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On Plurigenera of Normal Isolated Singularities II

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In this paper we prove some results on plurigenera of normal isolated singularities. This paper is a continuation of [19].

In Section 1, we recall some preliminary facts related to the concept of plurigenera of normal isolated singularities. In Section 2, we prove the δ_m-formula for non-degenerate hypersurface isolated singularities, which is a generalization of Theorem 1.13 [19, p. 71]. In Section 3, we determine the "type" of purely elliptic singularities of hypersurfaces. In the last section we show a criterion for a singularity to be Du Bois: In the case where a singularity is quasi-Gorenstein, a singularity (X, x) is Du Bois if and only if $0 \leqq \delta_m(X, x) \leqq 1$. Finally we give examples of Du Bois singularities with possibly positive geometric genera, which are not quasi-Gorenstein.

§ 1. Plurigenera of normal isolated singularities

We need to recall a few preliminaries related to the concept of plurigenera of normal isolated singularities. For more details we refer to [19].

Let (X, x) be a normal isolated singularity of an n-dimensional analytic space X. Let V be a (sufficiently small) Stein neighborhood of x and let K be the canonical line bundle of $V-\{x\}$. For convenience, we denote the line bundle $K^{\otimes m}$ by mK. An element of $\Gamma(V-\{x\}, \mathcal{O}(mK))$ is considered as a holomorphic m-ple n-form. Let ω be a holomorphic m-ple n-form on $V-\{x\}$. We write ω as

$$\omega = \phi(z)(dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n)^m,$$

using local coordinates $(z_1, z_2, \cdots, z_n)$. We associate with ω the continuous (n, n)-form $(\omega \wedge \bar{\omega})^{1/m}$ given by

$$|\phi(z)|^{2/m}(\sqrt{-1}/2)dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}.$$

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Definition 1.1. ω is called integrable ($L^{2/m}$-integrable) if

$$\int_{W-\{x\}} (\omega \wedge \bar{\omega})^{1/m} < \infty$$

for any sufficiently small relatively compact neighborhood W of x in X.

Let $L^{2/m}(V-\{x\})$ be the set of all integrable holomorphic m-ple n-form on $V-\{x\}$, which is a linear subspace of $\Gamma(V-\{x\}, \mathcal{O}(mK))$. Then $\Gamma(V-\{x\}, \mathcal{O}(mK))/L^{2/m}(V-\{x\})$ is a finite dimensional vector space.

Definition 1.2. The plurigenus (m-genus), m being a positive integer, of a normal isolated singularity (X, x) is

$$\delta_m(X, x) = \dim \Gamma(V-\{x\}, \mathcal{O}(mK))/L^{2/m}(V-\{x\}).$$

These integers $\{\delta_m\}$ are determined independently of the choice of the Stein neighborhood.

In particular $\delta_1(X, x) = p_g(X, x)$ (See Laufer [10], Yau [21]).

Example 1.3. Let (X, x) be a normal surface singularity defined by the polynomial $x^8 + y^8 + z^8 + (xyz)^2$. Then $\delta_m(X, x) = 48m^2 - 36m + 20$; see Example 2.8.

Let $\pi: \tilde{X} \to X$ be a resolution of (X, x) and $U = \pi^{-1}(V)$ and $E = \pi^{-1}(x)$. By Sakai [15, Theorem 2.1, p. 243],

$$L^{2/m}(V-\{x\}) \cong L^{2/m}(U-E) = \Gamma(U, \mathcal{O}(mK + (m-1)E))$$

if the exceptional set is a divisor which has at most normal crossings.

Thus

$$\delta_m(X, x) = \dim \Gamma(U-E, \mathcal{O}(mK))/\Gamma(U, \mathcal{O}(mK + (m-1)E)).$$

This formula provides a practical means to compute δ_m in many cases.

§ 2. Plurigenera of hypersurface singularities

Next we consider an isolated singularity defined by a non-degenerate holomorphic function f. Such singularities are always normal.

In the following we give an effective method for calculating δ_m via the combinatorial data of the Newton polyhedron $\Gamma_+(f)$ of a function f under the assumption that f is nondegenerate with respect to $\Gamma(f)$. It is based on the computation of resolutions of hypersurface singularities using the technique of toric varieties.

Before stating our result, we refer to some results obtained by using the technique of toric varieties.

First Ehlers and Lo [2] has computed the minimal characteristic exponent of a non-degenerate holomorphic map germ with an isolated critical point at 0. Secondly Bernstein et al. [1] and Hovanskii [4] has computed the Euler-Poincaré characteristic of the zero locus of a function, and Varchenko [18] has computed the characteristic polynomial of the monodromy of a critical point of an analytic function.

It is known that there is a canonical resolution by the torus embedding associated with a simplicial subdivision of the dual Newton diagram of f. So we need to recall a few preliminaries related to the repetition of the torus embedding associated with a simplicial subdivision of the dual Newton diagram of f. For more details we refer to [14].

