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Congruences between Hilbert Cusp Forms

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§ 0. Introduction

This is a continuation of our previous paper [14]. In that paper, we reported an example of a congruence relation between Hilbert cusp forms over a real quadratic field. In this paper, we study such a congruence relation in a more general setting, and add several examples. More precisely, let F be a totally real algebraic number field and K a quadratic extension of F with the relative discriminant $\mathfrak{q}$. For simplicity, we assume $\mathfrak{q}$ is a prime ideal not dividing 2. Mainly, we treat the case where K has two real archimedean places, namely rank $E_K = \text{rank } E_F + 1$. Here E_K (resp. E_F) denote the group of units in K (resp. F). When the class number of F in the narrow sense is odd, we can show that E_K is generated by E_F and a unit η of K. We define a certain polynomial $H_\nu(X)$ with rational integral coefficients associated with η and a positive integer ν (for the definition of $H_\nu(X)$, see the text § 2 (2.4)). Under a condition on η (see the text § 2 (2.6)), for each prime p which satisfies $p \mid H_\nu(1)$ and $p \nmid H_{\nu-1}(1)$, we can construct characters λ of K with the conductor $\mathfrak{Q}^{2\nu}\mathfrak{P}\tilde{w}$, where $\mathfrak{Q}$ is the prime ideal of K lying above $\mathfrak{q}$ and $\mathfrak{P}$ is an ideal of K such that $\mathfrak{P}\mathfrak{P}^\sigma = (p)$ and $(\mathfrak{P}, \mathfrak{P}^\sigma) = 1$ with the non-trivial automorphism σ of K over F. $\tilde{w}$ is one of real archimedean places of K. The Hilbert cusp form f_λ over F associated with λ is of weight 1 and of level $\mathfrak{q}^{2\nu+1}(p)$. Under some assumptions on the special value of L-functions of F (see the text § 2 (2.7)), we can show that there exists a primitive cusp form f over F of weight 2 and of level $\mathfrak{q}^{2\nu+1}$, which is congruent to f_λ modulo a prime ideal P lying above p. On the prime p, we note the following. When $F = \boldsymbol{Q}$ the rational number field, K is a real quadratic field $\boldsymbol{Q}(\sqrt{q})$ and η is the fundamental unit ε of K. We see in this case

$$H_\nu(1) = -\text{tr } \varepsilon^{q^\nu}.$$

So the value $H_\nu(1)$ is a natural generalization of $\text{tr } \varepsilon$ in Shimura [16] and of $\text{tr } \varepsilon^{q^\nu}$ in Doi-Yamauchi [6] and Ishii [8]. In Section 3, we give examples of the above results and also include the examples in the case where

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rank E_K = rank E_F and rank E_K = rank E_F+2. In the former case, we find a congruence relation between two Hilbert cusp forms, one of which is associated with a Grossencharacter of K. In the latter case, as the examples suggest, the situation seems to be different.

§ 1. Hilbert modular forms and Hecke operators

Let F be a totally real algebraic number field of degree n, and $\mathfrak{o}_F=\mathfrak{o}$, $\mathfrak{d}$ the ring of integers, the different of F respectively. In this paper, we assume $n\geq 2$. For a place v of F, F_v denotes the completion of F at v, and when v is a finite place, $\mathfrak{o}_v$ denotes the ring of v-adic integers in F_v. To each finite place v, we fix a prime element ϖ_v of $\mathfrak{o}_v$. Let F_A and $F_A^\times$ be the adele ring and the idele group of F respectively, and $\mathfrak{U}_F=\prod_v \mathfrak{o}_v^\times \times \prod_w F_w^\times$, where v and w run through all finite and infinite places respectively. For $a\in F_A^\times$, let $|a|$ be the module of a with respect to a Haar measure of F_A and $a\mathfrak{o}$ the ideal of F determined by $(a\mathfrak{o})\mathfrak{o}_v=a_v\mathfrak{o}_v$ for finite v. Here a_v is the v-component of a. We choose a non-trivial additive character $\tau=\prod_v \tau_v$ of F_A, trivial on F. We assume that

$$\tau_w(x)=e^{-2\pi\sqrt{-1}x}$$

for infinite places w. For each finite v, let $\delta(v)$ be the integer so that τ_v is trivial on $\varpi_v^{-\delta(v)}\mathfrak{o}_v$ but not on $\varpi_v^{-\delta(v)-1}\mathfrak{o}_v$, and d the element of $F_A^\times$ given by $d_v=\varpi_v^{-\delta(v)}$ for finite v and $d_w=1$ for infinite w. Then we have $d\mathfrak{o}=\mathfrak{d}$.

Let $G=GL(2)$ be the general linear group in 2 variables, considered as an algebraic group over F, and Z the center of G. We write G_A, Z_A for the corresponding adelized groups. Z_A can be identified with $F_A^\times$. We denote by G_0 and G_∞ the finite part and the infinite part of G_A respectively. For a place v, let $G_v=GL(2, F_v)$ and let $G_F=GL(2, F)$.

In this section, we fix an integral ideal $\mathfrak{c}$ of F. Let $\psi=\prod_v \psi_v$ be a character of $F_A^\times/F^\times$ of finite order such that $\mathfrak{f}(\psi)$ divides $\mathfrak{c}$, where $\mathfrak{f}(\psi)$ is the finite part of the conductor of ψ. To each infinite place w, we choose a positive integer $\kappa(w)$ satisfying $(-1)^{\kappa(w)}=\psi_w(-1)$, and put $\bar{\kappa}=(\kappa(w))$. To $\mathfrak{c}$, ψ, and $\bar{\kappa}$, we define a compact subgroup $\boldsymbol{K}=\boldsymbol{K}(\mathfrak{c})$ of G_A and a 1-dimensional representation ρ of $\boldsymbol{K}$. To a finite place v not dividing $\mathfrak{c}$, put $\boldsymbol{K}_v=GL(2, \mathfrak{o}_v)$, and to an infinite place w, put $\boldsymbol{K}_w=\mathrm{SO}(2, \boldsymbol{R})$. For a finite place v dividing $\mathfrak{c}$, let $\nu_v=\mathrm{ord}_v(\mathfrak{c})=\mathrm{ord}_v(c_v)$, where c_v is an element of F_v such that $\mathfrak{c}\mathfrak{o}_v=c_v\mathfrak{o}_v$ and ord_v is the additive valuation of F_v normalized as $\mathrm{ord}_v(\varpi_v)=1$. Let

$$\boldsymbol{K}_v=\left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in GL(2, \mathfrak{o}_v)\,\middle|\,\mathrm{ord}_v(c)\geq\nu_v\right\},$$

and let $\boldsymbol{K}=\prod_v \boldsymbol{K}_v$. For a positive integer m, let ρ_m be the representation of $SO(2, \boldsymbol{R})$ given by

$$\rho_m\left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right) = e^{m\theta\sqrt{-1}}.$$

For $k=(k_v) \in \boldsymbol{K}$, we define

$$\rho(k) = \prod_{v \mid \mathfrak{c}} \psi_v(d_v) \prod_{w \text{ infinite}} \rho_{\kappa(w)}(k_w),$$

where

$$k_v = \begin{bmatrix} a_v & b_v \\ c_v & d_v \end{bmatrix}.$$

Now we define a space of Hilbert modular forms associated with the triple $(\mathfrak{c}, \psi, \tilde{\kappa})$. We call a $\boldsymbol{C}$-valued continuous function f on G_A a Hilbert modular form (over F) of type $(\mathfrak{c}, \psi, \tilde{\kappa})$ if f satisfies the following conditions:

(1.1) $f(\gamma gzk) = \psi(z)\rho(k)f(g)$ for $\gamma \in G_F$, $z \in Z_A$, and $k \in \boldsymbol{K}$;

(1.2) as a function of $g_w \in GL(2, F_w)$ for infinite w, $f(gg_w)$ is of C^∞-class and satisfies $Xf(gg_w)=0$ for $g \in G_A$, where $X = \begin{bmatrix} 1 & -\sqrt{-1} \\ \sqrt{-1} & -1 \end{bmatrix}$ in the complex Lie algebra of $GL(2, F_w)$;

(1.3) for any compact set $S(\subset G_A)$ and a positive integer c, there exist constants C and N so that

$$\left| f\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g\right)\right| \leq C|a|^N,$$

for all $g \in S$ and $a \in F_A^\times$ with $|a|>c$.

