

CHAPTER 6

Integration on Locally Compact Spaces According to Bourbaki

For convenience of reference the following two references will be abbreviated: Bourbaki (1965) by “B,” and Taylor (1965, 1985) by “T.” The reader is reminded of the symbol $\mathcal{K}(X)$ introduced in Section 2.2 for the family of real valued continuous functions with compact support on the locally compact (l.c.) space X .

6.1. The Daniell method. There are basically two very different theories of measure and integration on a given space X . In the first one, which will be called “classical” here, the starting point is a family of subsets of X , called measurable, on which a measure is defined as a set function with certain properties. This theory is documented very well in Halmos (1950). The second approach is due to Daniell (1917–18, 1919–20) and consists of first defining the integral as a linear functional, with a certain monotonicity and continuity property, on a family of “nice” functions; then extending the integral to a wider family of functions called “integrable,” and finally defining measurable functions and sets. Thus, in the Daniell approach integrable functions and their integrals come first, measurable sets come last, in contrast to the classical approach. An advantage of the Daniell approach is that certain properties of the integral already follow from

the integrals of the “nice” functions, which are easier to handle than arbitrary integrable functions. For instance, this will be used in the definition of a product measure. If X is l.c., then the “nice” functions will be taken to be the family $\mathcal{K}(X)$. The Bourbaki theory of integration follows essentially the Daniell approach, although with a variation. Below we shall very briefly outline the Daniell method, without details or proofs. For a full account the reader is referred to the excellent expository account by Taylor (1965, 1985), Chapter 6.

The general Daniell method does not assume any structure on X , but assumes that there is a nonempty family \mathcal{F} of real valued functions on X that is closed under linear operations (sums and multiplication by real numbers) and is closed under the formation of the maximum and the minimum of two functions. Such a family is called a **vector lattice**. On \mathcal{F} is defined a linear functional with some additional properties, as follows.

6.1.1. DEFINITION. *Let \mathcal{F} be a vector lattice of functions $X \rightarrow R$, then a function $I : \mathcal{F} \rightarrow R$ is called an **elementary integral** on \mathcal{F} if it satisfies*

$$(6.1.1) \quad I(c_1 f_1 + c_2 f_2) = c_1 I(f_1) + c_2 I(f_2), \quad f_1, f_2 \in \mathcal{F}, \quad c_1, c_2 \in R,$$

$$(6.1.2) \quad I(f) \geq 0 \quad \text{if} \quad f \geq 0,$$

$$(6.1.3) \quad I(f_n) \rightarrow 0 \quad \text{if} \quad f_n \downarrow 0.$$

All the usual elementary properties of an integral are satisfied by I . The condition (6.1.2) implies monotonicity of I , whereas (6.1.3) is a substitute for continuity. If X is l.c., then \mathcal{F} will be taken to be $\mathcal{K}(X)$. For instance, if $X = R^n$, then an example of an elementary integral I is the Riemann integral of $f \in \mathcal{K}(X)$.

In order to extend the integral from the members of \mathcal{F} to a wider family, two families that both contain \mathcal{F} are introduced: the family \mathcal{F}° of **over-functions** and \mathcal{F}_u of **under-functions**. Here $f \in \mathcal{F}^\circ$ [\mathcal{F}_u]

if it is an upward [downward] pointwise limit of members of \mathcal{F} . Then f need not be finite everywhere. However, $f \in \mathcal{F}^\circ$ cannot have $-\infty$ values, nor $f \in \mathcal{F}_u + \infty$ values. Now for $f \in \mathcal{F}^\circ$ and $f_n \uparrow f$ with $f_n \in \mathcal{F}$ define $I(f) = \lim I(f_n)$. Then $I(f)$ is either finite or $+\infty$. Similarly, $I(f)$ is defined for $f \in \mathcal{F}_u$, and $I(f)$ is finite or $-\infty$. It can be shown that $I(f)$ does not depend on the choice of sequence f_n that converges to f (T, Lemma 6-2II).

Next, for arbitrary f whose values may be real numbers or $\pm\infty$ define the **upper integral** $\bar{I}(f) = \inf\{I(h) : h \in \mathcal{F}^\circ, f \leq h\}$, and the **lower integral** $\underline{I}(f) = \sup\{I(g) : g \in \mathcal{F}_u, g \leq f\}$. It can be shown that $\underline{I}(f) \leq \bar{I}(f)$. If $-\infty < \underline{I}(f) = \bar{I}(f) < \infty$, then we say that f is **integrable**, and its integral $I(f)$ is the common value of $\bar{I}(f)$ and $\underline{I}(f)$. The family of integrable functions is denoted \mathcal{L} . A subset A of X is called integrable if the indicator of A is in \mathcal{L} . Its integral is denoted $\mu(A)$. In particular, if X is l.c. and $\mathcal{F} = \mathcal{K}(X)$, then all compact sets are integrable. The functions in \mathcal{L} satisfy the same familiar properties as the integrable functions of classical measure theory; for instance, the Lebesgue dominated convergence theorem.

Finally, an arbitrary real valued f is called **measurable** if for every $g, h \in \mathcal{L}$ with $g \leq 0 \leq h$ we have $\max(g, \min(f, h)) \in \mathcal{L}$. This latter function can be interpreted as f truncated above by h and below by g . The definition can be modified in equivalent ways, e.g., by writing f as the difference $f^+ - f^-$ of its positive and negative part. A subset of X is called measurable if its indicator is measurable. Denote by S the family of measurable sets. It is shown in T, Theorem 6-5III, that S is a σ -ring.

An immediate consequence of the definition of integrability is

6.1.2. THEOREM (T, Thm. 6-3 I(a)). *$f \in \mathcal{L}$ if and only for every $\varepsilon > 0$ there exist functions $g \in \mathcal{F}_u$ and $h \in \mathcal{F}^\circ$ such that $I(h) < \infty$, $g \leq f \leq h$, and $I(h) - I(g) < \varepsilon$.*

This theorem states that an integrable function can be approximated arbitrarily closely (in terms of its integral) by an over-function and by an under-function. The next theorem shows that the approximation can even be done by a function in \mathcal{F} .

