MEASURE-VALUED PROCESSES: TECHNIQUES AND APPLICATIONS

Kenneth J. Hochberg[1]
Bar-Ilan University

Abstract

Several techniques are presented for the analysis of measure-valued stochastic processes. These methods are then applied to a number of examples so as to determine the behavior of the processes at fixed times, in the long term, and in the renormalization limit.

I. Introduction. Interest in the theory of measure-valued stochastic processes, which were first introduced by Dawson (1975) in the study of branching diffusion systems, has recently increased as a result of the work of Dynkin (1988, 1989) and Perkins (1988, 1989, 1990) on superprocesses. In the interim, measure-valued processes have been used to describe the dynamics of populations whose underlying distributions are continuously changing, and which are therefore described via a distribution or random measure at each fixed time. They also arise as the diffusion approximation to certain real-valued processes describing spatially-distributed systems. Applications that lead to measure-valued processes in the diffusion limit include models that describe the behavior of systems of branching and diffusing particles (Dawson (1977), Dawson-Hochberg (1979), Hochberg (1980, 1983), Iscoe (1986, 1988), Dawson-Iscoe-Perkins (1989)); models describing frequency distributions of alleles in neutral, non-neutral and interactive populations (Fleming-Viot (1979), Dawson-Hochberg (1982, 1983), Hochberg (1986), Ethier-Kurtz (1987), Ethier-Griffiths (1987, 1990), Vaillancourt (1990a,b)); and the continuous limit of hierarchically-structured branching and branching diffusion systems (Dawson-Hochberg-Wu (1990), Dawson-Hochberg (1991)).

In what follows, we review several techniques for analyzing measure-valued stochastic processes, present examples in which such processes arise, and show how these techniques are used to study the behavior of such processes at fixed times, in the long term, and in the renormalization limit.

1. Research partially supported by U.S. National Security Agency grant MDA 904-88-H-2044 and U. S. National Science Foundation grant DMS - 8800289.

II. Techniques. As is the case for any general Markov process, a measure-valued process can be described via an infinitesimal generator G, defined so as to incorporate the evolutionary forces that lead to changes in state, and a domain $D(G)$ on which the generator is defined. In the diffusion approximation, one considers an appropriately rescaled limiting version G of a generator G_N, defined for fixed values of N. One shows the existence and uniqueness of the resulting process as a Markov process with values in an appropriate space of measures via semigroup techniques or by solving an initial-value martingale problem.

For a locally compact complete separable metric space S, let $M(S)$ denote the space of bounded Radon measures on S, furnished with an appropriate metric so as to be a complete, separable metric space with a topology equivalent to that of weak convergence of measures. For a closed subset E of $M(S)$, let $B(E)$ denote the σ-algebra of Borel subsets of E, $C_b(E)$ denote the space of continuous bounded functions on E, and $L_\infty(E)$ denote the space of bounded measurable functions on E. Let $\Omega_c^E = C([0,\infty),E)$ be the space of continuous functions mapping $[0,\infty)$ into E, and let $\Omega_D^E = D([0,\infty),E)$ be the space of functions mapping $[0,\infty)$ into E that are right continuous with limits from the left. A *measure-valued* (or *E-valued*) stochastic process $\{X(t)\colon t \geq 0\}$ is then given by a mapping $X\colon\ [0,\infty] \times \Omega_C^E \text{ (or } \Omega_D^E) \to E$ defined by the canonical process $X(t,\omega) \equiv \omega(t)$ for $t \geq 0$, $\omega \in \Omega_C^E \text{ (or } \Omega_D^E)$. The distribution of a measure-valued stochastic process $\{X(t)\colon t \geq 0\}$ is determined by a measurable mapping $\mu \to P_\mu$ from E into $P(\Omega_C^E)$ or $P(\Omega_D^E)$, the space of probability measures on Ω_C^E and Ω_D^E, respectively, furnished with the topology of weak convergence.

An E-valued stochastic process $\{X(t)\colon t \geq 0\}$ with time-homogeneous transition probabilities is uniquely determined by the characteristic functional of the initial distribution $X(0) = \nu \in E$, defined for $f \in C_b(S)$ by

$$\phi(f) \equiv \int_E \exp\left(i \int_S f(x)\,\nu(dx)\right) P(d\nu), \tag{2.1}$$

and the characteristic functional of the probability transition function, given by

$$\phi_{t,\nu}(f) \equiv E\left\{ exp\left(i \int_S f(x)\, X(t,dx)\right) \middle|\, X(0) = \nu \right\}. \tag{2.2}$$

Equivalently, one can use Laplace functionals of the form

$$L_{t,\nu}(f) = E\{exp\left(-\int_S f(x)X(t,dx)\right) \Big| X(0) = \nu\}. \tag{2.3}$$

Under these circumstances, the *cumulant generating function* $u(\cdot,\cdot)$ will satisfy

$$L_{t,\nu}(f) = exp\left(-\int_S u(t,x)\nu(dx)\right), \tag{2.4}$$

where $u(t,x)$ can be obtained as the solution to a (generally nonlinear) initial-value problem of the form

$$\frac{\partial u(t,x)}{\partial t} = Au(t,x) + h(u) \tag{2.5}$$

$$u(0,x) = f(x).$$

In the semigroup approach to proving existence, the goal is now to show that the solution $u(t,x)$ can be expressed as

$$u(t,x) = \mathrm{U}_t f(x) \quad , \tag{2.6}$$

where $\{U_t: t \geq 0\}$ is a semigroup of nonlinear operators on $C_b(S)$; it is then related to the generator G of the Markov process via the relationship $Gf = \lim_{t\ 0}(\mathrm{U}_t f - f)/t$.

Another approach to proving existence, due to Stroock and Varadhan, is to solve an appropriate martingale problem. Specifically, for a subset A of $C_b(E) \times L_\infty(E)$, a family of probability distributions $\{P_\mu: \mu \in E\}$ is a $P(\Omega_C^E)$ – (*or* $P(\Omega_D^E)$ –) *solution to the initial-value martingale problem* for A if for every pair $(f,g) \in A$, the following hold:

(i) $P_\mu \in P(\Omega_C^E)$ (or $P(\Omega_D^E)$) for each $\mu \in E$;

(ii) $P_\mu\{\omega : X(0,\omega) = \mu\} = 1$ for each $\mu \in E$;

(iii) for each P_μ, the canonical process $\{X(t): t \geq 0\}$ is a *solution to the martingale problem* for A in the sense that for every $(f,g) \in A$,

$$Z(t) \equiv f(X(t)) - \int_0^t g(X(s))\,ds$$

is a P_μ-martingale.