Let $M(\mathfrak{c}, \psi, \tilde{\kappa})$ denote the space of Hilbert modular forms of type $(\mathfrak{c}, \psi, \tilde{\kappa})$. f in $M(\mathfrak{c}, \psi, \tilde{\kappa})$ is called a Hilbert cusp form of type $(\mathfrak{c}, \psi, \tilde{\kappa})$ if f satisfies the condition:

(1.4) $\displaystyle\int_{F\backslash F_A} f\left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\right) da = 0$ for all $g \in G_A$, where da is the Haar measure of $F\backslash F_A$.

We denote by $S(\mathfrak{c}, \psi, \tilde{\kappa})$ the space of Hilbert cusp forms of type $(\mathfrak{c}, \psi, \tilde{\kappa})$. In the rest of this section, we assume $\kappa(w)=\kappa$ for all infinite w, and put $\tilde{\kappa}=(\kappa, \cdots, \kappa)$. Let f be an element of $M(\mathfrak{c}, \psi, \tilde{\kappa})$, then f has a Fourier expansion;

$$f\left(\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}\right)=C_0(y)|y|^{\kappa/2}+|y|^{\kappa/2}\sum_{\substack{\xi\in F^\times \\ \xi\ll 0}} C(\xi d y \mathfrak{o})e^{-2\pi \mathrm{Tr}(\xi y_\infty)}\tau(\xi x),$$

for $x\in F_A$ and $y\in F_{A+}^\times=\{a\in F_A | a_w>0 \text{ for infinite } w\}$. Here the sum is extended over totally negative $\xi\in F^\times$. For $y\in F_A^\times$, y_∞ is the infinite component of y and $\mathrm{Tr}(y_\infty)=\sum_w y_w$. $C(y\mathfrak{o})=0$ unless $y\mathfrak{o}$ is integral and $C_0(y)$ satisfies $C_0(yu)=C_0(y)$ for $u\in F^\times \prod_v \mathfrak{o}_v^\times \prod_w F_{w+}^\times$ with $F_{w+}^\times=\{a\in F_w^\times | a>0\}$. If $f\in S(\mathfrak{c}, \psi, \tilde{\kappa})$, then $C_0(y)=0$. For f, we set

$$L(s,f)=\sum_{\mathfrak{m}} C(\mathfrak{m})N(\mathfrak{m})^{-s}$$

where $\mathfrak{m}$ runs through all integral ideals of F.

To each integral ideal $\mathfrak{a}$, the Hecke operator $T_{\mathfrak{c}}(\mathfrak{a})$ on $M(\mathfrak{c}, \psi, \tilde{\kappa})$ or $S(\mathfrak{c}, \psi, \tilde{\kappa})$ is defined in the following way. For finite $v\nmid\mathfrak{c}$(resp. $v|\mathfrak{c}$). Put

$$\Xi_v(\mathfrak{a})=\{g\in M_2(\mathfrak{o}_v) | \mathrm{ord}_v(\det g)=\mathrm{ord}_v(\mathfrak{a})\}$$

$$\left(\text{resp. } \Xi_v(\mathfrak{a})=\left\{g\in \begin{bmatrix} \mathfrak{o}_v & \mathfrak{o}_v \\ \mathfrak{c}\mathfrak{o}_v & \mathfrak{o}_v^\times \end{bmatrix} \middle| \mathrm{ord}_v(\det g)=\mathrm{ord}_v(\mathfrak{a})\right\}\right)$$

and put $\Xi(\mathfrak{a})=\prod_{v:\text{ finite}} \Xi_v(\mathfrak{a})$. Let $\Xi(\mathfrak{a})=\bigcup_{i=1}^d g_i K_0$ $(K_0=G_0\cap K)$ be a disjoint union. For $f\in M(\mathfrak{c}, \psi, \tilde{\kappa})$, we define

$$(T_{\mathfrak{c}}(\mathfrak{a})f)(g)=N(a)^{(\kappa-2)/2}\sum_{i=1}^d \bar{\psi}(d_i)f(gg_i),$$

where $g_i=\begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$. Let $C(\mathfrak{m})$(resp. $C'(\mathfrak{m})$) be the Fourier coefficients of f(resp. $T_{\mathfrak{c}}(\mathfrak{a})f$), then it holds

$$C'(\mathfrak{m})=\sum_{\mathfrak{l}|\mathfrak{a},\mathfrak{m}} \psi^*(\mathfrak{l})C(\mathfrak{m}\mathfrak{a}/\mathfrak{l}^2)N(\mathfrak{l})^{\kappa-1}.$$

Here $\psi^*(\mathfrak{l})=0$ unless $\mathfrak{l}$ is prime to $\mathfrak{c}$, and when $\mathfrak{l}$ is prime to $\mathfrak{c}$, let $l\in F_A^\times$ so that $l_\infty=1$, $l_v=1$ for $v\nmid c$, $l\mathfrak{o}=\mathfrak{l}$, and put $\psi^*(\mathfrak{l})=\psi(l)$. It is shown in Shimura [17] that $M(\mathfrak{c}, \psi, \tilde{\kappa})$ and $S(\mathfrak{c}, \psi, \tilde{\kappa})$ are spanned by forms for which $C(\mathfrak{m})$ are integers in an algebraic number field and that the eigenvalues for $T_{\mathfrak{c}}(\mathfrak{a})$ are algebraic integers. Let f and h be Hilbert cusp forms with coefficients $C(\mathfrak{m})$ and $C'(\mathfrak{m})$ in the localization of the ring of integers of an algebraic number field M at a prime ideal P of M. We say f is congruent to h modulo P if $C(\mathfrak{m})\equiv C'(\mathfrak{m})$ modulo P for all integral ideals $\mathfrak{m}$.

We introduce Eisenstein series following Hida [7]. Let $\chi=\prod_v \chi_v$ be a character of $F_A^\times/F^\times$ of finite order. Assume $\chi_w(x)=\mathrm{sgn}(x)=x/|x|$ for all infinite w or $\chi_w=$trivial for all infinite w. We also assume $\mathfrak{c}=\mathfrak{f}(\chi)\neq\mathfrak{o}$

and $\chi_w(-1)=(-1)^\kappa$. Then there exists $E_{\kappa,\chi}$ in $M(\mathfrak{c}, \chi, \tilde{\kappa})$, whose Fourier expansion is given by

$$E_{\kappa,\chi}\left(\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}\right)=|y|^{\kappa/2}+\frac{2^n|y|^{\kappa/2}}{L(1-\kappa, \chi)}\cdot \sum_{\substack{\xi\in F^\times \\ \xi\ll 0}} C_{\kappa,\chi}(\xi dy\mathfrak{o})e^{-2\pi \mathrm{Tr}(\xi y_\infty)}\tau(\xi x),$$

where

$$C_{\kappa,\chi}(\mathfrak{m})=\begin{cases} \sum_{\mathfrak{a}|\mathfrak{m}} \chi^*(\mathfrak{m}/\mathfrak{a})N(\mathfrak{a})^{\kappa-1} & \text{if } \mathfrak{m} \text{ is integral} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tilde{S}$ be the set of prime factors of $\mathfrak{c}$. For $S\subset\tilde{S}$, put

$$(W_S f)(g)=f(gw_S),$$

where $w_S=\begin{bmatrix} 0 & -1 \\ x & 0 \end{bmatrix}$ with $x_v=\varpi_v^{\nu_v}$ for $v\in S$ and $x_v=1$ for $v\notin S$. For $v\in\tilde{S}$, put

$$G(\bar{\chi}_v)=\sum_{a\in(\mathfrak{o}_v/\varpi_v^{\nu_v})^\times} \bar{\chi}_v(a)\tau_v(a/(d_v\varpi_v^{\nu_v}))$$