Let $f(z_0, z_1, \cdots, z_n)$ be a germ of an analytic function at the origin such that $f(0)=0$ and let f has an isolated critical point at the origin. Then f can be developed in a convergent Taylor series $f(z_0, z_1, \cdots, z_n)=\sum_\lambda a_\lambda z^\lambda$ where $z^\lambda = z_0^{\lambda_0}\cdots z_n^{\lambda_n}$. Recall that the Newton boundary $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$ where $\Gamma_+(f)$ is the convex hull of the union of the subsets $\{\lambda+(\boldsymbol{R}^+)^{n+1}\}$ for λ such that $a_\lambda\neq 0$. For any closed face Δ of $\Gamma(f)$, we associate the polynomial $f_\Delta(z)=\sum_{\lambda\in\Delta} a_\lambda z^\lambda$. We say that f is non-degenerate if f_Δ has no critical point in $(\boldsymbol{C}^*)^{n+1}$ for any $\Delta\in\Gamma(f)$. We assume that f has a non-degenerate Newton boundary. Let X be the germ of the hypersurface $f^{-1}(0)$. Let $\Gamma^*(f)$ be the dual Newton diagram and let Σ^* be a simplicial subdivision of $\Gamma^*(f)$. It is known that there is a canonical resolution $\pi\colon \tilde{X}\to X$ which is associated with Σ^*.

Let N^+ be the space of positive vectors in the dual spaces of $\boldsymbol{R}^{n+1}$. For any vector $A=(a_0, a_1, \cdots, a_n)$ of N^+, we associate the linear function $A(\lambda)=\sum_i a_i\lambda_i$ on $\Gamma_+(f)$ and let $d(A)$ be the minimal value of $A(\lambda)$ on $\Gamma_+(f)$ and let $\Delta(A)=\{\lambda\in\Gamma_+(f);\, A(\lambda)=d(A)\}$. We introduce an equivalence relation $\sim$ on N^+ by $A\sim B$ if and only if $\Delta(A)=\Delta(B)$. For any face Δ of $\Gamma_+(f)$, let

$$\Delta^*=\{A\in N^+;\, \Delta(A)=\Delta\}.$$

The collection of Δ^* gives a polyhedral decomposition of N^+ which we call the dual Newton polyhedron of f. We denote it by $\Gamma^*(f)$. $\Delta(A)$ is a compact face of $\Gamma(f)$ if and only if A is strictly positive. We say that a subdivision Σ^* of $\Gamma^*(f)$ is a simplicial subdivision if the following conditions are satisfied.

(i) Σ^* is a subdivision by the cones over a simplicial polyhedron whose simplexes are spanned by primitive integral vectors with determinant $+1$.

(ii) Let σ be an n-simplex spanned by $\{A_0, A_1, \cdots, A_n\}$, which is denoted by $\langle A_0, A_1, \cdots, A_n\rangle$. Then

$$\bigcap_{i=0}^{n} \Delta(A_i) = \{\text{one point}\}.$$

(iii) Assume that $\Gamma(f)^I$ is non-empty where

$$\Gamma(f)^I = \{x \in \Gamma(f);\ x_i \neq 0 \text{ only if } i \in I\}$$

and I is a subset of $\{0, \cdots, n\}$. Then $\sigma_I = \{A \in N^+;\ a_i = 0 \text{ if } i \text{ is in } I\}$ is a simplex.

Let $\rho\colon N^+ \to I_n$ be the natural projection onto an n-dimensional simplex $I_n = N^+/\boldsymbol{R}_{>0}$. Then $\rho(\Sigma^*)$ gives a simplicial subdivision of I_n.

Since every cone in Σ^* is non-singular, the associated torus-embedding Z is non-singular.

Let $E(A)$ be a divisor of Z associated with a one-dimensional cone generated by $A \in N^+$, i.e., using the notations of Oda [13] $E(A) = \overline{\mathrm{orb}\,(\boldsymbol{R}_0 A)}$. Since Σ^* is an r.p.p. decomposition of N^+, Z is a modification of $\boldsymbol{C}^{n+1}$. Let π be its birational morphism from Z to $\boldsymbol{C}^{n+1}$.

More precisely, let Σ^* be a simplicial subdivision of $\Gamma^*(f)$. For each n-simplex $\sigma = \langle A_0, \cdots, A_n \rangle$, $A_i = (a_{i0}, \cdots, a_{ij}, \cdots, a_{in})$, we associate an $(n+1)$-dimensional Euclidean space $\boldsymbol{C}^{n+1}$ with coordinates $(u_{\sigma,0}, u_{\sigma,1}, \cdots, u_{\sigma,n})$ and a birational mapping $\pi_\sigma\colon \boldsymbol{C}^{n+1} \to \boldsymbol{C}^{n+1}$ which is defined by

$$z_j = (u_{\sigma,0})^{a_{0,j}} (u_{\sigma,1})^{a_{1,j}} \cdots (u_{\sigma,n})^{a_{n,j}}.$$