Dawson and Kurtz (1982) proved that the uniqueness, measurability, and strong Markov property of a $P\{\Omega_C{}^{M(S^*)}\}$-valued solution $\{P_\mu: \mu \in M(S^*)\}$ of the initial-value martingale problem associated with the pair (F_f, GF_f) for $f \in D = \bigcup_{N=0}^{\infty} D(S^{*N})$, where S^* denotes the one-point compactification $S \cup \{\infty\}$ of S, $F_f(\mu)$ is the *monomial*

$$F_f(\mu) \equiv \int_S \ldots \int_S f(x_1, \ldots x_{N(f)})\, \mu(dx_1) \ldots \mu(dx_{N(f)}) \tag{2.7}$$

on $M(S^*)$, $N(f) = N$ if $f \in D(S^{*N})$, and S^{*N} denotes the N-fold Cartesian product of S, follows from the existence of a solution to the martingale problem associated with the function-valued dual process $\{Y(t): t \geq 0\}$ with generator G^d defined via the relationship

$$GF_f(\mu) = G^d F_\mu(f) + V(N(f))\, F_\mu(f) \tag{2.8}$$

for a function V defined on the set $\{0,1,2,\ldots\}$, where $F_f(u) \equiv F_\mu(f)$. $X(t)$ and $Y(t)$ are said to form a pair of *dual processes* if the martingale problem associated with $\{(F_f, G\,F_f) : f \in D(G)\}$ has an $M(S^*)$-valued solution $\{X(t): t \geq 0\}$ and the martingale problem associated with $\{(F_\mu, G^d F_\mu): \mu \in E\}$ has a $\Pi^D(E)$-valued solution $\{Y(t): t \geq 0\}$, where $\Pi^D(E)$ denotes the smallest algebra of functions on E which contains $\Pi_1^D(E)$, the set of monomials with coefficients $f \in D$ restricted to E.

The function-valued process $Y(t)$ has a state space $\Pi^D(E)$ that is sufficiently rich so as to generate $L_\infty(E)$ under bounded pointwise convergence, so one can apply the theorem of Stroock and Varadhan (1979) that asserts that there will be at most one solution to the initial-value martingale problem for A if for each $t \geq 0$ and $F \in \Pi^D(E)$, there is a function $H_{F,t} \in L_\infty(E)$ such that for $\mu \in E$,

$$E_\mu\{F_f(X(t))\} = E\{F_\mu(Y(t))\, exp\left(\int_0^t V(Y(u))\, du\right) \Big|\, Y(0) = f\}. \tag{2.9}$$

for any solution $\{P_\mu: \mu \in E\}$ of the martingale problem for A. From the duality relationship (2.8) above, it follows that the function $H_{F,t}(\mu)$ will be given by

$$E_\mu\{F_f(X(t))\} = E\{F_\mu(Y(t)) \exp\left(\int_0^t V(Y(u))\,du\right) \Big| Y(0) = f\}. \tag{2.10}$$

III. Examples. A. *Measure-valued branching diffusion process.* Branching diffusion processes consist of individuals or particles that move in space according to some deterministic diffusion, and independently, at random times, each particle either splits into k ($k = 2,3,\ldots$) particles with probability $p_k \geq 0$ or disappears with probability $p_0 = 1 - \sum_{k=2}^{\infty} p_k \geq 0$. The process is called *critical* if the expected number of particles remains constant, as occurs, for example, in a binary branching situation with $p_0 = p_2 = 1/2$.

Now consider a critical branching diffusion process where the particles move in R^d according to a symmetric stable diffusion of index α, where $0 < \alpha \leq 2$ (e.g. , $\alpha = 2$ corresponds to Brownian motion diffusion), and, independently, after exponentially distributed holding times, either branch or die. The "high density" measure-valued limit of this process has been called the *stochastic measure diffusion process* and is obtained by considering a succession of such branching diffusions with progressively increasing numbers of particles of successively smaller individual mass in such a way that the expected total mass remains constant.

The basic construction of this process, together with some of its fundamental properties, appears in Dawson (1975,1977); the local structure of the resulting random measures is presented in Dawson and Hochberg (1979). Here the semigroup approach is used, and it is shown via the Trotter product formula that the semigroup $\{U_t: t \geq 0\}$ of nonlinear operators defined in (2.6) can be well approximated by the alternation of the Markov semigroup $\{S_t: t \geq 0\}$ of contraction operators on $C_k(R^d)$ associated with the symmetric stable process on R^d of index α with the semigroup $\{T_t: t \geq 0\}$ of operators that determines the branching mechanism, defined for $f \in C_k(R^d)$ by

$$T_t f(x) = f(x)/[1 - i\gamma t f(x)], \tag{3.1}$$

successively over small intervals of time. Thus, the value of the process at time t is approximated by the result of successively creating particles according to the branching mechanism defined by (3.1) and then smearing them out via the symmetric stable diffusion, each over small time intervals of length t/m.

The cumulant generating function (2.6) here satisfies the nonlinear equation

$$\frac{\partial u(t,x)}{\partial t} = G_\alpha u(t,x) - \gamma u^2(t,x)\,, \tag{3.2}$$

where G_α is the infinitesimal generator of the Markov semigroup $\{S_t\colon t > 0\}$. From the scaling property

$$S_t f(0) = S_{rt} f_{\alpha,r}(0) \tag{3.3}$$

for symmetric stable diffusions, where $f_{\alpha,r}(u) \equiv f(r^{-1/\alpha}u)$ and $r > 0$, it follows that over a time interval of length t, particles will tend to diffuse only within a region whose diameter is of the order $t^{1/\alpha}$.

B. *Multilevel branching process.* Measure-valued processes arising in the study of dynamic multilevel information structures have been introduced by Dawson and Hochberg (1990, 1991). At each level, individual information units undergo a Galton-Watson-type branching process in which they can be copied or removed. In addition, collections of information units at a given level comprise information units at the next higher level which also, independently, undergo Galton-Watson branching. Such multilevel systems arise in the description of replication of digitized data banks and in the evolution of animal and plant populations, in which individuals evolve both at the local level and at the colony level, in the sense that genetic structures of individuals may be altered through births and deaths within a given colony, while the colony itself may disappear at random times due to natural calamities, or it may replicate, say to establish a copy of itself at a new habitat. These mutlilevel processes also arise in the description of new mutations in mitochondrial DNA, because sampling processes are taking place at both the organelle and the individual levels.

In the two-level critical binary branching case, we start with a finite system of branching particles on R^d, which can be represented as a point in $M(R^d)$. These particles then replicate to create superparticles or die, and each particle then continues to evolve independently and undergo further branching, thus yielding a random number of copies. The two-level branching process $\{X_t\colon t \geq 0\}$ can then be viewed as an $N(Z^+)$-valued pure jump Markov process, where $N(Z^+)$ denotes the set of integer-valued measures on Z^+. A measure in $N(Z^+)$ can be represented by a vector $\{(n_i)\colon i \in Z^+\}$, where n_i denotes the number of superparticles consisting of exactly i particles. The four possible transitions of state in the process involve the birth or death in a superparticle of size i or the copy or disappearance of a superparticle of size i. The generator G of this