6.1.3. THEOREM (T, Thm. 6-4 VI). *If $f \in \mathcal{L}$ and $\varepsilon > 0$, then there exists $h \in \mathcal{F}$ such that $I(|f - h|) < \varepsilon$. If $f \geq 0$, h may be chosen ≥ 0 .*

An important subfamily of \mathcal{L} consists of those f for which $\bar{I}(|f|) = 0$. Then f and $|f|$ are in \mathcal{L} and $I(|f|) = 0$. Such an f is called a **null function**. A **null set** is a subset of X whose indicator is a null function. This allows the notion of a statement that is true **almost everywhere** (a.e. or a.e. μ), i.e., true for all $x \in X$ except for x in a null set. It can be shown that f is a null function if and only if $f = 0$ a.e. (T, Thm. 6-4 II(c)) and that if $f \in \mathcal{L}$ and $g = f$ a.e., then $g \in \mathcal{L}$ and $I(g) = I(f)$ (T, Thm. 6-4 III). By the definition of null set, a subset of a null set is also a null set. Therefore, the measure μ defined on the integrable sets by I is always **complete**, i.e., if $\mu(A) = 0$ and $B \subset A$, then B is measurable and $\mu(B) = 0$.

The space \mathcal{L} is linear and we try to make it into a normed linear space by defining, for $f \in \mathcal{L}$, $\|f\| = I(|f|)$. This does not quite work, since $\|f\| = 0$ does not imply $f = 0$ but only $f = 0$ a.e. Thus, $\|\cdot\|$ is a semi-norm but not a norm (see Section 2.1). This situation can be remedied by considering the equivalence class $[f]$ consisting of all functions equal a.e. to f . These equivalence classes are preserved under addition and scalar multiplication so that we may take them as points of a linear space L with norm $\|[f]\| = I(|f|)$ (observe that the right-hand side does not depend on the choice of representative from $[f]$). It can be shown that L is complete (T, Thm. 6-4 IV). Thus, L is a Banach space (Section 2.2). If F is the space of equivalence classes $[f]$ with f restricted to the vector lattice \mathcal{F} with which the Daniell process was started, then $F \subset L$. Theorem 6.1.3 shows then that F is dense in L , i.e., $\bar{F} \supset L$. More can be said. Consider the linear space \mathcal{M} of real valued functions on X with finite semi-norm $\|f\| = \bar{I}(|f|)$. Then $\mathcal{F} \subset \mathcal{L} \subset \mathcal{M}$ and Theorem 6.1.3 shows that $\mathcal{L} \subset \bar{\mathcal{F}}$. But the completeness of \mathcal{L} (T, Thm. 6-4 IV) can be used to show that \mathcal{L} is closed in \mathcal{M} . Hence $\mathcal{L} \subset \bar{\mathcal{F}} \subset \bar{\mathcal{L}} = \mathcal{L}$, and therefore we have

6.1.4. THEOREM. *Consider \mathcal{F} and \mathcal{L} to be subspaces of the linear space \mathcal{M} of real valued functions f with semi-norm $\|f\| = \bar{I}(|f|) < \infty$.*

Then $\bar{\mathcal{F}} = \mathcal{L}$. For the corresponding Banach spaces M, L, F , we have $\bar{F} = L$.

A priori there is no guarantee in the Daniell definition of the class S of measurable sets that $X \in S$. However, this desirable property is guaranteed if the following condition is satisfied: $f \in \mathcal{F}$ implies $\min(f, 1) \in \mathcal{F}$, where 1 here stands for the function that equals 1 identically (T, Thm. 6-7 IV(c)). This condition was proposed by Stone (1948, 1949) and is termed **Stone's axiom**. From now on we shall assume X to be l.c. and take \mathcal{F} to be $\mathcal{K}(X)$. Then Stone's axiom is satisfied so that $X \in S$; therefore S is a σ -algebra (= σ field) rather than merely a σ -ring.

6.2. Comparison between Daniell and classical method.

It is of interest to compare the results of the classical theory of integration on l.c. spaces (Halmos, 1950, Chapter X) with those of the Daniell theory. However, comparison is made difficult by the fact that the two theories start out with different structures. In the classical theory the measurable sets are taken either as the σ -ring \mathcal{B} generated by the compact sets, or the σ -ring \mathcal{B}_0 generated by the G_δ compact sets (Halmos, 1950, §51). These are called the **Borel sets** and **Baire sets**, respectively. A priori there is no guarantee that $X \in \mathcal{B}$, let alone $\in \mathcal{B}_0$. A **Borel measure** is a measure on \mathcal{B} that is finite on compact sets. Similarly, a **Baire measure** is finite on G_δ compact sets. This distinction between Borel and Baire sets and measures vanishes if X is second countable (Section 2.2). In that case we have $\mathcal{B}_0 = \mathcal{B}$ and $X \in \mathcal{B}$ so that the σ -ring is a σ -algebra. Furthermore, every Borel measure is **regular**, which means that a set of finite measure can be approximated in measure arbitrarily closely from above by an open set and from below by a compact set (Halmos, 1950, Thm. 52G). Comparison between classical and Daniell theory is also easiest if X is second countable and we shall assume this in the following. It turns out then that $\mathcal{B} \subset S$. Hence, for any elementary integral I on $\mathcal{K}(X)$ and the extension of I to a measure μ on the integrable sets of S , we can consider the restriction of μ to \mathcal{B} ; denote this restriction by ν . This turns out to be a Borel measure. The question then is in what

way the integrable functions and measurable functions (in particular, measurable sets) differ in the two theories. The answer is as follows: (S, μ) is the completion of (\mathcal{B}, ν) , and if f is S -measurable, then there exists a \mathcal{B} -measurable function g such that $f = g$ a.e. μ , with an analogous statement for integrable functions. Thus, the end products of the two theories differ only by null sets and null functions. But it should be kept in mind that we have assumed second countability of X . In the general case we have no results on comparison except for an example in Section 6.4. A somewhat different comparison can be found in Taylor (1965, 1985), Section 6-7. See also his Sections 6-9 and 6-10.

6.3. The Bourbaki method. The Bourbaki theory of integration is basically the Daniell theory, with a small difference that will be mentioned later in this section. The space X is assumed to be l.c. and the vector lattice \mathcal{F} is taken to be $\mathcal{K}(X)$. For $f \in \mathcal{K}(X)$ the elementary integral $I(f)$ will now be written $\mu(f)$, or $\int f d\mu$, or $\int f(x)\mu(dx)$ (and the same notation will be retained after μ has been extended to the integrable functions). Let $\mathcal{K}_+(X)$ be the nonnegative functions in $\mathcal{K}(X)$. It is also convenient to have the notation $\mathcal{K}(X, K)$ for all functions $f \in \mathcal{K}(X)$ with $\text{supp } f \subset K$ compact. Bourbaki's definition of a measure μ seems at first different from the elementary integral I satisfying (6.1.1)–(6.1.3), but turns out to be equivalent. Bourbaki's definition of measure is embedded in that of a *signed measure* (following the terminology of Taylor, 1965, and others) whose values on $\mathcal{K}_+(X)$ can be negative as well as positive. (Unfortunately, there is a slight discrepancy with Bourbaki's terminology: the terms "signed measure" and "measure" here are the equivalents of "measure" and "positive measure," respectively, in Bourbaki.)

