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The Duality of the Exponents of Free Deformations Associated with Unitary Reflection Groups

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§ 0. Introduction

In this paper we prove the duality between the two exponents of some type of deformation of an isolated singularity, called the D-duality here, and apply it to the deformations associated with the finite unitary reflection groups. This scenario has been mentioned by K. Saito [3, 6.3] and T. Yano [11, p. 8]. But this is, as far as we know, the first paper to prove it.

Our motivation was as follows: Orlik-Solomon [1] observed the duality between the two exponents of some unitary reflection groups, called the UR-duality here. One exponent is defined as the degrees of invariant polynomials under the unitary reflection group. In [7] [8], H. Terao showed that the other exponents are nothing but the degrees of logarithmic vector fields along the set of reflecting hyperplanes. Our interest in the UR-duality led us to find the D-duality which is studied here.

There have been some case-by-case explanations for the reason of the UR-duality, but, as far as we know, there is no general explanation for it so far. Although our explanation here is not applicable to all the UR-duality, one can intrinsically and rather generally explain the UR-duality in many cases by applying the D-duality.

At first in Section 1, we review the concept of free and (GTQ)-deformations introduced in [10] [9]. Next we shall define the two exponents (c-exponents and v-exponents) of a weighted homogeneous, free and (GTQ) deformations. The D-duality between these two exponents is discussed and proved in Section 2. The large part of Section 3 owes to T. Yano [10] [11] who constructed deformations of an isolated hypersurface singularity associated with some finite unitary reflection groups. Those deformations enjoy the conditions for the D-duality and thus explain why the UR-duality holds for some class of finite unitary reflection groups.

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§ 1. The two exponents of a deformation

We, at first, briefly review the definition of free and (GTQ)-deformations. Let $f_0 \in \mathbf{C}\{x_1, \cdots, x_n\}$ such that $f_0^{-1}(0)$ is a germ of an isolated singularity. Assume that

$$f\colon X=(\mathbf{C}^{n+k-1}, 0) \longrightarrow Y=(\mathbf{C}^k, 0)$$

is a deformation of $f_0^{-1}(0)$: The coordinates of X and Y are

$$(x_1, \cdots, x_n, y_2, \cdots, y_k) \quad \text{and} \quad (y_1, y_2, \cdots, y_k)$$

respectively. Define

$$x=(x_1, \cdots, x_n), \quad y'=(y_2, \cdots, y_k) \quad \text{and} \quad y=(y_1, \cdots, y_k)=(y_1, y')$$

for simplicity. Put

$$f_1(x, y')=f^*y_1.$$

Then $f_1(x, 0)=f_0(x)$. Define

$$\mathcal{O}_{C,0}=\mathcal{O}_{X,0}/(\partial f_1/\partial x_1, \cdots, \partial f_1/\partial x_n),$$

where $\mathcal{O}_{X,0}$ is the sheaf of germs of holomorphic functions on X near 0; $\mathcal{O}_{X,0} \cong \mathbf{C}\{x, y'\}$. The germ of the critical set $(C, 0)$ of f is (as a set) the support of $\mathcal{O}_{C,0}$. The direct image $f_*\mathcal{O}_{C,0}$ is an $\mathcal{O}_{Y,0}$-module and the germ of the discriminant $(D, 0)$ is defined by the 0-th fitting ideal of $f_*\mathcal{O}_{C,0}$. Define a $\mathbf{C}$-algebra homomorphism

$$s\colon \mathbf{C}\{x, y\} \longrightarrow \mathbf{C}\{x, y'\}=\mathcal{O}_{X,0}$$

such that

$$s(x)=x, \quad s(y_1)=f_1, \quad s(y_i)=y_i \quad (i \geqq 2).$$

Define a $\mathbf{C}$-linear map

(1.1) $$\varphi_0\colon \mathrm{Der}_{Y,0} \longrightarrow f_*\mathcal{O}_{C,0}$$

by

$$\varphi_0(\theta)=[s\{\theta(f^*y_1-y_1)\}],$$

where $\theta \in \mathrm{Der}_{Y,0}$ and [ ] denotes the residue class in

$$\mathcal{O}_{C,0}=\mathcal{O}_{X,0}/(\partial f_1/\partial x_1, \cdots, \partial f_1/\partial x_n)).$$