$N(Z^+)$-valued Markov processes and its domain $D(G)$ are given by

$D(G) = \{F: F(\mu) \equiv f(\Sigma\varphi_i n_i) = f(\int \varphi d\mu) \equiv f(<\varphi,\mu>)\}$ where $\mu = \Sigma n_i \cdot \delta_i$,

$$GF(\mu) = \gamma_2(1 + c_2/2)\int [f(<\varphi,\mu> + \varphi(x)) - f(<\varphi,\mu>)]\,\mu(dx) \tag{3.4}$$

$$+ \gamma_2 (1 - c_2/2)\int [f(<\varphi,\mu> - \varphi(x)) - f(<\varphi,\mu>)]\mu(dx)$$

$$+ \gamma_1 (1 + c_1/2)\int x[f(<\varphi,\mu> - \varphi(x) + \varphi(x + 1)) - f(<\phi,\mu>)]\mu(dx)$$

$$+ \gamma_1 (1 - c_1/2)\int x[f(<\varphi,\mu> - \varphi(x) + \varphi(x - 1)) - f(<\phi,\mu>)]\mu(dx)$$

where γ_1 and γ_2 are constants denoting the particle and superparticle branching rates, and the constants c_1 and c_2 are determined by the mean offspring sizes of the level-1 and level-2 branching processes, so that $c_i = 0$ ($> 0, < 0$) corresponds to critical (supercritical, subcritical) behavior at level i. An analysis of the set of moment measures $\{m_n: n = 1,2,...\}$ given by

$$m_n(t,\mu: A_1, ..., A_n) \equiv E\{\prod_{i=1}^{n} X(t, A_i) \mid X(0) = \mu\} \tag{3.5}$$

yields the characterization of the process $\{X_t\}$ as the unique solution of the $N(Z^+)$-valued martingale problem associated with G.

The continuous diffusion limit of this multilevel system is given by the $M(R^d)$-valued process $\{Y_t: t > 0\}$ obtained as the weak limit of the process

$$Y_t^\varepsilon(A) = \varepsilon X_t^\varepsilon(A/\varepsilon), \tag{3.6}$$

where $\{X_t^\varepsilon: t \geq 0\}$ is the two-level process with $\gamma_1^\varepsilon = \gamma_1/\varepsilon$, $\gamma_2^\varepsilon = \gamma_2/\varepsilon$, $c_1^\varepsilon = \varepsilon c_1$, $c_2^\varepsilon = \varepsilon c_2$. The process $\{Y_t\}$ is characterized as the unique solution to the martingale problem associated with the generator G_c given by

$$G_c F(\mu) = f'(<\phi,\mu>) < L\phi,\mu> + \gamma_2 f'(<\phi,\mu>) < c_2\phi,\mu> \tag{3.7}$$

$$+ \frac{1}{2}\gamma_2 f''(<\phi, \mu>) <\phi^2, \mu>$$

where

$$L\phi = \frac{1}{2}\gamma_1 x\phi'' + \gamma_1 c_1 x\phi'. \tag{3.8}$$

Here, the cumulant generating function u(. , .) in the Laplace functional (2.4) that determines the transition probabilities on $M(R^+)$ satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\gamma_1 x \frac{\partial^2 u}{\partial x^2} + c_1\gamma_1 x \frac{\partial u}{\partial x} - \frac{1}{2}\gamma_2 u^2 + c_2\gamma_2 u \tag{3.9}$$

$$u(0,x) = \phi(x).$$

A non-explosion criterion to ensure existence and uniqueness for a general class of multilevel birth-and-death processes is given in Dawson, Hochberg and Wu (1990).

C. *Multilevel branching diffusion process.* We now consider a finite system of branching random walks on R^d and again assume that the entire system performs critical binary branching after exponentially-distributed holding times. Each replica then develops independently according to the branching random walk. This addition of a spatial structure to the lowest level yields a multilevel branching random walk, which can be represented as a random atomic measure $Y_a(t)$ on $M(R^d)$. This process and its continuous diffusion limit $Y(t)$ are $M_2(R^d) = M(M(R^d))$-valued processes. $Y(t)$ is characterized via the martingale problem for the limiting generator G given by

$$GF(\nu) = \langle\langle \mathcal{L}F'(\nu,\cdot), \nu\rangle\rangle + \frac{1}{2}\int\int F''(\nu;\mu_1,\mu_2)\,\delta_{\mu_1}(d\mu_2)\,\nu(d\mu_1) \tag{3.10}$$

where

$$\langle\langle H,\nu\rangle\rangle = \int H(\mu)\,\nu(d\mu) \tag{3.11}$$

for $H \in C(M(R^d))$, and the test functions $F(\nu)$ on $M_2(R^d)$ have the form

$$F(\nu) = f(\langle\langle h_1(\langle h_2,\cdot\rangle), \nu\rangle\rangle) \tag{3.12}$$

where $h_1, f \in C_b^2(R), h_2 \in C^2(R^d), \nu \in M_2(R^d)$, $\langle h,\mu\rangle = \int h\,d\mu$,

$$F'(\nu,\mu) = \frac{\delta F(\nu)}{\delta\nu(\mu)} = \frac{d}{d\varepsilon}[F(\nu+\varepsilon\delta_\mu)]_{\varepsilon=0} = f'(\langle\langle h_1(\langle h_2,\rangle),\nu\rangle\rangle)h_1(\langle h_2,\mu\rangle), \tag{3.13}$$

and $\mathcal{L}$ denotes the generator of the $M(R^d)$-valued branching process, so

$$\mathcal{L}F'(\nu,\mu) = f'(\langle\langle h_1(\langle h_2,.\rangle),\nu\rangle\rangle)h'_1(\langle h_2,\mu\rangle)\langle\Delta h_2,\mu\rangle \tag{3.14}$$

$$+\frac{1}{2}f'(\langle\langle h_1(\langle h_2,.\rangle),\nu\rangle\rangle)h''_1(\langle h_2,\mu\rangle)\langle h_2^2,\mu\rangle.$$

The Laplace functional of the M_2-valued process $Y(.,.)$ is given by

$$L_{t,v}(H) = E[exp(-\int_{M(R^d)} Y(t, d\mu)) \mid Y(0) = v] = exp(-\int U(t,\mu)v(d\mu)) \tag{3.15}$$

for $H \in C_+ \ (M(R^d))$, where

$$\frac{\partial u}{\partial t} = \mathcal{L}U - U^2. \tag{3.16}$$

D. *Fleming-Viot process and other genetical models.* Fleming and Viot (1979) introduced a model to describe the distribution of multi-dimensional genetic characteristics in large natural populations. Here, each coordinate represents a different observable genotypic or phenotypic characteristic of the individuals in the population. This model is analyzed in Dawson and Hochberg (1982, 1983), where it is shown that it arises as the weak limit of the continuous-time Ohta-Kimura ladder or stepwise-mutation model for selectively neutral allelic populations evolving under random genetic drift, via multinomial sampling from the empirical distribution of allelic frequencies in the host population, and a symmetric mutation structure, where the large population limit is taken so that the mutation rate remains constant while the incremental effect of each mutation is assumed to decrease.

