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On the Symmetric-Square Zeta Functions Attached to Hilbert Modular Forms

Shin-ichiro Mizumoto

In this note we present new proofs of properties of the "second" L-functions attached to modular forms without using the Rankin-Selberg method. Detailed proofs (for Hilbert modular cases) are contained in [7].

§ 1. Elliptic Modular Case

Let

$$f(z)=\sum_{n=1}^{\infty} a(n)e(nz)$$

be a normalized eigen cusp form of weight k with respect to $SL(2, \boldsymbol{Z})$. Here k is a positive integer, $e(x)=\exp(2\pi ix)$, and z is a variable on the upper half plane $\mathfrak{H}$. The "second" L-function we consider here is defined by:

$$L_2(s,f)=\prod_p (1-\alpha_p^2 p^{-s})^{-1}(1-\alpha_p\beta_p p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1}$$

where p runs over all prime numbers, and α_p, $\beta_p \in \boldsymbol{C}$ are taken so that $\alpha_p+\beta_p=a(p)$, $\alpha_p\beta_p=p^{k-1}$; this infinite product converges absolutely and uniformly for $\mathrm{Re}(s)>k$.

The following properties are known:

(i) (Shimura [9], Zagier [11], Gelbart-Jacquet [5])

$L_2(s,f)$ has a holomorphic continuation to the whole s-plane and satisfies a functional equation under $s\mapsto 2k-1-s$.

(ii) (Zagier [11], Sturm [10]) For each even integer m with $k\leqq m\leqq 2k-2$, the value $L_2(m,f)/\pi^{2m-k+1}(f,f)$ belongs to the totally real number field $\boldsymbol{Q}(f)=\boldsymbol{Q}(a(n)\,|\,n\geqq 1)$; here (,) is the Petersson inner product (cf. (2.2) below).

Most of the known proofs of (i) (ii) depend on the Rankin-Selberg method. The main purpose of this note is to give proofs of (i) (ii) not using the Rankin-Selberg method. Poincaré series and Kloosterman sums

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play a fundamental role in our proofs; this method was suggested by a remark in Zagier [11, pp. 141–142].

Let

$$G_r(z)=\frac{1}{2}\sum_{\substack{(c,d)\in \boldsymbol{Z}^2\\(c,d)=1}}(cz+d)^{-k}\boldsymbol{e}\left(r\cdot\frac{a_0z+b_0}{cz+d}\right)\quad (z\in\mathfrak{H})$$

be the Poincaré series with $0<r\in\boldsymbol{Z}$, which is a cusp form of weight k with respect to $SL(2,\boldsymbol{Z})$. For $\mathrm{Re}(s)>1$, we put

$$\Psi_s(z)=\sum_{n=1}^{\infty}n^{k-1-s}G_{n^2}(z). \tag{1.1}$$

This series converges absolutely and uniformly on any compact subset of $\{(s,z)\,|\,\mathrm{Re}(s)>1,\ z\in\mathfrak{H}\}$. By

$$(G_r,f)=\frac{\Gamma(k-1)}{(4\pi r)^{k-1}}a(r)$$

we have

$$\zeta(2s)(\Psi_s,f)=\frac{\Gamma(k-1)}{(4\pi)^{k-1}}L_2(s+k-1,f) \tag{1.2}$$

for $\mathrm{Re}(s)>1$, since

$$L_2(s,f)=\zeta(2s-2k+2)\sum_{n=1}^{\infty}a(n^2)n^{-s}.$$

Let

$$\Psi_s(z)=\sum_{m=1}^{\infty}b(m,s)\boldsymbol{e}(mz) \tag{1.3}$$

be the Fourier expansion of Ψ_s. To compute $b(m,s)$, we use the following Fourier expansion of G_r:

$$\begin{aligned}G_r(z)=\sum_{m=1}^{\infty}\Bigg\{\delta_{r,m}&+2\pi(-1)^{k/2}\left(\frac{m}{r}\right)^{(k-1)/2}\\&\times\sum_{c=1}^{\infty}\frac{1}{c}K_c(r,m)J_{k-1}\left(\frac{4\pi}{c}\sqrt{rm}\right)\Bigg\}\boldsymbol{e}(mz).\end{aligned} \tag{1.4}$$

Here $\delta_{r,m}$ is the Kronecker delta,

$$K_c(r,m)=\sum_{\substack{x \bmod c\\(x,c)=1}}\boldsymbol{e}\left(\frac{rx+mx^{-1}}{c}\right)$$

(x^{-1} denotes an integer such that $xx^{-1} \equiv 1 \pmod{c}$) the Kloosterman sum, and J_{k-1} the Bessel function of order $k-1$.

