Advanced Studies in Pure Mathematics 6, 1985
Algebraic Groups and Related Topics
pp. 139–154

Holonomic Systems on a Flag Variety Associated to Harish-Chandra Modules and Representations of a Weyl Group

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Dedicated to Professor Hirosi Nagao on his 60th birthday

§ 1. Introduction

1.1. In [KT] we studied the characteristic cycles of holonomic systems on a flag variety associated to highest weight modules of a complex semisimple Lie algebra, and investigated its relation to the representations of a Weyl group.

In this paper we consider Harish-Chandra modules instead of highest weight modules, and prove a theorem similar to the main theorem of [KT] (Theorem 1 below). The main theorem of [KT] turns out to be a special case of Theorem 1 and this paper gives a generalization of the result of [KT], although the proof is essentially the same as the one in [KT].

1.2. Let G be a connected complex semisimple algebraic group and $G_{\mathbf{R}}$ a real form of G. We assume that $G_{\mathbf{R}}$ is connected for simplicity. We fix a maximal compact subgroup $K_{\mathbf{R}}$ of $G_{\mathbf{R}}$ and denote its complexification in G by K.

We consider the abelian category $\mathscr{H}(\mathfrak{g}, K)$ whose objects are $(\mathfrak{g}, K)$-modules of finite length with trivial central character. Here $\mathfrak{g}$ is the Lie algebra of G. The result of Beilinson-Bernstein [BB] implies that $\mathscr{H}(\mathfrak{g}, K)$ is equivalent to the abelian category $\mathscr{M}(\mathfrak{g}, K)$ consisting of coherent $\mathscr{D}$-Modules on the flag variety X with K-actions. Using the fact that X is the union of finitely many K-orbits (Matsuki [M]), it is easily shown that any irreducible component of the characteristic variety $\mathrm{Ch}(\mathfrak{M})$ of $\mathfrak{M} \in \mathscr{M}(\mathfrak{g}, K)$ is the closure of the conormal bundle $T^*_O X$ of a K-orbit O, and in particular $\mathfrak{M}$ is holonomic (actually regular holonomic). We take the multiplicity of $\mathfrak{M}$ along $\overline{T^*_O X}$ into account and consider the characteristic cycle $\mathbf{Ch}(\mathfrak{M}) \in \bigoplus_O \mathbf{Z}_{\geq 0}[\overline{T^*_O X}]$, where O is running through the K-orbits on

Received March 26, 1984

X. Let $K(\mathscr{M}(\mathfrak{g}, K))$ be the Grothendieck group of $\mathscr{M}(\mathfrak{g}, K)$. By the additivity of **Ch** we have a **Z**-linear homomorphism

$$(*) \quad K(\mathscr{M}(\mathfrak{g}, K)) \xrightarrow{\mathbf{Ch}} \bigoplus_{O} \mathbf{Z}[\overline{T_O^* X}].$$

One can define natural actions of the Weyl group W on $K(\mathscr{M}(\mathfrak{g}, K))$ and $\bigoplus_O \mathbf{Z}[\overline{T_O^* X}]$ (see Section 3 below). Then our main theorem is the following.

Theorem 1. *The* **Z**-*linear homomorphism*

$$K(\mathscr{M}(\mathfrak{g}, K)) \xrightarrow{\mathbf{Ch}} \bigoplus_{O} \mathbf{Z}[\overline{T_O^* X}]$$

is W-equivariant.

1.3. The contents of this paper are as follows. In Section 2 we summarize the known results concerning the Beilinson-Bernstein theory, the Riemann-Hilbert correspondence, Harish-Chandra modules and K-orbits on the flag variety. In Section 3 we give the definitions of W-actions and prove Theorem 1. Additional remarks are stated in Section 4.

1.4. On this occasion we give a remark on the paper [KT]. After writing it up, we learned that Theorem 6 in [KT] was already conjectured by Joseph [J], and V. Ginsburg informed us that he also proved the same theorem by a different method (letter dated January 25, 1984).

1.5. The author expresses his hearty thanks to Professor M. Kashiwara and Professor R. Hotta for valuable suggestions.

§ 2. Harish-Chandra modules and holonomic systems

2.1. The Beilinson-Bernstein theory

Let G be a connected semisimple algebraic group over the complex number field **C** with Lie algebra $\mathfrak{g}$. We denote the flag variety by X. X is naturally identified with the set of all the Borel subalgebras of $\mathfrak{g}$. We denote the sheaf of regular functions and the sheaf of differential operators on X by $\mathscr{O}_X$ and $\mathscr{D}_X$, respectively. The natural action of G on X induces an algebra homomorphism $U(\mathfrak{g}) \xrightarrow{D} \Gamma(X, \mathscr{D}_X)$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. Let $\mathfrak{z}(\mathfrak{g})$ be the center of $U(\mathfrak{g})$ and χ_0 the trivial central character of $\mathfrak{z}(\mathfrak{g})$. χ_0 is the algebra homomorphism from $\mathfrak{z}(\mathfrak{g})$ onto **C** given by $\mathfrak{z}(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \to U(\mathfrak{g})/\mathfrak{g}U(\mathfrak{g}) = \mathbf{C}$. A $U(\mathfrak{g})$-module M is said to have the trivial central character if $z.m = \chi_0(z)m$ for all $z \in \mathfrak{z}(\mathfrak{g})$ and $m \in M$.

Theorem 2. (Beilinson-Bernstein [BB])

(i) *D is surjective with* $\mathrm{Ker}\, D = U(\mathfrak{g})\, \mathrm{Ker}\, \chi_0 = U(\mathfrak{g})(\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}U(\mathfrak{g}))$.

(ii) *The abelian category of finitely generated $U(\mathfrak{g})$-modules M with trivial central character and that of coherent $\mathscr{D}_X$-modules $\mathfrak{M}$ are naturally equivalent to each other. The correspondence is given by $M = \Gamma(X, \mathfrak{M})$ and $\mathfrak{M} = \mathscr{D}_X \otimes_{U(\mathfrak{g})} M$.*

2.2. The Riemann-Hilbert correspondence

Let $\mathcal{O}_{X_{\mathrm{an}}}$ be the sheaf of holomorphic functions. We set

$$\mathscr{D}_{X_{\mathrm{an}}} = \mathcal{O}_{X_{\mathrm{an}}} \underset{\mathcal{O}_X}{\otimes} \mathscr{D}_X \quad \text{and} \quad \mathfrak{M}_{\mathrm{an}} = \mathscr{D}_{X_{\mathrm{an}}} \underset{\mathscr{D}_X}{\otimes} \mathfrak{M} = \mathcal{O}_{X_{\mathrm{an}}} \underset{\mathcal{O}_X}{\otimes} \mathfrak{M}$$

for a $\mathscr{D}_X$-Module $\mathfrak{M}$. A coherent $\mathscr{D}_X$-Module $\mathfrak{M}$ is said to be regular holonomic if $\mathfrak{M}_{\mathrm{an}}$ is a holonomic $\mathscr{D}_{X_{\mathrm{an}}}$-Module with regular singularity in the sence of [KK]. For a regular holonomic $\mathscr{D}_X$-Module $\mathfrak{M}$, we set $\mathscr{DR}(\mathfrak{M}) = \mathbf{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{an}}}}(\mathcal{O}_{X_{\mathrm{an}}}, \mathfrak{M}_{\mathrm{an}})$. $\mathscr{DR}(\mathfrak{M})$ is a bounded complex of $\mathbf{C}_X$-Modules which is an object of the derived category. Furthermore it is known that $\mathscr{DR}(\mathfrak{M})$ is a perverse sheaf, that is, $\mathscr{K} := \mathscr{DR}(\mathfrak{M})$ satisfies the following conditions.