A general characterization of the Fleming-Viot process that includes the possibility of selective advantage of some allelic types over others is provided in Ethier and Kurtz (1987). Let the compact metric space S denote the set of types, let the mutation rates be described by the generator A of a Feller semigroup on the space $C(S)$ of continuous functions on S, and let selection intensities be specified by a symmetric, bounded, Borel function σ on $S \times S$. The single-locus measure-valued diffusion $\{X_t\colon t \geq 0\}$ is the Markov process in $P(S)$, the space of Borel probability measures on S, associated with the generator G with domain $D(G)$ given by

$$D(G) = \{\phi\colon \phi(\mu) \equiv \prod_{i=1}^{n} \langle f_i,\mu\rangle, f_1,\ldots f_n \in D(A), n \geq 1\},$$

$$G\phi(\mu) = \sum_{i<j} [\ \langle f_i f_j,\mu\rangle - \langle f_i,\mu\rangle \ \langle f_j,\mu\rangle\ ] \prod_{l \neq i,j} \langle f_l, \mu\rangle \tag{3.17}$$

$$+ \sum_i \langle Af_i,\mu\rangle \prod_{j \neq i} \langle f_j,\mu\rangle \ + \sum_i [\ \langle \sigma f_i,\mu^2\rangle - \langle f_i,\mu\rangle \ \langle \sigma, \mu^2\rangle\ ] \prod_{j \neq i} \langle f_j,\mu\rangle,$$

where $\langle f,\mu\rangle = \int f d\mu$. Specifically, A can be taken to be the infinitesimal generator G_α of a symmetric stable diffusion of index α, or, for the case $\alpha = 2$, the Laplace operator Δ.

Vaillancourt (1990) has extended our study of this process to include a class of weakly-interacting Fleming-Viot processes. Specifically, he considers the case where offspring choose their genotype or phenotype from the empirical frequency distribution of types in some neighboring population, rather than from the parent or host population.

Ethier and Kurtz (1987) have analyzed the infinitely-many-alleles model of Kimura and Crow using a similar measure-valued approach necessitated by the introduction of selection into the model. Ethier and Griffiths (1987) provide the corresponding measure-valued analysis for the infinitely-many-sites model of Kimura. These last two models take into consideration the possibility that mutations do not always lead to previously existing states. Such dynamic models arising from molecular population genetics theory have gained more credence among geneticists as a result of the recent recognition of the gene as a sequence of nucleotides and the resulting advances in nucleotide sequencing.

E. *Multilocus Fleming-Viot process with recombination.* The multilocus measure-valued Fleming-Viot model incorporates the possibility of recombination between genes at different loci. In the two-locus diffusion, the state space is taken to be $R^{d_1} \times R^{d_2}$ and (3.17) is replaced by

$$
\begin{aligned}
G\prod_{i=1}^{n} &<f_i \times g_i, \mu> \\
&= \sum_{i<j} [<f_i f_j \times g_i g_j, \mu> - <f_i \times g_i, \mu> <f_j \times g_j, \mu>] \prod_{l \neq i,j} <f_l \times g_l, \mu> \qquad (3.18) \\
&+ (1/2) D_1 \sum_{i=1}^{n} <\Delta_{d_1} f_i \times g_i, \mu> \prod_{l \neq i} <f_l \times g_l, \mu> \\
&+ (1/2) D_2 \sum_{i=1}^{n} <f_i \times \Delta_{d_2} g_i, \mu> \prod_{l \neq i} <f_l \times g_l, \mu> \\
&+ (1/2) \rho \sum_{i=1}^{n} [<f_i \times 1, \mu> <1 \times g_i, \mu> - <f_i \times g_i, \mu>] \prod_{l \neq i} <f_l \times g_l, \mu>.
\end{aligned}
$$

where Δ_d denotes the d-dimensional Laplace operator, D_1 and D_2 are mutation rates, and ρ, $0 \leq \rho \leq \infty$, is a recombination rate such that $\rho = 0$ corresponds to complete linkage and $\rho = \infty$ corresponds to zero linkage.

Ethier and Griffiths (1990) have discussed the general two-locus theory in the case where the two mutation semigroups (one for each locus) are ergodic. Uniqueness of the measure-valued two-locus problem follows using the function-valued dual approach, since the recombination term introduces a linear birth rate while the random genetic drift (final) term introduces a quadratic death rate. The special case of complete linkage can also be considered as a particular example of (3.17).

IV. Results. Analysis of the measure-valued processes introduced in the last section has produced results of three distinct types: fixed-time behavior of the process, including analysis of the topological support set of the associated random measures, clustering phenomena for individuals or mass, coherent translation or dispersive behavior of mass over the state space, and distribution of the numbers and mass of surviving particles; long-term behavior of the process, including ergodic limits, stationarity of the limiting distribution, rates of growth or decay, and the wandering or coherent nature of the processes as time increases; and renormalization theorems, in which an appropriate scaling factor is incorporated into both the time and space variables, and the resulting rescaled process is shown to converge to a limit as the scaling factor increases.

A. *Fixed-time behavior.* At each fixed time t, the value of a measure-valued process is given by a random measure $X_t(\omega)$. This measure is said to be *singular* if there exists a random support for the process that has Lebesgue measure zero. In general, a set is said to be *singular* if it has zero Lebesgue measure.

The *Hausdorff-Besicovitch dimension of support* of a Borel set E is defined by

$$H-\dim\ E = \sup\{\beta > 0 : \lim_{\delta \downarrow 0} \inf_{E} \sum_i [d(E_i)]^{\beta} = \infty\} \tag{4.1}$$

where $d(E_i)$ is the diameter of the set E_i and

$$E = \{\{E_i\} : E \subset \bigcup_i E_i, d(E_i) < \delta \text{ for each } i\},$$

the set of all coverings of the set E by sets of diameter less than δ.

A fundamental relationship between the Hausdorff-Besicovitch dimension of support and singularity of a set lies in the following result:

THEOREM 4.1. *If E is a subset of R^d and the Hausdorff-Besicovitch dimension of E is less than d, then the set E is singular.*

A set B in R^d is called a *generalized random Cantor set* if B can be expressed as the intersection of an infinite decreasing sequence of sets $\{B_n$: $n = 0,1,2,...\}$, such that B_0 is a unit cube in R^d, and each B_n is the union of some number, say Λ_n, of disjoint subcubes of volume $(\Gamma_n)^{-d}$, where $\{\Gamma_n\}$ is an increasing sequence of non-negative integers. In other words, a generalized random Cantor set is a limit of sets B_n which, as n increases, themselves are the unions of progressively increasing numbers of subsets, each of which is successively smaller in volume. A generalized random Cantor set differs from the Cantor ternary set in that the sequence of numbers $\{\Lambda_n\}$ and $\{\Gamma_n\}$ are not necessarily deterministic but may be random, as are the locations of the surviving intervals or subcubes in the successive B_n's.

