metric $\tilde{g}^\circ$ on $\widehat{L}$. The Kähler metric $\tilde{g}^\circ$ is an Einstein metric if and only if $E = 0$.*

Remark 2.3. The Kähler metric $\tilde{g}^\circ$ is not extremal in the sense of Calabi [1]. A Kähler metric is extremal if and only if the gradient of its scalar curvature is holomorphic. In our case, it is equivalent to that $(\varphi f')''$ is constant, and we get a unique solution

$$\varphi(x) = -(1/Q(x)) \int_{-D_{\min}}^{x} \big(C(D_{\min} + x)(D_{\max} - x) + 2x\big) Q(x) dx,$$

where the constant C is chosen so that $\varphi(D_{\max}) = 0$. This function φ defines an extremal Kähler metric, but it is a quasi-Einstein metric if and only if $E = 0$ and $C = 0$, i.e., if they are Einstein metrics.

Now we consider modified Hamilton's equation (H*K). By (2.1.2), equation (H*K) reduces to

$$2\frac{d}{dt}(\varphi_t \circ U_t) = \varphi_t \circ U_t \cdot \big\{P(\varphi_t)' \circ U_t - E\varphi_t{}' \circ U_t\big\}. \tag{2.3.1}$$

On the other hand, from the equality : $\varphi \circ U \cdot d/dU = H$, we get

$$\frac{d}{dt}(\varphi \circ U \cdot \frac{d}{dU} U^\circ) = 0.$$

Combining them, we get

$$2\frac{d}{dU}(\frac{d}{dt}U) = P(\varphi)' \circ U - E\varphi' \circ U,$$

and so

$$2\frac{d}{dt}U = (P(\varphi) - E\varphi) \circ U$$

modulo constant. But here both sides of this equality vanishes at the boundary $U = -D_{\min}$. Therefore it holds without modulo factor. We define a new function θ by $\varphi = \varphi^\circ \cdot (1 + \theta)$. Then from (2.1.3), $P(\varphi) = E\varphi + \varphi^\circ\theta' - 2x\theta$. Therefore,

$$2\frac{d}{dt}U = (\varphi^\circ\theta') \circ U - 2U\theta \circ U. \tag{2.3.2}$$

Substituting it into (2.3.1), we get

$$2\varphi^\circ\frac{d}{dt}\theta = \varphi^\circ\varphi\theta'' - (\varphi^\circ\theta')^2 - 2x\varphi^\circ\theta' - 2\big(\varphi^\circ - x(\varphi^\circ)'\big)(1 + \theta)\theta. \tag{2.3.3}$$

Remark that the function θ vanishes on the boundary $x = -D_{\min}$, $D_{\max}$.

§3. Convergence of the solution φ

To prove the convergence of θ, we need the following

Lemma 3.1. *$\varphi^\circ(x) - x(\varphi^\circ)'(x) > 0$ on $[-D_{\min}, D_{\max}]$.*

Proof. Put

$$\xi(x) = xe^{-Ex}Q(x) \text{ and } \eta(x) = \int_{-D_{\min}}^{x} \xi(x)dx.$$

Remark that $(\exp(-Ex)Q\varphi^\circ)' = -2\xi$ and $\varphi^\circ = -2(\exp(Ex)/Q)\eta$. Therefore,

$$e^{-Ex}Q \cdot \left(\varphi^\circ - x(\varphi^\circ)'\right) = 2(e^{Ex}/Q) \cdot (\xi^2 - \eta\xi').$$

Since we know that $\varphi^\circ - x(\varphi^\circ)' = 2D_{\min}$ (resp. $2D_{\max}$) > 0 at $x = -D_{\min}$ (resp. $D_{\max}$), it suffices to prove that $\xi^2 - \eta\xi' > 0$ on $(-D_{\min}, D_{\max})$. Then $\varphi^\circ > 0$ and so $\eta < 0$. Moreover, since $\xi(0) = 0$, $\eta(0) < 0$ and $Q(0) = 1$, we see that $\xi^2 - \eta\xi' > 0$ at $x = 0$. In the following, we consider only on the interval $(0, D_{\max})$. A similar proof holds on $(-D_{\min}, 0)$.