6.3.1. DEFINITION. *A signed measure on the l.c. space X is a linear functional μ satisfying the condition that for every compact $K \subset X$ there is a real number $c_K < \infty$ such that*

$$(6.3.1) \quad |\mu(f)| \leq c_K \|f\| \quad \text{for every } f \in \mathcal{K}(X, K),$$

in which

$$(6.3.2) \quad \|f\| = \sup\{|f(x)| : x \in X\}.$$

A measure is a signed measure μ satisfying

$$(6.3.3) \quad \mu(f) \geq 0 \quad \text{if} \quad f \in \mathcal{K}_+(X).$$

The norm in (6.3.2) is often called the “sup norm.” (Here the supremum is actually a maximum.) It turns out that a linear functional μ on $\mathcal{K}(X)$ for which (6.3.3) holds automatically satisfies (6.3.1). In order to show this we shall make use of the following lemma.

6.3.2. LEMMA. *If X is l.c. and $K \subset U \subset X$ with K compact and U open, then there exists open V with compact closure such that $K \subset V \subset \bar{V} \subset U$. Furthermore, there exists continuous $g : X \rightarrow [0, 1]$ with $g = 1$ on K and $g = 0$ off \bar{V} .*

PROOF. Kelley, Chap. 5, Thm. 18 (use the fact that with the Bourbaki definition of l.c. X above is regular). The first part of the lemma can also be found in Halmos (1950), Thm. 50D, in a slightly stronger version. \square

6.3.3. THEOREM (B, III, §1.6, Thm. 1). *Let X be l.c. If μ is a linear functional on $\mathcal{K}(X)$ and satisfies (6.3.3), then μ satisfies (6.3.1) so that μ is a measure on X .*

PROOF. Let K compact $\subset X$ and $f \in \mathcal{K}(X, K)$. According to Lemma 6.3.2 there exists $g \in \mathcal{K}(X)$ such that $g \geq 0$ and $g = 1$ on K . Since $f = 0$ off K we have $f = fg$ so that by (6.3.2) $|f(x)| \leq \|f\|g(x)$ for $x \in X$. The linearity of μ together with (6.3.3) implies that μ is monotone. Then $\mu(|f|) = \mu(|f|g) \leq \mu(\|f\|g) = \|f\|\mu(g)$. Hence, (6.3.1) is valid with $c_K = \mu(g) < \infty$. \square

6.3.4 PROPOSITION. *Let X be l.c., then the Bourbaki definition 6.3.1 of a measure μ on $\mathcal{K}(X)$ is equivalent to the Daniell definition 6.1.1 of an elementary integral I on $\mathcal{K}(X)$.*

PROOF. Let I be an elementary integral according to Definition 6.1.1 and put $\mu = I$. Then μ is linear by (6.1.1) and satisfies (6.3.3) by (6.1.2), hence μ is a measure by Theorem 6.3.3. Conversely, let μ be a measure according to Definition 6.3.1 and put $I = \mu$, then (6.1.2) follows from (6.3.3). In order to show (6.1.3) let $\text{supp } f_1 \subset K$ compact, then $\text{supp } f_n \subset K$ for all n since $f_n \downarrow 0$. By Dini's theorem (Kelley, 1955, Chap. 7, Probl. E) the point-wise convergence $f_n \downarrow 0$ is uniform, i.e., $\|f_n\| \rightarrow 0$ with $\|\cdot\|$ defined in (6.3.2). Consider the Banach space $\mathcal{K}(X, K)$ with norm (6.3.2), then the condition (6.3.1) is equivalent to the statement that the linear functional $\mu : \mathcal{K}(X, K) \rightarrow R$ is continuous (Section 2.2). Therefore, $\|f_n\| \rightarrow 0$ implies $\mu(f_n) \rightarrow 0$, which is (6.1.3). \square

The condition (6.3.1) implies that for every fixed compact $K \subset X$ a signed measure μ (in particular, a measure) is a continuous linear functional on the Banach space $\mathcal{K}(X, K)$ with norm (6.3.2). Now $\mathcal{K}(X)$, whose members have compact but unspecified support, is also a Banach space with norm (6.3.2). However, it is not true that μ satisfying Definition 6.3.1 is a continuous linear functional on the Banach space $\mathcal{K}(X)$. For instance, let $X = R$ and $\mu =$ Lebesgue measure on R . Take f_n continuous, equal to $1/n$ on an interval A_n of length $> n$, equal to 0 outside a finite interval containing A_n , and $\leq 1/n$ everywhere. Then $\|f_n\| \rightarrow 0$, but $\int f_n d\mu > 1$ for all n . It is possible to introduce on $\mathcal{K}(X)$ a topology that is finer than the topology induced by the norm (6.3.2) in such a way that a signed measure is continuous. For details see Bourbaki (1966a), Chap. II, §4.4, in particular Exemple II. Thus, a signed measure can be defined as a continuous linear functional on $\mathcal{K}(X)$, relative to the aforementioned topology. Equivalently, this continuity can be defined by requiring μ to be a continuous linear functional on the Banach space $\mathcal{K}(X, K)$, for each compact K . These considerations also determine when a subfamily \mathcal{F} of $\mathcal{K}(X)$ will be called dense in $\mathcal{K}(X)$: for any $f \in \mathcal{K}(X)$ there should be a compact set $K \supset \text{supp } f$ such that within the family $\mathcal{K}(X, K)$ there are members that come arbitrarily close to f in the sup norm. More precisely,

6.3.5 DEFINITION. Let $\mathcal{F} \subset \mathcal{K}(X)$, then \mathcal{F} is called **dense** in $\mathcal{K}(X)$ if for every $f \in \mathcal{K}(X)$ there exists compact K such that $\text{supp } f \subset K$ and $\inf\{\|f-g\| : g \in \mathcal{F} \cap \mathcal{K}(X, K)\} = 0$, with $\| \cdot \|$ defined in (6.3.2).