(i) $\mathscr{H}^i(\mathscr{K})$ is constructible for each i.

(ii) $\mathscr{H}^i(\mathscr{K}) = 0$ for $i < 0$.

(iii) $\mathrm{codim}(\mathrm{supp}\,(\mathscr{H}^i(\mathscr{K}))) \geq i$ for $i \geq 0$.

(iv) $\mathrm{codim}(\mathrm{supp}\,(\mathscr{H}^i(\mathscr{K}^*))) \geq i$ for $i \geq 0$, where $\mathscr{K}^* = \mathbf{R}\mathscr{H}om_{\mathbf{C}_X}(\mathscr{K}, \mathbf{C}_X)$.

Theorem 3 (Kashiwara, Mebkhout see [Ka]). *$\mathscr{DR}$ gives an equivalence between the abelian category of regular holonomic $\mathscr{D}_X$-Modules and that of perverse sheaves on X.*

2.3. Harish-Chandra modules

Let $G_{\mathbf{R}}$ be a connected real form of G. We fix a maximal compact subgroup $K_{\mathbf{R}}$ of $G_{\mathbf{R}}$ and denote its complexification by K.

Definition. A $\mathfrak{g}$-module M which has also a K-module structure is called a $(\mathfrak{g}, K)$-module if the following conditions hold.

(i) Any $m \in M$ is contained in a finite-dimensional K-invariant subspace M_0 and the induced homomorphism $K \to \mathrm{GL}(M_0)$ is a homomorphism of algebraic groups.

(ii) If $\mathfrak{k}$ is the Lie algebra of K, then the $\mathfrak{k}$-module structure on M obtained by differentiating the K-action coincides with the one obtained by restricting the $\mathfrak{g}$-module structure.

(iii) $k.(X.\, m) = (\mathrm{Ad}(k)X).\, (k.\, m)$ for $k \in K$, $X \in \mathfrak{g}$ and $m \in M$.

Let $\mathscr{H}(\mathfrak{g}, K)$ be the abelian category consisting of $(\mathfrak{g}, K)$-modules of finite length which have the trivial central character as $U(\mathfrak{g})$-modules. By the correspondence of Theorem 2 $\mathscr{H}(\mathfrak{g}, K)$ is equivalent to the category $\mathscr{M}(\mathfrak{g}, K)$ consisting of coherent $\mathscr{D}_X$-Modules with K-actions. We say that a coherent $\mathscr{D}_X$-Module $\mathfrak{M}$ has a K-action if an isomorphism $p^*\mathfrak{M} \simeq q^*\mathfrak{M}$ of $\mathscr{D}_{K\times X}$-Modules which satisfies the usual cocycle condition is given, where $K\times X \xrightarrow{q} X$ and $K\times X \xrightarrow{p} X$ are defined by $q(k, x)=k.x$ and $p(k, x)=x$.

In order to investigate $\mathscr{M}(\mathfrak{g}, K)$ we need the following.

Proposition 1 (Matsuki [M], see also Vogan [V] and 2.4 below).

(i) *There exist finitely many K-orbits on X.*

(ii) *For $x \in X$ let K_x be the stabilizer of x in K and $(K_x)_0$ its identity component. Then the order of any element of $K_x/(K_x)_0$ is at most 2. In particular $K_x/(K_x)_0$ is an abelian group.*

We denote the set of the K-orbits on X by $\mathscr{C}$.

Lemma 1. *For $\mathfrak{M} \in \mathscr{M}(\mathfrak{g}, K)$ any irreducible component of the characteristic variety $\mathrm{Ch}(\mathfrak{M})$ is the closure of the conormal bundle T_O^*X of some $O \in \mathscr{C}$. In particular $\mathfrak{M}$ is holonomic.*

Proof. Since $\mathrm{Ch}(\mathfrak{M})$ is an involutive subvariety of T^*X, it is sufficient to show that $\mathrm{Ch}(\mathfrak{M})$ is contained in $\coprod_{O\in\mathscr{C}} T_O^*X$. Set $M=\Gamma(X, \mathfrak{M}) \in \mathscr{M}(\mathfrak{g}, K)$. Take a finite-dimensional K-invariant subspace M_0 of M so that $M=U(\mathfrak{g})M_0$ and set $M_i=U_i(\mathfrak{g})M_0$. Then $\mathrm{gr}\, M=\bigoplus_{i\in\mathbf{Z}}(M_i/M_{i-1})$ is a finitely generated $S(\mathfrak{g})$-module and the support of the associated coherent sheaf $\widetilde{\mathrm{gr}M}$ on $\mathfrak{g}^*$ is contained in $\mathfrak{k}^\perp=\{x \in \mathfrak{g}^* \mid \langle x, \mathfrak{k}\rangle=0\}$. Let $T^*X \xrightarrow{\gamma} \mathfrak{g}^*$ be the natural map. Then we have $\mathrm{Ch}(\mathfrak{M}) \subset \gamma^{-1}(\mathrm{supp}(\widetilde{\mathrm{gr}M})) \subset \gamma^{-1}(\mathfrak{k}^\perp)=\coprod_{O\in\mathscr{C}} T_O^*X$. Here the first inclusion follows from the definition since $\mathfrak{M}=\mathscr{D}_X \otimes_{U(\mathfrak{g})} M$.

Moreover we have the following.

Proposition 2 (Beilinson-Bernstein [BB], see also Vogan [V]). *If $\mathfrak{M} \in \mathscr{M}(\mathfrak{g}, K)$, then $\mathfrak{M}$ is regular holonomic.*

Hence by Theorem 3 $\mathscr{M}(\mathfrak{g}, K)$ is equivalent to the abelian category $\mathscr{F}(\mathfrak{g}, K)$ consisting of the perverse sheaves on X with K-actions. Thus we have the following equivalence of the abelian categories:

$$(*) \quad \mathscr{H}(\mathfrak{g}, K) \simeq \mathscr{M}(\mathfrak{g}, K) \simeq \mathscr{F}(\mathfrak{g}, K).$$

Next we describe the simple objects of these categories. For $O \in \mathscr{C}$ and a one-dimensional local system (locally constant sheaf whose stalks are one-dimensional $\mathbf{C}$-vector spaces) γ on O with a K-action, let ${}^\pi\gamma$ be the DGM-extension of γ to $\bar{O}$. We also use the same notations for the zero extensions of γ and ${}^{\pi}\gamma$ to X. Then γ[-codim O] and ${}^{\pi}\gamma$[-codim O] are objects of $\mathscr{F}(\mathfrak{g}, K)$ and the latter is a simple object. Furthermore any simple object in $\mathscr{F}(\mathfrak{g}, K)$ is isomorphic to some ${}^{\pi}\gamma$[-codim O]. Hence the set of the simple objects is parametrized by

$$\mathscr{S}=\{(O, \gamma)\mid O \in \mathscr{C} \text{ and } \gamma \text{ is a } K\text{-equivariant one-dimensional local system on } O\}.$$

We remark here that the set of K-equivariant one-dimensional local systems on O is parametrized by the set of irreducible (one-dimensional) representations of $K_x/(K_x)_0$ for a fixed $x \in O$.