The following relationship between generalized random Cantor sets and Hausdorff dimensions is proven in Dawson and Hochberg (1982, Lemma 3.1) :

THEOREM 4.2. *If B is a generalized random Cantor set, then*

$$H\text{-dim } B \leq \liminf_{n\to\infty} \log \Lambda_n / \log \Gamma_n.$$

For the measure-valued branching diffusion, it is shown in Dawson and Hochberg (1979) that the approximating sequence of smeared particle processes described in section IIIA above can be viewed as a hierarchy of smeared clusters at different scales. The n-th scale is obtained by dividing the unit cube $V \subset R^d$ into $\Gamma_n{}^d$ equal subcubes of volume $\Gamma_n{}^{-d}$. For $m_n = [2\gamma t\, \Gamma_n{}^{\alpha}]$, where $[x]$ denotes the greatest integer less than or equal to x, one then obtains a generalized random Cantor set for which the random measure $X(t/m_n)$ consists of a Poisson number of clusters, each with total mass that is exponentially distributed with mean $\gamma t/m_n$. It then follows from Theorem 4.2 that the Hausdorff-Besicovitch dimension of the topological support of the random measure $X_t(\omega)$ is bounded above by the index α of the stable diffusion in dimension $d \geq \alpha$. Theorem 4.1 then implies that the random measure $X_t(\omega)$ is singular if $d > \alpha$.

Singularity of $X_t(\omega)$ for $d = \alpha$ ($\alpha = 1,2$) follows from the self-similarity property

$$X(t,A) \stackrel{d}{=} X(k^{\alpha}t, A_k)/k^{\alpha}, \tag{4.2}$$

where $x \in A_k$ if and only if $x/k \in A$. Taking A compact and $k = t^{-1/\alpha}$, this says that

$$X(t,A) \stackrel{d}{=} X(1,A_{t^{-1/\alpha}})/(1/t). \tag{4.3}$$

This expression may be thought of as describing a telescopic effect; it relates the relative density of the measure over a shrinking set at a fixed time to the measure over a fixed compact set as t increases. Since $X(t,A)$ converges to zero in probability for large t in one and two dimensions (Dawson (1977)), $X(1,\cdot)$ can have no absolutely continuous component.

For the Fleming-Viot process on R^d, let $S = R^d \cup \{\infty\}$, the one-point compactification of R^d. At fixed times t, the process is described by a random probability measure $X = X_t(\omega)$ on S. The *k-th moment measure* $M_k\,(dx_1,...,dx_k)$ is a probability measure on S^k defined by

$$E\{\prod_{j=1}^{k} <\phi_j, X> \} = \int_S ... \int_S \left(\prod_{i=1}^{k} \varphi_i(x_i)\right) M_k(dx_1,...,dx_k) \tag{4.4}$$

for $\phi_i \in C(S)$. It is proven in Dawson and Hochberg (1982, section 6) that a random probability measure X can be related via the system of moment measures $\{M_k(dx_1,...,dx_k) : k = 1,2,...\}$ to an infinite system of interacting particles via the following technique, which we shall call the *de Finetti representation of a random probability measure.*

The collection $\{M_k\}$ forms a consistent family of probability measures; moreover, the measure M_k is an exchangeable probability law on S^k, i.e., for $A_1,..., A_k \in B(S)$,

$$M_k(A_1,...,A_k) = M_k(A_{\pi(1)},...,A_{\pi(k)}) \tag{4.5}$$

for every permutation π. It follows from the Kolmogorov extension theorem that there exists a probability measure P^* on (S^∞, F^*), where F^* denotes the P^*-completion of the product σ-algebra, such that

$$P^*(A_1 \times ... \times A_k) = M_k(A_1,...,A_k); \tag{4.6}$$

moreover, P^* is the probability law of a sequence $\{Z_k: k = 1,2,...\}$ of exchangeable S-valued random variables, which can be viewed as the locations of a countable collection of particles in S. For ω in S^∞ and A in a countable algebra that generates $B(S)$, the de Finetti theorem for exchangeable random variables implies the existence of

$$Y^*(A,\omega) = \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N} I_A(Z_j(\omega)), \tag{4.7}$$

where $I_A(x)$ equals one if x is an element of the set A and zero otherwise. Y^* is a regular conditional probability given the σ-algebra F^*, so

$$P^*\{Z_k \in A_k, k = 1,2 \ldots \mid F^*\} = \prod_{k=1}^{\infty} Y^*(\omega, A_k) \quad P^* - \text{a.s.} \tag{4.8}$$

Moreover, there exists an extension of Y^* to a random measure Y such that Y is a version of X; i.e., for any set $B \in B(P(S))$,

$$P^* \{Y \in B\} = P\{X \in B\}. \tag{4.9}$$

For the Fleming-Viot process, the random motion of this related infinite particle system $\{Z_n\}$ is described by the motion of its k-particle subsystems, as follows: each particle performs an independent symmetric diffusion of index α on R^d, and, at a constant rate, one particle disappears and another splits into two particles, each of which continues to move independently according to the same symmetric diffusion law. If we look at the infinite system at time t_0 and trace back the genealogy of these particles to a time $t_0 - T_n^\infty$ at which the infinitely-many particles had exactly n common ancestors and now look only at these n "surviving family trees," we find that they form n random clusters. Moreover, within each cluster the radius is bounded, with the bound determined from the scaling property (3.2). As n increases, we obtain increasing numbers of such clusters, each with progressively smaller radii, so the clusters form a generalized random Cantor set. We then proceed as detailed earlier and conclude that the Fleming-Viot random measure has Hausdorff dimension bounded above by $min(d,\alpha)$ and is singular in dimensions greater than α, indicating a highly clustered distribution and, thus, a high correlation between the genetic traits.

The fact that $d = \alpha$ is in fact the exact lower bound for the Hausdorff dimension of support of these random measures X_t in dimensions $d \geq \alpha$ at fixed times is a consequence of the following result of Zähle (1988):

THEOREM 4.3. *Let X be a random measure with second moment measure K(x,dy) and assume that for E(X)-a.e.* $x \in R^d$,

$$\int_{S(x,\varepsilon)} |x-y|^{-D} K(x,dy) < \infty$$

where $S(x,\varepsilon)$ *is some sphere of radius* $\varepsilon > 0$ *centered at x. Then,*

$$P\{X(B) > 0 \setminus \{\dim(\Phi \cap B) \geq D\}\} = 0,$$

where Φ denotes the closed support of X.

Since the second moment measure $K(x,dy)$ for both the measure-valued branching diffusion and Fleming-Viot processes satisfies

$$K(x,y) \sim |x-y|^{-(d-\alpha)} \tag{4.10}$$

for small $|x-y|$, we have

$$\int \frac{1}{|x-y|^D} \frac{1}{|x-y|^{d-\alpha}} dx \approx \int \frac{r^{d-1}}{r^{D+d-\alpha}} dr = \int_{0^+} \frac{1}{r^{D-\alpha+1}} dr \tag{4.11}$$

which converges if and only if $D < \alpha$, so by Theorem 4.3, dim $\Phi \geq D$ for all $D < \alpha$, so dim $\Phi \geq \alpha$.

The same technique applied to the multilevel branching diffusion system in four or more dimensions yields a lower bound of 2α for the Hausdorff dimension of the support of the random measure Y_t.