Then we have by [7]

$$(W_{\tilde{S}}E_{\kappa,\chi})\left(\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}\right)=(-1)^{\kappa g}\cdot 2^n\cdot L(1-\kappa, \chi)^{-1}N\mathfrak{c}^{\kappa/2-1}\left(\prod_{v|\mathfrak{c}} \chi_v(-\varpi_v^{\nu_v})G(\bar{\chi}_v)\right)$$
$$\times\left(\delta_\kappa|y|^{\kappa/2}\chi(yd)C_{\bar{\chi}}(y)+|y|^{\kappa/2}\sum_{\substack{\xi\in F^\times \\ \xi\ll 0}}\chi(yd)C'_{\kappa,\chi}(\xi dy\mathfrak{o})e^{-2\pi\mathrm{Tr}(\xi y_\infty)}\tau(\xi x)\right),$$

where $\delta_\kappa=0$ if $\kappa>1$ and $\delta_\kappa=1$ if $\kappa=1$. $C_{\bar{\chi}}(y)=2^{-n}L(0, \bar{\chi})$, and $C'_{\kappa,\bar{\chi}}(\mathfrak{m}) =\sum_{\mathfrak{a}|\mathfrak{m}}\bar{\chi}^*(\mathfrak{m}/\mathfrak{a})N(\mathfrak{a})^{\kappa-1}$ if $\mathfrak{m}$ is integral and $C'_{\kappa,\bar{\chi}}(\mathfrak{m})=0$ if $\mathfrak{m}$ is not integral. Later, we need the action of W_S for $S\subsetneq\tilde{S}$.

Proposition 1.1. *Let $f\in M(\mathfrak{c}, \psi, \tilde{\kappa})$ be an eigenfunction for all $T_\mathfrak{c}(\mathfrak{a})$. Assume $(W_S f)\bar{\psi}(\det)\in M(\mathfrak{c}, \bar{\psi}, \tilde{\kappa})$ is also an eigenfunction for all $T_\mathfrak{c}(\mathfrak{a})$ with eigenvalues $\lambda(\mathfrak{a})$. Let S be a proper subset of $\tilde{S}$. Assume $\mathfrak{f}(\psi)\mathfrak{o}_v=\mathfrak{c}\mathfrak{o}_v$ for $v\in S$ and $(\psi\prod_{v\in S}\bar{\psi}_v)(\mathfrak{o}^\times)\neq\{1\}$. Then the Fourier expansion of $h=W_S f$ is given by*

$$h\left(\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}\right)=\prod_{v\in S}\lambda(\varpi_v\mathfrak{o})^{-\nu_v}|\varpi_v|^{-\nu_v(\kappa/2-1)}\psi_v(-\varpi_v^{\nu_v})G(\bar{\psi}_v)$$
$$\times\left(|y|^{\kappa/2}\sum_{\substack{\xi\in F^\times \\ \xi\ll 0}}\prod_{v\in S}\psi_v((\xi dy)_v)C'(\xi dy\mathfrak{o})e^{-2\pi\mathrm{Tr}(\xi y_\infty)}\tau(\xi x)\right).$$

For an integral ideal $\mathfrak{m}$, *let* $\mathfrak{m}=\mathfrak{m}_1\mathfrak{m}_2$ *be the decomposition into integral ideals* $\mathfrak{m}_1$, $\mathfrak{m}_2$ *so that* $\mathfrak{m}_1$ *is prime to* $\prod_{v\in S}\varpi_v\mathfrak{o}$ *and all the prime factor of* $\mathfrak{m}_2$ *is contained in S. Then* $C'(\mathfrak{m})=C(\mathfrak{m}_1)\lambda(\mathfrak{m}_2)$.

This can be verified in the same way as in Asai [1], and the proof will be omitted.

§ 2. Construction of characters λ and congruences

Let K be a quadratic extension of F, and $\mathfrak{f}_{K/F}$ its conductor. Let $\mathrm{Gal}(K/F)=\langle\sigma\rangle$. We assume the following;

(2.1) only one infinite place of F decomposes in K.

We denote it by w_1 and the other places by $w_2, w_3, \cdots$, and w_n. Let E_F and E_K be the groups of units of F and K respectively. Then by the above assumption, we have rank $E_K=\text{rank } E_F+1=n$. Let $E_K^1=\{\varepsilon\in E_K \mid N_{K/F}(\varepsilon)=1\}$, $E_F^+=\{\varepsilon\in E_F \mid w_i(\varepsilon)>0 \text{ for all } i\}$, and $\tilde{E}_F=\{\varepsilon\in E_F \mid w_i(\varepsilon)>0 \text{ for } i\geq 2\}$. Later, for the sake of simplicity, we assume the following conditions:

(2.2) $\mathfrak{f}_{K/F}$ is a prime ideal $\mathfrak{q}$ which is prime to 2 and $\mathfrak{d}_F$;

(2.3) the class number of F is odd and $[E_F : E_F^+]=2^n$.

Let $N\mathfrak{q}=q^\alpha$ with a rational prime q and a positive integer α.

Proposition 2.1. (1) *If* (2.2) *is satisfied, there exists* η_0 *in* E_K *so that* E_K *is generated by* η_0 *and* E_F.
(2) *If* (2.2) *and* (2.3) *are satisfied, then* $N_{K/F}(E_K)=\tilde{E}_F$ *and* E_K^1 *is generated by* ± 1 *and* $\eta_0^2 N_{K/F}(\eta_0)^{-1}$.

Proof. (1) It is enough to show that E_K/E_F is torsion-free. Let $\eta\in E_K$ and assume $\eta^m\in E_F$, then $(\eta^\sigma/\eta)^m=1$. Since K is not totally imaginary, $\eta^\sigma/\eta=\pm 1$. If $\eta^\sigma/\eta=-1$, then $\eta^2\in E_F$ and $K=F(\eta)$. This contradicts (2.2).

(2) Since $N_{K/F}(E_K)\subset\tilde{E}_F$ and $[\tilde{E}_F : E_F^+]=[\tilde{E}_F : E_F^2]$ by the condition (2.3), it is enough to show that $N_{K/F}(\eta_0)$ is not contained in $E_F^+=E_F^2$. Assume $\eta_0\eta_0^\sigma=\varepsilon^2$ with $\varepsilon\in E_F$, then $(\eta_0/\varepsilon)^\sigma(\eta_0/\varepsilon)=1$. Put $\mu=1+\eta_0/\varepsilon$, then μ satisfies $\mu=\mu^\sigma\eta_0/\varepsilon$. If μ is a unit, then

$$\eta_0/\varepsilon=\mu/\mu^\sigma=\mu\mu^\sigma/(\mu^\sigma)^2=(\varepsilon'/\mu^\sigma)^2,$$

with $\varepsilon'\in E_F$. But this contradicts the fact that η_0 gives a generator of E_K/E_F. Hence μ is not a unit. Let $\mathfrak{J}$ be the ideal of K generated by μ, then $\mathfrak{J}$ satisfies $\mathfrak{J}^\sigma=\mathfrak{J}$. Let α be an element of F so that $K=F(\sqrt{\alpha})$. Then, there exists an ideal $\mathfrak{a}$ of F so that $\mathfrak{J}=(\sqrt{\alpha})\mathfrak{a}\mathfrak{o}_K$. Since the class number of F is odd, $\mathfrak{a}$ is a principal ideal, which is generated by $a \in F$. Hence $\mu=\eta a\sqrt{\alpha}$ with a unit η of K, and we have

$$\eta_0/\varepsilon=\mu/\mu^\sigma=-\eta/\eta^\sigma=-(\varepsilon'/\eta^\sigma)^2,$$

with $\varepsilon'^2=\eta\eta^\sigma$. This is a contradiction, and the proof is completed.