Put

$$M_0=\operatorname{im}\varphi_0. \tag{1.2}$$

Then M_0 is an $\mathcal{O}_{Y,0}$-submodule of $f_*\mathcal{O}_{C,0}$ generated by $\{[g_1], \cdots, [g_k]\}$, where

$$g_1=-1$$
$$g_i=(\partial f_1/\partial y_i) \in \mathcal{O}_{X,0} \qquad (i=2, \cdots, k).$$

Let $Z=(\mathbf{C}^{k-1}, 0)$ with its coordinate $(y')=(y_2, \cdots, y_k)$. Let $\pi: (Y, 0)\to (Z, 0)$ be the natural projection induced from the inclusion $\mathbf{C}\{y'\}\subset\mathbf{C}\{y_1, y'\}$. Assume that $[g_1], \cdots, [g_k]$ are independent over $\mathcal{O}_{Z,0}$ in π_*M_0.

(1.3) **Definition.** The deformation f is said to be *free* if π_*M_0 is $\mathcal{O}_{Z,0}$-free with its basis $\{[g_1], \cdots, [g_k]\}$.

Let

$$F\colon U\longrightarrow V$$

be a representative for the germ f; U and V are domains containing the origins in $\mathbf{C}^{n+k-1}$ and $\mathbf{C}^k$ respectively and $F(0)=0$. The germ of F at the origin is equal to f. Let C be the critical set of F.

(1.4) **Definition.** We say that F is (GTQ) if there is an analytic subset A of U with $\operatorname{Sing}(C)\subset A\subset C$ and $\operatorname{codim}_U A\geqq 2$ such that

1. $(\partial\Delta/\partial y_1)(F(p))\neq 0$ (Δ is a reduced defining equation for the discriminant) for any $p \in C\backslash A$,
2. the germ of F at p is a deformation which gives a trivial family of a quasi-homogeneous singularity along the critical set for any $p \in C\backslash A$.

Example. Since a rational double point is quasi-homogeneous and has no parameter, a deformation whose generic singularities are rational double points is (GTQ). In particular, any deformation of a rational double point is (GTQ).

Let us return to a germ f: $X=(\mathbf{C}^{n+k-1}, 0)\longrightarrow Y=(\mathbf{C}^k, 0)$.

(1.5) **Definition.** The germ f is said to be (GTQ) if it is represented by a (GTQ) deformation.

(1.6) **Definition.** The germ f is said to be *weighted homogeneous of type* $(\alpha_1, \cdots, \alpha_n, e_1, \cdots, e_k) \in \mathbf{N}^{n+k}$ if

$$y_1-f_1(x, y')$$

is homogeneous (of degree e_1) when one defines

$$\deg(x_i)=\alpha_i \qquad (i=1,\cdots,n)$$
$$\deg(y_i)=e_i \qquad (i=1,\cdots,k)$$

with $e_1\geqq\cdots\geqq e_k$.

Next we have to slightly generalize the exponents of a free divisor [9] which was defined exclusively for a divisor defined by a homogeneous polynomial. Let $h\in \mathbf{C}\{y\}$ be a weighted homogeneous polynomial of type $(e_1,\cdots,e_k)$ with $e_1\geqq\cdots\geqq e_k$. Put $V=V(h)\subset(\mathbf{C}^k,0)=Y$. Assume that V is free, i.e., $\mathrm{Der}_Y(\log V)$ is $\mathcal{O}_Y$-free.

Let

$$\theta=\textstyle\sum_{j=1}^k f_j(y)\partial/\partial y_j.$$

Then θ is said to be weighted homogeneous of degree of type $(e_1,\cdots,e_k)$, denoted by $d=\deg\theta$, if

$$d=(\deg f_j)-e_j$$

or

$$f_j=0$$

for $j=1,\cdots,k$. Of course $\deg(f_j)$ is the degree of weighted homogeneous f_j of type $(e_1,\cdots,e_k)$. It is easy to see that one can choose a free basis $\{\theta_1,\cdots,\theta_k\}$ for $\mathrm{Der}_Y(\log V)$ such that each θ_i is weighted homogeneous of degree d_i of type $(e_1,\cdots,e_k)$ with $d_1\leqq\cdots\leqq d_k$. Then $(d_1,\cdots,d_k)$ depends only upon V and $(e_1,\cdots,e_k)$.

(1.7) **Definition.** We call $(d_1+1,\cdots,d_k+1)$ the *exponents* of V (of type $(e_1,\cdots,e_k)$).

(1.8) **Remarks.** 1. When V is defined by a homogeneous polynomial (i.e., $(e_1,\cdots,e_k)=(1,\cdots,1)$), the exponents above are the same as the exponents in [9].