Zagier [11, pp. 141–142] asked whether it is possible to obtain an explicit formula of $b(m, s)$ (reflecting the properties of $L_2(s, f)$) by substituting directly (1.4) into (1.1). In Section 3, we shall show that this is possible.

§ 2. Statement of Results

Let F be a totally real number field of degree g over $\boldsymbol{Q}$, with the class number one in the narrow sense. Let $\mathcal{O}$, $\mathfrak{d}$, $d(F)$ be the ring of integers in F, the different of $F/\boldsymbol{Q}$, and the discriminant of F, respectively. Let

$$(2.1) \qquad f(z) = \sum_{0 \ll \nu \in \mathfrak{d}^{-1}} a((\nu)\mathfrak{d})\boldsymbol{e}(\mathrm{tr}(\nu z)) \quad (z = (z_1, \cdots, z_g) \in \mathfrak{H}^g)$$

be a normalized eigen cusp form of weight k ($\in \boldsymbol{Z}$) with respect to $SL(2, \mathcal{O})$. For two modular forms g_1, g_2 of weight k with respect to $SL(2, \mathcal{O})$ such that $g_1 g_2$ is a cusp form, we put

$$(2.2) \qquad (g_1, g_2) = \int_{SL(2,\mathcal{O})\backslash \mathfrak{H}^g} g_1(z)\overline{g_2(z)}\, \mathrm{Im}(z)^k d\mu(z),$$

where

$$\mathrm{Im}(z) = \prod_{j=1}^{g} y_j \quad \text{and} \quad d\mu(z) = \prod_{j=1}^{g} y_j^{-2} dx_j dy_j \quad \text{if } z = (z_1, \cdots, z_g)$$

and $z_j = x_j + iy_j$ $(j = 1, \cdots, g)$. For $\mathrm{Re}(s) > k + \frac{1}{2}$, we put

$$L_2(s, f) = \prod_{\mathfrak{p}} (1 - \alpha_{\mathfrak{p}}^2 N(\mathfrak{p})^{-s})^{-1} (1 - \alpha_{\mathfrak{p}}\beta_{\mathfrak{p}} N(\mathfrak{p})^{-s})^{-1} (1 - \beta_{\mathfrak{p}}^2 N(\mathfrak{p})^{-s})^{-1},$$

where the product is over all non-zero prime ideals in $\mathcal{O}$ with $\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}} \in \boldsymbol{C}$ satisfying $\alpha_{\mathfrak{p}} + \beta_{\mathfrak{p}} = a(\mathfrak{p})$ and $\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}} = N(\mathfrak{p})^{k-1}$.

Theorem 1. *The notation being as above, suppose f is a normalized eigen cusp form of even weight $k \geqq 4$ with respect to $SL(2, \mathcal{O})$. Put*

$$(2.3) \qquad \Lambda_2(s, f) = d(F)^{3s/2} \left(2^{-s} \pi^{-3s/2} \Gamma(s) \Gamma\left(\frac{s-k+2}{2} \right) \right)^g L_2(s, f).$$

Then, $\Lambda_2(s, f)$ has a holomorphic continuation to the whole s-plane and satisfies the functional equation

$$\Lambda_2(s, f) = \Lambda_2(2k-1-s, f).$$

Theorem 2. *Let f be as in Theorem 1. Then, for each even integer m with $k \leqq m \leqq 2k-2$, we have*:

$$[L_2(m, f)/\pi^{g(2m-k+1)}(f, f)]^{\sigma} = L_2(m, f^{\sigma})/\pi^{g(2m-k+1)}(f^{\sigma}, f^{\sigma})$$

for all $\sigma \in \mathrm{Aut}(\boldsymbol{C})$. In particular, $L_2(m, f)/\pi^{g(2m-k+1)}(f, f)$ belongs to $\boldsymbol{Q}(f)$.

Here $\mathrm{Aut}(\boldsymbol{C})$ denotes the group of all ring automorphisms of $\boldsymbol{C}$. Each $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ acts on f with the Fourier expansion (2.1) by

$$f^{\sigma}(z) = \sum_{0 \ll \nu \in \mathfrak{d}^{-1}} a((\nu)\mathfrak{d})^{\sigma} \boldsymbol{e}(\mathrm{tr}(\nu z)).$$

We denote by $\boldsymbol{Q}(f)$ the totally real number field generated over $\boldsymbol{Q}$ by the eigenvalues of all Hecke operators on f.

