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REDISTRIBUTION OF VELOCITY: COLLISION TRANSFORMATIONS

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Dedicated to David Blackwell

Abstract. Suppose a continuum of identical particles collide in triples forming a complex. The total momentum and energy of each triple is redistributed according to some given redistribution law. It is shown that there is an invariant distribution of velocity such that for any initial distribution of velocity with all moments finite, the distribution of velocity obtained under iteration converges to this invariant distribution. We show that there are several natural redistribution of velocity laws whose invariant distribution is the normal law. These results are a continuation of some work of Blackwell and Mauldin who obtained similar results for the redistribution of energy.

Several years ago, David Blackwell and Dan Mauldin, the first author of this paper, wrote a note on a problem Ulam had raised concerning "toy" models for physics[2]. Ulam's redistribution of energy problem can be informally stated as follows. Suppose we have a large number of identical particles with an initial distribution of energy and to normalize matters, with total energy one. Assume the particles are randomly paired, forming a sort of "complex" and the total energy of each pair is redistributed according to some given redistribution of energy law. Now, iterate this procedure. Is there a limiting distribution of energy which is independent of the initial distribution of energy? We showed that this is indeed the case. We also showed that the limiting distribution attracts all initial distributions for which all moments exist. It turns out that these results are special cases of a theorem of Holley and Liggett[4]. See section 7 of their paper. Indeed, they had showed there is only one invariant distribution and it attracts all distributions which have a finite first moment. In addition, Blackwell and Mauldin showed there is a one-to-one correspondence between the redistribution of energy law and the limiting attractive distribution of energy. We also had conversations about Ulam's second stage toy model, in which both energy and momentum are conserved, but did not pursue it. The subject of this paper is an analysis of part of this second stage model.

Suppose we have a large number of particles of equal mass with an initial distribution of velocity. We assume that these particles undergo triple collisions at random and that the total velocity of each triple is redistributed according to some given redistribution law. We assume that the total energy and the total momentum of each triple are conserved. We show that

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for each given redistribution law there is an attractive invariant velocity distribution. There is a distribution of velocity such that for any nontrivial initial distribution with finite moments of all orders, the iterates converge weakly to this stable distribution. However, there is not a one-to-one correspondence between the redistribution of velocity law and the final invariant distribution. In fact, we shall give several redistribution laws which yield the normal or Maxwell-Boltzman distribution.

This work is a natural generalization of Ulam's redistribution of energy problem [2]. In fact, Ulam had speculated that such a theorem may be true. One of the major differences between the first stage and second stage models is that the transformation obtained in first stage model is linear in the i.i.d. random variables whereas the corresponding transformation in the second stage model is nonlinear. Thus, the analysis is somewhat more intricate. In developing our approach to the problem, we benefitted from computer studies which strongly indicated that the result holds in some cases. We thank Tony Warnock for conducting these studies. The formal setting is developed in sections 1 and 2 and the Main Theorem (Theorem 2.1) is stated. Moment recursion formulas and their convergence are developed in sections 3, 4, and 5. In section 6, the proof of Theorem 2.1 is completed and a partial converse (Theorem 6.1) is proven. Finally, in section 7, the uniform redistribution law and some others are shown to yield a normal velocity distribution(Theorem 7.3).

Also, let us comment about why we consider triple collisions. We could consider only binary collisions. If one follows the scheme described in this paper, then there is only one attractive limiting distribution—the normal distribution. One can prove this by following the proofs given here for triple collisions. In fact, the proofs for binary collisions are much easier. One could also consider a mixture of n-ary collisions, but we have not worked out all the details for this. Also, we guess in analogy with the work of Holley and Liggett[4] the condition on the initial distribution of velocity that all moments be finite is too strong, perhaps having first moments finite suffices. As the referee points out, it would also be natural to search for a metric under which the redistribution of momentum operator is a contraction. We have not yet been successful in finding such a metric.

1. The Setting

Consider three particles of equal mass which form a complex, and the velocities of the particles are redistributed with the constraints that the total energy and momentum are conserved. Thus, if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are the initial velocity vectors in $\mathbf{R}^3$ of the particles, then the new velocities $\mathbf{v}_1'$, $\mathbf{v}_2'$,and $\mathbf{v}_3'$ satisfy:

$$\mathbf{S}_1 := \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{v}_1' + \mathbf{v}_2' + \mathbf{v}_3' := \mathbf{S}_1', \tag{1.1}$$

and

(1.2) $$S_2 := \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + \|\mathbf{v}_3\|^2 = \|\mathbf{v}_1'\|^2 + \|\mathbf{v}_2'\|^2 + \|\mathbf{v}_3'\|^2 := S_2'.$$

We will consider redistribution of energy and velocity in the center of mass frame of reference. Let λ_i be the fraction of the total kinetic energy that the i^{th} particle has after collision and let $\mathbf{w}_i$ be the direction vector of the velocity of the i^{th} particle after collision. For convenience, we assume all particles have mass 1. Thus,

(1.3) $$0 \le \lambda_i = K_i/K$$

where K is the kinetic energy measured in the center of mass frame of reference, $K = S_2 - \|\mathbf{S}_1\|^2/3$. So,

(1.4) $$\lambda_1 + \lambda_2 + \lambda_3 = 1.$$

Since the total velocity in the center of mass system is zero, we have

(1.5) $$\sqrt{\lambda_1}\mathbf{w}_1 + \sqrt{\lambda_2}\mathbf{w}_2 + \sqrt{\lambda_3}\mathbf{w}_3 = 0.$$