Let Z be the union of $\boldsymbol{C}_\sigma^{n+1}$ which are glued "along the image of π_σ". Let π be the projection and let $\tilde{X}$ be the closure of $\pi^{-1}(X - \{x\})$, x being the origin. It is known that $\pi|_{\tilde{X}}\colon \tilde{X} \to X$ is a resolution of X. Let $d_i = d(A_i)$ and $\Delta_i = \Delta(P_i)$. We assume that $\bigcap_{i=0}^{n} \Delta(A_i) = \{\text{one point}\}$. We define $f_\sigma(u_\sigma)$ and $f_{\Delta_i,\sigma}(u_\sigma)$ by

$$f(\pi_\sigma(u_\sigma)) = f_\sigma(u_\sigma) \prod_i (u_{\sigma,i})^{d_i} \quad \text{and} \quad f_{\Delta_i}(\pi_\sigma(u_\sigma)) = f_{\Delta_i,\sigma}(u_\sigma) \prod_i (u_{\sigma,i})^{d_i}.$$

By the definition, $\tilde{X}$ is defined by $f_\sigma(u_\sigma) = 0$ and $\tilde{X} \cap \{u_{\sigma,i} = 0\}$ is $\{u_\sigma;\ u_{,\sigma i} = 0$ and $f_{\Delta_i,\sigma}(u_\sigma) = 0\}$.

Note that $f_{\Delta_i,\sigma}(u_\sigma)$ is a function of $\{u_{\sigma,j};\ \Delta_i - \Delta_j \neq \emptyset\}$. Thus $\tilde{X} \cap \{u_{\sigma,i} = 0\}$ is non-empty if and only if $\dim \Delta_i > 0$. Let $E(A_i; \sigma) = \{u_\sigma \in \tilde{X};\ u_{\sigma,i} = 0\}$. $\pi(E(A_i; \sigma)) = \{0\}$ if and only if A_i is strictly positive. The union of $E(A_i; \sigma)$ for simplexes σ which contain A_i is a divisor of $\tilde{X}$ and we denote it by $E(A_i)$. We say that vertices $A_0, \cdots, A_k$ in Σ^* are adjacent if there is an n-simplex σ of Σ^* which contains $A_0, \cdots, A_k$.

Let $\omega = \mathrm{Res}\,((1/f) dz_0 \wedge dz_1 \wedge \cdots \wedge dz_n)$. ω is locally written in the form

$$\omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n / (\partial f / \partial z_0).$$

Then ω is a nowhere vanishing holomorphic n-form on $X-\{x\}$.

Lemma 2.1. *$\pi^*(z^\lambda(dz/f)^m)$ has zeros of order $A_0(\lambda)+m(|A_0|-1-d(A_0))$ at a divisor $E(A_0)$, where $|A_0|=a_{00}+a_{01}+\cdots+a_{0n}$.*

Proof. Pick n primitive integral vectors $A_i=(a_{i,0}, a_{i,1}, \cdots, a_{i,n})$ of Σ^*, by which n-simplex σ is spanned. Then there exists the associated $(n+1)$-dimensional Euclidean space C_σ^{n+1} with coordinates $(u_{\sigma,0}, \cdots, u_{\sigma,n})$ and a birational mapping π_σ: $C^{n+1}\to C^{n+1}$ which is defined by

$$z_j=(u_{\sigma,0})^{a_{0,j}}(u_{\sigma,1})^{a_{1\,j}}\cdots(u_{\sigma,n})^{a_{n,j}}.$$

Then

$$\begin{aligned}\pi^*(z^\lambda)&=\pi^*(z_0^{\lambda_0}z_1^{\lambda_1}\cdots z_n^{\lambda_n})\\&=(u_{\sigma,0}^{a_{0,0}}\cdots u_{\sigma,n}^{a_{n,0}})^{\lambda_0}\cdots(u_{\sigma,0}^{a_{0,n}}\cdots u_{\sigma,n}^{a_{n,n}})^{\lambda_n}\\&=(u_{\sigma,0})^{A_0(\lambda)}(u_{\sigma,1})^{A_1(\lambda)}\cdots(u_{\sigma,n})^{A_n(\lambda)}\\ \pi^*dz&=\pi^*(dz_0\wedge dz_1\wedge\cdots\wedge dz_n)\\&=(u_{\sigma,0})^{|A_n|-1}\cdots(u_{\sigma,n})^{|A_n|-1}du_{\sigma,0}\wedge\cdots\wedge du_{\sigma,n}\\ \pi^*f&=(u_{\sigma,0})^{d(A_0)}\cdots(u_{\sigma,n})^{d(A_n)}f_\sigma(u_{\sigma,0}, \cdots, u_{\sigma,n}).\end{aligned}$$

The desired result follows immediately from the equations above.

When (X, x) is defined by a non-degenerate holomorphic function, the numbers $\{\delta_m\}$ are expressed in terms of the Newton diagram. We denote by $\Gamma_-(f)$ the cone over $\Gamma(f)$ with cone point the origin.

Theorem 2.2. $\delta_m(X, x)=\#(m\Delta(f))-\#\{\lambda\in N^{n+1}; (\lambda+\Gamma_+(f))\cap m\Delta(f)\neq\phi\}$ *where* $m\Delta(f)=\{\lambda\in N^{n+1}; \lambda/m+(1, 1, \cdots, 1)\in\Gamma_-(f)\}$.