Remark 1. As in Zagier [11], our method yields also the trace formula for the Hecke operators acting on the space of cusp forms with respect to $SL(2, \mathcal{O})$, which is a special case of the formula of Shimizu [8].

Remark 2. Theorem 2 is used in Furusawa [4].

§ 3. Proofs

We sketch our method of proofs in the case $F = \boldsymbol{Q}$. We substitute (1.4) into (1.1) to obtain:

$$(3.1) \qquad b(m, s) = (-1)^{k/2} \pi m^{(k-1)/2} S + \begin{cases} m^{(k-1-s)/2} & \text{if } m \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

Here

$$S = 2 \sum_{c=1}^{\infty} \sum_{n=1}^{\infty} n^{-s} c^{-1} K_c(m, n^2) J_{k-1}\left(\frac{4\pi n \sqrt{m}}{c}\right).$$

Suppose $1 < \mathrm{Re}(s) < k-1$ or $s = k-1$, and put

$$A(x) = |x|^{-s} J_{k-1}\left(\frac{4\pi \sqrt{m}\,|x|}{c}\right).$$

So

$$A(0) = \begin{cases} 0 & \text{if } \mathrm{Re}(s) < k-1, \\ \dfrac{(2\pi)^{k-1}}{\Gamma(k)} \left(\dfrac{\sqrt{m}}{c}\right)^{k-1} & \text{if } s = k-1, \end{cases}$$

and

$$
\begin{aligned}
S &= \sum_{c=1}^{\infty} \sum_{n \in Z} c^{-1} K_c(m, n^2) A(n) - \sum_{c=1}^{\infty} c^{-1} K_c(m, 0) A(0) \\
&= \sum_{c=1}^{\infty} \sum_{r \bmod c} c^{-1} K_c(m, r^2) \sum_{\substack{n \equiv r \pmod c \\ n \in Z}} A(n) \\
&\quad - \delta_{s,k-1} \frac{(2\pi)^{k-1}}{\Gamma(k)} m^{(k-1)/2} \sum_{c=1}^{\infty} c^{-k} K_c(m, 0).
\end{aligned} \tag{3.2}
$$

Here it is easy to see:

$$
\sum_{c=1}^{\infty} c^{-k} K_c(m, 0) = \frac{\sigma_{1-k}(m)}{\zeta(k)} \tag{3.3}
$$

where $\sigma_s(m) = \sum_{\substack{d \mid m \\ d>0}} d^s$. We compute

$$
\sum_{\substack{n \equiv r \pmod c \\ n \in Z}} A(n) = c^{-s} \sum_{n \in Z} \left| n + \frac{r}{c} \right|^{-s} J_{k-1}\left(4\pi\sqrt{m} \left| n + \frac{r}{c} \right| \right) \tag{3.4}
$$

with c, r fixed. If we put

$$
B(x) = \sum_{n \in Z} |n + x|^{-s} J_{k-1}(4\pi\sqrt{m}\, |n + x|),
$$

(3.4) is equal to $c^{-s} B(r/c)$. Let $B(x) = \sum_{l \in Z} c_l e(lx)$ be the Fourier expansion of $B(x)$. By the Poisson summation formula, we have:

$$
c_l = \int_{-\infty}^{+\infty} e(-lx) |x|^{-s} J_{k-1}(4\pi\sqrt{m}\, |x|) dx. \tag{3.5}
$$

(We denote this integral by $I(l, m, s)$.)

Thus

$$
\sum_{n \equiv r \pmod c} A(n) = c^{-s} \sum_{l \in Z} I(l, m, s) e\left(\frac{lr}{c}\right).
$$

We note that $I(l, m, s)$ is independent of c and r. Hence by (3.2),

$$
\begin{aligned}
S &= \sum_{l \in Z} \left(\sum_{c=1}^{\infty} \sum_{r \bmod c} c^{-1-s} K_c(m, r^2) e\left(\frac{lr}{c}\right) \right) I(l, m, s) \\
&\quad - \delta_{s,k-1} \frac{(2\pi)^{k-1}}{\Gamma(k)} m^{(k-1)/2} \frac{\sigma_{1-k}(m)}{\zeta(k)},
\end{aligned} \tag{3.6}
$$

for $1 < \mathrm{Re}(s) < k-1$ or $s = k-1$.