We derive another form of the constraints on the $\lambda's$ which are more suitable for our purposes. From (1.5), we get

(1.6) $$\lambda_1 + 2\sqrt{\lambda_1\lambda_2}\,\langle \mathbf{w}_1, \mathbf{w}_2\rangle + \lambda_2 = \lambda_3.$$

Using (1.4), we have

(1.7) $$|1 - 2(\lambda_1 + \lambda_2)| \le 2\sqrt{\lambda_1\lambda_2}.$$

After some algebra,

(1.8) $$4\lambda_1^2 - 4\lambda_1 + 4\lambda_2^2 - 4\lambda_2 + 1 + 4\lambda_1\lambda_2 \le 0.$$

From this inequality, we derive

(1.9) $$(1/3)(18\lambda_1^2 - 18\lambda_1 + 18\lambda_2^2 - 18\lambda_2 + 6 + 18\lambda_1\lambda_2) \le 1/2.$$

Now, (1.9) can be expressed as

(1.10)
$$(1/3)(9\lambda_1^2 - 6\lambda_1 + 9\lambda_2^2 - 6\lambda_2 + 9(1 - \lambda_1 - \lambda_2)^2 - 6(1 - \lambda_1 - \lambda_2) + 3) \le 1/2.$$

Using the identity (1.4), we get the inequality

(1.11) $$(\lambda_1 - 1/3)^2 + (\lambda_2 - 1/3)^2 + (\lambda_3 - 1/3)^2 \le 1/6.$$

Thus, conditions (1.4) and (1.11) imply that the point $[\lambda_1, \lambda_2, \lambda_3]$ must lie on the circular disk, D, of radius $\frac{1}{\sqrt{6}}$, center $\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$ which lies in the plane given by equation (1.4). The mutually orthogonal vectors $\left[\frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}\right]$ and $\left[0, \frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right]$ both have length $\frac{1}{\sqrt{6}}$, and are both orthogonal to the normal, $[1,1,1]$, of equation (1.4). Thus for each $\mathbf{x} \in D \backslash \{\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]\}$, there exists a unique $r \in (0,1]$ and a unique $\theta \in [0, 2\pi)$, so that

(1.12) $$\mathbf{x} = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right] + r\cos\theta \left[\frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}\right] + r\sin\theta \left[0, \frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right].$$

Conversely, if $[\lambda_1, \lambda_2, \lambda_3]$ lies on the disk D, then tracing backwards from (1.11) (using (1.4) when necessary) it is easy to see that $[\lambda_1, \lambda_2, \lambda_3]$ also satisfies (1.7). Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be unit vectors in $\mathbf{R}^3$ with $2\sqrt{\lambda_1\lambda_2}\,\langle \mathbf{w}_1, \mathbf{w}_2\rangle = 1 - 2(\lambda_1 + \lambda_2)$. The vector $[\lambda_1, \lambda_2, \lambda_3]$ satifies $\left\|\sqrt{\lambda_1}\mathbf{w}_1 + \sqrt{\lambda_2}\mathbf{w}_2\right\|^2 = \lambda_3$. Letting $-\mathbf{w}_3$ denote the direction vector for $\sqrt{\lambda_1}\mathbf{w}_1 + \sqrt{\lambda_2}\mathbf{w}_2$, we have $\sqrt{\lambda_1}\mathbf{w}_1 + \sqrt{\lambda_2}\mathbf{w}_2 = -\sqrt{\lambda_3}\mathbf{w}_3$. Thus all points in this disk, D, are realizable as values of λ_1, λ_2 and λ_3 which satisfy (1.4) and (1.5) for some set of unit vectors.

A redistribution of energy law is a probability measure $\widetilde{\mu}$ supported on the disk D which is symmetric, or invariant under permutations of the coordinates. This last condition signifies that the particles are indistinguishable. The direction vectors for the velocity of the particles are chosen, independent of $\widetilde{\mu}$, as follows. First, $\mathbf{w}_1$ is chosen from the unit sphere according to the uniform distribution. Next, a unit vector $\mathbf{z}$ which is perpendicular to $\mathbf{w}_1$ is chosen according to the uniform distribution on the great circle which is the intersection of the unit sphere and the plane normal to $\mathbf{w}_1$. Thus, $\mathbf{w}_1$ and $\mathbf{z}$ determine a plane which contains $\mathbf{w}_2$ and $\mathbf{w}_3$. Finally, $\mathbf{w}_2$ and $\mathbf{w}_3$ are determined up to the reflection $\mathbf{y} \to -\mathbf{y}$ in this plane from equations (1.4), (1.5), and the equality $\|\mathbf{w}_2\| = \|\mathbf{w}_3\| = 1$.

We will study this process under iteration. We note that it is immaterial whether we chose $\mathbf{w}_1$ at random first. Thus, we have not violated the indistinguishability of the particles. Also, we note that there are 9 velocity component variables which have 5 degrees of freedom in view of the conservation laws. In our scheme, we have 2 degrees of freedom in choosing the $\lambda's$, 2 degrees in choosing $\mathbf{w}_1$ and one degree in choosing $\mathbf{z}$, a total of five.