The difference between the Bourbaki and Daniell methods lies in the definition of over- and under-functions. For the definition of integrable functions it is sufficient to consider only $f \geq 0$ since an arbitrary function can be written as $f^+ - f^-$. For the extension of μ to integrable $f \geq 0$ only the over-functions are needed. In the Bourbaki theory the family of nonnegative over-functions coincides with the family of all nonnegative lower semicontinuous functions on X (Section 2.2). If $f \geq 0$ is l.s.c., then $f = \sup\{g \in \mathcal{K}_+(X) : g \leq f\}$ (B, IV, §1.1, Lemme 1) and one defines $\mu(f) = \sup \mu(g)$ (which may be $+\infty$). On the other hand, in the Daniell theory an over-function is defined as any upward limit of a *sequence* of functions in $\mathcal{K}_+(X)$. Such a limit is l.s.c. (Section 2.2) but an arbitrary l.s.c. function is not necessarily so obtainable. That is, the set of Daniell over-functions could possibly be a proper subset of the Bourbaki ones. If this happens then there could conceivably be more integrable and measurable functions and sets according to Bourbaki than according to Daniell. Certain conditions on X may prevent this. For instance, if X is metric with distance function d having the property that every set of the form $\{y \in X : d(x, y) \leq c\}$, $x \in X$, $c \in R_+$, is compact, then every nonnegative l.s.c. function is an upward limit of elements of $\mathcal{K}_+(X)$. This follows from Taylor (1965, 1985), Theorem 6-9 III, after observing that the proof of this theorem remains valid for metric X with the above compactness condition on the metric d .

There is another difference between the Daniell and Bourbaki methods that is more apparent than real. Bourbaki (B, IV, §4.1) extends a measure μ on $\mathcal{K}(X)$ to \mathcal{L} by defining \mathcal{L} as the closure of $\mathcal{K}(X)$ in \mathcal{M} . But it follows from Theorem 6.1.4 (with $\mathcal{F} = \mathcal{K}(X)$) that this leads to the same integrable functions as does the Daniell method, provided of course that the meaning of the upper integral \bar{I} is the same in both methods. This will be the case if the nonnegative integrable over-functions are the same. Another way of verifying the

equivalence of the two methods is provided by

6.3.6. THEOREM (B, IV, §4.4, Thm. 3). *If $f \geq 0$, then $f \in \mathcal{L}$ if and only if for every $\varepsilon > 0$ there exists $g \geq 0$, u.s.c. with compact support, and h l.s.c. integrable such that $g \leq f \leq h$ and $\int(h - g)d\mu < \varepsilon$.*

This is the same as Theorem 6.1.2 for $f \geq 0$ provided the function h in Theorem 6.3.6 is in \mathcal{F}° of Daniell. (Note that in Theorem 6.1.2 for $f \geq 0$ we may choose $g \geq 0$ and then g has compact support since it is the downward limit of functions in $\mathcal{K}(X)$.)

In the remainder of this monograph a measure on a l.c. space X will mean a continuous linear functional on $\mathcal{K}(X)$ that is nonnegative on $K_+(X)$, i.e., satisfying Definitions 6.1.1 or 6.3.1 (equivalent by Proposition 6.3.4), and then extended uniquely by the Bourbaki method to a wider class \mathcal{L} of functions called integrable. By continuity a measure is already defined by its value on a dense subset of $\mathcal{K}(X)$ (Definition 6.3.5) and this will be used in the definition of product measure in Section 6.5.

We mention without proof that in the Bourbaki theory for any chosen measure μ the compact sets are integrable (B, IV, §4.6, Prop. 10, Cor. 1) and the measurable sets form a σ -algebra including the open sets (and therefore the closed sets) (B, IV, §5.4, Thm. 2, Cor. 2 and IV, §5.1, Prop. 3, Cor. 1). Furthermore, every measure is regular (B, IV, §4.6, Thm. 4).

Effect of a proper mapping. Let X and Y be l.c. spaces and $h : X \rightarrow Y$. If h is a **proper** mapping (Definition 2.2.1) then for every compact $K \subset Y$, $h^{-1}(K)$ is compact $\subset X$ (Theorem 2.2.3). It follows that if $f \in \mathcal{K}(Y)$, then $f \circ h \in \mathcal{K}(X)$. If μ is a measure on X , define the **induced** measure $h(\mu)$ on Y (also written μh^{-1}) by:

$$(6.3.4) \quad h(\mu)(f) = \mu(f \circ h), \quad f \in \mathcal{K}(Y).$$

This induced measure is also called the **image** of μ under h .

6.4. Comparison of the three integration methods in an example. In an attempt to find an example of a space where the

three theories of integration—classical, Daniell, and Bourbaki—would lead to different results, the following space was tried. Let Y be an uncountable index set (e.g., an interval of R). For each $y \in Y$ let R_y be a copy of the real line R . Define $X = \cup\{R_y : y \in Y\}$ with each R_y open in X and the relative topology on R_y being the usual topology of R . (Thus, each R_y is a component of X ; see Section 2.2. X is also called the **sum** of the topological spaces R_y ; see Bourbaki, 1966b, I, §2.4, Example 3.) Then X is l.c. but not metric nor second countable. Any compact $K \subset X$ must be of the form $\cup\{K_y \text{ compact } \subset R_y : y \in Y_1\}$, with Y_1 finite. Define the elementary integral on $\mathcal{K}(X)$ by $\mu(f) = \sum_{y \in Y_1} \int_{K_y} f dm$, where m is Lebesgue measure.

It turns out that a function $f \geq 0$ is an over-function according to Daniell if and only if there is a countable subset $Y_0 \subset Y$ such that f is l.s.c. on R_y for $y \in Y_0$, and f vanishes on R_y for $y \notin Y_0$. On the other hand, any $f \geq 0$ with f l.s.c. on each R_y is an over-function according to Bourbaki. Therefore, the class of Daniell over-functions is properly contained in that of Bourbaki. However, for the definition of integrable functions only the over-functions f with $\mu(f) < \infty$ matter, and those are the same for Bourbaki as for Daniell. Therefore, in this example Daniell and Bourbaki lead to the same integrable functions. It can be shown that this implies that the measurable functions are also the same under both theories. Hence, this example does not lead to different results under Daniell and Bourbaki. On the other hand, the classical integration theory leads to a different result. Every compact K is a G_δ (because this is true for every $K_y \subset R_y$, with $y \in Y_1$ finite) so that the Borel sets and the Baire sets coincide (Section 6.2): $\mathcal{B} = \mathcal{B}_0$. It is easy to show that \mathcal{B} , i.e., the smallest σ -ring containing the compact sets, consists of all sets of the form $\cup_{y \in Y_0} B_y$, Y_0 some countable subset of Y and B_y a Borel subset of R_y for each $y \in Y_0$. This implies that X is not a Borel set, i.e., X is not measurable according to classical integration theory, in contrast to Daniell or Bourbaki theory. Thus, in this example there is an essential difference between classical integration theory on one hand and Daniell and Bourbaki on the other. It should be remarked, though, that the

difference between the two sides is not nearly as large when we only consider integrable (rather than measurable) functions and sets. In each of the three theories a necessary condition for f to be integrable is that it vanish on all but a countable number of R_y . The only difference lies in the measurability of f on each such R_y , which is as in the comparison between classical and Daniell theories in section 6.2.