Proof. If θ is any holomorphic m-ple n-form on $X-\{x\}$, $g=\theta/\omega^m$ is a holomorphic function on $X-\{x\}$ and hence extends to be holomorphic also at x. The singular point x is normal, so there exists a holomorphic function $G(z)$ in $C\{z_0, z_1, \cdots, z_n\}$ such that $G|_X=g$. Expand $G(z)$ in a power series:

$$G=\sum c_{\lambda_0,\lambda_1,\cdots,\lambda_n}z_0^{\lambda_0}z_1^{\lambda_1}\cdots z_n^{\lambda_n}.$$

Then $\pi^*(z^\lambda\omega^m)\in\Gamma(\tilde{X}, \mathcal{O}(mK+(m-1)E))$ if and only if $A(\lambda)+m(|A|-1-d(A))+(m-1)\geq 0$ for any strictly positive integral vector A of Σ^*. So, $\pi^*(z^\lambda\omega^m)\notin\Gamma(\tilde{X}, \mathcal{O}(mK+(m-1)E))$ if and only if $A(\lambda)\leq m(d(A)-|A|)$ for some strictly positive integral vector A of Σ^*, i.e., $\lambda/m+(1, 1, \cdots, 1)\in\Gamma_-(f)$.

An arbitrary polynomial F can be uniquely divided into two parts: $F=(F)_++(F)_-$, where $(F)_-$ consists of monomials z^λ such that $\lambda\in m\Delta(f)$, and we denote $F-(F)_-$ by $(F)_+$.

Hence $\theta \sim \tilde{g}\omega^m$ (mod $L^{2/m}(X-\{x\})$), where $g=(G)_-|_X$. Moreover, assume moreover $\tilde{g}\omega^m \in L^{2/m}(X-\{x\})$. Then there exists $H=H_+$ in $C\{z_0, z_1, \cdots, z_n\}$ such that $H_+|X=G_-|X$. Hence $G_- - H_+ = P f = (\sum c_\lambda z^\lambda)f = \sum c_\lambda(z^\lambda f) = \sum c_\lambda((z^\lambda f)_+ + (z^\lambda f)_-)$ for some polynomial P. Therefore $G_- = \sum c_\lambda (z^\lambda f)_-$. Thus the proof is complete.

Example 2.5. Let (X, x) be a normal surface singularity defined by the polynomial $x^2+y^3+z^7-\lambda yz^5$. Then

$$x^2\omega^{42} \notin L^{2/42}(X-\{x\}), \quad y^3\omega^{42} \notin L^{2/42}(X-\{x\}) \quad \text{and}$$
$$z^7\omega^{42} \notin L^{2/42}(X-\{x\}), \quad \text{but}$$
$$(x^2+y^3+z^7)\omega^{42} = \lambda yz^5\omega^{42} \in L^{2/42}(X-\{x\}).$$

Corollary 2.6 (Merle-Teissier [11]). $p_g(X, x) = \sharp(\Delta(f))$.

Corollary 2.7. *It is the Newton boundary of a non-degenerate holomorphic function that determines δ_m completely.*

Example 2.8. Let (X, x) be a normal surface singularity defined by the polynomial $x^8+y^8+z^8+x^2y^2z^2$. Then

$$\sharp(m\Delta(f)) = 8m^3+15m^2+8m+1,$$

and

$$\sharp\{\lambda \in N^{n+1}; (\lambda+\Gamma_+(f)) \cap m\Delta(f) = \phi\} \neq 8(m-2)^3+15(m-2)^2+8(m-2)+1.$$

Therefore

$$\delta_m(X, x) = 48m^2-36m+20.$$

Remark. Using Kato's theorem [9, p. 246], we can calculate δ_m of the above example.

One can easily check that Theorem 2.2 gives the following:

Proposition 2.9 (Watanabe-Higuchi [20], Yumiba [23]). *Under similar conditions,*

$$(1, 1, \cdots, 1) \in (\Gamma_-(f))^\circ \Longleftrightarrow \delta_m = 0, \quad \textit{for } m \geq 1,$$
$$(1, 1, \cdots, 1) \in \Gamma(f) \Longleftrightarrow \delta_m = 1, \quad \textit{for } m \geq 1,$$
$$(1, 1, \cdots, 1) \in (\Gamma_+(f))^\circ \Longleftrightarrow \limsup_{m\to\infty} \delta_m/m^n > 0,$$

where ( )° means the interior of ( ).

§ 3. (0, s)-type purely elliptic singularities

In this section we consider natural generalization of purely elliptic singularities of surfaces to higher dimensions.

Definition 3.1. A normal isolated singularity (X, x) is purely elliptic if $\delta_m(X, x)=1$, for $m\geqq 1$, where $\delta_m(X, x)$ is the m-genus of (X, x).

They are the next most reasonable class of singularities after rational singularities. These purely elliptic singularities have a theory very similar to the theory for simple elliptic singularities and cusp singularities. They are also useful in answering some questions about other types of normal n-dimensional isolated singularities. Basically, we apply the result of Ishii [8] to this situation, our tool being the technique of toric varieties due to Varchenko [18], Ehlers-Lo [2] and Oka [14]. Several examples of such singularities are found, especially a certain class of hypersurface singularities with the minimal characteristic exponent equal to 1, and all cusp singularities.