We denote the objects in $\mathfrak{M}(\mathfrak{g}, K)$ (resp. $\mathscr{H}(\mathfrak{g}, K)$) corresponding to γ[-codim O] and ${}^{\pi}\gamma$[-codim O] under the equivalence (*) by $\mathfrak{M}_{(O,\gamma)}$ and $\mathfrak{L}_{(O,\gamma)}$ (resp. $M_{(O,\gamma)}$ and $L_{(O,\gamma)}$). Then we have the following decomposition of the Grothendieck groups:

$$K(\mathscr{H}(\mathfrak{g}, K))=\bigoplus_{(O,\gamma)} \mathbf{Z}[M_{(O,\gamma)}]=\bigoplus_{(O,\gamma)} \mathbf{Z}[L_{(O,\gamma)}],$$
$$K(\mathscr{M}(\mathfrak{g}, K))=\bigoplus_{(O,\gamma)} \mathbf{Z}[\mathfrak{M}_{(O,\gamma)}]=\bigoplus_{(O,\gamma)} \mathbf{Z}[\mathfrak{L}_{(O,\gamma)}].$$

2.4. K-orbits on X

We give a parametrization of K-orbits on X and other informations for the convenience of the readers. The reader is referred to Matsuki [M] and Vogan [V] for the proofs and other results.

We denote the Lie algebras of $G_{\mathbf{R}}$, $K_{\mathbf{R}}$ and K by $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{k}$, respectively. Let $\mathfrak{g}_0=\mathfrak{k}_0\oplus\mathfrak{p}_0$ and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}_0$ and its complexification. Let θ be the involution on $\mathfrak{g}$ defined by $\theta(x+y)=x-y$ ($x \in \mathfrak{k}$, $y \in \mathfrak{p}$). we also denote its restriction to $\mathfrak{g}_0$ by θ.

For a θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ and a positive root system Δ^+ of $(\mathfrak{g}, \mathfrak{h}_0\otimes_{\mathbf{R}}\mathbf{C})$ let $\mathfrak{b}(\mathfrak{h}_0, \Delta^+)$ be the corresponding Borel subalgebra of $\mathfrak{g}$.

Proposition 3 (Matsuki [M]). (i) *Any Borel subalgebra of* $\mathfrak{g}$ *is* K-*conjugate to* $\mathfrak{b}(\mathfrak{h}_0, \Delta^+)$ *for some* θ-*stable Cartan subalgebra* $\mathfrak{h}_0$ *of* $\mathfrak{g}_0$ *and a positive root system* Δ^+ *of* $(\mathfrak{g}, \mathfrak{h}_0\otimes_{\mathbf{R}}\mathbf{C})$.

(ii) *Let* $\mathfrak{h}_0$ *and* $\mathfrak{h}_0'$ *be* θ-*stable Cartan subalgebras of* $\mathfrak{g}_0$. *Let* Δ^+ *and* Δ'^+ *be positive root systems of* $(\mathfrak{g}, \mathfrak{h}_0\otimes_{\mathbf{R}}\mathbf{C})$ *and* $(\mathfrak{g}, \mathfrak{h}_0'\otimes_{\mathbf{R}}\mathbf{C})$, *respectively. Then* $\mathfrak{b}(\mathfrak{h}_0, \Delta^+)$ *is* K-*conjugate to* $\mathfrak{b}(\mathfrak{h}_0', \Delta'^+)$ *if and only if there exists an element* $k \in K_{\mathbf{R}}$ *so that* $k.\mathfrak{h}_0=\mathfrak{h}_0'$ *and* $k.\mathfrak{b}(\mathfrak{h}_0, \Delta^+)=\mathfrak{b}(\mathfrak{h}_0', \Delta'^+)$.

For a θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ let $W(\mathfrak{h}_0)$ be the Weyl group of $(\mathfrak{g}, \mathfrak{h}_0\otimes_{\mathbf{R}}\mathbf{C})$. Set $W(\mathfrak{h}_0, K_{\mathbf{R}})=(N_G(\mathfrak{h}_0)\cap K_{\mathbf{R}})/(Z_G(\mathfrak{h}_0)\cap K_{\mathbf{R}})$ $(\subset W(\mathfrak{h}_0))$. Since the set of $G_{\mathbf{R}}$-conjugacy classes of Cartan subalgebras of $\mathfrak{g}_0$ and the set of $K_{\mathbf{R}}$-conjugacy classes of θ-stable Cartan subalgebras of $\mathfrak{g}_0$ are in one-to-one correspondence, we have the following.

Corollary (Matsuki [M]). *Let* $\{\mathfrak{h}_0^{(i)} \mid i \in I\}$ *be a set of representatives of the* $G_{\mathbf{R}}$*-conjugacy classes of Cartan subalgebras of* $\mathfrak{g}_0$ *so that each* $\mathfrak{h}_0^{(i)}$ *is* θ*-stable. We fix a positive root system* $\Delta^{(i)+}$ *of* $(\mathfrak{g}, \mathfrak{h}_0^{(i)} \otimes_{\mathbf{R}} \mathbf{C})$ *for each* $i \in I$. *Then the set of K-orbits on X (K-conjugacy classes of Borel subalgebras in* $\mathfrak{g}$) *is parametrized by the set* $\coprod_{i \in I} W(\mathfrak{h}_0^{(i)}, K_{\mathbf{R}}) \backslash W(\mathfrak{h}_0^{(i)})$, *and the K-conjugacy class corresponding to* $W(\mathfrak{h}_0^{(i)}, K_{\mathbf{R}})w$ *is the one containing* $\mathfrak{b}(\mathfrak{h}_0^{(i)}, w\Delta^{(i)+})$.

For the classification of the Cartan subalgebras of $\mathfrak{g}_0$ we refer the reader to Sugiura [Su] and Warner [W]. In particular, since the number of the conjugacy classes of Cartan subalgebras is finite, the number of K-orbits on X is finite.

Let $\mathfrak{h}_0$ be a θ-stable Cartan subalgebra and Δ^+ a positive root system of $(\mathfrak{g}, \mathfrak{h}_0 \otimes_{\mathbf{R}} \mathbf{C})$. Let O be the K-orbit on X containing $\mathfrak{b} = \mathfrak{b}(\mathfrak{h}_0, \Delta^+)$. We denote the Borel subgroup corresponding to $\mathfrak{b}$ by B. Then O is isomorphic to $K/K_{\mathfrak{b}}$ with $K_{\mathfrak{b}} = \{k \in K \mid k.\mathfrak{b} = \mathfrak{b}\} = K \cap B$. Note that the set of the irreducible K-equivariant local systems on O is in one-to-one correspondence with the set of irreducible representations of the component group $K_{\mathfrak{b}}/(K_{\mathfrak{b}})_0$. This group is described as follows.