Since the function $Q(x)$ is a product of polynomials of first order, the second derivative $(\log\xi)''$ is negative on $(0, D_{\max})$, which implies that the first derivative ξ' crosses the x-axis at most once. If ξ' does not cross the x-axis, then $\xi^2 - \eta\xi' > 0$ on $(0, D_{\max})$, which completes the proof. Assume that ξ' crosses the x-axis at $x = a$. Then $\xi^2 - \eta\xi' > 0$ on $(0, a]$ and so we may consider only on the interval $(a, D_{\max})$.

Thus it suffices to prove that $\xi^2/\xi' - \eta < 0$ on $(a, D_{\max})$, because $\xi' < 0$. But we see that

$$(\xi^2/\xi' - \eta)' = \xi \cdot \left((\xi')^2 - \xi\xi''\right)/(\xi')^2,$$

here,
$$0 > (\log\xi)'' = \left(\xi\xi'' - (\xi')^2\right)/\xi^2.$$

Therefore, the function $\xi^2/\xi' - \eta$ is increasing. Moreover, at $x = D_{\max}$, $\xi' < 0$, $\xi^2 \geq 0$ and $\eta = 0$. Hence $\xi^2/\xi' - \eta < 0$ on $(a, D_{\max})$. Q.E.D.

Lemma 3.2. *The functions θ and $\varphi^\circ\theta'$ converges uniformly to* 0 *in exponential order.*

Proof. From the definition of θ, we see that $1 + \theta > 0$. Therefore (2.3.3) and Lemma 3.1 imply that the minimum of θ is increasing. By Lemma 3.1 we can choose a positive number c_1 smaller than

$\min\{(\varphi^\circ - x(\varphi^\circ)')(1+\theta^\circ)/\varphi^\circ\}$. Then by (2.3.3), we get

$$\begin{aligned}2\varphi^\circ \frac{d}{dt}(e^{c_1 t}\theta) =&\varphi^\circ\varphi \cdot (e^{c_1 t}\theta)'' - e^{-c_1 t}\big(\varphi^\circ \cdot (e^{c_1 t}\theta)'\big)^2 \\ &-2x\varphi^\circ \cdot (e^{c_1 t}\theta)' - 2\Big(\big(\varphi^\circ - x(\varphi^\circ)'\big)(1+\theta) - c_1\varphi^\circ\Big)e^{c_1 t}\theta,\end{aligned}$$

from which we conclude that the function $\exp(c_1 t)\theta$ is bounded by the maximum principle.

Put $\xi = \varphi^\circ\theta' + c_2 x\theta$, where c_2 is a constant. Then we see that

$$2\varphi^\circ \frac{d}{dt}\xi = \varphi^\circ \cdot \big(2\varphi^\circ \frac{d}{dt}\theta\big)' - \big((\varphi^\circ)' - c_2 x\big) - 2\varphi^\circ \frac{d}{dt}\theta.$$

Here,

$$\begin{aligned}2\varphi^\circ \frac{d}{dt}\theta =&\varphi\xi' - \xi^2 - \Big(\big((\varphi^\circ)' - c_2 x\big)\theta + (\varphi^\circ)' + (c_2+2)x\Big)\xi \\ &- \Big((c_2+2)\big(\varphi^\circ - x(\varphi^\circ)'\big)(1+\theta) - c_2(c_2+2)x^2\Big)\theta.\end{aligned}$$

and so,

$$\begin{aligned}&\varphi^\circ \cdot (2\varphi^\circ\theta')' \\ =&\varphi^\circ\varphi\xi'' + (\xi' \text{ term}) - \big((\varphi^\circ)' - c_2 x\big)\xi^2 \\ &- \big\{(c_2+2)\big(2\varphi^\circ - x(\varphi^\circ)'\big) - c_2(c_2+2)x^2 + \varphi^\circ(\varphi^\circ)'' + (\theta \text{ term})\big\}\xi \\ &+ (\theta \text{ term}).\end{aligned}$$