Let η_0 be as in Proposition 2.1. We note $N_{K/Q}(\eta_0)=-1$. For each positive integer ν, we define a polynomial in $\boldsymbol{Z}[X]$ of degree 2^n associated with η_0. Let $f_\nu(X)=X^2-sX+m$ be the minimal polynomial of $\eta_0^{q^\nu}$ over F. For $a \in F$, let $a^{(i)}$, $1\leq i\leq n$, be all distinct conjugates of a over $\boldsymbol{Q}$, and let

$$X^2-s^{(i)}X+m^{(i)}=(X-\alpha_{i1})(X-\alpha_{i2}).$$

Let S be the n-tuple products of the set $\{1, 2\}$. We define $H_\nu(X)$ by

$$H_\nu(X)=\prod_{(s_1,s_2,\cdots,s_n)\in S}\left(X-\prod_{i=1}^{n}\alpha_{is_i}\right) \tag{2.4}$$

Then we see $H_\nu(X) \in \boldsymbol{Z}[X]$ and $\deg H_\nu=2^n$. It is easy to see that $|H_\nu(1)|$ is unchanged if we replace η_0 by η_0^{-1} or by $\eta_0\varepsilon$ for $\varepsilon \in E_F$. To each prime divisor p of $H_\nu(1)$ satisfying the following condition, we construct idele class characters of K. Let p be a prime satisfying

$$p|H_\nu(1), \quad p\nmid H_{\nu-1}(1), \quad \text{and} \quad \mathrm{ord}_p(H_\nu(1))=1. \tag{2.5}$$

Let $\boldsymbol{C}_p$ be the completion of the algebraic closure $\bar{\boldsymbol{Q}}_p$ of $\boldsymbol{Q}_p$. We fix embeddings $\iota_\infty : \bar{\boldsymbol{Q}}\to\boldsymbol{C}$ and $\iota_p : \bar{\boldsymbol{Q}}\to\boldsymbol{C}_p$. By means of ι_∞ and ι_p, algebraic numbers can be seen as elements of $\boldsymbol{C}$ and as elements of $\boldsymbol{C}_p$. Let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be all the distinct embeddings of F into $\bar{\boldsymbol{Q}}$. We assume w_i corresponds to $\iota_\infty\sigma_i$ for each i. Let us consider sets of embeddings $T=\{\tau_1, \tau_2, \cdots, \tau_n\}$ of K into $\bar{\boldsymbol{Q}}$ such that the restriction of τ_i to F coincides with σ_i for all i. There exists 2^n such T's. We denote them by T_i, $1\leq i\leq 2^n$. For $x \in K$, we set $x^T=\prod_{i=1}^n x^{\tau_i}$. Then $H_\nu(1)=\prod_T (1-\eta_0^{q^\nu})^T$. We consider K and F as subfields of $\bar{\boldsymbol{Q}}$ by fixing an embedding $\tau : K\to\bar{\boldsymbol{Q}}$ such that $\tau|F=\sigma_1$. Let $\tilde{K}$ and $\tilde{F}$ be the Galois closure of K and F over $\boldsymbol{Q}$ respectively. Then it is easy to see $[\tilde{K} : \tilde{F}]=2^n$ by (2.1). Let $\tilde{G}=\mathrm{Gal}(\tilde{K}/\boldsymbol{Q})$, $\tilde{H}=\mathrm{Gal}(\tilde{K}/K)$ and $H=\mathrm{Gal}(\tilde{K}/F)$. Then there exists a natural one to one correspondence between the left cosets $\tilde{H}\backslash\tilde{G}$ (resp. $H\backslash\tilde{G}$) and the embeddings of K (resp. F) into $\bar{\boldsymbol{Q}}$, and ρ in $\tilde{G}$ induces a permutation among T_i by the multiplication on the right. The embedding ι_p determines a prime ideal $\mathfrak{P}$ of $\tilde{K}$ lying above p. We denote the decomposition group of $\mathfrak{P}$ by D.

Lemma 2.2. *Let* $\mathfrak{T}=\{T \mid T\gamma=T$ *for* $\gamma \in D\}$, *and* $\mathfrak{T}'=\{T \mid T\gamma \neq T$ *for some* $\gamma \in D\}$. *Then*
(1) $\iota_p(x^T) \in \boldsymbol{Q}_p$ *for all* $x \in K$ *if and only if* $T \in \mathfrak{T}$.
(2) *Let* p *be a prime satisfying* (2.5) *and assume* p *is unramified in* K. *Then, there exists only one* $T \in \mathfrak{T}$ *such that* $\iota_p((1-\eta_0^{q^\nu})^T)$ *is divided by* p, *and all prime ideals* $\mathfrak{p}$ *of* F *lying above* p *decomposes in* K.

Proof. (1) is obvious, because $\iota_p(x^T) \in \boldsymbol{Q}_p$ for all $x \in K \Leftrightarrow x^{T\gamma}=x^T$ for all $x \in K$ and all $\gamma \in D \Leftrightarrow T\gamma=T$ for all $\gamma \in D$. If p divides $\iota_p((1-\eta_0^{q^\nu})^T)$ for T in $\mathfrak{T}'$, then p^2 divides $H_\nu(1)$. The first assertion of (2) follows from this. Let T be the set in $\mathfrak{T}$ satisfying the above condition. Then there exists g_i, $1 \leq i \leq d$, in $\tilde{G}$ so that T corresponds to the cosets $\bigcup_{i=1}^d \tilde{H}\backslash\tilde{H}g_iD$ in the correspondence stated above, where we assume $\bigcup_{i=1}^d \tilde{H}\backslash\tilde{H}g_iD$ is a disjoint union. By the condition on T, we have $\tilde{G}=\bigcup_i Hg_iD$. Now it holds

$$|\tilde{H}\backslash\tilde{G}| \leq \sum_i |\tilde{H}\backslash Hg_iD| \leq 2\sum_i |\tilde{H}\backslash\tilde{H}g_iD| = 2[F:\boldsymbol{Q}] = |\tilde{H}\backslash\tilde{G}|.$$

Hence the union $\bigcup_i Hg_iD$ is disjoint, and $|\tilde{H}\backslash Hg_iD|=2|\tilde{H}\backslash\tilde{H}g_iD|$. The second assertion follows from this.

Let T be as in Lemma 2.2 (2). The mappings $\iota_p\tau_i : x \to \iota_p(x^{\tau_i})$ can be extended to a homomorphism of $K\otimes\boldsymbol{Q}_p$ into $\boldsymbol{C}_p$ and the mapping $x \to \iota_p(x^T)$ can be extended to a homomorphism as multiplicative groups of $(K\otimes\boldsymbol{Q}_p)^\times$ into $\boldsymbol{Q}_p^\times$. We denote it by ι_T. Let P be a prime ideal of $\boldsymbol{Q}(1^{1/(p-1)})$ lying above p and ω_p the character of $\boldsymbol{Z}_p^\times$ of order $p-1$ such that $\omega_p(a) \equiv a \bmod p$ for $a \in \boldsymbol{Z}$ prime to p. Then q^ν divides $p-1$ and the order of $\omega_p(\iota_T(\eta_0))$ is q^ν. Let ω_T be the character of $(\mathfrak{o}_K\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p)^\times$ given by $\omega_T(a)=\omega_p(\iota_T(a))$, where $\mathfrak{o}_K$ is the ring of integer of K. Then we obtain the following by virtue of Lemma 2.2.

Corollary 2.3. *Let* $\mathfrak{P}_T$ *be the conductor of* ω_T, *then* $\mathfrak{P}_T$ *is prime to* $\mathfrak{P}_T^\sigma$ *and* $\mathfrak{P}_T\mathfrak{P}_T^\sigma=(p)$.

Let $\mathfrak{Q}$ be the prime ideal of K lying above $\mathfrak{q}$ and $\mathfrak{o}_{K,\mathfrak{Q}}$ be the completion of $\mathfrak{o}_K$ at $\mathfrak{Q}$. Let Π be a prime element of $\mathfrak{o}_{K,\mathfrak{Q}}$ and let $\eta_0=a+b\Pi$ with $a, b \in \mathfrak{o}_\mathfrak{q}$. We consider the following condition on η_0.

(2.6) $$\eta_0=a+b\Pi \text{ with a unit } b.$$

Lemma 2.4. *If* (2.6) *is satisfied, then the order of the class* $\tilde{\eta}_0$ *of* η_0 *in* $(\mathfrak{o}_{K,\mathfrak{Q}}/\mathfrak{q}^\nu\mathfrak{o}_{K,\mathfrak{Q}})^\times$ *is* q^ν *and* $\langle\tilde{\eta}_0\rangle \cap (\mathfrak{o}_q/\mathfrak{q}^\nu)^\times=\langle\tilde{\eta}_0^{q^\nu}\rangle$, *where* $\langle\tilde{\eta}_0\rangle$ *is the subgroup generated by* $\tilde{\eta}_0$.