Proposition 4 (see Vogan [V]). *In the above notations set* $H_{\mathbf{R}} = Z_{G_{\mathbf{R}}}(\mathfrak{h}_0)$ *and* $H = Z_G(\mathfrak{h}_0 \otimes_{\mathbf{R}} \mathbf{C})$. *Then we have*:

$$K_{\mathfrak{b}}/(K_{\mathfrak{b}})_0 = (K \cap B)/(K \cap B)_0 \simeq (K \cap H)/(K \cap H)_0 \simeq (K_{\mathbf{R}} \cap H_{\mathbf{R}})/(K_{\mathbf{R}} \cap H_{\mathbf{R}})_0$$
$$\simeq H_{\mathbf{R}}/(H_{\mathbf{R}})_0 \simeq (\mathbf{Z}/2\mathbf{Z})^N$$

for some non-negative integer N with $0 \leq N \leq \dim_{\mathbf{R}}(\mathfrak{h}_0 \cap \mathfrak{p}_0)$.

§ 3. *W*-module structures

3.1. W-module structure on $K(\mathscr{M}(\mathfrak{g}, K))$

Set $G_1 = G \times G$, $\mathfrak{g}_1 = \mathfrak{g} \oplus \mathfrak{g}$ and $K_1 = \Delta G = \{(g, g) \in G_1 \mid g \in G\}$. We first consider $\mathscr{M}(\mathfrak{g}_1, K_1) = \mathscr{M}(\mathfrak{g} \oplus \mathfrak{g}, \Delta G)$. The flag variety of G_1 is $X \times X$, where X is the flag variety of G, and its decomposition into ΔG-orbits is given by $X \times X = \coprod_{w \in W} O(w)$, where W is the Weyl group of G and $O(w) = \Delta G.(eB, wB)$. Here we identify X with G/B for a fixed Borel subgroup B. Since each $O(w)$ is simply-connected, we have:

$$K(\mathscr{M}(\mathfrak{g} \oplus \mathfrak{g}, \Delta G)) = \bigoplus_{w \in W} \mathbf{Z}[\mathfrak{M}_w] = \bigoplus_{w \in W} \mathbf{Z}[\mathfrak{L}_w],$$

with $\mathfrak{M}_w = \mathfrak{M}_{(O(w),1)}$ and $\mathfrak{L}_w = \mathfrak{L}_{(O(w),1)}$.

Let $X \times X \times X \xrightarrow{p_{ij}} X \times X$ $(1 \leq i < j \leq 3)$ be the natural projection. For

$\mathfrak{M}_1, \mathfrak{M}_2 \in \mathcal{M}(\mathfrak{g}\oplus\mathfrak{g}, \Delta G)$ we have

$$\mathcal{H}^i\left(\int_{p_{13}} (p_{12}^*\mathfrak{M}_1) \overset{L}{\underset{\mathcal{O}_{X\times X\times X}}{\otimes}} (p_{23}^*\mathfrak{M}_2)\right) \in \mathcal{M}(\mathfrak{g}\oplus\mathfrak{g}, \Delta G)$$

for each i. Hence we can define a multiplication on $K(\mathcal{M}(\mathfrak{g}\oplus\mathfrak{g}, \Delta G))$ by

$$[\mathfrak{M}_1].[\mathfrak{M}_2]=\sum_i(-1)^i\left[\mathcal{H}^i\left(\int_{p_{13}} (p_{12}^*\mathfrak{M}_1) \overset{L}{\underset{\mathcal{O}_{X\times X\times X}}{\otimes}} (p_{23}^*\mathfrak{M}_2)\right)\right].$$

Proposition 5 (see Lusztig-Vogan [LV] and Springer [Sp]). *The above multiplication defines a ring structure on $K(\mathcal{M}(\mathfrak{g}\oplus\mathfrak{g}, \Delta G))$ so that*

$$\begin{array}{ccc} K(\mathcal{M}(\mathfrak{g}\oplus\mathfrak{g}, \Delta G)) & \simeq & \mathbf{Z}[W]. \\ \cup\!\!\!\!\cup & & \cup\!\!\!\!\cup \\ [\mathfrak{M}_w] & \longleftrightarrow & w \end{array}$$

Remark 1. By the Riemann-Hilbert correspondence one can translate this proposition into topological language, and this is actually the approach given in [LV] and [Sp]. Since they consider the Hecke algebra of W, we must specialize the indeterminant q to 1 to get the above result.

Now we define a W-action on $K(\mathcal{M}(\mathfrak{g}, K))$. By Proposition 5 we have only to define an action of the ring $K(\mathcal{M}(\mathfrak{g}\oplus\mathfrak{g}, \Delta G))$ on $K(\mathcal{M}(\mathfrak{g}, K))$. Let $X\times X\xrightarrow{q_i}X$ $(i=1, 2)$ be the projection onto the i-th factor. For $\mathfrak{M} \in \mathcal{M}(\mathfrak{g}\oplus\mathfrak{g}, \Delta G)$ and $\mathfrak{N} \in \mathcal{M}(\mathfrak{g}, K)$ we have

$$\mathcal{H}^i\left(\int_{q_1} \mathfrak{M} \overset{L}{\underset{\mathcal{O}_{X\times X}}{\otimes}} (q_2^*\mathfrak{N})\right) \in \mathcal{M}(\mathfrak{g}, K)$$

for each i.

Proposition 6 (Lusztig-Vogan [LV]). *An action of $K(\mathcal{M}(\mathfrak{g}\oplus\mathfrak{g}, \Delta G))$ on $K(\mathcal{M}(\mathfrak{g}, K))$ is defined by*

$$[\mathfrak{M}].[\mathfrak{N}]=\sum_i(-1)^i\left[\mathcal{H}^i\left(\int_{q_1} \mathfrak{M} \overset{L}{\underset{\mathcal{O}_{X\times X}}{\otimes}} (q_2^*\mathfrak{N})\right)\right],$$

where $\mathfrak{M} \in \mathcal{M}(\mathfrak{g}\oplus\mathfrak{g}, \Delta G)$ and $\mathfrak{N} \in \mathcal{M}(\mathfrak{g}, K)$.

Hence $K(\mathcal{M}(\mathfrak{g}, K))$ is endowed with a W-module structure.

In particular $K(\mathcal{M}(\mathfrak{g}\oplus\mathfrak{g}, \Delta G))$ $(\simeq\mathbf{Z}[W])$ has a $W\times W$-module structure. Note that this action of $W\times W$ coincides with the two-sided regular representation of $W\times W$ on $\mathbf{Z}$[W].

For a simple reflection s of W let X_s be the variety of semisimple-rank 1 parabolic subalgebras of $\mathfrak{g}$ corresponding to s. Write $X\xrightarrow{\pi_s}X_s$ for the natural map.

Proposition 7. *For $\mathfrak{M} \in \mathscr{M}(\mathfrak{g}, K)$ we have:*

$$s.[\mathfrak{M}]=[\mathfrak{M}]+\sum_i (-1)^i \left[\mathscr{H}^i\left(\mathbf{L}\pi_s^* \int_{\pi_s} \mathfrak{M}\right)\right].$$

This is proved by the same method as in the proof of Proposition 5 in [KT], so we omit the proof.

3.2. W-module structure on $\bigoplus_O \mathbf{Z}[\overline{T_O^* X}]$

We first review the Springer representations of W. We follow the approach of Lusztig [L] using DGM-extensions (see also Borho-MacPherson [BM]).