Proposition.

$$\sum_{c=1}^{\infty}\sum_{r \bmod c} c^{-1-s}K_c(m, r^2)e\left(\frac{lr}{c}\right)=\frac{L(s, l^2-4m)}{\zeta(2s)}$$

for each $l, m \in \boldsymbol{Z}$. Here

$$L(s, l^2-4m)=\begin{cases}\zeta(2s-1) & \text{if } l^2=4m,\\ L\left(s, \left(\frac{D}{*}\right)\right)\sum_{\substack{d\mid \mathfrak{f}\\ d>0}}\mu(d)\left(\frac{D}{d}\right)d^{-s}\sigma_{1-2s}(\mathfrak{f}d^{-1}) & \text{if } l^2\neq 4m.\end{cases}$$

In the latter case, we write $l^2-4m=D\mathfrak{f}^2$ with $0<\mathfrak{f} \in \boldsymbol{Z}$ and the discriminant D of the field $\boldsymbol{Q}(\sqrt{l^2-4m})$; $\left(\frac{D}{*}\right)$ denotes the Kronecker symbol, and μ the Möbius function.

Proof of Proposition. By the definition of the Kloosterman sums,

$$\sum_{r \bmod c} e\left(\frac{lr}{c}\right)K_c(m, r^2)=\sum_{\substack{x \bmod c\\ (x,c)=1}}\sum_{r \bmod c} e\left(\frac{1}{c}(lr+r^2x^{-1}+mx)\right).$$

In the inner summation, we replace r by rx to find that this is equal to

$$\sum_{\substack{x \bmod c\\ (x,c)=1}}\sum_{r \bmod c} e\left(\frac{x}{c}(r^2+lr+m)\right).$$

This value is calculated by counting the numbers of the solutions of quadratic congruences. We omit the details.

Proof of Theorem 1. For $1<\mathrm{Re}(s)<k-1$, we have

$$\zeta(2s)S=\sum_{l\in \boldsymbol{Z}} I(l, m, s)L(s, l^2-4m)$$

by (3.6) and Proposition. So, in this region, by (3.1) we have

$$\zeta(2s)b(m, s)=(-1)^{k/2}m^{(k-1)/2}\pi\sum_{\substack{l^2\neq 4m\\ l\in \boldsymbol{Z}}} I(l, m, s)L(s, l^2-4m)+H(s) \tag{3.7}$$

with

$$H(s)=0 \qquad \text{if } m \text{ is not a square,}$$

$$H(s)=(-1)^{k/2}2^s\pi^{s+(1/2)}m^{(k+s-2)/2}$$

$$\times\frac{\Gamma\left(\frac{k-s}{2}\right)\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(\frac{k+s}{2}\right)\Gamma\left(\frac{k+s-1}{2}\right)\Gamma\left(\frac{1-k+s}{2}\right)}\zeta(2s-1)+m^{(k-1-s)/2}\zeta(2s), \tag{3.8}$$

if m is a square.

By Erdélyi et al. [3, 1.12, (13)] [2, 2.1.4, (22)],

$$I(l,m,s)=\begin{cases} 2^{k-1}m^{(k-1)/2}\pi^{s-1}(4m-l^2)^{(s-k)/2}\Gamma\left(\dfrac{k-s}{2}\right)\Gamma\left(\dfrac{k+s}{2}\right)^{-1} \\ \qquad\times F\left(\dfrac{k-s}{2},\ \dfrac{k+s-1}{2};\ \dfrac{1}{2};\ \dfrac{l^2}{l^2-4m}\right) \quad \text{if } 4m>l^2, \\ 2^s m^{(k-1)/2}\pi^{s-1}(l^2-4m)^{(s-k)/2}\Gamma(k-s)\Gamma(k)^{-1}\cos\left(\dfrac{\pi}{2}(k-s)\right) \\ \qquad\times F\left(\dfrac{k-s}{2},\ \dfrac{k+s-1}{2};\ k;\ \dfrac{4m}{4m-l^2}\right) \quad \text{if } 4m<l^2, \end{cases} \tag{3.9}$$

where $F(a, b; c; x)$ is the hypergeometric function.