2. The Redistribution Operator

We now formalize the redistribution operator, $T_{\widetilde{\mu}}$. Let ν be a probability measure on $\mathbf{R}^3$ and assume we have a vast number of particles with velocity distribution ν. We imagine that these particles are partitioned into triples at random. For each triple the velocity is redistributed as described in section one, which yields a new velocity distribution, $T_{\widetilde{\mu}}(\nu)$. So, if $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ are independent random velocity vectors each distributed as ν, then $T_{\widetilde{\mu}}(\nu)$ will be the distribution of $\mathbf{X}'_1$, since $\mathbf{X}'_1$, $\mathbf{X}'_2$, $\mathbf{X}'_3$ all have the same distribution.

From this point on, we will suppress the subscript $\widetilde{\mu}$ in $T_{\widetilde{\mu}}$. Thus, $T(\nu)$ is the distribution of

$$\frac{\mathbf{S}_1}{3} + \sqrt{K\lambda}\mathbf{u},$$

where (1) $\mathbf{u}$ has the uniform distribution, π_2, on the unit sphere, (2) λ is distributed as $\mu = \widetilde{\mu} \circ p^{-1}$ on $[0, 2/3]$, where $\widetilde{\mu}$ is the redistribution law on the disk D and p is the projection map of $\mathbf{R}^3$ onto the first coordinate, and, (3) K is the kinetic energy in the center of mass frame of reference; *i.e.*, $K = S_2 - \|\mathbf{S}_1\|^2/3$. Therefore, $T(\nu)$ is determined by the functional equation:

$$T\nu(f) = \int_{\mathbf{R}^3} f(\mathbf{x})dT\nu(\mathbf{x}) = \tag{2.1}$$

$$\int f\left(\frac{\sum_{i=1}^{3}\mathbf{x}_i}{3} + \sqrt{\lambda}\sqrt{\sum_{i=1}^{3}\|\mathbf{x}_i\|^2 - \frac{\left\|\sum_{i=1}^{3}\mathbf{x}_i\right\|^2}{3}}\,\mathbf{u}\right) d\nu(\mathbf{x}_1)d\nu(\mathbf{x}_2)d\nu(\mathbf{x}_3)d\pi_2(\mathbf{u})d\mu(\lambda)$$

where the integral is over $[0, \frac{2}{3}] \times S^2 \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$.

We will abbreviate integrals like this last one unless the domain of integration or integrators are not clear. Thus, (2.1) could be written as

$$(T\nu)(f) = \int \cdots \int f\left(\frac{\mathbf{S}_1}{3} + \sqrt{\lambda}\sqrt{S_2 - \frac{\| \mathbf{S}_1 \|^2}{3}}\,\mathbf{u}\right) d\nu d\nu d\nu d\pi_2 d\mu.$$

We note a few basic properties of the non-linear operator $T : Prob(\mathbf{R}^3) \to Prob(\mathbf{R}^3)$. First, T is weakly continuous and commutes with the translation operators:

$$T(\nu(\cdot) + \mathbf{x}_0) = T\nu(\cdot + \mathbf{x}_0), \tag{2.2}$$

for any $\mathbf{x}_0 \in \mathbf{R}^3$. This follows from using (2.1) and the fact that the function $\sqrt{S_2 - \|\mathbf{S}_1\|^2/3}$ is invariant under translation in $\mathbf{R}^3$. The operator T preserves energy:

$$\nu(\|\mathbf{x}\|^2) = T\nu(\|\mathbf{x}\|^2) \tag{2.3}$$

and momentum:

$$\nu(\mathbf{x}) = T\nu(\mathbf{x}). \tag{2.4}$$

Equations (2.3) and (2.4) can be verified by using (2.1) and the facts that

$$\int_D x_1 d\widetilde{\mu}(\mathbf{x}) = \int_{[0,2/3]} \lambda d\mu(\lambda) = 1/3, \tag{2.5}$$

and

$$\int_{S^2} u_i(\mathbf{u})d\pi_2(\mathbf{u}) = 0, \tag{2.6}$$

where u_i is the i^{th} coordinate of $\mathbf{u}$.

In order to see that (2.5) holds, we will use Choquet's representation theorem [5]. We will also use this representation later on. Our measure $\widetilde{\mu}$ is simply a probability measure on the disk D which is invariant under the symmetries of the circumscribing triangle with vertices $\mathbf{e}_1 = [1, 0, 0]$, $\mathbf{e}_2 = [0, 1, 0]$, and $\mathbf{e}_3 = [0, 0, 1]$. Therefore, $\widetilde{\mu}$ can be expressed uniquely as an integral over the set of all extreme points of $\mathbf{C}$, the compact convex set of all probability measures which are invariant under these symmetries. Thus, there is a unique probability measure P on $ext(\mathbf{C})$ such that

$$\tilde{\mu}(f) = \int_{ext(\mathbf{C})} \int_D f(\mathbf{x}) d\tau(\mathbf{x}) dP(\tau). \tag{2.7}$$

Now, an extreme point in this case is simply a probability measure on D which is ergodic under the action of the symmetry group. It is easy to see that τ is ergodic if and only if there is a point $\mathbf{z}$ of D such that

$$\tau = \frac{1}{6} \sum_{i=0}^{5} \delta_{\xi_i(\mathbf{z})}, \tag{2.8}$$

where $\xi_0, \xi_1, ..., \xi_5$ are the symmetries and δ_x is point mass at x.

Of course, for each such τ,

$$\int_D x_1 d\tau(\mathbf{x}) = 1/3. \tag{2.9}$$

Since $\int_{[0,2/3]} \int_D g d\mu = \int_D g \circ p d\tilde{\mu}$, (2.5) follows from (2.9).