Combining these equalities, we get

$$\begin{aligned}&2\varphi^\circ \frac{d}{dt}\xi \\ =&\varphi^\circ\varphi\xi'' + (\xi' \text{ term}) \\ &-\big\{(c_2+2)\big(2\varphi^\circ - x(\varphi^\circ)'\big) + \varphi^\circ(\varphi^\circ)'' - 2x(\varphi^\circ)' - \big((\varphi^\circ)'\big)^2 + (\theta \text{ term})\big\}\xi \\ &+ (\theta \text{ term}).\end{aligned}$$

If we remark that $2\varphi^\circ - x(\varphi^\circ)' \geq \varphi^\circ - x(\varphi^\circ)' > 0$ and choose c_2 sufficiently large, then we can choose a positive constant c_3 so that the function $\exp(c_3 t)\xi$ is bounded by a similar way to the first part. We know that θ converges to 0, so is $\varphi^\circ\theta'$. Q.E.D.

Lemma 3.3. *The functions $(\varphi^\circ\theta' - 2x\theta)/\varphi$ and θ' are bounded and the function θ/φ° converges to 0 in L^1-norm.*

Proof. Put $\xi = (\varphi^\circ\theta' - 2x\theta)/\varphi$. Remark that ξ is a C^∞-function on $[-D_{\min}, D_{\max}] \times [0,\infty)$. Since $\xi' = 2(1/\varphi^2)\cdot d\varphi/dt$,

$$2\frac{d}{dt}\xi = -(\varphi^\circ\theta' - 2x\theta)\xi' + 2(1/\varphi)\frac{d}{dt}(\varphi^\circ\theta' - 2x\theta).$$

On the other hand,

$$\varphi^\circ\theta' - 2x\theta = \varphi' - \big((\varphi^\circ)' + 2x\big)\theta.$$

Therefore,

$$2\frac{d}{dt}\xi = \varphi\xi'' + \big\{-\varphi\xi + 2\varphi' - \big((\varphi^\circ)' + 2x\big)(1+\theta)\big\}\xi'.$$

In particular, $2d\xi/dt = 2(D_{\min}+1)\xi'$ at $x = -D_{\min}$ and $2d\xi/dt = -2(D_{\max}+1)\xi'$ at $x = D_{\max}$. Thus by the maximum principle, ξ is bounded.

Put $\eta = \theta/\varphi^\circ$. Since θ converges uniformly to 0, it suffices to prove that η is bounded on a neighbourhood of $x = -D_{\min}, D_{\max}$. In fact, then θ' is bounded by the boundedness of ξ and so η converges to 0 in L^1-norm by Lemma 3.2. Remark that ξ is bounded and so is $(1+\theta)\xi$. But

$$(1+\theta)\xi = \varphi^\circ\eta' + \big((\varphi^\circ)' - 2x\big)\eta.$$

Thus we can choose a positive constant c_1 so that

$$-c_1 - \big((\varphi^\circ)' - 2x\big)\eta < \varphi^\circ\eta' < c_1 - \big((\varphi^\circ)' - 2x\big)\eta.$$

If we choose a sufficiently small neighbourhood $(a, D_{\max}]$ of $x = D_{\max}$, then we can select a positive constant c_2 so that $(\varphi^\circ)' - 2x < -c_2$ on $(a, D_{\max}]$. Therefore, if $\eta > c_1/c_2$, then $\eta' > 0$, and if $\eta < -c_1/c_2$, then $\eta' < 0$. On the other hand, substituting $x = D_{\max}$ into the above inequality, we see that

$$-\frac{1}{2}c_1/(D_{\max}+1) < \eta(D_{\max}) < \frac{1}{2}c_1/(D_{\max}+1).$$

Thus $|\eta| < \max\{c_1(D_{\max}+1), c_1/c_2\}$ on $(a, D_{\max}]$. We can prove the boundedness of η for $[-D_{\min}, b)$ by the same way. Q.E.D.