6.5. Product measure. Let X and Y be two l.c. spaces, so that $X \times Y$ is also l.c. (Section 2.2). Let λ be a measure on X , μ a measure on Y . In classical measure theory it is shown that if λ and μ are σ -finite, then on $X \times Y$ there is a unique measure ν , called **product measure**, such that on a product set $A \times B$ we have $\nu(A \times B) = \lambda(A)\mu(B)$ (Halmos, 1950, Chap. VII). The analogue in the Bourbaki theory is a measure ν such that for functions f on $X \times Y$ of the form $f(x, y) = g(x)h(y)$, $g \in \mathcal{K}(X)$, $h \in \mathcal{K}(Y)$, we have $\nu(f) = \lambda(g)\mu(h)$. It remains to be shown that such a measure ν on $X \times Y$ exists and is unique. This will be the subject of Theorem 6.5.1 below. Note that in this theorem λ and μ are not assumed to be σ -finite. The proof of the theorem will be given in some detail since the theorem is of central importance to the main topic of this monograph, and since the various needed lemmas are scattered in Bourbaki (1965) and some of them stated in greater generality than needed here. Furthermore, one of the lemmas will again be needed in Section 6.6. Recall from Section 2.1 the notation $f_1 \otimes f_2$.

6.5.1. THEOREM (B, III, §4.1, Thm. 1) *Let X and Y be l.c. and let there be given measures λ on X , μ on Y . Then there exists on $X \times Y$ a unique measure ν such that for every $g \in \mathcal{K}(X)$, $h \in \mathcal{K}(Y)$, we have*

$$(6.5.1) \quad \int g \otimes h \, d\nu = \int g \, d\lambda \int h \, d\mu.$$

The theorem is equally valid with “measure” replaced by “signed measure.”

Before giving the proof, several lemmas are needed and some additional notation has to be established. If Z and E are arbitrary

spaces, then $\mathcal{F}(Z; E)$ shall stand for the family of all functions $Z \rightarrow E$ (note that this \mathcal{F} bears no relation to the vector lattice \mathcal{F} of the Daniell theory). Our choices for Z will be X , or Y , or $X \times Y$, and E will be either R or a space of real valued functions. If $E = R$ it may be omitted in the notation. If E is a space of continuous functions with fixed compact support, then we shall take the sup norm as a norm on E so that E becomes a Banach space. If Z and E are topological and K compact $\subset Z$, then $\mathcal{K}(Z, K; E)$ stands for the family of all continuous functions $Z \rightarrow E$ that have support contained in K . Finally, $\mathcal{F}(Z_1; R) \otimes \mathcal{F}(Z_2; R)$ stands for the space of all functions of the type $\sum g_i \otimes h_i$ (finite sum) with $g_i \in \mathcal{F}(Z_1; R)$, $h_i \in \mathcal{F}(Z_2; R)$.

Now consider the following spaces of functions:

- (A) $\mathcal{F}(X \times Y; R)$; (B) $\mathcal{F}(X; \mathcal{F}(Y; R))$;
 (C) $\mathcal{K}(X \times Y, K \times L; R)$; (D) $\mathcal{K}(X, K; \mathcal{K}(Y, L; R))$;
 (E) $\mathcal{K}(X, K; R) \otimes \mathcal{K}(Y, L; R)$

in which K and L are compact subsets of X, Y respectively. There is a natural 1-1 correspondence between the elements of (A) and those of (B), since a function of (x, y) can be regarded as associating to each $x \in X$ a function $Y \rightarrow R$. Write this correspondence as a bijection ω :

$$(6.5.2) \quad \omega : \mathcal{F}(X \times Y; R) \rightarrow \mathcal{F}(X; \mathcal{F}(Y; R)).$$

The most important part of the proof of Theorem 6.5.1 is the proof that (E) is dense in (D). This will be presented in Lemma 6.5.5 below. First several other lemmas are needed.

6.5.2. LEMMA. *Let X be l.c., K compact $\subset X$, and U_1, \dots, U_n a finite open cover of K . Then there exists an open cover V_1, \dots, V_n of K such that $\bar{V}_i \subset U_i$ and \bar{V}_i is compact, $i = 1, \dots, n$.*

PROOF. For every $x \in K$ there is a compact neighborhood contained in one of the U_i . By compactness of K there is a finite number of these neighborhoods, say W_1, \dots, W_k , such that $K \subset \cup_{j=1}^k W_j^\circ$. For $i = 1, \dots, n$ take $\bar{V}_i =$ union of all W_j contained in U_i , then \bar{V}_i is compact, $\bar{V}_i \subset U_i$, and their interiors V_1, \dots, V_n is an open cover of K since each W_j° is contained in some V_i . \square

6.5.3. LEMMA (B, III, §1.2, Lem. 1). *Let X be l.c., K compact $\subset X$, and U_1, \dots, U_n an open cover of K . Then there exist real valued continuous functions g_1, \dots, g_n such that g_i has compact support contained in U_i , $g_i \geq 0$, $\sum_1^n g_i \leq 1$ on X , and $\sum_1^n g_i = 1$ on K .*

PROOF. Take the V_i as in Lemma 6.5.2. Using Lemma 6.3.2, for each i there exists open W_i with compact closure such that $\bar{V}_i \subset W_i \subset \bar{W}_i \subset U_i$, and a continuous function $f_i : X \rightarrow [0, 1]$ that equals 1 on \bar{V}_i and 0 off \bar{W}_i . Thus, $\text{supp } f_i \subset \bar{W}_i \subset U_i$. Since $K \subset \cup_1^n V_i$, $\sum_1^n f_i \geq 1$ on K . Put $m(x) = \max(1, \sum_1^n f_i(x))$, $x \in X$, so that $m = \sum f_i$ on K . Define $g_i = f_i/m$ on X , then g_i is continuous, $\sum_1^n g_i \leq 1$ on X and $= 1$ on K . Moreover, $\text{supp } g_i = \text{supp } f_i \subset \bar{W}_i \subset U_i$. \square

REMARK. A similar proof is given in Chevalley (1946), Chap. 5, §VII, Lemma 1 (attributed to Dieudonné) for X a manifold, in which case the functions f_i can be constructed explicitly. The set of functions g_i is called a **partition of unity** subordinate to the cover U_1, \dots, U_n . (This device in more general form will be used again in Section 13.3.)

The following lemma is a special case of B, III, §1.2, Lemme 2.