Also, since the function $\sqrt{S_2 - \|\mathbf{S}_1\|^2/3}$ is positive homogeneous, T commutes with positive scaling:

$$T(\nu(c\cdot)) = T\nu(c\cdot). \tag{2.10}$$

Facts concerning weak convergence of probability measures in metric spaces such as $\mathbf{R}^3$ can be found in [1]. Our main theorem concerns the properties of the fixed point and its domain of attraction.

Theorem 2.1 (Main). *For each symmetric probability measure $\tilde{\mu}$, on D, there exists a unique radially symmetric probability measure $\hat{\mu}$ on $\mathbf{R}^3$ with total energy one and moments of all orders so that $\hat{\mu} = T_{\tilde{\mu}}(\hat{\mu})$. Further, if ν is any probability measure on $\mathbf{R}^3$ with finite moments of all orders and ν is not point mass, then*

$$\{T^n(\nu)\}_{n=0}^{\infty} \text{ converges weakly to } \hat{\mu} \left[\frac{\cdot - \nu(\mathbf{x})}{\left(\nu\left(\|\mathbf{x}\|^2\right) - \|\nu(\mathbf{x})\|^2\right)^{1/2}} \right]. \tag{2.11}$$

Moreover, the invariant measure $\hat{\mu}$ is determined totally by the marginal of $\tilde{\mu}$ with respect to the projection onto the x-axis.

Let us note that each point mass measure is a fixed point of T. Also, if ν is not concentrated at $E(\mathbf{x}) = \nu(\mathbf{x}) = \mathbf{m}$, then

$$\sigma^2 = E\left(\| \mathbf{x} - E(\mathbf{x}) \|^2\right) = \nu\left(\| \mathbf{x} \|^2\right) - \| \nu(\mathbf{x}) \|^2 > 0. \tag{2.12}$$

In this case, $\tau(A) = \nu(\sigma A + \mathbf{m})$, is a probability measure with momentum zero and energy one. If $\{T^n(\tau)\}_{n=1}^{\infty}$converges weakly to $\hat{\mu}$, then by the commutativity properties of T, $T^n\tau((\cdot - \mathbf{m})/\sigma)) = T^n(\nu)$would converge weakly to $\hat{\mu}((\cdot - \nu(\mathbf{x}))/(\nu(\|\mathbf{x}\|^2) - \|\nu(\mathbf{x})\|^2)^{1/2}$. Therefore, to prove the theorem, we only need to prove (2.11) under the conditions that $\nu(\mathbf{x}) = 0$ and $\nu(\|\mathbf{x}\|^2) = 1$.

Our strategy is to first obtain a recursion formula for the moments of $T^n(\nu)$. Second, to show convergence to these moments and finally, to show that the limits are the moments of a unique element of $Prob(\mathbf{R}^3)$. This is the same strategy employed in [2].

3. The Moment Recursion Formulas.

Temporarily fix a probability measure ν on $\mathbf{R}^3$ with $\nu(\mathbf{x}) = \mathbf{0}$, $\nu(\|\mathbf{x}\|^2) = 1$. Let $\mathbf{Z}_+$be the set of all non-negative integers. For each multi-index $\mathbf{k} = [k_1, k_2, k_3] \in \mathbf{Z}_+^3$, consider the mixed moment of order $\mathbf{k}$ of the nth iterate of ν under T:

$$m_{n,\mathbf{k}} = \int_{\mathbf{R}^3} \mathbf{x}^k dT^n\nu(\mathbf{x}) = \int_{\mathbf{R}^3} \prod_{i=1}^{3} x_i^{k_i} dT^n\nu(\mathbf{x}). \tag{3.1}$$

We will first find a formula relating the moments of the $(n+1)^{\text{th}}$ iterate to those of the nth. From (2.1), we have

$$m_{n+1,\mathbf{k}} = \int_{\mathbf{R}^3} \mathbf{x}^{\mathbf{k}} dT\left(T^n(\nu)(\mathbf{x})\right) \tag{3.2}$$

$$= \int \cdots \int \left[\frac{\mathbf{S}_1}{3} + \sqrt{\lambda}\left(S_2 - \frac{\|\mathbf{S}_1\|^2}{3}\right)^{1/2} \mathbf{u}\right]^{\mathbf{k}} dT^n(\nu)dT^n(\nu)dT^n(\nu)d\pi_2(\mathbf{u})d\mu(\lambda)$$

$$= \int \cdots \int \prod_{i=1}^{3} \left[\frac{S_{1,i}}{3} + \sqrt{\lambda}\left(S_2 - \frac{\|\mathbf{S}_1\|^2}{3}\right)^{1/2} u_i\right]^{k_i} dT^n(\nu)...d\mu(\lambda)$$

$$= \int \cdots \int \prod_{i=1}^{3} \left[\sum_{j=0}^{k_i} \binom{k_i}{j} \sqrt{\lambda^j} \left(S_2 - \frac{\|\mathbf{S}_1\|^2}{3}\right)^{\frac{j}{2}} u_i^j \left(\frac{S_{1,i}}{3}\right)^{k_i - j}\right] dT^n(\nu)...d\mu(\lambda).$$

Since π_2 is the uniform distribution on the unit sphere,

$$\int_{S^2} \mathbf{u}^{\mathbf{j}} d\pi_2(\mathbf{u}) = 0, \tag{3.3}$$

unless $\mathbf{j} = [j_1, j_2, j_3]$ is even (each j_i is even). Let $\lceil\cdot\rceil$ denote the greatest integer function. Thus,

$$m_{n+1,\mathbf{k}} =$$

$$\int \prod_{i=1}^{3} \sum_{j=0}^{\lceil k_i/2 \rceil} \binom{k_i}{2j} \lambda^j \left(S_2 - \frac{\|\mathbf{S}_1\|^2}{3}\right)^j u_i^{2j} \left(\frac{S_{1,i}}{3}\right)^{k_i - 2j} dT^n(\nu)...d\mu(\lambda). \tag{3.4}$$