Proof. For $u+v\Pi$ with $u, v \in \mathfrak{o}_q$, put $(u+v\Pi)^q = u'+v'\Pi$ with $u', v' \in \mathfrak{o}_q$. It is easy to see that if u is a unit in $\mathfrak{o}_q$ and $\mathrm{ord}_q(v)=m$, then u' is a unit in $\mathfrak{o}_q$ and $\mathrm{ord}_q(v')=m+1$. Our assertion easily follows from this.

Let $(\ /\mathfrak{Q})$ be the quadratic residue symbol of $(\mathfrak{o}_q/\mathfrak{q})^\times \cong (\mathfrak{o}_{K,\mathfrak{Q}}/\mathfrak{Q})^\times$, then the infinite place $\tilde{w}_1$ of K lying above w_1 is uniquely determined by

$$\mathrm{sgn}\ \tilde{w}_1(\eta_0) = \left(\frac{\eta_0}{\mathfrak{Q}}\right).$$

Let ω be the Dirichlet character modulo p of order $p-1$ given by $\omega(a) \equiv a \bmod P$. Now we prove the following theorem.

Theorem 2.5. *Let K be a quadratic extension of F satisfying* (2.1), (2,2), *and* (2.3), *and assume the condition* (2.6). *Let p be a prime number satisfying* (2.5) *for a positive integer ν, and assume p is unramified in K. Let T be as in Lemma* 2.2 *for ν, then for each k, $1 \leq k \leq p-1$, the character ω_T^k of $(\mathfrak{o}_K \otimes_Z Z_p)^\times$ can be extended in $(N\mathfrak{q}/q)^\nu h_K/h_F$ ways to idele class characters λ of K so that the conductor of λ is $\mathfrak{Q}^{2\nu}\mathfrak{P}_T\tilde{w}_1$ and the restriction to F is $\omega N_{F/\mathfrak{Q}}\chi_{K/F}$ where $\chi_{K/F}$ is the quadratic character of F corresponding to the extension K/F. h_K and h_F are the class numbers of K and F respectively.*

Proof. For $\tilde{\eta}_0^\alpha a \in \langle\tilde{\eta}_0\rangle(\mathfrak{o}_\mathfrak{q}/\mathfrak{q}^\nu)^\times$, put

$$\lambda_1(\tilde{\eta}_0^\alpha a) = (\overline{\omega_T(\eta_0)}^k\ \mathrm{sgn}\ \tilde{w}_1(\eta_0))^\alpha \left(\frac{a}{\mathfrak{Q}}\right).$$

Then, by Lemma 2.2, λ_1 is well-defined and gives a character of $\langle\tilde{\eta}_0\rangle(\mathfrak{o}_\mathfrak{q}/\mathfrak{q}^\nu)^\times$, since the order of $\omega_T(\eta_0)$ is q^ν and

$$(\overline{\omega_T(\eta_0)}^k\ \mathrm{sgn}\ \tilde{w}_1(\eta_0))^{q^\nu} = \left(\frac{\eta_0}{\mathfrak{Q}}\right)^{q^\nu}.$$

λ_1 can be extended to characters of $(\mathfrak{o}_{K,\mathfrak{Q}}/\mathfrak{q}^\nu\mathfrak{o}_{K,\mathfrak{Q}})^\times$ of conductor $\mathfrak{Q}^{2\nu}$ in $(N\mathfrak{q}/q)^{\nu-1}$ ways. Let λ_2 be one of such characters. For $(a, b, c) \in \mathfrak{o}_{K,\mathfrak{Q}}^\times \times (\mathfrak{o}_K \otimes_Z Z_p)^\times \times K_{\tilde{w}_1}^\times$, put

$$\lambda_3((a, b, c)) = \lambda_2(a)\omega_T(b)^k\ \mathrm{sgn}\ (c).$$

Then, by the definition, we see

$$\lambda_3((\eta_0, \eta_0, \eta_0)) = \overline{\omega_T(\eta_0)}^k\ \mathrm{sgn}\ \tilde{w}_1(\eta_0)\omega_T(\eta_0)^k\ \mathrm{sgn}\ \tilde{w}_1(\eta_0) = 1.$$

For $\varepsilon \in E_F$ with $N_{F/Q}(\varepsilon)=1$, we have

$$\lambda_3((\varepsilon, \varepsilon, \varepsilon)) = \left(\frac{\varepsilon}{\mathfrak{Q}}\right) \operatorname{sgn} w_1(\varepsilon).$$

Since $\prod_{i=1}^n \operatorname{sgn} w_i(\varepsilon)=1$, and $\chi_{K/F}(\varepsilon)=(\varepsilon/\mathfrak{Q}) \prod_{i=2}^n \operatorname{sgn} w_i(\varepsilon)=1$, we see $\lambda_3((\varepsilon, \varepsilon, \varepsilon))=1$. By Proposition 2.1, E_K is generated by η_0 and ε in E_F with $N_{F/Q}(\varepsilon)=1$, hence we have $\lambda_3((\eta, \eta, \eta))=1$ for all $\eta \in E_K$. We can conclude from this that λ_3 can be extended to characters of $K_A^\times/K^\times$ of conductor $\mathfrak{Q}^{2\nu}\mathfrak{P}_T\tilde{w}_1$ of finite order in h_K ways. But in these extensions, h_K/h_F characters satisfy the second condition, since $\mathfrak{U}_K K^\times \cap F_A^\times = \mathfrak{U}_F F^\times$ by (2.3). Here $\mathfrak{U}_K = \prod_{\tilde{v}} \mathfrak{o}_{K,\tilde{v}}^\times \times \prod_{\tilde{w}} K_{\tilde{w}}^\times$, where $\tilde{v}$ and $\tilde{w}$ run through all finite and infinite places respectively. This completes the proof.

Let λ be as in Theorem 2.5. Then by a result of Jacquet-Langlands [9], there exists a cusp form f_λ in $S(\mathfrak{q}^{2\nu+1}(p), \omega \cdot N_{F/Q}, \tilde{1})$ such that $L(s, f_\lambda) = L(s, \lambda)$. Following the argument of Koike [10], we will show that there exists a cusp form in $S(\mathfrak{q}^{2\nu+1}, 1, \tilde{2})$ congruent to f_λ modulo a prime ideal dividing p under the following assumption (2.7) on $L(0, \overline{\omega N_{F/Q}})$ and $L(0, \omega N_{F/Q})$.

(2.7) $L(0, \overline{\omega N_{F/Q}}) \equiv a/p \bmod p$ with $a \in \boldsymbol{Z}$ prime to p and $L(0, \omega N_{F/Q})$ is prime to p.

Let $S^0(\mathfrak{q}^{2\nu+1}, 1, \tilde{2})$ denote the subspace of new forms in $S(\mathfrak{q}^{2\nu+1}, 1, \tilde{2})$ (for definition cf. Miyake [12]). Under (2.7), we can prove the following theorem.