§4. Convergence of the metric

By (2.3.2) and Lemma 3.2, the derivative dU/dt converges uniformly to 0 in exponential order, and so the function U converges uniformly to a function U_∞. Since the function φ also converges to the function φ°, the function U_∞ satisfies the equation $H[U_\infty] = \varphi^\circ \circ U_\infty$, and thus the pair $(U_\infty, \varphi^\circ)$ defines a quasi-Einstein metric by Lemma 2.2. We replace U° by U_∞ so that U converges uniformly to U°. Note that the pair (U°, φ°) corresponds to a quasi-Einstein metric $\tilde{g}^\circ$. Since $\varphi \circ U \cdot d/dU = \varphi^\circ \circ U^\circ \cdot d/dU^\circ = H$, there exists a function $a(t)$ of t such that

$$\int_0^x \frac{dy}{\varphi(y)} = \int_0^{x^\circ} \frac{dy}{\varphi^\circ(y)} + a(t).$$

Remark that $a(t)$ converges to 0.

Lemma 4.1. *The function $\varphi \circ U/(\varphi^\circ \circ U^\circ)$ converges uniformly to* 1 *and the function $(U - U^\circ)/(\varphi^\circ \circ U^\circ)$ converges uniformly to* 0.

Proof. First we see that

$$\begin{aligned}
&\left|\int_0^x \frac{dy}{\varphi^\circ(y)} - \int_0^{x^\circ} \frac{dy}{\varphi^\circ(y)}\right| \\
&\quad\leq \left|\int_0^x \frac{dy}{\varphi^\circ(y)} - \int_0^x \frac{dy}{\varphi(y)}\right| + \left|\int_0^x \frac{dy}{\varphi(y)} - \int_0^{x^\circ} \frac{dy}{\varphi^\circ(y)}\right| \\
&\quad= \left|\int_0^x \frac{1}{1+\theta(y)} \frac{\theta(y)}{\varphi^\circ(y)} dy\right| + |a(t)|,
\end{aligned}$$

and the last line converges uniformly to 0 by Lemmas 3.2 and 3.3. Put $c = \max|(\varphi^\circ)'(x)|$ and let I be the closed interval between x and x°. If

$$\left|\int_0^x \frac{dy}{\varphi^\circ(y)} - \int_0^{x^\circ} \frac{dy}{\varphi^\circ(y)}\right| < \varepsilon,$$

then

$$\varepsilon > \left|\int_{x^\circ}^x \frac{dy}{\varphi^\circ(y)}\right| \geq |x - x^\circ| \cdot \min\{\frac{1}{\varphi^\circ(y)}; y \in I\},$$

and so

$$|x - x^\circ| \leq \varepsilon \cdot \max\{\varphi^\circ(y); y \in I\} \leq \varepsilon \cdot \big(\varphi^\circ(x^\circ) + c \cdot |x - x^\circ|\big).$$

Therefore, if ε is sufficiently small, then $|x - x^\circ| \le 2\varepsilon\varphi^\circ(x^\circ)$. Thus

$$\begin{aligned}\Big|\frac{\varphi(x)}{\varphi^\circ(x^\circ)} - 1\Big| &\le \Big|\frac{\varphi^\circ(x)}{\varphi^\circ(x^\circ)}(1+\theta(x)) - \frac{\varphi^\circ(x)}{\varphi^\circ(x^\circ)}\Big| + \Big|\frac{\varphi^\circ(x)}{\varphi^\circ(x^\circ)} - 1\Big| \\ &= |\theta(x)|\Big|\frac{\varphi^\circ(x)}{\varphi^\circ(x^\circ)}\Big| + \frac{|\varphi^\circ(x) - \varphi^\circ(x^\circ)|}{\varphi^\circ(x^\circ)} \\ &\le |\theta(x)|\cdot\frac{\varphi^\circ(x^\circ) + c|x - x^\circ|}{\varphi^\circ(x^\circ)} + \frac{c|x - x^\circ|}{\varphi^\circ(x^\circ)} \\ &\le |\theta(x)|(1 + 2c\varepsilon) + 2c\varepsilon.\end{aligned}$$

Q.E.D.