Set $\tilde{\mathfrak{g}}=\{(x, \mathfrak{b}) \in \mathfrak{g}\times X \mid x \in \mathfrak{b}\}$ and let $\tilde{\mathfrak{g}}\xrightarrow{p}\mathfrak{g}$ be the natural map. We denote the set of regular semisimple elements (resp. nilpotent elements) in $\mathfrak{g}$ by $\mathfrak{g}_{\mathrm{rs}}$ (resp. N) and set $\tilde{\mathfrak{g}}_{\mathrm{rs}}=p^{-1}(\mathfrak{g}_{\mathrm{rs}})$ (resp. $\tilde{N}=p^{-1}(N)$). Let $\tilde{\mathfrak{g}}_{\mathrm{rs}}\xrightarrow{p_{\mathrm{rs}}}\mathfrak{g}_{\mathrm{rs}}$ and $\tilde{N}\xrightarrow{p_N}N$ be the restrictions of p. Since p_{rs} is a W-principal bundle, we have an action of W on the local system $p_{\mathrm{rs}*}(\mathbf{Q}_{\tilde{\mathfrak{g}}_{\mathrm{rs}}})$ on $\mathfrak{g}_{\mathrm{rs}}$, where $\mathbf{Q}_{\tilde{\mathfrak{g}}_{\mathrm{rs}}}$ is the constant sheaf on $\tilde{\mathfrak{g}}_{\mathrm{rs}}$ whose stalks are the rational number field $\mathbf{Q}$. By the functoriality of the DGM-extension we have an action of W on ${}^{\pi}(p_{\mathrm{rs}*}(\mathbf{Q}_{\tilde{\mathfrak{g}}_{\mathrm{rs}}}))$. Since ${}^{\pi}(p_{\mathrm{rs}*}(\mathbf{Q}_{\tilde{\mathfrak{g}}_{\mathrm{rs}}}))$ is isomorphic to $\mathbf{R}p_*(\mathbf{Q}_{\tilde{\mathfrak{g}}})$ as an object in the derived category (Lusztig [L]) and since $\mathbf{R}p_{N*}(\mathbf{Q}_{\tilde{N}})$ is isomorphic to $\mathbf{R}p_*(\mathbf{Q}_{\tilde{\mathfrak{g}}})\mid N$ by the base change theorem, we have an action of W on $\mathbf{R}p_{N*}(\mathbf{Q}_{\tilde{N}})$.

For $x \in N$ set $X_x=p^{-1}(x)=\{\mathfrak{b} \in X \mid x \in \mathfrak{b}\}$. Then the action of W on $\mathbf{R}p_*(\mathbf{Q}_{\tilde{N}})$ induces its action on $H^i(X_x, \mathbf{Q})=R^i p_{N*}(\mathbf{Q}_{\tilde{N}})_x$ for each i. This is the Springer representation of W in the usual sence.

For $O \in \mathscr{C}$ we set $Z_O=\overline{T_O^* X}$ and $Z=\bigcup_{O\in\mathscr{C}} Z_O$. Z is an algebraic variety of pure dimension $d=\dim X$. We identify T^*X with $\tilde{N}$ via the Killing form on $\mathfrak{g}$. Then we have $Z=p_N^{-1}(N(\mathfrak{p}))$ with $N(\mathfrak{p})=N\cap\mathfrak{p}$. Hence we have an action of W on $H_c^i(N(\mathfrak{p}), \mathbf{R}p_{N*}(\mathbf{Q}_{\tilde{N}})\mid N(\mathfrak{p}))=H_c^i(Z, \mathbf{Q})$. Since the dual space of the top cohomology group $H_c^{2d}(Z, \mathbf{Q})$ has a natural basis $\{[\overline{T_O^* X}]\}_{O\in\mathscr{C}}$, we have a W-action on the vector space $(H_c^{2d}(Z, \mathbf{Q}))^*=\bigoplus_{O\in\mathscr{C}} \mathbf{Q}[\overline{T_O^* X}]$.

Remark 2. In order to define a W-action we can use the method of Kazhdan-Lusztig [KL] in place of the above approach. The coincidence of these two approaches is proved in Hotta [H1; Appendix] though it is not exactly of this form.

Next we review a geometric description of the action of simple reflections of W on the space $(H_c^{2d}(Z, \mathbf{Q}))^*$ due to Hotta [H1], [H2]. We fix a simple reflection s. We define natural maps ρ_s, ϖ_s and τ_s by the following commutative diagram:

$$\begin{tikzcd}
T^*X_s \times_{X_s} X \arrow[r, hook, "\rho_s"] \arrow[d, "\varpi_s"'] & T^*X = \tilde{N} \arrow[r, hook] & X \times N \arrow[d, "\tau_s"] \\
T^*X_s \arrow[rr, hook] & & X_s \times N.
\end{tikzcd}$$

For $O \in \mathscr{C}$ we say that O is s-vertical (resp. s-horizontal) if $\tau_s^{-1}(\tau_s(Z_O)) = Z_O$ (resp. $\tau_s^{-1}(\tau_s(Z_O)) \supsetneqq Z_O$). Let $\mathscr{C}_v^s$ and $\mathscr{C}_h^s$ be the set of s-vertical and s-horizontal K-orbits on X, respectively. Set $O_s = \pi_s(O)$ for $O \in \mathscr{C}$. The following is obvious.

Lemma 2.

(i) *O is s-vertical if and only if $\pi_s^{-1}(\pi_s(x)) \cap O$ is open dense in $\pi_s^{-1}(\pi_s(x))$ for any $x \in O$.*

(ii) *$X_s = \bigcup_{O \in \mathscr{C}_v^s} O_s$ (disjoint union).*

(iii) *$\rho_s^{-1}(Z) = \bigcup_{O \in \mathscr{C}_v^s} Z_O$ (irreducible decomposition).*

(iv) *$\varpi_s(Z_O) = \overline{T^*_{O_s} X_s}$ for $O \in \mathscr{C}_v^s$.*

Proposition 8 (Hotta [H1], [H2]). *Let O be a K-orbit on X.*

(i) *If O is s-vertical, we have*

$$s.[Z_O] = -[Z_O].$$

(ii) *If O is s-horizontal, we have*

$$s.[Z_O] = [Z_O] + (\varpi_s^* \circ \varpi_{s*} \circ \rho_s^*)([Z_O]),$$

where ϖ_s^*, ρ_s^* and ϖ_{s*} are pull-back and direct image of algebraic cycles. Furthermore $(\varpi_s^* \circ \varpi_{s*} \circ \rho_s^*)([Z_O]) \in \bigoplus_{O'} \mathbf{Z}_{\geq 0}\,[Z_{O'}]$, where O' is running through $O' \in \mathscr{C}_v^s$ with $O' \subset \overline{\pi_s^{-1}(\pi_s(O))}$.

By the above proposition we see that the $\mathbf{Z}$-lattice $\bigoplus_{O \in \mathscr{C}} \mathbf{Z}[\overline{T_O^* X}]$ in $(H_c^{2d}(Z, \mathbf{Q}))^*$ is W-invariant. Hence we have an action of W on $\bigoplus_{O \in \mathscr{C}} \mathbf{Z}[\overline{T_O^* X}]$.

3.3. We prove the following in this subsection

Theorem 1 (repeated). *The $\mathbf{Z}$-linear homomorphism*

$$K(\mathscr{M}(\mathfrak{g}, K)) \xrightarrow{\mathbf{Ch}} \bigoplus_{O \in \mathscr{C}} \mathbf{Z}[\overline{T_O^* X}]$$

is W-equivariant.