6.5.4. LEMMA. *Let X be l.c., K compact $\subset X$, E a Banach space with norm $\| \cdot \|$, $f \in \mathcal{K}(X, K; E)$. Then for every $\varepsilon > 0$ there exist functions $g_1, \dots, g_n \in \mathcal{K}(X, K; R)$ such that if x_i is an arbitrary point of $\text{supp } g_i$, $i = 1, \dots, n$, then for every $x \in X$ we have*

$$(6.5.3) \quad \left\| f(x) - \sum_{i=1}^n g_i(x)f(x_i) \right\| < \varepsilon.$$

PROOF. Let ∂K be the boundary of K . For every $y \in \partial K$ we have $f(y) = 0$ so that there exists an open neighborhood V_y of y such that $\|f(z)\| < \varepsilon/2$ if $z \in V_y$. Define $K' = K - \cup\{V_y : y \in \partial K\}$, then K' is compact and $\subset K^\circ$. Every $x \in K'$ has an open neighborhood $U_x \subset K$ such that $z \in U_x$ implies $\|f(z) - f(x)\| < \varepsilon/4$ and therefore $z_1, z_2 \in U_x$ implies $\|f(z_1) - f(z_2)\| < \varepsilon/2$. By compactness, K' may be covered by a finite number of these open sets, say U_1, \dots, U_n . By Lemma 6.5.3 there exist nonnegative continuous

functions $g_1, \dots, g_n : X \rightarrow R$ such that $\text{supp } g_i \subset U_i$, $\sum_1^n g_i(x) \leq 1$ for $x \in X$, and $\sum_1^n g_i(x) = 1$ for $x \in K'$. For $i = 1, \dots, n$ choose any $x_i \in \text{supp } g_i$, then for any $x \in U_i$:

$$(6.5.4) \quad \|f(x)g_i(x) - f(x_i)g_i(x)\| = g_i(x)\|f(x) - f(x_i)\| < \frac{\varepsilon}{2}g_i(x)$$

since x and x_i are both in U_i . This is still valid if $x \notin U_i$ since then $g_i(x) = 0$. Thus, (6.5.4) holds for every $x \in X$. Sum (6.5.4) over $i = 1, \dots, n$, and put $\sum_1^n g_i(x) = s(x)$, which is ≤ 1 on X and $= 1$ on K' , then we obtain

$$(6.5.5) \quad \left\| f(x)s(x) - \sum_1^n f(x_i)g_i(x) \right\| \leq \frac{\varepsilon}{2}s(x).$$

For $x \in K'$, $s(x) = 1$ so that (6.5.5) implies (6.5.3). For $x \notin K$, (6.5.3) is true also since then $f(x) = 0$ and all $g_i(x) = 0$. For $x \in K - K'$ we have

$$(6.5.6) \quad \|f(x)(1 - s(x))\| \leq \|f(x)\| \leq \frac{\varepsilon}{2}.$$

Combine (6.5.5) and (6.5.6):

$$\begin{aligned} \left\| f(x) - \sum_1^n g_i(x)f(x_i) \right\| &\leq \left\| f(x)s(x) - \sum_1^n f(x_i)g_i(x) \right\| \\ &\quad + \|f(x)(1 - s(x))\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which is (6.5.3). \square

6.5.5. LEMMA. *The space (E) is dense in the space (D) .*

PROOF. In Lemma 6.5.4 take $E = \mathcal{K}(Y, L; R)$. Let $\varepsilon > 0$ be arbitrary. Take g_i and $f(x_i)$ of Lemma 6.5.4 then $f(x_i)$ is an element of $\mathcal{K}(Y, L; R)$, say $f(x_i) = h_i$, $i = 1, \dots, n$. By (6.5.3) of Lemma 6.5.4 we have

$$(6.5.7) \quad \left\| f(x) - \sum_1^n g_i(x)h_i \right\| < \varepsilon, \quad x \in X,$$

where the norm $\| \cdot \|$ is that of $\mathcal{K}(Y, L; R)$. In (6.5.7) take the sup over $x \in X$ to obtain $\|f - \sum_1^n g_i \otimes h_i\| < \varepsilon$, where this time the norm $\| \cdot \|$ refers to (D) . Since $\sum_1^n g_i \otimes h_i \in (E)$ the lemma is proved. \square

6.5.6. LEMMA. (i) *The restriction of ω defined in (6.5.2) to (C) is an isometry of (C) with (D) ; (ii) (E) , considered as a subspace of (C) , is dense in (C) ; hence, $\mathcal{K}(X) \otimes \mathcal{K}(Y)$ is dense in $\mathcal{K}(X \times Y)$.*

PROOF. (i) We have to show that $f \in (C)$ considered as a function in (B) is actually a function in (D) , i.e., has compact support and is continuous as a function $X \rightarrow \mathcal{K}(Y, L; R)$. Conversely, if $f \in (D)$ is regarded as a function in (A) we have to show that it actually is in (C) . The question about the supports is trivial, so only the continuity part will be handled.

First, take $f \in (C)$, regarded as $\in (B)$. In order to verify the continuity of $f : X \rightarrow \mathcal{K}(Y, L; R)$, we have to check that as $z \rightarrow x \in X$, then $\sup_y |f(z, y) - f(x, y)| \rightarrow 0$. This follows from the continuity of f as a function on $X \times Y$ and the compactness of L by a standard argument. Next, let $f \in (D)$ be regarded as $\in (A)$. Since $f \in (D)$, $f(z, \cdot) \rightarrow f(x, \cdot)$ uniformly as $z \rightarrow x$. This, together with the continuity of $f(x, \cdot)$ (as a function of y) shows that f is jointly continuous in (x, y) , hence $f \in (C)$.

So far we have shown that ω is a bijection $(C) \rightarrow (D)$. It remains to be shown that it is an isometry. The norm of $f \in (C)$ is by definition

$$(6.5.8) \quad \|f\|_{(C)} = \sup\{|f(x, y)| : (x, y) \in K \times L\}.$$

For fixed $x \in X$, the norm of $f(x, \cdot) \in \mathcal{K}(Y, L; R)$ is $\sup\{|f(x, y)| : y \in L\}$, so that the norm of $f \in (D)$ is

$$(6.5.9) \quad \|f\|_{(D)} = \sup_{x \in K} \sup_{y \in L} |f(x, y)|.$$

But the right-hand sides of (6.5.8) and (6.5.9) are equal, so that $\|f\|_{(C)} = \|f\|_{(D)}$.