In the case where $\dim_x X=2$, (X, x) is a simple elliptic singularity or a cusp singularity if (X, x) is a purely elliptic Gorenstein singularity. In higher dimensions, however, the condition is in some sense less restrictive than in dimension 2, as the following example shows:

Example 3.2. Let (X, x) be the n-dimensional normal isolated singularity obtained by blowing down the zero section, denoted by M, of a negative line bundle. If the canonical line bundle of M is trivial, then (X, x) is purely elliptic; see [20].

Example 3.3. Let (X, x) be the n-dimensional normal isolated singularity defined by a quasihomogeneous polynomial of type $(r_0, r_1, \cdots, r_n)$ with $r(f)=r_0+r_1+\cdots+r_n=1$. Then by Proposition 2.9 (X, x) is purely elliptic; see also [20].

Now we derive a criterion for (X, x) to be purely elliptic.

Definition 3.4. Let (X, x) be a normal isolated singularity. We say (X, x) is quasi-Gorenstein if there exists a holomorphic n-form ω defined on a deleted neighborhood of x, which is nowhere vanishing on this neighborhood.

Theorem 3.5. *Let (X, x) be a normal isolated quasi-Gorenstein singularity and let ω be an n-form satisfying the condition of Definition* 3.4 *and let V be a Stein neighborhood of x. Then (X, x) is purely elliptic if and only if $\omega \notin L^2(V-\{x\})$ and $f\omega^m \in L^{2/m}(V-\{x\})$ for any $f \in \mathfrak{m}$, the maximal ideal in $\mathcal{O}_{X,x}$.*

One can easily check that the Theorem above gives the following:

Theorem 3.6. *Let (X, x) be a normal isolated singularity. Suppose that ω is a holomorphic n-form defined on a deleted neighborhood of x, which is nowhere vanishing on this neighborhood and that there exists a resolution $\pi: \tilde{X} \to X$ such that the exceptional set $A = \pi^{-1}(x)$ is a divisor which has at most normal crossings. Then (X, x) is purely elliptic if and only if ω is not L^2-integrable and $(\pi^*\omega) + A \geqq 0$, i.e., any multiplicity of $\pi^*\omega$ on each component of A is greater than or equal to -1 and there exists at least one component where the multiplicity of $\pi^*\omega$ is exactly -1.*

Remark. Let (X, x) be a normal isolated singularity whose local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay. If the singularity (X, x) is quasi-Gorenstein, then the local ring $\mathcal{O}_{X,x}$ is Gorenstein.

Problem 3.7. Find a purely elliptic singularity which is not quasi-Gorenstein. There are no known examples of this type.

From now on, we study the exceptional sets of a good resolution of a purely elliptic singularity (X, x) of a hypersurface. By Theorem 3.5, $K_{\tilde{X}} \sim \sum_{i \in I} m_i E_i - \sum_{j \in J} E_j$ where $m_i \geqq 0$ for $i \in I$. Put $E_I = \sum_{i \in I} E_I$, $E_J = \sum_{j \in J} E_j$. We call E_J the essential part of the exceptional set, which plays, in fact, an essential role as we see below.

Definition 3.8. A quasi-Gorenstein purely elliptic singularity (X, x) is of $(0, i)$-type $(i = 0, 1, \cdots, n-1)$ if $H^{n-1}(E_J, \mathcal{O}_E)$ consists of $(0, i)$-Hodge component.

Remark. Since $C \cong H^{n-1}(E_J, \mathcal{O}_{E_J}) = \mathrm{Gr}_F^0 H^{n-1}(E_J) = \bigoplus_{i=0}^{n-1} H_{n-1}^{0,i}(E_J)$, $H^{n-1}(E_J, \mathcal{O}_{E_J})$ coincides with one of $H_{n-1}^{0,i}(E_J)$.

Theorem 3.9 (Ishii [8]). *Let (X, x) be a 3-dimensional Gorenstein purely elliptic singularity with a good resolution $\pi: \tilde{X} \to X$ such that $K_{\tilde{X}} = -E_J$. Then E_J is either;*

(i) *a non-singular $K3$-surface if (X, x) is of type* $(0, 2)$,

(ii) *a chain of surfaces $E_1, E_2, \cdots, E_r$ where E_1 and E_r are rational and E_i $(i = 2, \cdots, r-1)$ are elliptic ruled and any intersection curves are elliptic, if (X, x) is of type* $(0, 1)$, *or*

(iii) *the dual graph Γ_{E_J} of E_J is a triangulation of the real 2-dimensional sphere, any component of E_J is a rational surface and any intersection curve is rational if (X, x) is of type* $(0, 0)$.

Let (X, x) be a normal n-dimensional isolated singularity defined by a non-degenerate holomorphic function f. In the following, we study the "type" of purely elliptic singularities. From the result of Ishii [8] we need not obtain all the information about the exceptional set. We only have to know the dimension of the dual graph of the essential divisors.

Theorem 3.10 (Ishii [8]). *Let $\pi\colon \tilde{X}\to X$ be a good resolution of an n-dimensional purely elliptic singularity (X, x) of type $(0, s)$ with the essential divisor E_J. Assume moreover that (X, x) is quasi-Gorenstein. Then the dual graph of Γ_{E_J} of E_J is an $(n-s-1)$-dimensional simplicial complex.*

We can easily see that the converse of Theorem 3.10 is true. Let Δ_0 be the compact face which contains the point $(1, 1, \cdots, 1)$ as a (relatively) interior point.