Remark 3. Set $\partial O=\bar{O}-O$. Since $\mathfrak{M}_{(O,\gamma)}|X-\partial O$ and $\mathfrak{L}_{(O,\gamma)}|X-\partial O$ are isomorphic to $\mathscr{H}_O^{\operatorname{codim} O}(\mathcal{O}_{X-\partial O})\otimes_{\mathbf{C}_{X-\partial O}}\gamma$, we see that $\mathbf{Ch}(\mathfrak{M}_{(O,\gamma)})$ and $\mathbf{Ch}(\mathfrak{L}_{(O,\gamma)})$ belong to $[\overline{T_O^*X}]+(\bigoplus_{O'\subsetneqq\bar{O}}\mathbf{Z}_{\geq 0}[\overline{T_{O'}^*X}])$. Hence $\mathbf{Ch}$ is surjective.

Proof of Theorem 1. It is sufficient to show $\mathbf{Ch}(s.[\mathfrak{M}])=s.\mathbf{Ch}(\mathfrak{M})$ for any simple reflection s and $\mathfrak{M}\in\mathscr{M}(\mathfrak{g},K)$. By [Sa] (see [KT; Theorem 7]) there exist integers $m_s(O',O)$ for $O\in\mathscr{C}$ and $O'\in\mathscr{C}_v^s$ so that

$$\mathbf{Ch}(\mathfrak{M})=\sum_{O\in\mathscr{C}}n_O[\overline{T_O^*X}]$$

implies

$$\mathbf{Ch}\Big(\int_{\pi_s}\mathfrak{M}\Big):=\sum_i(-1)^i\,\mathbf{Ch}\Big(\mathscr{H}^i\Big(\int_{\pi_s}\mathfrak{M}\Big)\Big)=\sum_{O\in\mathscr{C}}n_O\Big(\sum_{O'\in\mathscr{C}_v^s}m_s(O',O)[\overline{T_{O'_s}^*X_s}]\Big).$$

Furthermore we have

$$\sum_{O'\in\mathscr{C}_v^s}m_s(O',O)[\overline{T_{O'_s}^*X_s}]=(\varpi_{s*}\circ\rho_s^*)([\overline{T_O^*X}])$$

for $O\in\mathscr{C}_h^s$. Hence we have

$$\mathbf{Ch}\Big(\mathbf{L}\pi_s^*\int_{\pi_s}\mathfrak{M}\Big)=\varpi_s^*\Big(\mathbf{Ch}\Big(\int_{\pi_s}\mathfrak{M}\Big)\Big)=\sum_{O\in\mathscr{C}}n_O\Big(\sum_{O'\in\mathscr{C}_v^s}m_s(O',O)[\overline{T_{O'}^*X}]\Big).$$

Thus by Proposition 7

$$\begin{aligned}\mathbf{Ch}(s.[\mathfrak{M}])=&\sum_{O\in\mathscr{C}_v^s}n_O\Big([\overline{T_O^*X}]+\sum_{O'\in\mathscr{C}_v^s}m_s(O',O)[\overline{T_{O'}^*X}]\Big)\\&+\sum_{O\in\mathscr{C}_h^s}n_O([\overline{T_O^*X}]+(\varpi_s^*\circ\varpi_{s*}\circ\rho_s^*)([\overline{T_O^*X}])).\end{aligned}$$

On the other hand by Proposition 8

$$\begin{aligned}s.\mathbf{Ch}([\mathfrak{M}])=&\sum_{O\in\mathscr{C}_v^s}n_O(-[\overline{T_O^*X}])\\&+\sum_{O\in\mathscr{C}_h^s}n_O([\overline{T_O^*X}]+(\varpi_s^*\circ\varpi_{s*}\circ\rho_s^*)([\overline{T_O^*X}])).\end{aligned}$$

Thus we have only to prove that if O and O' are distinct elements of $\mathscr{C}_v^s$, then $m_s(O',O)=0$ and $m_s(O,O)=-2$.

For $O\in\mathscr{C}_v^s$ set $\hat{O}=\pi_s^{-1}(\pi_s(O))$. If we set $\hat{K}=\langle K,P\rangle$, the decomposition of X into $\hat{K}$-orbit is given by $X=\coprod_{O\in\mathscr{C}_v^s}\hat{O}$, and hence this decomposition satisfies the Whitney condition. For $O\in\mathscr{C}_v^s$ we define $\mathfrak{M}_O\in\mathscr{M}(\mathfrak{g},K)$ by $\mathscr{D}\mathscr{R}(\mathfrak{M}_O)=\mathbf{C}_{\hat{O}}[-\operatorname{codim} O]$. Then

$$\mathbf{Ch}(\mathfrak{M}_O)\in[\overline{T_O^*X}]+\sum_{O'}\mathbf{Z}_{\geq 0}[\overline{T_{O'}^*X}],$$

where O' is running through $O' \in \mathscr{C}_v^s$ with $O' \subset \partial O$. Hence using induction on dim O we see that it is sufficient to show

$$\mathbf{Ch}\left(\mathbf{L}\pi_s^* \int_{\pi_s} \mathfrak{M}_O\right) = -2\mathbf{Ch}(\mathfrak{M}_O)$$

for $O \in \mathscr{C}_v^s$. By the Riemann-Hilbert correspondence we have

$$\begin{aligned}\mathscr{D}\mathscr{R}\left(\mathbf{L}\pi_s^* \int_{\pi_s} \mathfrak{M}_O\right) &= \pi_s^{-1}(\mathbf{R}\pi_{s*}(\mathscr{D}\mathscr{R}(\mathfrak{M}_O)))[1] \\ &= \pi_s^{-1}(\mathbf{R}\pi_{s*}(\mathbf{C}_{\hat{O}}))[-\operatorname{codim} O + 1].\end{aligned}$$

Since π_s is a $\mathbf{P}^1$-bundle, we have the following triangle

$$\begin{tikzcd} & \mathbf{C}_{\hat{O}} \arrow[dl, "+1"'] & \\ \mathbf{C}_{\hat{O}}[-2] \arrow[rr] & & \pi_s^{-1}(\mathbf{R}\pi_s^*(\mathbf{C}_{\hat{O}})). \arrow[ul] \end{tikzcd}$$

By the Riemann-Hilbert correspondence we have

$$\begin{tikzcd} & \mathfrak{M}_O[1] \arrow[dl, "+1"'] & \\ \mathfrak{M}_O[-1] \arrow[rr] & & \mathbf{L}\pi_s^* \int_{\pi_s} \mathfrak{M}_O. \arrow[ul] \end{tikzcd}$$

Hence

$$\mathbf{Ch}\left(\mathbf{L}\pi_s^* \int_{\pi_s} \mathfrak{M}_O\right) = \mathbf{Ch}(\mathfrak{M}_O[1]) + \mathbf{Ch}(\mathfrak{M}_O[-1]) = -2\mathbf{Ch}(\mathfrak{M}_O)$$

and we are done.

§ 4. Complements

4.1. We describe the W-module structure of $K(\mathscr{M}(\mathfrak{g}, K))$ and $(H_c^{2d}(Z, \mathbf{Q}))^*$ more explicitly.

The following is a generalization of a result of Kazhdan-Lusztig [KL; (6.1) and (6.2)]. Since the proof is the same as that of [KL], we omit it.