In each of these processes, there are two forces acting in opposing ways on the distribution: branching (and random genetic drift in the Fleming-Viot process) leads on occasion to disappearance of individuals from the population, whereas diffusion (and mutation in the Fleming-Viot case) leads to a spread of the process to new members. The mean distance between survivors in a branching process is of the order $t^{1/d}$ (and $t^{2/d}$ in the case of two-level branching), whereas a stable diffusion of index α spreads at the rate $t^{1/\alpha}$, so we see changes of behavior at the critical dimension $d = \alpha$ for the single-level cases and $d = 2\alpha$ for the two-level situation.

The topological support properties for the two-locus Fleming-Viot model with recombination given by (3.18) have not yet appeared in the literature. What is known to date can be summarized in the following theorem, obtained with S.N. Ethier:

THEOREM 4.4. *The topological support of the random measure $X_t^\rho(\omega)$ associated with the two-locus Fleming-Viot process with combination rate ρ characterized by the generator* (3.18) on $R \times R$ and $R^2 \times R^2$ *has the properties summarized in the following chart, where H*-dim *A is the Hausdorff-Besicovitch dimension of the set A and "singular" and "absolutely continuous" denote a.e.-properties of the random measure:*

space	complete linkage $\rho = 0$	partial linkage $0 < \rho < \infty$	zero linkage $\rho = \infty$
$R \times R$	H-dim $X_t^0(\omega) = 2$ singular	H-dim $X_t^\rho(\omega) = 2$	H-dim $X_t^\infty(\omega) = 2$ absolutely continuous
$R^2 \times R^2$	H-dim $X_t^0(\omega) = 2$ singular	$2 \leq H$-dim $X_t^\rho(\omega) \leq 4$ singular	H-dim $X_t^\infty(\omega) = 4$ singular

Proof: In the case of complete linkage there is no recombination, so we have one Fleming-Viot process in $2d$ dimensions ($d = 1,2$). Applying the result that H-dim $X_t(\omega) = min(d,2)$ for the Fleming-Viot process with Brownian diffusion parameter $\alpha = 2$, we get H-dim $X_t^0(\omega) = min(2d,2) = 2$ for $d = 1,2$. For $d = 1$ we have a two-dimensional Fleming-Viot random measure, which was shown above to be singular. For $d = 2$ the random measure $X_t^0(\omega)$ is singular by Theorem 4.1, since we are in a $2 \times 2 = 4$-dimensional situation, and the Hausdorff dimension of the support set of $X_t^0(\omega)$ is only two, which is less than this dimension.

In the case of zero linkage there is complete recombination, so the Hausdorff dimension of support is $min(d,2) \times min(d,2) = min(4,2d)$. For $d = 1$ we get H-dim $X_t^\infty(\omega) = 2$, and the random measure is absolutely continuous with respect to Lebesgue measure, as is each one-dimensional Fleming-Viot measure. For $d = 2$, the Hausdorff dimension of support is four, and the measure is singular because it is formed from two singular two-dimensional Fleming-Viot measures. It is then clear that for $0 < \rho < \infty$, the random measure $X_t^\rho(\omega)$ will have Hausdorff dimension of support equal to two in the case $d = 1$ as it is for the extremal cases $\rho = 0$ and $\rho = \infty$, and dimension bounded below by two and above by four in the case $d = 2$. Similarly, the measure $X_t^\rho(\omega)$ will be singular in the case $d = 2$. □

REMARK: It seems reasonable to conjecture that the exact Hausdorff dimension of $X_t^\rho(\omega)$ in the case of partial linkage will be a function of the recombination rate ρ.

The Fleming-Viot process satisfies an additional property as well, that of long-term compact coherence. A probability-measure-valued process $\{X(t): t > 0\}$ is said to be *coherent* if for every $\varepsilon > 0$ there exists t_0, $0 \leq t_0 < \infty$, with the property that for each $t \geq t_0$ there is a random sphere $S_\varepsilon(t,\omega)$ centered at

$$x(t) = \int_S xX(t,dx)\,, \tag{4.12}$$

with radius

$$R_\varepsilon(t) = \int_S |x| X(t,dx) < \infty \tag{4.13}$$

that is a stationary stochastic process, such that

$$P\{X(t,\omega,S_\varepsilon(t,\omega)) \geq 1-\varepsilon\} = 1. \tag{4.14}$$

The *wandering motion* of a coherent distribution can be described by the process $\{x(t)\colon t \geq 0\}$. The process $\{X(t)\}$ is said to be *compactly coherent* if (4.14) also holds for $\varepsilon = 0$ with some centering, not necessarily that prescribed by (4.12); thus, the sphere $S_0(t,\omega)$ contains all the mass of the process $X(t,\cdot)$ with probability one. A process that is not coherent is said to be *dispersive.*

The results of this section, excluding those already included in Theorem 4.4, can be summarized as follows:

THEOREM 4.5. *Let $X_t^{BD}(\omega)$ and $X_t^{FV}(\omega)$ denote the measure-valued branching diffusion and Fleming-Viot random measures, respectively , on R^d with spatial diffusion governed by a symmetric stable process of index α, $0 < \alpha \leq 2$, and let $X_t^{MLB}(\omega)$ and $X_t^{MBD}(\omega)$ denote the two-level branching and two-level branching diffusion random measures, respectively, at a fixed time t. Assume that each of these random measures has compact support at time $t = 0$ with probability one. Then the following hold:*

(i) *In any spatial dimension, each of these random measures has compact support, with probability one, for each fixed* $t > 0$.

(ii) *The Hausdorff-Besicovitch dimension of the topological supports of $X_t^{BD}(\omega)$ and $X_t^{FV}(\omega)$ both equal $\min(d,\alpha)$.*

(iii) *The random measures $X_t^{BD}(\omega)$ and $X_t^{FV}(\omega)$ are singular in R^d for $d \geq \alpha$, with probability one.*

(iv) *The Fleming-Viot random measure $X_t^{FV}(\omega)$ is compactly coherent.*

(v) *The Hausdorff-Besicovitch dimension of the topological supports of $X_t^{MLB}(\omega)$ and $X_t^{MBD}(\omega)$ are bounded below by 2α in dimensions $d \geq 2\alpha$.*

B. *Long-term behavior.* We have already noted that the measure-valued branching diffusion process $X(t,A)$ on a compact set A converges to zero in probability for large t in the case of recurrent diffusion (e.g., in one and two dimensions for $\alpha = 2$). When the diffusion is transient, it is shown in Dawson (1977) that there exists a limiting steady-state random measure.

In Dawson and Hochberg (1982,1983), the single-locus Fleming-Viot process $\{X(t): t \geq 0\}$ on $S = R^d \cup \{\infty\}$ with $A = \Delta_d$, the d-dimensional Laplace operator, is related to a function-valued dual process $\{Y(t): t \geq 0\}$ with infinitesimal generator G^d corresponding to that of a Markov process on $C = \bigcup_{N=0}^{\infty} C(S^N)$, evolving according to the following mechanisms:

(i) $Y(t)$ jumps from $C(S^N)$ to $C(S^{N-1})$ for $N = 2,3...$

(ii) at the time of a jump from $C(S^N)$ to $C(S^{N-1})$, a pair $\{j,k\}$ is picked at random from $\{1,2,...,N\}$, and f is replaced by a function $\Phi_{jk}f$ of the N-1 variables $x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_N$, the k - th variable x_k having merged with the j-th-variable x_j;

(iii) between jumps, $Y(t)$ evolves deterministically on $C(S^N)$ according to the Brownian motion semigroup H_t^N on $(R^d)^N$ given by

$$H_t^N f(x) = (2\pi t)^{-(Nd)/2} \int e^{-|x-y|^2/(2t)} f(y)\, dy; \tag{4.15}$$

(iv) once $Y(t)$ is a continuous function of only one variable, no further jumps occur.