(ii) By Lemma 6.5.5 (E) is dense in (D) , and by part (i) (D) is homeomorphic to (C) ; therefore, (E) is dense in (C) . It remains to be shown that $\mathcal{K}(X) \otimes \mathcal{K}(Y)$ is dense in $\mathcal{K}(X \times Y)$; i.e., we have to show (by Definition 6.3.5) that for every $f \in \mathcal{K}(X \times Y)$ there exists a compact subset, say M , of $X \times Y$ such that $f \in \mathcal{K}(X \times Y, M)$ and such

that for every $\varepsilon > 0$ there exists $g \in \mathcal{K}(X) \otimes \mathcal{K}(Y)$ with $\text{supp } g \subset M$ and $\|f - g\| < \varepsilon$, with $\| \cdot \|$ defined in (6.3.2). Let $\text{supp } f = J$ compact, then J can be covered by a finite number of sets of the form $K_i \times L_i$, K_i compact $\subset X$, L_i compact $\subset Y$. Take $K = \cup K_i$, $L = \cup L_i$, then $M = K \times L$ is compact and $\text{supp } f \subset M$ so that $f \in (C)$. Since (E) is dense in (C) there are functions $g \in (C)$ with $\|f - g\|$ arbitrarily close to 0. Moreover, each such g is in $\mathcal{K}(X) \otimes \mathcal{K}(Y)$ and has $\text{supp } g \subset M$. \square

PROOF OF THEOREM 6.5.1. Uniqueness. Let ν be a signed measure satisfying (6.5.1), then for $f = \sum_1^n g_i \otimes h_i$ with $g_i \in \mathcal{K}(X)$, $h_i \in \mathcal{K}(Y)$ we have $\nu(f) = \sum_1^n \int g_i d\lambda \int h_i d\mu$. Thus, ν is uniquely defined on $\mathcal{K}(X) \otimes \mathcal{K}(Y)$ and since the latter is dense in $\mathcal{K}(X \times Y)$ by Lemma 6.5.6(ii), ν is uniquely defined on $\mathcal{K}(X \times Y)$.

Existence. Let $f \in \mathcal{K}(X \times Y)$, then as in the proof of Lemma 6.5.6 there are compact subsets K, L of X, Y , respectively such that $f \in \mathcal{K}(X \times Y, K \times L; R)$. Therefore, for fixed $y \in Y$, $f(\cdot, y) \in \mathcal{K}(X, K; R)$ so that $h(y) = \int f(x, y)\lambda(dx)$ is well defined. Moreover, $\text{supp } h \subset L$. Furthermore, $h : Y \rightarrow R$ is continuous since it is the composition of $y \rightarrow f(\cdot, y)$ which is continuous by Lemma 6.5.6(i) (after reversing the roles of X and Y) and $f(\cdot, y) \rightarrow \lambda(f(\cdot, y))$ which is continuous by definition of signed measure. Hence $h \in \mathcal{K}(Y, L; R)$ and therefore $\mu(h)$ is well defined. Now define ν on $\mathcal{K}(X \times Y)$ by

$$(6.5.10) \quad \nu(f) = \mu(h), \quad h(y) = \int f(x, y)\lambda(dx),$$

for $f \in \mathcal{K}(X \times Y)$. By taking $f = g \otimes h$ it is easily seen that ν satisfies (6.5.1). Obviously, ν is a linear functional on $\mathcal{K}(X \times Y)$. If λ and μ are measures, then by (6.5.10) ν is nonnegative on $\mathcal{K}_+(X \times Y)$ so that ν is a measure on $X \times Y$ by Theorem 6.3.3. If λ and μ are signed measures it remains to be shown that ν satisfies the boundedness condition (6.3.1). That is, we have to show that if compact $M \subset X \times Y$, then there exists $c_M < \infty$ such that for every $f \in \mathcal{K}(X \times Y, M)$ we have

$$(6.5.11) \quad |\nu(f)| \leq c_M \|f\|$$

in which $\|f\| = \sup\{|f(x, y)| : (x, y) \in M\}$. As in the proof of Lemma 6.5.6(ii), we may WLOG assume $M = K \times L$, K and L compact. Since λ and μ are signed measures, there exist constants $a_K < \infty$, $b_L < \infty$, such that $|\lambda(g)| \leq a_K \|g\|$ for every $g \in \mathcal{K}(X, K)$ and $|\mu(h)| \leq b_L \|h\|$ for every $h \in \mathcal{K}(Y, L)$. Take h as defined in (6.5.10), then for fixed $y \in Y$, $|h(y)| = |\lambda(f(\cdot, y))| \leq a_K \|f(\cdot, y)\| = a_K \sup_x |f(x, y)|$, so $\|h\| \equiv \sup_y |h(y)| \leq a_K \|f\|$. Using the definition (6.5.10) of ν we have $|\nu(f)| = |\mu(h)| \leq b_L \|h\| \leq a_K b_L \|f\|$, hence we may take $c_M = a_K b_L < \infty$, so that (6.5.11) has been proved. \square

The (signed) measure ν defined by (6.5.10) is called the **product** of λ and μ , and will be denoted by $\lambda \otimes \mu$, following Bourbaki. Then (6.5.1) reads

$$(6.5.12) \quad (\lambda \otimes \mu)(g \otimes h) = \lambda(g)\mu(h), \quad g \in \mathcal{K}(X), \quad h \in \mathcal{K}(Y).$$

The defining equation (6.5.10) can also be written in the form

$$(6.5.13) \quad \int f d(\lambda \otimes \mu) = \int \mu(dy) \int f(x, y) \lambda(dx),$$

and by symmetry,

$$(6.5.14) \quad \int f d(\lambda \otimes \mu) = \int \lambda(dx) \int f(x, y) \mu(dy),$$

for $f \in \mathcal{K}(X \times Y)$. These are Fubini-type equations that follow here simply from the definition of $\lambda \otimes \mu$ provided $f \in \mathcal{K}(X \times Y)$. We shall need those equations also for integrable $f : X \times Y \rightarrow R$ and quote without proof the relevant theorem.

6.5.7. THEOREM (Bourbaki). *Let X and Y be l.c., λ a signed measure on X , μ on Y . If $f : X \times Y \rightarrow R$ is $(\lambda \otimes \mu)$ -integrable, then the set of points $y \in Y$ such that $f(\cdot, y)$ is not λ -integrable has μ -measure 0 and on the remaining set $\int f(x, \cdot) \lambda(dx)$ is μ -integrable. Moreover, (6.5.13) holds. An analogous statement holds with X and Y interchanged, resulting in (6.5.14).*

PROOF. Bourbaki (1967), V, §8.4, Thm. 1. \square

6.6. Integration on a manifold with respect to a differential form. Let M be a d -dimensional C^1 manifold (Chapter 3). It was already hinted in Section 4.3 that a d -form ω could be used to define a measure on M . This will be made more precise now. It is sometimes assumed (e.g., in Chevalley, 1946, V, §VII) that M is orientable (Section 4.4) but this is unnecessary if we are only interested in measures rather than signed measures. The only assumption we shall make on ω is that if the expression of ω in a chart with local coordinates $x = (x_1, \dots, x_n)$ is (4.4.1), then $|\alpha|$ is continuous. For convenience, we make the following definition.