The notation being as in Section 2, the following propositions are trivial from Lemma 2.1.

Proposition 3.11. *A meromorphic $(n+1)$-form $dz_0\wedge dz_1\wedge\cdots\wedge dz_n/f$ has a pole of order one along $E(A)$ if and only if $\Delta(A)\ni(1, 1, \cdots, 1)$.*

Proposition 3.12. *$\tilde{X}\cap E(A)$ is non-empty if and only if* $\dim \Delta(A)>0$.

Theorem 3.13. *Let s be the dimension of Δ_0. Then there are $n+1-s$ vertices $A_1, \cdots, A_{n+1-s}$ of Σ^* such that $\Delta_0=\bigcap_{i=0}^{n+s-1}\Delta(A_i)$, and the dual graph Γ_{E_J} of E_J is a subdivision of the $(n-s)$-dimensional simplicial complex spanned by $A_1, A_2, \cdots, A_{n+1-s}$ if $s>0$, and a $(n-1)$-dimensional simplicial subdivision of the boundary of the n-dimensional simplicial complex spanned by $A_1, A_2, \cdots, A_{n+1}$ if $s=0$.*

Corollary 3.14. *Let (X, x) be an n-dimensional purely elliptic singularity defined by a non-degenerate holomorphic function, and let s be the dimension of Δ_0. Then (X, x) is of type* $(0, s-1)$ *if* $s>0$, *and of type* $(0, 0)$ *if* $s=0$.

Remark. This corollary is due to S. Iida [5] in the case where all faces of $\Gamma(f)$ are simplicial.

Now consider non-degenerate holomorphic functions $f(z_0, z_1, \cdots, z_n)$ with the property that the Newton boundary contains the point $(1, 1, \cdots, 1)$. All such functions fall into $n+1$ classes according to the dimension of the compact face Δ_0 which contains the point $(1, 1, \cdots, 1)$ as a (relatively) interior point.

The following are examples of polynomials of 4 variables, which makes Δ_0.

Example 3.15 (Yonemura [22]).

$\dim \Delta_0 = 0$ $xyzw$.
$\dim \Delta_0 = 1$ $x^2 + y^2z^2w^2$, and others.
$\dim \Delta_0 = 2$ $x^3 + y^3 + z^3w^3$, and others.
$\dim \Delta_0 = 3$ $x^p + y^q + z^r + w^s$ $(1/p + 1/q + 1/r + 1/s = 1)$,
and other quasihomogeneous polynomials of type (a, b, c, d) with $a + b + c + d = 1$.

Remark. The case of $\dim \Delta_0 = 1$ is reduced to the case of $\dim \Delta_0 = 0$, e.g., $x^2 + y^2z^2w^2 = (x + yzw)^2 - 2xyzw$.

§ 4. A criterion for a singularity to be Du Bois

The purpose of this section is to prove the following:

Theorem 4.1. *If a purely elliptic singularity (X, x) is quasi-Gorenstein, then (X, x) is a Du Bois singularity.*

Remark. This Theorem is due to S. Ishii [8] in the case of Gorenstein singularities, for the Cohen-Macaulay property implies $H^i(X, \mathcal{O}_X) = 0$ for $i \neq 0, n-1$.

In fact, by Theorem 4.1, Ishii's Theorem [8, Theorem 2.3] holds in the case of quasi-Gorenstein singularities. For the reader's convenience we shall restate it in a generalized form.

Theorem 4.2. *Let $\pi: \tilde{X} \to X$ be a good resolution of a normal isolated quasi-Gorenstein singularity (X, x) of dimension $n \geqq 2$. Denote $\pi^{-1}(x)_{\mathrm{red}}$ by E and decompose E into irreducible components E_i $(i = 1, 2, \cdots, n)$. Then the following three conditions are equivalent:*

(i) $\delta_m(X, x) \leqq 1$ *for any* $m \in N$.
(ii) (X, x) *is a Du Bois singularity.*
(iii) $K_{\tilde{X}} = \pi^* K_X + \sum m_i E_i$ *with* $m_i \geqq -1$ *for all* i.

Normal isolated Du Bois singularities are characterized as follows: Let (X, x) be a normal n-dimensional isolated singularity and $\pi: \tilde{X} \to X$ be a good resolution. Then (X, x) is a Du Bois singularity if and only if the natural maps $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to H^i(E, \mathcal{O}_E)$ are isomorphisms for all $i > 0$ where $E = \pi^{-1}(x)_{\mathrm{red}}$ (see Steenbrink [16]).

Assume moreover that X is a (contractible) Stein space. Let $\pi: \tilde{X} \to X$ be a resolution of X such that $E = \pi^{-1}(x)_{\mathrm{red}}$ is a divisor with normal crossings on $\tilde{X}$.