Theorem 2.6. *Let K, p and λ be as in Th.* 2.5. *Assume* (2.7). *Then there exist a prime $\tilde{P}$ of $\bar{\boldsymbol{Q}}$ lying above $P(\subset \boldsymbol{Q}(1^{1/p-1}))$ and a primitive form h in $S^0(\mathfrak{q}^{2\nu+1}, 1, \tilde{2})$ which is congruent to f_λ modulo $\tilde{P}$.*

Proof. Let $f' = f_\lambda E_{1,\overline{\omega N}}^{(\nu)}$, where $E_{1,\overline{\omega N}}^{(\nu)}(g) = E_{1,\overline{\omega N}}\left(g\begin{bmatrix}1 & 0\\ 0 & \varpi_{\mathfrak{q}}^\nu\end{bmatrix}\right)$ with $N = N_{F/Q}$, and let P' be a prime ideal lying above P of the field generated by the value of λ over $\boldsymbol{Q}(1^{1/(p-1)})$. Then $f' \in S(\mathfrak{q}^{2\nu+1}, 1, \tilde{2})$ and $f \equiv f' \bmod P'$ by (2.7). Put

$$\tilde{f} = \operatorname{Tr}(f') = \sum_{a \in \prod_{\mathfrak{p}|p} \left(\begin{bmatrix}\mathfrak{o}_\mathfrak{p} & \mathfrak{o}_\mathfrak{p}\\ p\mathfrak{o}_\mathfrak{p} & \mathfrak{o}_\mathfrak{p}\end{bmatrix}^\times \backslash GL(2, \mathfrak{o}_\mathfrak{p})\right)} f'(ga)$$

Let $\tilde{S}$ be the set of prime divisors of p in F, and for a subset S of $\tilde{S}$, put $T_S = \prod_{\mathfrak{p} \in S} T_\mathfrak{c}(\mathfrak{p})$ for $\mathfrak{c} = \mathfrak{q}^{2\nu+1}(p)$ and $\psi^*(S) = \prod_{\mathfrak{p} \in S} \psi_\mathfrak{p}(\varpi_\mathfrak{p})$ with $\psi = \omega N$. Then we see $\tilde{f}$ is contained in $S(\mathfrak{q}^{2\nu+1}, 1, \tilde{2})$, and

$$\tilde{f} = f' + \sum_{S \subset \tilde{S}} \psi^*(S) T_S W_S f'.$$

By Proposition 1.1 and (2.7), it is easy to see that $\tilde{f}\equiv f'$ mod P', and $\tilde{f}$ is a common eigen function for all $T_{\mathfrak{c}'}(\mathfrak{a})$ modulo $\tilde{P}$ with $\mathfrak{c}'=\mathfrak{q}^{2\nu+1}$. Let χ be a character of $F_{\mathfrak{q}}^{\times}$ such that $\chi|\mathfrak{o}_{\mathfrak{q}}^{\times}=(\ /\mathfrak{q})$, and U_{χ} be the operator defined in [15]. Then in the same way as in Corollary 4.2 [13], we see $U_{\chi}f'=(U_{\chi}f_{\lambda})E_{1,\overline{\omega N}}^{(\nu)}=c_{\chi}f'$ with a non-zero constant c_{χ}. Since Tr and U commute with each other, it follows from Theorem 1.4 in [15] that $\tilde{f}=\mathrm{Tr}(f')$ is contained in $S^0(\mathfrak{q}^{2\nu+1}, 1, \tilde{2})$. They by a Lemma of Deligne-Serre [4], we obtain our result.

Remark 2.7. The condition (2.7) is always satisfied if $F=\boldsymbol{Q}$ and $p\geq 5$, since $L(0, \bar{\omega})\equiv -\zeta(2-p)\equiv 1/p \bmod \boldsymbol{Z}_p$ and $L(0, \omega)\equiv\zeta(-1)=-1/12 \bmod P$. For $n\geq 2$, assume p is prime to $\mathfrak{d}_F$. Then we have

$$(2.8)\qquad \begin{aligned} L(0, \overline{\omega N_{F/Q}})&\equiv -\zeta_F(2-p) \bmod \boldsymbol{Z}_p,\\ L(0, \omega N_{F/Q})&\equiv\zeta_F(-1) \bmod P.\end{aligned}$$

Here ζ_F is the Dedekind zeta function of F. Hence the condition (2.7) can be stated as $\zeta_F(2-p)/\zeta(2-p)$ and $\zeta_F(-1)/\zeta(-1)$ are p-units. (2.8) can be shown in the following way. Let $L_p(\lambda, s)$ be the p-adic L-function of a ray class character λ of F constructed in Deligne-Ribert [5] and Cassou-Nouguès [3]. Then for a suitable ideal $\mathfrak{c}$, $(\lambda(\mathfrak{c})(N\mathfrak{c}/\omega(N\mathfrak{c}))^{1-s}-1)L_p(\lambda, s)$ is an Iwasawa function. Hence

$$(\lambda(\mathfrak{c})(N\mathfrak{c}/\omega(N\mathfrak{c}))-1)L_p(\chi, 0)\equiv(\lambda(\mathfrak{c})(N\mathfrak{c}/\omega(N\mathfrak{c}))^{p-1}-1)L_p(\chi, 2-p) \bmod P,$$

where P is the prime ideal of $\boldsymbol{Q}_p(\lambda)$ the field generated by the values of λ over $\boldsymbol{Q}_p$. If $\mathfrak{d}_F$ is prime to $\mathfrak{p}$, we can choose as $\mathfrak{c}$ an integral ideal such that

$$\mathrm{ord}_p\left((N\mathfrak{c}/\omega(N\mathfrak{c}))-1\right)=1.$$

The first congruence follows from this taking $\lambda=$trivial. The second one can be shown in the same way taking $\lambda=(\omega N_{F/Q})^2$. Furthermore, if F is an abelian extension of $\boldsymbol{Q}$, for a prime p with $(p, 2d_F n)=1$ it is known by Leopoldt [11] that

$$\zeta_F(2-p)/\zeta(2-p)\equiv\frac{2^{n-1}h_F R_p}{\sqrt{d_F}} \bmod p,$$

where $R_p=\det(Q_p(\varepsilon_i^{\sigma}))_{1\leq i\leq n-1, \sigma\in\mathrm{Gal}(F/Q)}$ for a system of fundamental units $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n-1}$ of F, and Q_p is the Fermatquotient mod p, namely for an integer A of F prime to p,

$$Q_p(A)\equiv\frac{A^{q-1}-1}{p} \bmod p.$$

Here q is the norm of a prime divisor of $\mathfrak{p}$ in F. Hence in this case, the condition on $\zeta_F(2-p)/\zeta(2-p)$ can be checked by h_F and R_p.

§ 3. Numerical Examples

In this section, we will discuss a few examples of Theorem 2.5 and Theorem 2.6, and examples of different type. Before giving them, we explain some notations. Let χ and U_χ be as in Section 2. Then $S^0(\mathfrak{q}^{2\nu+1}, 1, \tilde{2})$ decomposes into a direct sum of four subspaces S_{I}, S_{II}, S_{II_χ}, and S_{III}. Each subspace is given as follows;

$$\begin{aligned}
S_{\mathrm{I}} &= \{f \in S^0(\mathfrak{q}^{2\nu+1}, 1, \tilde{2}) \,|\, Wf=f,\ U_\chi f=f\},\\
S_{\mathrm{II}} &= \{f \in S^0(\mathfrak{q}^{2\nu+1}, 1, \tilde{2}) \,|\, Wf=f,\ U_\chi f=-f\},\\
S_{\mathrm{II}_\chi} &= \{f \in S^0(\mathfrak{q}^{2\nu+1}, 1, \tilde{2}) \,|\, Wf=-f,\ U_\chi f=-f\},\\
S_{\mathrm{III}} &= \{f \in S^0(\mathfrak{q}^{2\nu+1}, 1, \tilde{2}) \,|\, Wf=-f,\ U_\chi f=f\}.
\end{aligned}$$