6.6.1. DEFINITION. *Let M be a d -dimensional C^1 manifold and ω a d -form on M . We shall say that $|\omega|$ is continuous on M if $|\alpha|$ is continuous on every chart, where α represents ω via equation (4.4.1). If M is orientable and α continuous on every chart, then ω will be called continuous on M .*

A manifold, being locally Euclidean, is a special case of a locally compact space so that the Bourbaki integration theory applies. Therefore, it suffices to define a measure on $\mathcal{K}(M)$. Let $f \in \mathcal{K}(M)$ and suppose first that there is an open set U on which there is a chart with coordinates $x = (x_1, \dots, x_d)$ such that $\text{supp } f \subset U$. For short we shall say that the support of f is contained in a chart. Let ω on U have the expression (4.4.1) and assume that $|\omega|$ is continuous on M . Let V be the open subset of R^d corresponding to $U \subset M$. Then on V the function $x \rightarrow f(p(x))|\alpha(x)|$ is continuous with compact support contained in V so that we can define

$$(6.6.1) \quad \mu(f) = \int f(p(x))|\alpha(x)|dx_1 \dots dx_d$$

as a Riemann integral. The right-hand side does not depend on the particular chosen parametrization because under a change of variables the expression for ω changes in the same way (disregarding sign) according to (4.3.5) as the volume element $dx_1 \cdots dx_d$ on the right-hand side of (6.6.1). From the linearity in f of the Riemann integral it follows immediately that if f_1 and f_2 are both in $\mathcal{K}(M)$ and have their supports contained in the same chart, then $\mu(f_1 + f_2) = \mu(f_1) + \mu(f_2)$.

Now suppose $f \in \mathcal{K}(M)$ and $\text{supp } f \subset K$ compact, but K not necessarily contained in a chart. Then, by compactness, K can be covered by a finite number of open sets U_1, \dots, U_n , on each of which there is a chart. Choose a partition of unity with functions g_1, \dots, g_n according to Lemma 6.5.3, then $f = \sum_i f g_i$, and each term $f g_i$ is continuous with compact support contained in U_i so that $\mu(f g_i)$ is well defined by (6.6.1) applied to $f g_i$ instead of f . Then define

$$(6.6.2) \quad \mu(f) = \sum_{i=1}^n \mu(f g_i).$$

However, the sets U_i and the functions g_i are not unique, and it remains to be shown that for another choice, say $U'_j, g'_j, j = 1, \dots, m$ we have

$$(6.6.3) \quad \sum_{i=1}^n \mu(f g_i) = \sum_{j=1}^m \mu(f g'_j).$$

In order to show this consider the expression $\sum_{ij} \mu(f g_i g'_j)$. For fixed i , the functions $f g_i g'_j$ are all continuous with compact support contained in U_i . Therefore $\sum_j \mu(f g_i g'_j) = \mu(\sum_j f g_i g'_j) = \mu(f g_i)$ since $\sum_j g'_j = 1$ on K . Thus, $\sum_{ij} \mu(f g_i g'_j) = \sum_i \mu(f g_i)$ which is the left-hand side of (6.6.3). By interchanging the roles of g_i and g'_j we obtain the right-hand side of (6.6.3) and the latter has therefore been shown to hold. Therefore, $\mu(f)$ has been defined unambiguously for all $f \in \mathcal{K}(M)$. Instead of $\mu(f)$ one usually writes $\int f \omega$. It is obvious that $\int f \omega$ is a linear functional on $\mathcal{K}(M)$ and that it is nonnegative on $\mathcal{K}_+(M)$. Therefore, $\int f \omega$ defines a measure on $\mathcal{K}(M)$ by Theorem 6.3.3. We summarize the result in

6.6.2. THEOREM. *Let M be a d -dimensional C^1 manifold and ω a d -form on M such that $|\omega|$ is continuous on M (Definition 6.6.1). Then ω defines a unique measure μ on M , where $\mu(f)$ is also written $\int f \omega$, by the formula*

$$(6.6.4) \quad \int f \omega = \sum_{i=1}^n \mu(f g_i)$$

in which the terms on the right-hand side are defined by (6.6.1) and the functions g_1, \dots, g_n form a partition of unity subordinate to a finite open cover U_1, \dots, U_n of the support of f such that on each U_i there is a chart.

Although not needed in the sequel, we remark that if M is orientable and ω continuous (Definition 6.6.1), then (6.6.4) defines a signed measure if in (6.6.1) one replaces $|\alpha(x)|$ by $\alpha(x)$. The boundedness condition (6.3.1) is easily established by taking $c_K = \sum_i \mu(g_i)$, with the original definition (6.6.1) of μ .

Invariance under a diffeomorphism. If there is a diffeomorphism between two manifolds of dimension d , then a function and a d -form on one induces in a natural way a function and a d -form on the other. The next result shows that the two resulting integrals are equal.

6.6.3 PROPOSITION. *Let $\phi : M \rightarrow N$ be a diffeomorphism between the C^1 manifolds M and N and let ω be a d -form on N , with $|\omega|$ continuous. If $f : N \rightarrow R$ is integrable with respect to the measure defined by ω , then*

$$(6.6.5) \quad \int_N f\omega = \int_M (f \circ \phi)\delta\phi(\omega).$$

PROOF. It is sufficient to consider $f \in \mathcal{K}(N)$ with support contained in an open set U on which there is a chart with local coordinates $x = (x_1, \dots, x_d)$. Then $f \circ \phi \in \mathcal{K}(M)$ and has its support contained in $V = \phi^{-1}(U)$. On V choose the same chart as on U , so that if we write $p \in V$ and $q \in U$ as functions of x we have $q(x) = \phi(p(x))$. Then $f(q(x))$ on the left-hand side of (6.6.5) equals $f(\phi(p(x)))$ on the right-hand side; denote the common value by $f^*(x)$. Moreover, ω on U and $\delta\phi(\omega)$ on V have the same expression in terms of x , of the form (4.4.1), by (4.5.3) with $x_i = y_i$. Thus, both sides of (6.6.5) equal $\int f^*(x)|\alpha(x)|dx_1 \dots dx_d$. \square