Now, the weight of $\mathbf{k}$, $wt(\mathbf{k}) := k_1 + k_2 + k_3$. Noting that each additive term, associated with the upper summand k_i, is a polynomial in $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ of degree k_i and with coefficients only a function of λ and $\mathbf{u}$, the product must be a polynomial of degree $wt(\mathbf{k})$ in the variables $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, with coefficients a function of λ and $\mathbf{u}$. Thus, integrating, we obtain an equation

$$m_{n+1,\mathbf{k}} = \sum_{\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3 \in \mathbf{Z}_+^3 : wt(\sum \mathbf{s}_i) = wt(\mathbf{k})} a_{\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3}(\mathbf{k}) m_{n,\mathbf{s}_1} m_{n,\mathbf{s}_2} m_{n,\mathbf{s}_3}, \tag{3.5}$$

where the coefficients $a_{\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3}(\mathbf{k})$ do not depend on ν, but only on the redistribution law $\widetilde{\mu}$. Also, if τ is a permutation of $\{1,2,3\}$, then

$$a_{\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3}(\mathbf{k}) = a_{\mathbf{s}_{\tau(1)},\mathbf{s}_{\tau(2)},\mathbf{s}_{\tau(3)}}(\mathbf{k}). \tag{3.6}$$

There is only one multi-index $\mathbf{k}$ with $wt(\mathbf{k}) = 0$, and

$$m_{n,[0,0,0]} = 1,\ n = 0,1,2.... \tag{3.7}$$

The canonical unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the only multi-indices of weight one. It is easy to check that

$$m_{n,\mathbf{e}_i} = 0,\ n = 0,1,2.... \tag{3.8}$$

Now, according to (3.5)

$$m_{n+1,\mathbf{k}} = \sum_{I(\mathbf{k})} a_{\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3}(\mathbf{k}) m_{n,\mathbf{s}_1} m_{n,\mathbf{s}_2} m_{n,\mathbf{s}_3} + 3 \left(\sum_{wt(\mathbf{s})=wt(\mathbf{k})} a_{\mathbf{s},0,0}(\mathbf{k}) m_{n,\mathbf{s}} \right), \tag{3.9}$$

where the first sum is over the index set $I(\mathbf{k}) = \{(\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3) \in \mathbf{Z}_+^3 : \sum_{i=1}^{3} wt(\mathbf{s}_i) = wt(\mathbf{k})$,and $\forall i,\ wt(\mathbf{s}_i) < wt(\mathbf{k})\}$.
Define

$$\gamma_n(\mathbf{k}) = \sum_{I(\mathbf{k})} a_{\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3}(\mathbf{k}) m_{n,\mathbf{s}_1} m_{n,\mathbf{s}_2} m_{n,\mathbf{s}_3}$$

and, for $wt(\mathbf{k}) = wt(\mathbf{s})$,

$$\mathbf{A}_{\mathbf{k},\mathbf{s}} = a_{\mathbf{s},0,0}(\mathbf{k}).$$

Thus, (3.9) becomes

$$m_{n+1,\mathbf{k}} = \gamma_n(\mathbf{k}) + 3 \sum_{wt(\mathbf{s})=wt(\mathbf{k})} \mathbf{A}_{\mathbf{k},\mathbf{s}} m_{n,\mathbf{s}}. \tag{3.10}$$

Or, defining $y_n(p) = \{m_{n,\mathbf{s}}\}_{wt(\mathbf{s})=p}$, for $n = 0,1,2,...$; $p = 0,1,2,....$, $\gamma_n(p) = \{\gamma_n(\mathbf{k})\}_{wt(\mathbf{k})=p}$ and $\mathbf{A}(p) = \{\mathbf{A}_{\mathbf{k},\mathbf{s}} : wt(\mathbf{k}) = wt(\mathbf{s}) = p\}$, we have

$$y_{n+1} = \gamma_n + 3\mathbf{A} y_n, \tag{3.11}$$

for $n = 0,1,2,....$ We have suppressed the argument p in (3.11) and juxtaposition signifies matrix multiplication.

Note that it follows from (3.8) that $\gamma_n(2) = 0$.

We will need to know the entries of $\mathbf{A}$. In order to compute these, note that for each $\mathbf{k}$ with weight p,

$$\sum_{wt(\mathbf{h})=p} \mathbf{A}_{\mathbf{k},\mathbf{h}}\mathbf{x}^{\mathbf{h}} = \int\int \left[\frac{\mathbf{x}}{3} + \sqrt{\lambda}\sqrt{2/3}\|\mathbf{x}\|\mathbf{u}\right]^{\mathbf{k}} d\pi_2(\mathbf{u})d\mu(\lambda). \tag{3.12}$$