2. Because of the existence of the Euler vector field

$$\textstyle\sum e_i y_i(\partial/\partial y_i)\in \mathrm{Der}_Y(\log V),$$

$1=0+1$ is an exponent.

(1.9) **Proposition.** *Let $p\colon A=(\mathbf{C}^k,0)\longrightarrow B=(\mathbf{C}^k,0)$ be a finite holomorphic map germ. Let C and D be the critical set and the discriminant respectively. Assume that $p=(p_1,\cdots,p_k)$ and that each $p_i\in\mathcal{O}_{\mathbf{C}^k,0}$ is homogeneous of degree e_i. Assume that D is free. Let $(d_1+1,\cdots,d_k+1)$*

be the exponents of D of type $(e_1, \cdots, e_k)$. Then C is free and its exponents are also $(d_1+1, \cdots, d_k+1)$ of type $(1, \cdots, 1)$.

Proof. Note that D is defined by a reduced weighted homogeneous polynomial of type $(e_1, \cdots, e_k)$. Let $\{\theta_1, \cdots, \theta_k\}$ be a free basis for $\mathrm{Der}_B(\log D)$. Assume that each θ_i is weighted homogeneous of degree d_i of type $(e_1, \cdots, e_k)$. By Theorem B in [9], we know that $\theta_1, \cdots, \theta_k$ are liftable by p and that $\{p^{-1}\theta_1, \cdots, p^{-1}\theta_k\}$ is a free basis for $\mathrm{Der}_A(\log C)$. It is obvious that $p^{-1}\theta_i$ is homogeneous of degree d_i of type $(1, \cdots, 1)$. □

Let $f: X \to Y$ be a free and (GTQ) deformation. Assume that f is weighted homogeneous of type $(\alpha_1, \cdots, \alpha_n, e_1, \cdots, e_k)$ as in (1.6).

Let $(D, 0)$ be the discriminant of F. Then $(D, 0)$ is free (Theorem D [9]) and defined by a reduced weighted homogeneous polynomial of type $(e_1, \cdots, e_k)$. Thus we can define the exponents $(d_1+1, \cdots, d_k+1)$ of D of type $(e_1, \cdots, e_k)$ as in (1.7).

(1.10) **Definition.** The c(coordinate)-*exponents* and v(vector field)-*exponents* are defined by $(e_1-1, \cdots, e_k-1)$ and by $(d_1+1, \cdots, d_k+1)$ respectively.

In the next Section, we shall prove the duality (D-duality) between the c-exponents and v-exponents.

§ 2. Duality

In this section we shall fix a free (GTQ) deformation

$$f: X=(\mathbf{C}^{n+k-1}, 0) \longrightarrow Y=(\mathbf{C}^k, 0)$$

of an isolated singularity. Assume that f is weighted homogeneous of type $(\alpha_1, \cdots, \alpha_n, e_1, \cdots, e_k)$. Let $(e_1-1, \cdots, e_k-1)$ and $(d_1+1, \cdots, d_k+1)$ be the c-exponents and the v-exponents respectively in this section.

Note that $e_1-1 \geqq \cdots \geqq e_k-1$ and $1=d_1+1 \leqq \cdots \leqq d_k+1$.

(2.1) **Theorem** (D-duality). *The duality holds between $(e_1-1, \cdots, e_k-1)$ and $(d_1+1, \cdots, d_k+1)$, in other words,*

$$e_i+d_i$$

are constant equal to e_1 for $i=1, \cdots, k$.

Recall the definitions of M_0 (1.2) and φ_0: $\mathrm{Der}_{Y,0} \longrightarrow M_0$ (1.1). For the proof of (2.1), we need

(2.2) **Proposition** [9, in the proof of Theorem D]. *The sequence*

$$0 \longrightarrow \mathrm{Der}_Y(\log D)_0 \longrightarrow \mathrm{Der}_{Y,0} \xrightarrow{\varphi_0} M_0 \longrightarrow 0$$

is exact. □

Since $M_0 \otimes_{\mathcal{O}_{Y,0}} \mathcal{O}_{Y,0}/(y')$ is a **C**-vector space with its basis $\{[g_1], \cdots, [g_k]\}$, the 0-th fitting ideal of M_0 is generated by a reduced equation $\Delta(y)=0$ [9, the last Remark] with $\Delta(y_1, 0)=y_i^k$ because the operation of taking the fitting ideal commutes with the basis change [6].