For the analysis of the situation, two exact sequences are of fundamental importance:

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(-E) \longrightarrow \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_E \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(K) \longrightarrow \mathcal{O}_{\tilde{X}}(K+E) \longrightarrow \mathcal{O}_E(K_E) \longrightarrow 0.$$

Lemma 4.3 (Steenbrink [16]). *Let (X, x) be an isolated singularity where X is a contractible Stein Space; let $\pi\colon \tilde{X} \to X$ be a resolution of X such that $E = \pi^{-1}(x)^{\mathrm{red}}$ is a divisor with normal crossings on X. Then for all $i \geqq 0$ the natural map*

$$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \longrightarrow H^i(E, \mathcal{O}_E)$$

is surjective.

From this lemma, we need only to verify the natural map

$$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \longrightarrow H^i(E, \mathcal{O}_E)$$

is injective for all $i > 0$, i.e., we only have to prove the following Lemma and Proposition. From the assumption that (X, x) is a quasi-Gorenstein singularity there exists a non-vanishing holomorphic n-form defined on a deleted neighborhood of $x \in X$. We denote $\pi^*\omega$ by $\tilde{\omega}$. The notations being as above, since (X, x) is purely elliptic, $(\tilde{\omega}) + E \geqq 0$. Then by cupping, or wedging with $\tilde{\omega}$ we have a sheaf morphism from $\mathcal{O}_{\tilde{X}}$ to $\mathcal{O}_{\tilde{X}}(K+E)$.

Lemma 4.4. *The associated morphism*

$$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \longrightarrow H^i(\tilde{X}, O_{\tilde{X}}(K+E))$$

is injective for $i < n$.

Proof. Recall the point of view of [10], which we will be using below. In particular, consider the sheaf cohomology with support at infinity. Then we have the following commutative diagram, where the vertical arrows are cupping, of wedging, with $\tilde{\omega}$:

$$\begin{array}{ccccccc}
\cdots \to H^i_c(U, \mathcal{O}_U) & \longrightarrow & H^i(U, \mathcal{O}_U) & \xrightarrow{\psi_i} & H^i_\infty(U, \mathcal{O}_U) & \to \cdots \\
 & & \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \gamma_i} & \\
\cdots \to H^i_c(U, \mathcal{O}_U(K+E)) & \longrightarrow & H^i(U, \mathcal{O}_U(K+E)) & \xrightarrow{\phi_i} & H^i_\infty(U, \mathcal{O}_U(K+E)) & \to \cdots
\end{array}$$

The right hand side arrows γ_i are isomorphisms, since "at ∞" $\tilde{\omega} = \omega$ doesn't vanish. By the result of Grauert and Riemenschneider [3] and Serre duality $H^i_c(U, \mathcal{O}_U) = 0$ for $i < n$, so ψ_i is bijective for all $i < n-1$ and ψ_{n-1} is injective. Because $\gamma_i \circ \psi_i = \phi_i \circ \beta_i$ is injective, β_i must be injective too.

Proposition 4.5. *For all $i>0$, the natural map*

$$H^i(U, \mathcal{O}_U) \longrightarrow H^i(E, \mathcal{O}_E)$$

is injective.

Proof. Consider the following commutative diagram, where the vertical arrows are cupping, or wedging, with $\tilde{\omega}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_U(-E) \arrow[r] \arrow[d] & \mathcal{O}_U \arrow[r] \arrow[d] & \mathcal{O}_E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_U(K) \arrow[r] & \mathcal{O}_U(K+E) \arrow[r] & \mathcal{O}_E(K_E) \arrow[r] & 0.
\end{tikzcd}$$

We obtain long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(U, \mathcal{O}_U(-E)) \arrow[r] & H^i(U, \mathcal{O}_U) \arrow[r, "\lambda_i"] \arrow[d, "\beta_i"] & H^i(E, \mathcal{O}_E) \arrow[r] \arrow[d, "\alpha_i"] & \cdots \\
\cdots \arrow[r] & H^i(U, \mathcal{O}_U(K)) \arrow[r] & H^i(U, \mathcal{O}_U(K+E)) \arrow[r, "\mu_i"] & H^i(E, \mathcal{O}_E(K_E)) \arrow[r] & \cdots.
\end{tikzcd}$$

Just as in Lemma 4.4, $H^i(U, \mathcal{O}_U(K))=0$ for $i>0$, so μ_i is bijective for $i>0$. Because $\mu_i \circ \beta_i = \alpha_i \circ \lambda_i$ is injective for $i>0$, λ_i must be injective too. Thus the Proposition is proved, and therefore completes the proof of Theorem 4.1.

Let us illustrate some of implications among rational singularities, purely elliptic singularities and Du Bois singularities, which have appeared so far.

$\delta_m=0$ for $m>0$.

$\omega \in L^2(X-\{x\}) \cdot \longleftrightarrow p_g=\delta_1=h^{n-1}=0$

$h^i(X, x)=0$ for $i>0$. $\longrightarrow$ Cohen-Macaulay $\longrightarrow$ Buchsbaum

$(p_g=0)$

$\pi^*\omega \in \Gamma(\tilde{X}, \mathcal{O}(K+E)) \leftarrow H^{n-1}(\tilde{X}, \mathcal{O}_{\tilde{X}}) \rightarrow H^{n-1}(E, \mathcal{O}_E)$ is isomorphic. $\leftarrow$ Du Bois

$(p_g=1)$

$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \rightarrow H^i(E, \mathcal{O}_E)$ is isomorphic for $i>0$.