Here $(Wf)(g)=f\left(g\begin{bmatrix} 0 & -1 \\ \varpi_q^{2\nu+1} & 0 \end{bmatrix}\right)$. These subspaces are stable under Hecke operators. We write $G^{\mathrm{I}}_{T(\mathfrak{a})}$, $G^{\mathrm{II}}_{T(\mathfrak{a})}$, $G^{\mathrm{II}_\chi}_{T(\mathfrak{a})}$, and $G^{\mathrm{III}}_{T(\mathfrak{a})}$ for the characteristic polynomials of $T_{\mathfrak{c}}(\mathfrak{a})$ with $\mathfrak{c}=\mathfrak{q}^{2\nu+1}$ on S_{I}, S_{II}, S_{II_χ}, and S_{III} respectively. Let $h(X)=\Sigma a_i X^i$ be a polynomial with coefficients in an algebraic number field M. We set $(N_{M/Q}h)(X)=\prod_\sigma (\Sigma a_i^\sigma x^i)$, where σ runs through all the distinct embeddings of M into $\bar{\boldsymbol{Q}}$. For a prime q and j, $1\leq j\leq (q-3)/2$, let $\alpha_j=e^{2\pi j\sqrt{-1}/q}+e^{-2\pi j\sqrt{-1}/q}$, and $\alpha_0=1$. We express by $(a_0, \cdots, a_{(q-3)/2})$ the algebraic number $a_0\alpha_0+\cdots+a_{(q-3)/2}\alpha_{(q-3)/2}$ in the maximal real field F_q of $\boldsymbol{Q}(1^{1/q})$. Let η_0 be as in Proposition 2.1, and $\eta_1=\eta_0^2 N_{K/F}\eta_0^{-1}$. If we define the polynomial $H'_\nu(X)$ in the same way as $H_\nu(X)$ taking η_1 instead of η_0, then we find $H'_\nu(-1)=H_\nu(1)^2$. The formula for $H'_\nu(-1)$ is simpler than that of $H_\nu(1)$. For example, let X^2-sX+1 be the minimal polynomial of η_1^ν, then $H'_\nu(-1)=(s+s')^2$ for $n=2$, and $H'_\nu(-1)=(s^2+s'^2+s''^2+ss's''-4)^2$ for $n=3$. Here s' and s'' are the conjugates of s over $\boldsymbol{Q}$. Throughout the following examples, we assume $\nu=1$, namely the level is a cube of a prime ideal. The examples 1, 2 and 3 are the case where rank $E_K=$ rank E_F+1. The example 4 treats the case where rank $E_K=$rank E_F. In this case, K is a totally imaginary quadratic extension of F. The examples 5 and 6 concern the case where rank E_K $=$rank E_F+2. We note that this case does not occur for $F=\boldsymbol{Q}$. These examples are calculated by the formula in Saito [15].

Example 1. Let $F=\boldsymbol{Q}(\sqrt{5})$, and $\mathfrak{q}=(\theta)$ with $\theta=-1+2\sqrt{5}$, $N_{F/Q}(\theta)$ $=-19$. Then we find $\dim S_{\mathrm{I}}=36$ and $\dim S_{\mathrm{III}}=18$, and the followings:

$$G^{\mathrm{III}}_{T((2))}(X)=N_{F_{19}/Q}(X^2-A)$$
$$\text{with}\quad A=(4,\,-1,0,0,1,\,-1,\,-1,1,1),\quad N_{F_{19}/Q}(A)=419;$$
$$G^{\mathrm{III}}_{T((3))}(X)=N_{F_{19}/Q}(X^2-B)$$
$$\text{with}\quad B=(6,\,-5,\,-2,\,-5,\,-5,\,-2,\,-5,\,-4,\,-3),$$
$$N_{F_{19}/Q}(B)=37^2\cdot 419.$$

Let $K=F(\sqrt{\theta})$, then $\mathfrak{f}_{K/F}=\mathfrak{q}$ and F and K satisfy the conditions (2.1), (2.2), and (2.3). We remark that for a prime ideal $\mathfrak{l}$ of F which remains prime in K, the Fourier coefficient for $\mathfrak{l}$ of f_λ associated with an idele class character λ of K vanishes. Hence the modulus P in Theorem 2.6 should divide the constant of the characteristic polynomial of $T(\mathfrak{l})$ on S_{III}. In the above example, the prime ideals (2) and (3) in F remain prime in K, so P should divide 419. In fact, we may set

$$\eta_0=\frac{1-\sqrt{\theta}}{2},\quad \eta_1=\frac{5+\sqrt{5}}{4}-\frac{1+\sqrt{5}}{4}\sqrt{\theta}\,.$$

Then we find

$$\eta_1^{19}=\frac{-7815395405-3495151081\sqrt{5}}{4}+\frac{4194233399+1875718199\sqrt{5}}{4}\sqrt{\theta},$$

and

$$H_0'(-1)=H_0(1)^2=5^2,$$
$$H_1'(-1)=H_1(1)^2=(5\cdot 419\cdot 3730499)^2.$$

Since we have

$$Q_p\left(\frac{1+\sqrt{5}}{2}\right)\equiv 158\sqrt{5}\bmod p\quad\text{for}\quad p=419,$$

and $\zeta_F(-1)/\zeta(-1)=-2/5$, we see the prime 419 satisfies the conditions in Theorem 2.5 and Theorem 2.6. So the above example gives a verification of our Theorem 2.6. Now, the prime 3730499 in $H_1(1)^2$ appears in the remaining space S_{I} as follows:

$$G^{I}_{T((2))}(X)=N_{F_{19}/Q}(X^4-CX^2+D)$$
$$\text{with}\quad C=(13,1,0,0,1,1,3,1,1)$$
$$D=(28,7,0,\,-6,\,-1,9,15,5,7),$$
$$N_{F_{19}/Q}(D)=37^2\cdot 3730499.$$

Since we have $Q_p(1+\sqrt{5})/2\equiv 1640877\cdot\sqrt{5} \bmod p$ for $p=3730499$, the condition (2.7) is valid also in this case.

Example 2. Let $F=F_7$ and $\beta_j=e^{2\pi j\sqrt{-1}/7}+e^{-2\pi j\sqrt{-1}/7}$ for an integer j. We set $\mathfrak{q}=(\theta)$ with $\theta=-3+4\beta_1+\beta_2$ then $N_{F_7/Q}(\theta)=13$. We find

$$\dim S_{\mathrm{I}}=12, \quad \dim S_{\mathrm{III}}=0,$$

and the following:

$$\begin{aligned} G^I_{T((2))}(X) &= N_{F_{13}/Q}(X^2-A) \\ &\text{with} \quad A=(9,0,0,-2,-2,-4,-8) \\ & N_{F_{13}/Q}(A)=3^3\cdot 4447. \end{aligned}$$

Let $K=F(\sqrt{\theta})$, then F and K satisfy the conditions (2.1), (2.2), and (2.3). We note $\chi_{K/F}((2))=-1$. We may set

$$\eta_0=\frac{1+2\beta_2+\sqrt{\theta}}{2}, \quad \eta_1=\frac{-(1+\beta_1)-(1+\beta_1)\sqrt{\theta}}{2}.$$

Then we find

$$\eta_1^{13}=\frac{-(326+261\beta_1+117\beta_2)-(714+573\beta_1+255\beta_2)\sqrt{\theta}}{2}$$

and

$$H'_0(-1)=3^2, \quad H'_1(-1)=(3^4\cdot 4447)^2.$$

Since we have

$$\begin{aligned} R_p &\equiv \det\begin{pmatrix} Q_p(\beta_2) & Q_p(\beta_1) \\ Q_p(\beta_1) & Q_p(\beta_3) \end{pmatrix} \\ &\equiv 2613+1622\beta_1+2439\beta_1^2 \bmod p \quad \text{for} \quad p=4447, \end{aligned}$$

and $\zeta_F(-1)/\zeta(-1)=-4/7$, the condition (2.7) is satisfied.

Example 3. Let $F=Q(\beta)$, where β is the unique solution of $X^3-4X+2=0$ satisfying $0<\beta<1$. We take $\mathfrak{q}=(\theta)$ with $\theta=5+2\beta-3\beta^2$. Then $N_{F/Q}(\theta)=5$, and we find

$$\dim S_{\mathrm{I}}=4, \quad \dim S_{\mathrm{III}}=0,$$

and

$$G^{I}_{T(\mathfrak{p}_2)}(X)=X^4-8X^2+11,$$
$$G^{I}_{T(\mathfrak{p}_{13})}(X)=X^4-17X^2+11,$$

where $\mathfrak{p}_2=(\beta)$ with $N\mathfrak{p}_2=2$ and $\mathfrak{p}_{13}=(6-\beta-\beta^2)$ with $N\mathfrak{p}_{13}=13$. Let $K=F(\theta)$, then F and K satisfy the conditions (2.1), (2.2), and (2.3), and the prime ideals $\mathfrak{p}_2$ and $\mathfrak{p}_{13}$ remain prime in K. In this case we may take

$$\eta_0=\frac{1-\beta+\sqrt{\theta}}{2}, \text{ and } \eta_1=\frac{-3+\beta+\beta^2+(1-2\beta+\beta^2)\sqrt{\theta}}{2}$$

We find

$$\eta_1^5=\frac{-9+\beta+2\beta^2+(-5+2\beta+2\beta^2)\sqrt{\theta}}{2},$$

and

$$H_0'(-1)=2^2 \qquad H_1'(-1)=2^4 11^2.$$

We know by the table 8 in Cartier and Roy [2] that $\zeta_F(2-p)/\zeta(2-p)$ and $\zeta_F(-1)/\zeta(-1)$ are p-units for $p=11$, hence the condition (2.7) is satisfied.