Proposition 9. *As a W-module, we have*

$$(H_c^{2d}(Z, \mathbf{Q}))^* \simeq \bigoplus_{x \in K \backslash \mathscr{N}(\mathfrak{p})} (H_{2d_x}(X_x, \mathbf{Q}))^{C_K(x)},$$

where $d_x = \dim X_x$ *and* $C_K(x) = Z_K(x)/(Z_K(x))_0$.

Hence suitable information on the K-conjugacy classes of nilpotent elements in $\mathfrak{p}$ and the Springer correspondence of the group G give the irreducible decomposition of $(H_c^{2d}(Z, \mathbf{Q}))^*$ as a W-module.

Next we consider $K(\mathscr{M}(\mathfrak{g}, K))$. The explicit description of the action of a simple reflection with respect to the basis $\{[\mathfrak{M}_{(O,\gamma)}] \mid (O, \gamma) \in \mathscr{S}\}$ is given in [LV] (see also [V]). We include it here for the convenience of the readers.

Lemma 3 (Lusztig-Vogan [LV]). Fix a simple reflection s, $(O, \gamma) \in \mathscr{S}$ and $x \in O$. Set $\hat{O} = \pi_s^{-1}(\pi_s(O))$ and $L_x^s = \pi_s^{-1}(\pi_s(x))$.

(a) If $L_x^s \subset O$, then $s.[\mathfrak{M}_{(O,\gamma)}] = -[\mathfrak{M}_{(O,\gamma)}]$.

(b1) If $L_x^s \cap O = \{x\}$ and $O' = \hat{O} - O$ is a single K-orbit, then there exists a unique locally constant extension $\hat{\gamma}$ of γ to $\hat{O}$ and $s.[\mathfrak{M}_{(O,\gamma)}] = [\mathfrak{M}_{(O',\gamma')}]$ with $\gamma' = \hat{\gamma} \mid O'$.

(b2) If $L_x^s \cap O = L_x^s - \{\text{point}\}$, then $O' = \hat{O} - O$ is a single K-orbit, there exists a unique locally constant extension $\hat{\gamma}$ of γ to $\hat{O}$ and $s.[\mathfrak{M}_{(O,\gamma)}] = [\mathfrak{M}_{(O',\gamma')}]$ with $\gamma' = \hat{\gamma} \mid O'$.

(c1) If $L_x^s \cap O = \{x, y\}$, then $O' = \hat{O} - O$ is a single K-orbit, γ has two distinct extension $\hat{\gamma}_1$, $\hat{\gamma}_2$ to $\hat{O}$ and $s.[\mathfrak{M}_{(O,\gamma)}] = -[\mathfrak{M}_{(O,\gamma)}] + [\mathfrak{M}_{(O',\gamma_1')}] + [\mathfrak{M}_{(O',\gamma_2')}]$ with $\gamma_i' = \hat{\gamma}_i \mid O'$.

(c2) If $L_x^s \cap O = L_x^s - \{\text{two points in one } K\text{-orbit}\}$, then γ has at most one extension to $\hat{O}$. In the case γ has an extension $\hat{\gamma}$ to $\hat{O}$, $\hat{\gamma} \mid \hat{O} - O$ has a unique extension $\hat{\gamma}_*$ to $\hat{O}$ different from $\hat{\gamma}$ and $s.[\mathfrak{M}_{(O,\gamma)}] = [\mathfrak{M}_{(O,\gamma_*)}]$ with $\gamma_* = \hat{\gamma}_* \mid O$. In the case γ does not have an extension, $s.[\mathfrak{M}_{(O,\gamma)}] = [\mathfrak{M}_{(O,\gamma)}]$.

(d1) If $L_x^s \cap O = \{x\}$ and $\hat{O} - O$ is a union of two orbits O' and O'' with $\dim O = \dim O'' = \dim O' - 1$, then γ has a unique extension $\hat{\gamma}$ to $\hat{O}$ and $s.[\mathfrak{M}_{(O,\gamma)}] = [\mathfrak{M}_{(O',\gamma')}] - [\mathfrak{M}_{(O'',\gamma'')}]$ with $\gamma' = \hat{\gamma} \mid O'$ and $\gamma'' = \hat{\gamma} \mid O''$.

(d2) If $L_x^s \cap O = L_x^s - \{\text{two points in two } K\text{-orbits}\}$, then $s.[\mathfrak{M}_{(O,\gamma)}] = [\mathfrak{M}_{(O,\gamma)}]$.

Now we give an alternative description of the W-module $(H_c^{2d}(Z, \mathbf{Q}))^* \simeq (K(\mathscr{M}(\mathfrak{g}, K))/\mathrm{Ker}\,\mathbf{Ch}) \otimes_{\mathbf{Z}} \mathbf{Q}$ different from that of Proposition 9.

Proposition 10. *Let $\{\mathfrak{h}_0^{(i)}\}_{i \in I}$ be a set of representatives of the conjugacy classes of Cartan subalgebras of $\mathfrak{g}_0$ so that $\theta(\mathfrak{h}_0^{(i)}) = \mathfrak{h}_0^{(i)}$. We fix for each $i \in I$ a positive root system $\Delta^{(i)+}$ of the root system $\Delta^{(i)}$ of $(\mathfrak{g}, \mathfrak{h}_0^{(i)} \otimes_{\mathbf{R}} \mathbf{C})$. We identify Cartan subalgebras $\mathfrak{h}_0^{(i)} \otimes_{\mathbf{R}} \mathbf{C}$ via the Borel subalgebras $\mathfrak{b}(\mathfrak{h}_0^{(i)}, \Delta^{(i)+})$ and regard W as its Weyl group. Then there exist linear characters $W(\mathfrak{h}_0^{(i)}, K_{\mathbf{R}}) \xrightarrow{\chi_i} \{\pm 1\}$ so that*

$$(K(\mathscr{M}(\mathfrak{g}, K))/\mathrm{Ker}\,\mathbf{Ch}) \underset{\mathbf{Z}}{\otimes} \mathbf{Q} \simeq \bigoplus_{i \in I} \mathrm{Ind}_{W(\mathfrak{h}_0^{(i)}, K_{\mathbf{R}})}^{W}(\chi_i).$$

Remark 4. If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are Cartan subalgebras of $\mathfrak{g}$ contained in Borel subalgebras $\mathfrak{b}_1$ and $\mathfrak{b}_2$, respectively, then there exists $g \in G$ so that $\mathrm{Ad}(g)\mathfrak{h}_1=\mathfrak{h}_2$ and $\mathrm{Ad}(g)\mathfrak{b}_1=\mathfrak{b}_2$. Moreover $\mathrm{Ad}(g)|\mathfrak{h}_1: \mathfrak{h}_1 \to \mathfrak{h}_2$ is uniquely determined by $\mathfrak{b}_1$ and $\mathfrak{b}_2$.