$\pi^*\omega \in \Gamma(\tilde{X}, \mathcal{O}(K+E)) - \Gamma(\tilde{X}, \mathcal{O}(K))$

$\delta_m=1$ for $m>0$.

Theorem 4.6. *Let (X, x) be a normal isolated singularity with the property*

$$\delta_m(X, x)=0 \quad \text{if } m\not\equiv 0 \pmod p$$
$$\delta_m(X, x)=1 \quad \text{if } m\equiv 0 \pmod p$$

and with a non-vanishing holomorphic cross section of $\mathcal{O}(pK)$ on $X-\{x\}$. Then the singularity (X, x) is Du bois.

Proof. Recall that there exists a quasi-Gorenstein purely elliptic singularity (Y, y) and a finite group G of $\mathrm{Aut}(Y, y)$ with no fixed points except at the singularity y such that $(X, x)=(Y, y)/G$. Since a quotient singularity of a Du Bois singularity is also Du Bois, from the preceding lemma (X, x) is Du Bois.

Typical examples including the condition of Theorem 4.6 are the Tsuchihashi cusp singularities. For the definition of Tsuchihashi cusp singularities and their fundamental properties, see [17].

Let N be a free Z-module of rank $n>1$. Let (C, Γ) be a pair of an open convex cone in $N_{\boldsymbol{R}}=N\otimes_{\boldsymbol{Z}}\boldsymbol{R}$ which contains no line in $N_{\boldsymbol{R}}$ and a subgroup Γ of the automorphism group $\mathrm{GL}(N)$ of N such that C is Γ-invariant, the action of Γ on $D=C/\boldsymbol{R}_+$ is properly discontinuous and free, and has the compact quotient D/Γ.

For such a pair (C, Γ) there exists a normal isolated singularity, which is denoted by $\mathrm{Cusp}(C, \Gamma)$ and is called a Tsuchihashi cusp singularity.

The important properties to notice are as follows:

(i) A Tsuchihashi cusp singularity (X, x) has a resolution $\pi: \tilde{X}\to X$ whose exceptional set E consists of rational surfaces, crossing each other along rational curves, in such a way that the "dual graph" is a triangulation of a compact topological surface T.

(ii) According to $\Gamma\subset\mathrm{SL}(N)$ or not, a Tsuchihashi cusp singularity has a nonvanishing holomorphic cross section ω of $\mathcal{O}(K)$ or $\mathcal{O}(2K)$ so that $(\omega)=-E$ or $(2\omega)=-2E$ respectively, which is defined on a deleted neighborhood of the singularity x. This phenomenon corresponds to the fact that T is orientable or not.

(iii) Let $\mathfrak{m}_x$ be the maximal ideal at x. Then $\boldsymbol{R}\pi_*\mathcal{O}_{\tilde{X}}(-E)=\mathfrak{m}_x$ in the derived category of complexes of $\mathcal{O}_X$-modules bounded below and with coherent cohomology sheaves.

Corollary 4.7. *The Tsuchihashi cusp singularities are Du Bois singularities.*

Ishida [7] showed that Du Bois singularities are Buchsbaum.

Theorem 4.8. *Let (X, x) be a normal isolated Du Bois singularity. Then the local ring $\mathcal{O}_{X,x}$ is a Buchsbaum ring.*

Corollary 4.9 (Ishida). *Tsuchihashi cusp singularities are Buchsbaum.*

Remark. Ishida [6] proves this Corollary directly.

Finally we show that there is a Du Bois singularity, which is not purely elliptic. Let M be a compact complex manifold, and let F be a complex analytic line bundle over M.

Lemma 4.10 (Kodaira vanishing theorem). *If $F-K$ is positive in the sense of* [12], *then $H^q(M, \mathcal{O}(F))=0$ for $q \geqq 1$. Here K is the canonical line bundle of M.*

Assume that F is positive. We denote the total space of the dual line bundle F^* by $\tilde{X}$. The zero section of X is contractible. Then we get an n-dimensional normal isolated singularity (X, x) by blowing down $\pi: \tilde{X} \to X$. The Leray spectral sequence for $p: \tilde{X} \to M$, p the projection of F^*, shows

$$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^i(M, R^0 p_* \mathcal{O}_X) = \bigoplus_{k \geqq 0} H^i(M, \mathcal{O}(kF)).$$

Proposition 4.11. *If $F-K$ is positive, then (X, x) is Du Bois.*

Proof. From Lemma 4.10 it follows that the maps $H^i(\tilde{X}, \mathcal{O}_X) \to H^i(M, \mathcal{O}_M)$ are isomorphisms as soon as $kF-K$ is positive for all $k \geqq 1$.

Corollary 4.12. *If K is negative, then (X, x) is Du Bois.*

Corollary 4.13. *If $F-K$ is positive and $p_g(M)>1$, then (X, x) is a non-quasi-Gorenstein Du Bois singularity. Consequently (X, x) is not an "elliptic" singularity, much less a purely elliptic singularity. Here $p_g(M)$ is the geometric genus of M.*

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