Example 4. Let $F=\boldsymbol{Q}(\sqrt{2})$, and $\mathfrak{q}=(\theta)$ with $\theta=-7+4\sqrt{2}$. Here $N_{F/Q}(\theta)=17$. Then we find

$$\dim S_{\mathrm{I}}=8\cdot 8, \qquad \dim S_{\mathrm{III}}=4\cdot 8,$$

and

$$\begin{aligned} G^{\mathrm{III}}_{T((\sqrt{2}))}(X)&=N_{F_{17}/Q}((X-A)^2(X^2-BX+C)) \\ \text{with } A&=(0,0,1,0,0,0,0,0,0) \\ B&=(1,0,-1,1,1,0,0,0,0) \\ C&=(0,0,0,1,0,-1,-1,1,0) \\ G^{\mathrm{III}}_{T((3))}(X)&=N_{F_{17}/Q}((X^2-D)X^2) \\ D&=(10,-1,-2,1,0,3,2,1,0). \end{aligned}$$

Let $G^{0}_{T((\sqrt{2}))}(X)$ and $G^{0}_{T((3))}(X)$ be the second factors of $G^{\mathrm{III}}_{T((\sqrt{2}))}(X)$ and $G^{\mathrm{III}}_{T((3))}(X)$ respectively. Then we find

$$\begin{aligned} N_{F_{17}/Q}(G^{0}_{T((\sqrt{2}))}(A))&=953\cdot 1123 \\ N_{F_{17}/Q}(G^{0}_{T((3))}(\sqrt{D}))&=953\cdot 1123. \end{aligned} \tag{3.1}$$

Here we note A and $\sqrt{D}$ are the roots of the first factors of $G^{\mathrm{III}}_{T((\sqrt{2}))}(X)$

and $G^{\mathrm{III}}_{T((3))}(X)$ respectively. Let $K=F(\sqrt{\theta})=F(\sqrt{-7+4\sqrt{2}})$, then K is a totally imaginary quadratic extension of F with the conductor $\mathfrak{q}$, $h_K=1$, and $E_K=\langle\pm1, 1-\sqrt{2}\rangle$. Let σ_1 and σ_2 be the embeddings of K into $\boldsymbol{C}$ given by $\sigma_1(\beta)=\beta$ for $\beta\in K$ and

$$\sigma_2(\beta)=(a-b\sqrt{2})+(c-d\sqrt{2})\sqrt{\theta},$$

for $\beta=a+b\sqrt{2}+(c+d\sqrt{2})\sqrt{\theta}$ with $a, b, c, d\in\boldsymbol{Q}$. Then all the embeddings of K into $\boldsymbol{C}$ are given by σ_1, $\rho\sigma_1$, σ_2 and $\rho\sigma_2$ with the complex conjugation ρ. For $a\in\mathfrak{o}_K$ prime to $\mathfrak{q}$, let $a\equiv a'b \bmod (\sqrt{\theta})$ with $a'\in\mathfrak{o}$ and $b\in\mathfrak{o}_K$ congruent to 1 modulo $(\sqrt{\theta})$, and for b let $b\equiv1+u\sqrt{\theta} \bmod (\sqrt{\theta})^2$ with $u\in\mathfrak{o}$, and put $\psi(b)=e^{2\pi iu/17}$. Let χ be the quadratic residue symbol of $(\mathfrak{o}_K/(\sqrt{\theta}))^\times\simeq(\mathfrak{o}/\theta)^\times$. For $a\in\mathfrak{o}_K$ prime to q, define

$$\lambda_1((a))=\chi(a')\psi(b)\sigma_1(a)\rho\sigma_2(a),$$
$$\lambda_2((a))=\chi(a')\psi(b)\rho\sigma_1(a)\sigma_2(a),$$

then λ_1 and λ_2 give Grossencharacters of K with conductors $(\sqrt{\theta})^2$. Let f_1 and f_2 be the cusp forms satisfying $L(s,f_1)=L(s,\lambda_1)$ and $L(s,f_2)=L(s,\lambda_2)$, then we see f_1 and f_2 are contained in S_{III}. Let $C_1(\mathfrak{m})$ and $C_2(\mathfrak{m})$ be the Fourier coefficients of f_1 and f_2 respectively, then we find

$$N_{F_{17}/Q}(X-C_1((\sqrt{2}))(X-C_2((\sqrt{2})))=G^0_{T((\sqrt{2}))}(X)$$
$$N_{F_{17}/Q}((X-C_1((3))(X-C_2((3)))=G^0_{T((3))}(X),$$

namely $G^0_{T((\sqrt{2}))}(X)$ and $G^0_{T((3))}(X)$ correspond to the subspace spanned by the companions of f_1 and f_2. Hence (3) suggests that f_1 and f_2 are congruent to some cusp forms in S_{III} which are different from the companions of f_1 and f_2 modulo prime ideals lying above 953 and 1123.

Example 5. Let $F=\boldsymbol{Q}(\sqrt{5})$ and $\mathfrak{q}=(\theta)$ with $\theta=(11+\sqrt{5})/2$. Here $N_{F/Q}(\theta)=29$. Then we find

$$\dim S_{\mathrm{I}}=8\cdot14,\quad \dim S_{\mathrm{III}}=6\cdot14,$$

and

$G^I_{T((2))}(X)=N_{F_{29}/Q}(X^6-AX^4+BX^2-C)$
with $A=(13, 0, 0, 1, 1, 0, 0, -2, 1, 0, 0, 0, 1, 0, 1)$
$B=(40, 2, 1, 10, 3, 2, 2, -9, 7, 0, -1, 5, 5, 2, 7)$
$C=(28, 7, 5, 14, 6, 14, 3, 1, 14, 2, 8, 12, 5, 16, 10)$,
$G^{\mathrm{III}}_{T((2))}(X)=N_{F_{29}/Q}(X^8-DX^6+EX^4-FX^2+G)$

$D=(22, 0, 0, 1, 1, 0, 0, 4, 1, 0, 0, 0, 1, 2, 1)$

$E=(154, 5, 0, 6, 15, 1, 7, 60, 11, 7, 0, 0, 13, 29, 19)$

$F=(324, -17, 140, -68, -17, -44, 25, 160, -15, 25, -63, -61, -4, 78, 17)$

$G=(118, -103, -117, -156, -104, -108, -51, -11, -112, -60, -146, -126, -89, -14, -104).$

In the above two cases, we find $N_{F_{29}/Q}(C)=59^4\cdot 173^2$ and $N_{F_{29}/Q}(G)=33871^2\cdot 763223^2$ for the constant terms C and G of $G^{I}_{T((2))}(X)$ and $G^{\mathrm{III}}_{T((2))}(X)$ respectively.

We shall give one more example of the same type as Example 5.

Example 6. Let $F=\boldsymbol{Q}(\sqrt{29})$, and $\mathfrak{q}=(\theta)$ with $\theta=11+2\sqrt{29}$. Here $N_{F/Q}(\theta)=5$. Then we find $\dim S^{\mathrm{I}}=2$, $\dim S^{\mathrm{III}}=4$, and

$$G^{\mathrm{I}}_{T((2))}(X)=N_{F_5/Q}(X^4-AX^2+B)$$

$$\text{with } A=(15, 3)$$

$$B=(5, -5)^2$$

$$G^{\mathrm{III}}_{T((2))}(X)=N_{F_5/Q}(X^2-C)$$

$$C=(1, -1)^2$$

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