4. Convergence of Moments of weight 2.

From equation (3.12), we find

$$\phi(\mathbf{x},\mathbf{y}) := \sum_{wt(\mathbf{k})=wt(\mathbf{h})=2} \binom{2}{\mathbf{k}} \mathbf{A}_{\mathbf{k},\mathbf{h}}\mathbf{x}^{\mathbf{h}}\mathbf{y}^{\mathbf{k}}$$

$$= \int\int \sum_{wt(\mathbf{k})=2} \binom{2}{\mathbf{k}} \mathbf{y}^{\mathbf{k}} \left[\frac{\mathbf{x}}{3} + \left(\frac{2\lambda}{3}\|\mathbf{x}\|^2\right)^{1/2}\mathbf{u}\right]^{\mathbf{k}} d\pi_2(\mathbf{u})d\mu(\lambda)$$

$$= \int\int \left[\left\langle \left(\frac{\mathbf{x}}{3} + \sqrt{\frac{2\lambda}{3}}\|\mathbf{x}\|\mathbf{u}\right), \mathbf{y}\right\rangle\right]^2 d\pi_2(\mathbf{u})d\mu(\lambda). \tag{4.1}$$

But, since $\mathbf{u}$ is uniformly distributed on S^2,

$$\phi(\mathbf{x},\mathbf{y}) = \int\int \left[\frac{\langle \mathbf{x},\mathbf{y}\rangle}{3} + \sqrt{\lambda}\sqrt{\frac{2}{3}}\|\mathbf{x}\|\|\mathbf{y}\|u_1\right]^2 d\pi_2(\mathbf{u})d\mu(\lambda). \tag{4.2}$$

Since the integral of u_1 is zero,

$$\phi(\mathbf{x},\mathbf{y}) = \frac{\langle \mathbf{x},\mathbf{y}\rangle^2}{9} + \frac{2}{3}\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \int \lambda\ d\mu(\lambda) \int u_1^2 d\pi_2(\mathbf{u}). \tag{4.3}$$

Of course,

$$\int u_1^2 d\pi_2(\mathbf{u}) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \cos^2\theta\ \sin^3\phi\ d\phi\ d\theta = \frac{1}{3}. \tag{4.4}$$

Thus, from (2.5) we get

$$\phi(\mathbf{x},\mathbf{y}) = \frac{\langle \mathbf{x},\mathbf{y}\rangle^2}{9} + \frac{2}{27}\|\mathbf{x}\|^2\|\mathbf{y}\|^2. \tag{4.5}$$

The definition of ϕ and (4.5) yield the following matrix for $\mathbf{A} = \mathbf{A}(2)$:

$$\begin{bmatrix} \frac{5}{27} & \frac{2}{27} & \frac{2}{27} & 0 & 0 & 0 \\ \frac{2}{27} & \frac{5}{27} & \frac{2}{27} & 0 & 0 & 0 \\ \frac{2}{27} & \frac{2}{27} & \frac{5}{27} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{27} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{27} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{27} \end{bmatrix}$$

where the indices of the rows listed from top to bottom and the columns from left to right is the following sequence: $[2,0,0]$, $[0,2,0]$, $[0,0,2]$, $[1,1,0]$, $[1,0,1]$, $[0,1,1]$. Formula (3.8) implies $\gamma_n(2) = 0$, for $n = 1,2,3,\ldots$. Thus,

$$\mathbf{y}_{n+1}(2) = (3\mathbf{A})^{n+1}(\mathbf{y}_0). \tag{4.6}$$

Inspection of the block matrix $3\mathbf{A}$ shows that the upper $3x3$ block matrix has largest eigenvalue 1 and corresponding eigenvector $[1,1,1]$ and eigenvalue $1/3$ of multiplicity 2. The lower $3x3$diagonal matrix of $3\mathbf{A}$ has eigenvalue $1/3$ of multiplicity 3. Let $\mathbf{e} = 1/\sqrt{3}\,[1,1,1,0,0,0]$. Then by this analysis

$$\lim_{n\to\infty} 3\mathbf{A}^{n+1}(\mathbf{y}_0) = \langle \mathbf{y}_0, \mathbf{e}\rangle\, \mathbf{e}. \tag{4.7}$$

Since $\langle \mathbf{y}_0, e\rangle = \nu(\|\mathbf{x}\|^2)/\sqrt{3}$, we have

$$\lim_{n\to\infty} m_{n,2\mathbf{e}_i} := m_{2\mathbf{e}_i} = \frac{Energy}{3}, \tag{4.8}$$

and, if $wt(\mathbf{k}) = 2$, and $\mathbf{k}$ has an odd component, then $\lim_{n\to\infty} m_{n,\mathbf{k}} := m_{\mathbf{k}} = 0$.

5. Convergence of Higher Order Mixed Moments

Define a stochastic process $\{\mathbf{X}_n\}_{n=0}^{\infty}$ on $\mathbf{R}^3$ by letting $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of independent random variables all distributed as λ, $\{\mathbf{u}_n\}_{n=1}^{\infty}$ be a sequence of independent random vectors all distributed as $\mathbf{u}$ so that $\{\lambda_n, \mathbf{u}_n\}_{n=1}^{\infty}$ forms an independent family and set

$$\mathbf{X}_{n+1} = \frac{\mathbf{X}_n}{3} + \sqrt{\frac{2\lambda_{n+1}}{3}}\|\mathbf{X}_n\|\mathbf{u}_{n+1},\ n = 1,2,3,.... \tag{5.1}$$

Let $E_{\mathbf{x}_0}$ be the expectation operator where the process starts with $\mathbf{X}_0 = \mathbf{x}_0$, a.s. Also, let $W(p) = \{\mathbf{s} \in \mathbf{Z}_+^3 : wt(\mathbf{s}) = p\}$.