Define $a_{ij} \in \mathbf{C}\{y'\}=\mathcal{O}_{Z,0}$ $(1 \leqq i, j \leqq k)$ by

$$\varphi_0(f_1(\partial/\partial y_i))=[f_1 g_i]=\textstyle\sum_{j=1}^k a_{ij}[g_{ij}] \in M_0.$$

Put $A=(a_{ij})_{1\leqq i,j\leqq k}$. Then A is a $k\times k$-matrix with entries in $\mathcal{O}_{Z,0}$. Define

$$\theta_i = \textstyle\sum_{j=1}^k (y_i\delta_{ij}-a_j)(\partial/\partial y_j) \qquad (I=1, \cdots, k).$$

Then we have

(2.3) **Proposition.** *The set* $\{\theta_1, \cdots, \theta_k\}$ *is a basis for* Der $(\log D)_0$.

Proof. We have

$$\begin{aligned}\varphi_0\left(\textstyle\sum_{j=1}^k a_{ij}(\partial/\partial y_j)\right) &= \textstyle\sum_{j=1}^k a_{ij}[g_j] \\ &= [f_1 g_i] = y_1\varphi_0(\partial/\partial y_i) \\ &= \varphi_0(y_1(\partial/\partial y_i)) \quad (i=1, \cdots k).\end{aligned}$$

Thus

$$\begin{aligned}0 &= \varphi_0\left(y_1(\partial/\partial y_i) - \textstyle\sum_{j=1}^k a_{ij}(\partial/\partial y_j)\right) \\ &= \varphi_0(\theta_i) \qquad (i=1, \cdots, k).\end{aligned}$$

This implies by (2.2) that $\{\theta_1, \cdots, \theta_k\} \subset \mathrm{Der}_Y(\log D)_0$. Therefore we obtain

$$\begin{aligned}|\theta_1, \cdots, \theta_k| &= \det(y_1\delta_{ij}-a_{ij}) \\ &\equiv y_1^k \bmod (y').\end{aligned}$$

Note that

$$\Delta \| \theta_1, \cdots, \theta_k| \qquad ([3, 1.5.\mathrm{iii})])$$

and

$$\Delta(y_1, 0)=y_1^k.$$

From these we deduce that $|\theta_1, \cdots, \theta_k|$ is a reduced defining equation for

D. By Saito's criterion [3, 1.8.ii)], we have (2.3). □

Finally we shall prove (2.1). Since

$$f_1 g_j - \sum_{j=1}^{k} a_{ij} g_j \in (\partial f_1/\partial x_1, \cdots, \partial f_1/\partial x_n)$$

and $(\partial f_1/\partial x_1, \cdots, \partial f_1/\partial x_n)$ is a homogeneous ideal with respect to the weight $(\alpha_1, \cdots, \alpha_n, e_1, \cdots, e_k)$, we know that a_{ij} is weighted homogeneous of degree $e_1 - e_i + e_j$ of type $(e_1, \cdots, e_k)$. Thus θ_i is weighted homogeneous of degree $e_1 - e_i$ of type $(e_1, \cdots, e_k)$. Therefore the v-exponents and the c-exponents of f are

$$(1, e_1 - e_2 + 1, \cdots, e_1 - e_k + 1)$$

and

$$(e_1 - 1, \cdots, e_k - 1)$$

respectively. This proves the D-duality as desired.

§ 3. Free deformations associated with finite reflection groups

In [4], Shephard-Todd classified the finite unitary reflection groups into 37 types 1–37. Let G be such a group irreducibly acting a complex vector space V. Let $\ell = \dim_{\mathbf{C}} V$. Recently the interesting numerology for the groups, called the UR-duality here, was revealed by Orlik-Solomon [1, 5.5]: The two exponents "m_i" and "n_i" are defined for the group G. Then the numbers

$$m_i + n_{\ell - i + 1} \qquad (i = 1, \cdots, \ell)$$

are independent of i when G is of type 1, 2′ ($p=1$), 2″ ($r=2$), 3, 4, 5, 6, 8, 9, 10, 14, 16, 17, 18, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36 or 37. When G is defined on $\mathbf{R}$ (1(A_ℓ), 2′ ($p=1$, $r=2$) (B_ℓ), 2″ ($r=2$) (D_ℓ), 23 (H_3), 28 (F_4), 30 (H_4), 35 (E_6), 36 (E_7) or 37 (E_8)), one has $m_i = n_i$ ($i = 1, \cdots, \ell$). In this case, the UR-duality is the self duality ($m_i + n_{\ell-i+1}$ is independent of i) which can be explained by using the Coxeter transformation.