Lemma. (1) *Fix $0<m\in \boldsymbol{Z}$. For each $l\in \boldsymbol{Z}$ such that $l^2\neq 4m$, put*

$$Z_l(s)=2^{-s}\pi^{-3s/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma(s+k-1)I(l,m,s)L(s,l^2-4m),$$

which is a meromorphic function on $\boldsymbol{C}$. Then,

(i) *$Z_l(s)$ satisfies the functional equation: $Z_l(s)=Z_l(1-s)$,*

(ii) *$Z_l(s)$ is holomorphic in the strip $-\frac{1}{2}<\mathrm{Re}(s)<\frac{3}{2}$.*

(2) *Put*

$$H^*(s)=2^{-s}\pi^{-3s/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma(s+k-1)H(s).$$

Then,

(i) *$H^*(s)$ satisfies the functional equation: $H^*(s)=H^*(1-s)$,*

(ii) *$H^*(s)$ is holomorphic in the strip $-\frac{1}{2}<\mathrm{Re}(s)<\frac{3}{2}$.*

The assertion (1) follows from well-known properties of $L(s, l^2-4m)$ and the fact that $F(a, b; c; x)$ with $x<0$ is holomorphic in a, b and $F(a, b; c; x)=F(b, a; c; x)$. The assertion (2) is a simple consequence of properties of the Riemann zeta function.

The infinite sum in (3.7) converges uniformly and absolutely in the strip $-\frac{1}{2}<\mathrm{Re}(s)<k-1$. So (3.7) and (1.2) imply that $\Lambda_2(s,f)$ (in the notation of (2.3) with $F=\boldsymbol{Q}$) is a meromorphic function on $\boldsymbol{C}$ satisfying

$$\Lambda_2(s,f)=\Lambda_2(2k-1-s,f),$$

and holomorphic in $k-\frac{3}{2}<\mathrm{Re}(s)<2k-2$. But by the Euler product expression of $L_2(s,f)$ which converges absolutely and uniformly for $\mathrm{Re}(s)>k$, we see that $\Lambda_2(s,f)$ is also holomorphic for $\mathrm{Re}(s)>k-\frac{3}{2}$. Hence $\Lambda_2(s,f)$ is an entire function with the above functional equation.

Proof of Theorem 2. If $l^2>4m$, then by (3.9) we see: $I(l, m, r)=0$ for each odd integer with $3\leqq r\leqq k-1$. Moreover each

$$F\left(\frac{k-r}{2}, \frac{k+r-1}{2}; \frac{1}{2}; \frac{l^2}{l^2-4m}\right) \quad (l^2<4m)$$

has an expression in terms of the Gegenbauer polynomials. So, for each r, by (3.6) and Proposition we have:

$$\Psi_r=-2^{2r-2}\pi^{2r}\frac{\Gamma(k-r)}{\Gamma(k+r-1)}\frac{1}{\zeta(2r)}C_{k,r}-\delta_{r,k-1}\cdot\frac{1}{2}E_k. \tag{3.10}$$

Here

$$E_k(z)=1+(-1)^{k/2}\frac{(2\pi)^k}{\Gamma(k)\zeta(k)}\sum_{m=1}^{\infty}\sigma_{k-1}(m)e(mz)$$

is the Eisenstein series of weight k, and

$$C_{k,r}(z)=\sum_{m=0}^{\infty}\Big(\sum_{\substack{l\in Z\\ l^2\leqq 4m}}p_{k,r}(l, m)L(1-r, l^2-4m)\Big)e(mz)$$

with

$$p_{k,r}(l, m)=\text{coefficient of } x^{k-r-1} \text{ in } (1-lx+mx^2)^{-r}.$$

(In particular we obtain: $C_{k,r}$ is a modular form of weight k with respect to $SL(2, Z)$; if $r<k-1$, it is a cusp form. This is a result of Cohen [1] and Zagier [11].) By (3.10), the Fourier coefficients of Ψ_r ($3\leqq r\leqq k-1$, r odd) are rational numbers. So, by Shimura [9a, Lemma 4, p. 792], we obtain Theorem 2 by (1.2) for $k<m\leqq 2k-2$, m even. For $m=k$, $L_2(k,f)=2^{2k-1}\pi^{k+1}\Gamma(k)^{-1}(f,f)$ is a classical result.

Remark. As in Zagier [11], we obtain the trace formula of Hecke operators by putting $s=1$ in (3.7).

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Department of Mathematics
Tokyo Institute of Technology
Oh-Okayama, Meguro-ku
Tokyo 152, Japan