Lemma 5.1. *Let* $\mathbf{x}_0 \in \mathbf{R}^3$, $p = wt(\mathbf{k})$, *and* $\mathbf{A} = A(p)$. *Then*

$$E_{\mathbf{x}_0}(\mathbf{X}_n^{\mathbf{k}}) = \left(\mathbf{A}^n \mathbf{x}_0^{W(p)}\right)(\mathbf{k}),\ n = 0,1,2,... \tag{5.2}$$

where

$$\mathbf{x}_0^{W(p)} := [\mathbf{x}_0^{\mathbf{s}}]_{\mathbf{s}\in W(p)}\,. \tag{5.3}$$

Proof. Clearly (5.2) is true if $n = 0$. If (5.2) holds for n, we have

$$\begin{aligned} E_{\mathbf{x}_0}\left(\mathbf{X}_{n+1}^{\mathbf{k}}\right) &= E_{\mathbf{x}_0}\left[\left(\frac{\mathbf{X}_n}{3} + \sqrt{\frac{2\lambda_{n+1}}{3}}\|\mathbf{X}_n\|\mathbf{u}_{n+1}\right)\right] \\ &= E_{\mathbf{x}_0}\left[E_{\mathbf{x}_0}\left[\left(\frac{\mathbf{X}_n}{3} + \sqrt{\frac{2\lambda_{n+1}}{3}}\|\mathbf{X}_n\|\mathbf{u}_{n+1}\right)^{\mathbf{k}} |\mathcal{F}_n\right]\right], \end{aligned}$$

where $\mathcal{F}_n$ is the σ-algebra generated by $\{\lambda_j,\, u_j | j < n\}$.

According to (3.12),

$$E_{\mathbf{x}_0}(\mathbf{X}_{n+1}^{\mathbf{k}}) = E_{\mathbf{x}_0}\left[\sum_{\mathbf{s}\in W(p)} \mathbf{A}_{\mathbf{k},\mathbf{s}}\mathbf{X}_n^{\mathbf{s}}\right],$$

and by the induction hypothesis

$$= \sum_{\mathbf{s}\in W(p)} \mathbf{A}_{\mathbf{k},\mathbf{s}} \left[\mathbf{A}^n \mathbf{x}_0^{W(p)}\right](\mathbf{s}).$$

Or,

$$E_{\mathbf{x}_0}(\mathbf{X}_{n+1}^{\mathbf{k}}) = \left(\mathbf{A}^{n+1}\mathbf{x}_0^{W(p)}\right)(\mathbf{k}).$$

Lemma 5.2. $\mathbf{R}^{W(p)} = span\{\mathbf{x}^{W(p)} | \mathbf{x} \in \mathbf{R}^3\} := \mathbf{L}.$

Proof. Let $\mathbf{v} = \{v_\mathbf{k}\}_{\mathbf{k}\in W(p)}$ be such that $\langle \mathbf{v}, \mathbf{x}^{W(p)}\rangle = 0$, for every $\mathbf{x} \in \mathbf{R}^3$. Then $\sum_{\mathbf{k}\in W(p)} \mathbf{v}_\mathbf{k}\mathbf{x}^\mathbf{k}$ is a polynomial which is identically 0. Therefore, all of its coefficients are zero. So, $\mathbf{v}$ is the zero vector. This means $L = \mathbf{R}^{W(p)}$.■

In order to study the behavior of the iterates of $3\mathbf{A}(p)$ for $p > 2$, let us make the following notations. Define

$$H(\alpha) := E\left[\left\|\frac{\mathbf{e}_1}{3} + \sqrt{\frac{2}{3}}\sqrt{\lambda}\mathbf{u}\right\|^{2\alpha}\right]$$

$$= E\left[\left(\frac{1}{9} + 2\frac{\sqrt{\lambda}}{3}\sqrt{\frac{2}{3}}u_1 + \frac{2}{3}\lambda\right)^{\alpha}\right]. \tag{5.4}$$

We have

$$H(1) = \frac{1}{3}. \tag{5.5}$$

Also, since $0 \le \lambda \le \frac{2}{3}$, $H(\alpha)$ is decreasing. Thus,

$$H(\alpha) < \frac{1}{3}, \text{ if } 1 < \alpha. \tag{5.6}$$

Lemma 5.3. *Let C_p be the cardinality of $W(p)$. Then*

$$\|(3\mathbf{A})^n\mathbf{x}_0^{W(p)}\| \le \sqrt{C_p}\|\mathbf{x}_0\|^P(3H(p/2))^n. \tag{5.7}$$

Proof. From equation (5.2), we have

$$\left|\mathbf{A}^n\left(\mathbf{x}_0^{W(p)}\right)(\mathbf{k})\right| \le E_{\mathbf{x}_0}\left(|\mathbf{X}_n^\mathbf{k}|\right) \le E_{\mathbf{x}_0}(\|\mathbf{X}_n\|^p). \tag{5.8}$$

Setting $\alpha = p/2$,

$$E_{\mathbf{x}_0}\left(\|\mathbf{X}_n\|^{2\alpha}\right) = E_{\mathbf{x}_0}[\langle\mathbf{X}_n, \mathbf{X}_n\rangle^\alpha]$$

$$= E_{\mathbf{x}_0}\left[\left(\frac{\|\mathbf{X}_{n-1}\|^2}{9} + \sqrt{\frac{8\lambda_n}{27}}\|\mathbf{X}_{n-1}\|\langle\mathbf{u}_n, \mathbf{X}_{n-1}\rangle + \frac{2}{3}\lambda_n\|\mathbf{X}_{n-1}\|^2\right)^\alpha\right]$$

$$= E_{\mathbf{x}_0}\|\mathbf{X}_{n-1}\|^{2\alpha}\left(\frac{1}{9} + \frac{2}{3}\sqrt{\frac{2}{3}}\sqrt{\lambda_n}\left\langle\mathbf{u}_n, \frac{\mathbf{X}_{n-1}}{\|\mathbf{X}_{n-1}\|}\right\rangle + \frac{2}{3}\lambda_n\right)^\alpha.$$