The *empirical centered moments* of $\{X(t): t \geq 0\}$ are defined by

$$R_{k_1,...,k_d}(t) \equiv \int_{R^d} \prod_{i=1}^{d} (x_i - x_i(t))^{k_i} X(t, dx) \tag{4.16}$$

where the *empirical mean process* $x(t)$ is given by

$$x(t) \equiv (x_1(t), ..., x_d(t)) \tag{4.17}$$

where

$$x_i(t) = \int_{R^d} x_i X(t, dx), \quad i = 1, 2, ..., d. \tag{4.18}$$

If we apply the duality relationship (2.10) to $E_\mu\{\ R_{k_1,\ldots,k_d}(t)\ \}$ and note that the number $N(f)$ of variables in the dual process $Y(t)$ decreases according to a pure death process and thus converges to one in finite time with probability one, we obtain the following:

THEOREM 4.6. *For the Fleming-Viot process* $\{X(t)\colon t \geq 0\}$, *if*

$\int |x|^{N_0} \mu(dx) < \infty$ for $N_0 = \sum_{l=1}^{d} k_l$, *then*

(a) $E_\mu\{R_{k_1,\ldots,k_d}(t)\} < \infty$ *for* $t \geq 0$,

and

(b) $\lim_{t\to\infty} E_\mu\{R_{k_1,\ldots,k_d}(t)\} \equiv r_{k_1,\ldots,k_d}$

exists and is finite.

The random cluster

$$X^*(t, dx) = X(t, \{a - x(t) : a \in dx\}) \tag{4.19}$$

centered at the empirical mean $x(t)$ is a stationary process, and the Birkhoff ergodic theorem implies that

$$I(f) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{R^d} f(x)\, X^*(t, dx)dt \tag{4.20}$$

exists a.s. as a linear functional on $C(S)$ and is independent of the initial measure μ. We therefore have the following result:

THEOREM 4.7. *The centered Fleming-Viot random cluster* $X^*(t,dx)$ *satisfies*

$$\lim_{t\to\infty} E_\mu\ \{\int f(x)\, X^*(t, dx)\} = I(f) \tag{4.21}$$

for any $\mu \in P(S)$, *where* $I(f)$ *is given by* (4.20).

For example, we have the non-Gaussian limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T X^*(t)dt = E\{X^*(0)\} = \gamma \in P(R^d) \text{ a.s.}; \tag{4.22}$$

i.e.,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T X^*(t,A)dt = \upsilon(A), \tag{4.23}$$

where $\upsilon(A)$ is the expected value of the equilibrium random measure on the set A. The collection $\{r_{k_1,\ldots,k_d}\}$ forms the joint moment system of the expected steady-state distribution of the random cluster centered at the empirical mean.

For the continuous limit $Y(t)$ of the multilevel branching process, the long-term extinction problem is solved in Dawson and Hochberg (1991) for various combinations of the values of c_i and γ_i ($i = 1,2$) using the martingale-problem formulation. In addition, the following generalizations of the growth rate and conditional limit law (conditioned on non-extinction of the process) for ordinary critical continuous-state branching are obtained for the continuous multilevel process $Y(t)$:

THEOREM 4.8. (a) *Let Y_t denote the multilevel branching continuous limit process with $\gamma_1 = \gamma_2 = 2$, $c_1 = c_2 = 0$. Then*

$$\lim_{t\to\infty} t^2 P\{\langle x, Y_t\rangle > 0\} = c, \text{ a constant.} \tag{4.24}$$

(b) *For $A \in B((0,\infty))$ and $t > 0$, let*

$$Y_K(t,A) := Y(Kt,\{x\colon x/K \in A\})/K.$$

Then conditioned on $\langle x,Y(t)\rangle > 0$ and $Y(0) = \delta_x$, $Y_K(t)$ converges weakly to a random measure with Laplace functional

$$L(\phi) = 1 - xV(t,\phi) \tag{4.25}$$

where

$$V(t,\phi) = (c_0 x)^{-1} \lim_{K\to\infty} Kv(t,x/K) \tag{4.26}$$

for a constant c_0 *and function $v(t,x)$ satisfying the partial differential equation*

$$\frac{\partial v}{\partial t} = x\frac{\partial^2 v}{\partial x^2} - v^2 \tag{4.27}$$

with initial value $v(0) = \phi$, $\phi \in C_K(0,\infty)$.

(c) *For each $k \in Z^+$ there exists $c_k \in R$, called the k-th* limiting cumulant *of the total mass process, such that (formally) as $t \to \infty$,*

$$\log E_{\delta_x}\left(exp\left[\frac{-\theta < x,Y(t) >}{t^2}\right]\middle| < x,Y(t) > \quad > 0\right) \to \sum_{k=1}^{\infty} (-\theta^k)\, c_k/k!. \tag{4.28}$$

We note that a growth rate of t^2 in equations (4.24) and (4.28) for multilevel critical branching replaces the usual rate of t for ordinary critical branching.

In the two-level branching diffusion model, analysis of the dual process in [6] yields the following:

THEOREM 4.9. *Let $Y(t)$ denote the continuous limit of the two-level branching diffusion on R^d with spatial diffusion governed by a symmetric stable process of index α, and let the* total mass process $\{\bar{Y}(t) : t \geq 0\}$ *be defined by*

$$\bar{Y}(t) = \int \mu Y(t,d\mu). \tag{4.29}$$

Then, $E\{<\bar{Y}(t),\phi>\}$ remains bounded if and only if $d > 2\alpha$.