(Sketch of the proof of Proposition 10)

For $O \in \mathscr{C}$ set $A_O=\sum_{(O,\tau)\in\mathscr{S}}[\mathfrak{M}_{(O,\tau)}]$. Then we see from Lemma 3 that $V=\bigoplus_{O\in\mathscr{C}}\mathbf{Q}A_O$ is W-invariant. By Remark 3 V is isomorphic to

$$(K(\mathscr{M}(\mathfrak{g}, K))/\mathrm{Ker}\,\mathbf{Ch})\underset{\mathbf{Z}}{\otimes}\mathbf{Q}\simeq(H_c^{2d}(Z,\mathbf{Q}))^*.$$

Let $\mathscr{C}_i$ be the set of $O \in \mathscr{C}$ such that $\mathfrak{b}(\mathfrak{h}_0^{(i)}, w\Delta^+) \in O$ for some $w \in W$. For $i, j \in I$ we write $i \leq j$ if $i=j$ or $\dim(\mathfrak{h}_0^{(j)} \cap \mathfrak{p}_0) < \dim(\mathfrak{h}_0^{(i)} \cap \mathfrak{p}_0)$. Then

$$\bar{V}^{(i)}=\bigoplus_{j\leq i}\Big(\bigoplus_{O\in\mathscr{C}_j}\mathbf{Z}A_O\Big)$$

is W-invariant and if we set

$$V^{(i)}=\bar{V}^{(i)}\Big/\sum_{j\leq i, j\neq i}\bar{V}^{(j)},$$

we have $V\simeq\bigoplus_{i\in I}V^{(i)}$ as a W-module. We fix $i \in I$. For $w \in W$ let a_w be the class of A_O in $V^{(i)}$, where O is the K-orbit on X containing $\mathfrak{b}(\mathfrak{h}_0^{(i)}, w\Delta^{(i)+})$. Then we have $V^{(i)}=\bigoplus_{w\in W(\mathfrak{h}_0^{(i)},K_{\mathbf{R}})\backslash W}\mathbf{Q}a_w$. By Lemma 3 and the arguments as in [V] we see that $s.\,a_w=\pm a_{ws}$. Thus

$$V^{(i)}\simeq\mathrm{Ind}_{W(\mathfrak{h}_0,K_{\mathbf{R}})}^{W}(\chi_i)$$

for some χ_i and we are done.

Remark 5. χ_i is uniquely determined by $\chi_i(w)a_e=w.\,a_e$ for $w \in W(\mathfrak{h}_0^{(i)}, K_{\mathbf{R}})$. In particular, if $\mathfrak{h}_0^{(i)}$ is a fundamental Cartan subalgebra (i.e. $\dim(\mathfrak{h}_0^{(i)} \cap \mathfrak{k}_0)$ is maximal) we see that χ_i coincides with the restriction of the sign representation sgn of W to $W(\mathfrak{h}_0^{(i)}, K_{\mathbf{R}})=W(K)=$(the Weyl group of K).

4.2. As an example we treat the case when $\mathfrak{g}_0$ has only one conjugacy class of Cartan subalgebras. By Propositions 9 and 10 we have the following.

Proposition 11. *If $\mathfrak{g}_0$ has a unique conjugacy class of Cartan subalgebras, then we have*

$$\mathrm{Ind}_{W(K)}^{W}(1)\otimes\mathrm{sgn}\simeq\bigoplus_{x\in K\backslash N(\mathfrak{p})}H_{2d_x}(X_x, \mathbf{Q})^{C_K(x)}.$$

In the case $\mathfrak{g}_0$ is a complex semisimple Lie algebra viewed as a real semisimple Lie algebra, the above formula is just

$$\mathbf{Q}[W] \simeq \bigoplus_{x \in G\backslash N} (H_{2d_x}(X_x, \mathbf{Q}) \otimes H_{2d_x}(X_x, \mathbf{Q}))^{C_G(x)}$$

with $C_G(x) = Z_G(x)/(Z_G(x))_0$, from which the completeness theorem of Springer is obtained (see [KL]).

We see by [Su] that if $\mathfrak{g}_0$ is a non-compact real form of a complex simple Lie algebra, $\mathfrak{g}_0$ has only one conjugacy class of Cartan subalgebra if and only if the pair $(\mathfrak{g}, \mathfrak{k})$ is one of the following three types:

(I) $\mathfrak{g} = A_{2n-1}$, $\mathfrak{k} = C_n$ $(n \geq 2)$,
(II) $\mathfrak{g} = D_n$, $\mathfrak{k} = B_{n-1}$ $(n \geq 4)$,
(III) $\mathfrak{g} = E_6$, $\mathfrak{k} = F_4$.

Remark 6. Note that in the cases (I), (II) and (III) the automorphism θ of $\mathfrak{g}$ is obtained by extending the symmetry of the Dynkin diagram of $\mathfrak{g}$ using the usual presentation of $\mathfrak{g}$ by generators and relations.

In order to write down the formula in Proposition 11 explicitly, we need a classification of K-conjugacy classes of nilpotent elements in $\mathfrak{p}$. Consider the natural map $K\backslash N(\mathfrak{p}) \xrightarrow{\Phi} G\backslash N$. Then we have the following.

Proposition 12 (see [Se]). *For $e \in N$ the following conditions are equivalent.*

(i) *e is conjugate to an element of $N(\mathfrak{p})$ under the action of G.*

(ii) *Let n_i be the number attached to the vertex i of the weighted Dynkin diagram of e ($n_i = 0$, 1 or 2). Then in the Satake diagram of $\mathfrak{g}_0$ we have*:

$$n_i = n_j \quad \textit{if } \underset{i}{\circ} \longleftrightarrow \underset{j}{\circ},$$
$$n_i = 0 \quad \textit{if } \underset{i}{\bullet}.$$

Proposition 13. *If $\mathfrak{g}_0$ has a unique conjugacy class of Cartan subalgebras, then the map Φ is injective.*

Proposition 12 is stated in [Se] as a theorem of Antonyon and the proof is given for the case $\mathfrak{g}_0$ is a normal real form. But the proof in [Se] for normal forms applies to the general case under some modification. Proposition 13 is shown by using the results of [Ko] and [KR] if we note Remark 6.

Remark 7. We can prove more generally that the natural maps $K\backslash\mathfrak{p} \to G\backslash\mathfrak{g}$ and $K\backslash\mathfrak{k} \to G\backslash\mathfrak{g}$ are injective if $\mathfrak{g}_0$ has only one conjugacy class of Cartan subalgebras.

Using Propositions 12, 13 and the Springer correspondence (see [Sh] and [ALS]) we can write down the formula in Proposition 11 explicitly.

In the case (I) we have $\mathrm{Ind}_{W(C_n)}^{S_{2n}}(1)=\bigoplus \chi_\sigma$, where σ is running through the partitions of 2n whose parts are even and χ_σ is the irreducible representation corresponding to σ. A direct proof of this formula is given in [T]. I understand that this was originally conjectured by N. Iwahori.

In the case (II) we have $\mathrm{Ind}_{W(B_{n-1})}^{W(D_n)}(1)=1\oplus\chi$, where χ is the irreducible representation of $W(D_n)$ corresponding to the pair of partitions $((n-1>1), \emptyset)$ in the usual conventions. But this is trivial.

In the case (III) we have $\mathrm{Ind}_{W(F_4)}^{W(E_6)}(1)=1_p\oplus 20_p\oplus 24_p$.

Remark 8. Using the recent result of Matsuki describing the closure relations of K-orbits on X, we can construct for each K-orbit O a non-singular variety Y with a K-action and a K-equivariant proper surjective map $Y\xrightarrow{f}\bar{O}$ which is generically finite. This is an obvious generalization of the desingularization of Schubert varieties given in [D]. If $\mathfrak{g}_0$ has only one conjugacy class of Cartan subalgebras, the above f is birational and in this case we can calculate the dimension of each stalk of the intersection cohomology sheaf of the closure of any K-orbit by the method given in [Sp] (see [LV]).

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