Let G be one of finite unitary reflection groups of type 2′ ($p=1$), 4, 5, 8, 16, 20, 25, 26 and 32. In [11], T. Yano constructed a free deformation f of a rational double point whose discriminant is locally isomorphic (near the origin) to the discriminant D of the projection

$$\begin{array}{cccc} \pi: & V \longrightarrow V/G & \xrightarrow{\sim} & \mathbf{C}_\ell \\ & \cup & & \cup \\ & C & & D. \end{array}$$

(See Table 1.) The critical set C of π is exactly the union of reflecting hyperplanes of G. The map π is explicitly given by

$$(x_1, \cdots, x_\ell) \longmapsto (f_1(x_1, \cdots, x_\ell), \cdots, f_\ell(x_1, \cdots, x_\ell)),$$

where $f_1, \cdots, f_\ell$ are algebraically independent homogeneous invariant polynomials with

$$\mathbf{C}[x_1, \cdots, x_\ell]^G = \mathbf{C}[f_1, \cdots, f_\ell]$$

and $\deg f_1 \leqq \cdots \leqq \deg f_\ell$.

Note that

$$m_i = \deg f_i - 1 \qquad (i=1, \cdots, \ell).$$

The deformation f in Table 1 is weighted homogeneous of type $(\alpha_1, \cdots, \alpha_n, m_1+1, \cdots, m_\ell+1)$ in (1.6). Then D is defined by a weighted homogeneous polynomial of type $(m_1+1, \cdots, m_\ell+1)$. Thus the c-exponents of D are $(m_1+1, \cdots, m_\ell+1)$. It is known that D is a free divisor [9, Theorem C]. Let $(d_1+1, \cdots, d_\ell+1)$ be the v-exponents of D of type $(m_1+1, \cdots, m_\ell+1)$. Then, by (1.9), C is free with its exponents $(d_1+1, \cdots, d_\ell+1)$ of type $(1, \cdots, \ell)$. By [7], one has

$$n_i = d_i + 1 \qquad (i=1, \cdots, \ell).$$

After all, the c-exponents and the v-exponents of D are $(m_1, \cdots, m_\ell)$ and $(n_1, \cdots, n_\ell)$ respectively. Thus, by the D-duality (2.1), the numbers

$$m_i + n_{\ell-i+1}$$

are independent of i. This is nothing other than the UR-duality. This argument explains the UR-duality when G is of type $2'$ $(p=1)$, 4, 5, 8, 16, 20, 25, 26 or 32. Since $n_1=1$, one has

$$m_i + n_{\ell-i+1} = m_\ell + 1 \qquad (i=1, \cdots, \ell).$$

When G is defined on $\mathbf{R}$, there are the free deformations associated with G thanks to Brieskorn-Slodowy theory [5]. In this case, the D-duality implies the independence of i of the numbers

$$m_i + m_{\ell-i+1} \qquad (i=1, \cdots, \ell).$$

Table 1.

type	deformation	$\{m_i\}$	$\{n_i\}$
$2'$ $(p=1)$	$x^{r\ell}+yz+t_r x^{r\ell-r}+t_{2r}x^{r\ell-2r}+\cdots+t_{\ell r}$	$r-1, 2r-1, \cdots, (\ell-1)r-1, \ell r-1$	$1, r+1, \cdots, (\ell-1)r+1,$
4	$x^3+y^3+t_4x+t_6$	3, 5	1, 3
5	$x^4+y^3+t_6x^2+t_{12}$	5, 11	1, 7
8	$x^4+y^3+t_8y+t_{12}$	7, 11	1, 5
16	$x^5+y^3+t_{20}y+t_{30}$	19, 29	1, 11
20	$x^5+y^3+t_{12}x^3+(1/5)t_{12}x+t_{30}$	11, 29	1, 19
25	$x^4+y^3+t_6x^2+t_9x+t_{12}$	5, 8, 11	1, 4, 7
26	$x^3y+y^3+t_6y^2+t_{12}y+t_{18}$	5, 11, 17	1, 7, 13
32	$x^5+y^3+t_{12}x^3+t_{18}x^2+t_{24}x+t_{30}$	11, 17, 23, 29	1, 7, 13, 19

Here the t_i are the coordinates of the base space of deformation and the weight of t_i is i.

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