C. *Renormalization results.* The single-locus Fleming-Viot process $\{X(t): t \geq 0\}$ on R^d with $A = \Delta_d$ can be rescaled as follows. Let

$$X_\varepsilon(t,A) = X(t/\varepsilon^2, \{x{:}\varepsilon x \in A\}). \tag{4.30}$$

Then, $\{X_\varepsilon(t)\}$ has generator L_ε which is related via

$$L_\varepsilon F_f(\mu) = L_\varepsilon^d F_\mu(f) \tag{4.31}$$

to a function-valued dual process $\{Y_\varepsilon(t)\}$ with generator $L_\varepsilon{}^d$ given by

$$L_\varepsilon^d F_\mu(f) = F_\mu(\Delta_{N(f)} f) + \frac{c}{\varepsilon^2} \sum_{j=1}^{N(f)} \sum_{\substack{k=1 \\ j \neq k}}^{N(f)} [F_\mu(\phi_{jk} f) - F_\mu(f)]\,, \tag{4.32}$$

where Φ_{jk} is the operator defined in section B. For $f \in C(S^N)$,

$$\lim_{\varepsilon \to 0} E_{\delta_x}\{F_f(X_\varepsilon(t))\} = \lim_{\varepsilon \to 0} E_f\{F_{\delta_x}(Y_\varepsilon(t))\} \tag{4.33}$$

$$= H_t^1(\Psi^1 f)(x) = E\{F_f(\delta_{W(t)})\}\,,$$

where $H_t^N f(x)$ is defined in (4.15), $\{W(t): t \geq 0\}$ denotes the standard Wiener process in R^d, and $\psi^1 f(x) \equiv f(x,x,\ldots,x)$. We have thus proven the following:

THEOREM 4.10. *Let $\{X(t): t \geq 0\}$ be the Fleming-Viot process in R^d with diffusion generator $A = \Delta_d$. Assume that $X(0)$ has compact support. Then, the fi-*

nite-dimensional distributions of the rescaled process $\{X_\varepsilon(t): t \geq 0\}$ *defined by* (4.30) *converge to those of the probability-measure-valued process which consists of a single unit atom undergoing Brownian motion in* R^d.

In [2], this scaling limit is obtained, with convergence in the sense of weak convergence of probability measures on $\Omega_c^{P(S)}$, using martingale techniques. Vaillancourt (1990a, b) has similarly derived scaling limit results for a class of weakly-interacting Fleming-Viot processes.

References

[1] Dawson, D.A. (1975). Stochastic evolution equations and related measure processes. *J. Multivariate Analysis* **5** 1-52.

[2] Dawson, D.A. (1977). The critical measure diffusion process. *Z. Wahrsch. Verw .Geb.* **40** 125-145.

[3] Dawson, D.A. and Hochberg, K.J. (1979). The carrying dimension of a stochastic measure diffusion. *Ann. Probability* **7** 693-703.

[4] Dawson, D.A. and Hochberg, K.J. (1982). Wandering random measures in the Fleming-Viot model. *Ann. Probability* **10** 554-580.

[5] Dawson, D.A. and Hochberg, K.J. (1983). Qualitative behavior of a selectively neutral allelic model. *Theoretical Population Biology* **23** 1-18.

[6] Dawson, D.A. and Hochberg, K.J. (1985). Function-valued duals for measure-valued processes and applications. *Particle Systems, Random Media, and Large Deviations*, R. Durrett, ed., *Contemporary Mathematics* **41** 55-69.

[7] Dawson, D.A. and Hochberg, K.J. (1991). A multilevel branching model. *Adv. Appl. Prob.* In press.

[8] Dawson, D.A., Hochberg, K.J. and Wu, Y. (1990). Multilevel branching systems. *White Noise Analysis -- Mathematics and Applications*, T. Hida, H.H. Kuo, J. Potthoff, and L. Streit, eds., World Scientific Pub; Singapore - New Jersey - London, 93-107.

[9] Dawson, D.A., Iscoe, I. and Perkins, E.A. (1989). Super-Brownian motion: path properties and hitting probabilities. *Probab. Th. Rel. Fields* **83** 135-205.

[10] Dawson, D.A. and Kurtz, T.G. (1982). Application of duality to measure-valued diffusion processes. *Advances in Filtering and Optimal Stochastic Control. Lecture Notes in Control and Information Sciences*, Springer-Verlag, Berlin-Heidelberg-New York, **42** 91-105.

[11] Dynkin, E.B. (1988). Representation for functionals of superprocesses by multiple stochastic integrals, with applications to self-intersection local times. *Asterisque* **157-158** 147-171.

[12] Dynkin, E.B. (1989). Superprocesses and their linear functionals. *Trans. Am. Math. Soc.* **314** 255-282.

[13] Dynkin, E.B. (1989). Regular transition functions and regular superprocesses. *Trans. Am. Math. Soc.* **316** 623-634.

[14] Dynkin, E.B. (1989). Branching particle systems and superprocesses. Preprint.

[15] Ethier, S.N. and Griffiths, R.C. (1987). The infinitely-many-sites model as a measure-valued difffusion. *Ann. Probability* **15** 515-545.

[16] Ethier, S.N. and Griffiths, R.C. (1990). The neutral two-locus model as a measure-valued difffusion. *Adv. Appl. Prob.* **22** 773-786.

[17] Ethier, S.N. and Kurtz, T.G. (1987). The infinitely-many-alleles model with selection as a measure-valued diffusion. *Stochastic Methods in Biology. Lecture Notes in Biomathematics*, Springer-Verlag, Berlin-Heidelberg-N.Y., **70** 72-86.

[18] Fleming, W.H. and Viot , M. (1979). Some measure-valued Markov processes in population genetics theory. *Indiana Univ. Math. J.* **28** 817-843.

[19] Hochberg, K.J. (1980). Medidas de difusion estocasticas. *Conferencias Sobre Sistemas Estocasticos*, Centro de Investigacion del Inst. Politecnico Nac., Mexico, 97-107.

[20] Hochberg, K.J. (1983). Microscopic clustering of population processes. Population Biology. *Lecture Notes in Biomathematics*, Springer-Verlag, Berlin-Heidelberg- N.Y., **52** 18-24.

[21] Hochberg, K.J. (1986). Stochastic population theory: mathematical evolution of a genetical model. *The Merging of Disciplines: New Directions in Pure, Applied, and Computational Mathematics*, R.E. Ewing, K.I. Gross and C.F. Martin, eds., Springer-Verlag, N.Y., 101-115.

[22] Iscoe, I. (1986). A weighted occupation time for a class of measure-valued critical branching Brownian motion. *Probab. Th. Rel. Fields* **71** 85-116.

[23] Iscoe, I. (1986). Ergodic theory and a local occupation time for measure-valued branching processes. *Stochastics* **18** 197-243.

[24] Iscoe, I. (1988). On the supports of measure-valued critical branching Brownian motion. *Ann. Probab.* **16** 200-221.

[25] Perkins, E.A. (1988). A space-time property of a class of measure-valued branching diffusions. *Trans. Am. Math. Soc.* **305** 743-795.

[26] Perkins, E.A. (1989). The Hausdorff measure of the closed support of super Brownian motion. *Ann. Inst. H. Poincare* **25** 205-224.

[27] Perkins, E.A. (1990). Polar sets and multiple points for super Brownian motion. *Ann. Probab.* **18** 453 - 491.

[28] Stroock, D.W. and Varadhan, S.R.S. (1979). *Multidimensional Diffusion Processes.* Springer-Verlag, Berlin-Heidelberg-New York.

[29] Vaillancourt, J. (1990a). Interacting Fleming-Viot processes. *Stoch. Proc. Appl.* **36** 45-57.

[30] Vaillancourt, J. (1990b). On the scaling theorem for interacting Fleming-Viot processes. *Stoch. Proc. Appl.* **36** 263-267.

[31] Zähle, U. (1988). Self-similar random measures, I. Notion, carrying Hausdorff dimension, and hyperbolic distribution. *Probab. Th. Rel. Fields* **80** 79-100.