Put $\tilde{g} - \tilde{g}^\circ = \partial\bar{\partial}u$. Then

$$\Delta^\circ u = -\operatorname{tr}_{\tilde{g}^\circ}(\tilde{g} - \tilde{g}^\circ) = -(\varphi \circ U - \varphi^\circ \circ U^\circ)/(\varphi^\circ \circ U^\circ)$$

converges uniformly to 0 by Lemma 4.1, which implies that $\tilde{g}$ converges to $\tilde{g}^\circ$. Moreover,

$$\begin{aligned}&\frac{d}{dU^\circ}\Delta^\circ u = -\frac{d}{dU^\circ}\Big(\frac{\varphi \circ U}{\varphi^\circ \circ U^\circ}\Big) \\ &= \frac{\varphi \circ U}{\varphi^\circ \circ U^\circ}\frac{1}{\varphi^\circ \circ U^\circ} \\ &\quad \times \Big\{\Big(\frac{d(\varphi^\circ \circ U^\circ)}{dU^\circ} - \frac{d(\varphi^\circ \circ U)}{dU}\Big) + \Big(\frac{d(\varphi^\circ \circ U)}{dU} - \frac{d(\varphi \circ U)}{dU}\Big)\Big\},\end{aligned}$$

and the last line uniformly converges to 0 by Lemma 3.3, Lemma 4.1 and the mean value theorem. Therefore $\tilde{g}^\circ(d\Delta^\circ u, d\Delta^\circ u) = \varphi^\circ \circ U^\circ \cdot (d\Delta^\circ u/dU^\circ)^2$ converges to 0, which implies that $\tilde{g}_t$ converges to $\tilde{g}^\circ$ in C^1-topology. Combining it with Lemma 1.4, we complete a proof of the following

Theorem 4.2. *Let $\tilde{g}_0$ be as in section 2 and let $\tilde{g}_t$ be the solution of original Hamilton's equation* (H). *Then the family $\gamma_t^*\tilde{g}_t$ converges to a quasi-Einstein metric in C^∞-norm, where $\gamma_t = \exp(-(1/2)EtH)$.*

References

[1] E. Calabi, Extremal Kähler metrics, in Seminars on differential geometry (S.T. Yau ed.), Princeton Univ. Press & Univ. of Tokyo Press, Princeton, NJ.

[2] H.-D. Cao, Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds, Invent. math., **81** (1985), 359–372.

[3] A. Futaki, An obstruction to the existence of Einstein Kähler metrics, Invent. math., **73** (1983), 437–443.

[4] R.S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geometry, **17** (1982), 255–306.

[5] N. Koiso, On rotationally symmetric Hamilton's equation for Kähler–Einstein metrics, Max–Planck–Institut preprint series 87–16, (1987), Max–Planck–Institut für Mathematik, Bonn.

[6] N. Koiso and Y. Sakane, Non-homogeneous Kähler–Einstein metrics on compact complex manifolds, in Springer Lecture Notes in Math. 1201 (Curvature and Topology of Riemannian Manifolds, ed. K. Shiohama and others).

[7] N. Koiso and Y. Sakane, Non-homogeneous Kähler–Einstein metrics on compact complex manifolds II, Osaka J. Math., **25** (1988), to appear.

[8] S. Nishikawa, Deformation of riemannian metrics and manifolds with bounded curvature ratios, Proc. of Sympos. in Pure Math., **44** (1986), 343–352.

[9] Y. Sakane, Examples of compact Einstein Kähler manifolds with positive Ricci tensor, Osaka J. Math., **23** (1986), 585–616.

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