Expanding the last expression in terms of conditional expectations, we have

$$E_{\mathbf{x}_0}\left[\|\mathbf{X}_{n-1}\|^{2\alpha}E_{\mathbf{x}_0}\left[\left(\frac{1}{9}+\frac{2}{3}\sqrt{\frac{2}{3}}\sqrt{\lambda_n}\left\langle\mathbf{u}_n,\frac{\mathbf{X}_{n-1}}{\|\mathbf{X}_{n-1}\|}\right\rangle+\frac{2}{3}\lambda_n\right)^{\alpha}|\mathcal{F}_{n-1}\right]\right].$$

Now, $\mathbf{u}_n$ is uniformly distributed and independent of $\mathbf{X}_{n-1}$; so we can replace $\mathbf{X}_{n-1}/\|\mathbf{X}_{n-1}\|$ by $\mathbf{e}_1$. Also, the function $\left(\frac{1}{9}+\frac{2}{3}\sqrt{\frac{2}{3}}\sqrt{\lambda_n}\langle\mathbf{u}_n,\mathbf{e}_1\rangle+\frac{2}{3}\lambda_n\right)^{\alpha}$ is independent of $\mathcal{F}_{n-1}$ and its expected value is the same as the expected value of $\left(\frac{1}{9}+\frac{2}{3}\sqrt{\frac{2}{3}}\sqrt{\lambda}\langle\mathbf{u},\mathbf{e}_1\rangle+\frac{2}{3}\lambda\right)^{\alpha}$, which equals $\left\|\frac{\mathbf{e}_1}{3}+\sqrt{\frac{2}{3}}\sqrt{\lambda}\mathbf{u}\right\|^{2\alpha}$. Thus,

$$E_{\mathbf{x}_0}\left(\|\mathbf{X}_n\|^{2\alpha}\right)=E_{\mathbf{x}_0}\left[\|\mathbf{X}_{n-1}\|^{2\alpha}\right]H(\alpha). \tag{5.9}$$

By recursion on (5.9), we have

$$E_{\mathbf{x}_0}\left(\|\mathbf{X}_n\|^{2\alpha}\right)=\|\mathbf{x}_0\|^{2\alpha}\left(H(\alpha)\right)^n. \tag{5.10}$$

Therefore, from (5.8), (5.10) and the fact that if $|z(\mathbf{k})|\le L$, for $\mathbf{k}\in W(p)$, then $\|z\|\le\sqrt{C_p}L$, we have

$$\|(3\mathbf{A})^n\mathbf{x}_0^{W(p)}\|\le\sqrt{C_p}\|\mathbf{x}_0\|^p\left(3H(p/2)\right)^n.$$

From Lemmas 5.2 and 5.3, we obtain

Lemma 5.4. *There is a constant $D=D_p$ such that if $\mathbf{v}\in R^{W(p)}$*

$$\|(3\mathbf{A})^n(\mathbf{v})\|\le D\|\mathbf{v}\|^p(3H(p/2))^n. \tag{5.11}$$

We investigate the convergence properties of $\mathbf{y}_n$. Assume $p>2$ and for each $q<p$, $\lim_{n\to\infty}\mathbf{y}_n(q)=\mathbf{y}(q)$. In the previous sections we have demonstrated the convergence of $\mathbf{y}_n(0)$, $\mathbf{y}_n(1)$, and $\mathbf{y}_n(2)$. Since $p>2$, $3H(p/2)<1$; and, it follows from Lemma 5.4 that the spectral radius of $3\mathbf{A}\le 3H(p/2)<1$. This means that the operator $(I-3\mathbf{A})^{-1}$ exists and is equal to $\Sigma_{j=0}^{\infty}(3\mathbf{A})^j$. Next, a check of the definition of $\gamma_j(p)$ shows that there is a continuous function f_p such that for all j,

$$\gamma_j(p)=f_p(\mathbf{y}_j(q);\,q<p).$$

So, by our assumption

$$\lim_{j\to\infty}\gamma_j(p)=f_p(\mathbf{y}(q);\,q<p):=\gamma(p). \tag{5.12}$$

We claim now that

$$\lim_{n\to\infty}\mathbf{y}_n(p)=(I-3\mathbf{A})^{-1}(\gamma). \tag{5.13}$$

By recursion on (3.11), we have

$$\mathbf{y}_{n+1}=\gamma_n+3\mathbf{A}\gamma_{n-1}+(3\mathbf{A})^2\gamma_{n-2}+\ldots+(3\mathbf{A})^n\gamma_1+(3\mathbf{A})^{n+1}\mathbf{y}_0. \tag{5.14}$$

We have

$$\|\mathbf{y}_{n+1} - (I - 3\mathbf{A})^{-1}(\gamma)\| \le \|\gamma_n - \gamma\| + \|3\mathbf{A}(\gamma_{n-1} - \gamma)\| + \ldots \tag{5.15}$$

$$+ \|(3\mathbf{A})^n(\gamma_1 - \gamma)\| + \|(3\mathbf{A})^{n+1}(\mathbf{y}_0 - \gamma)\| + \|\sum_{j=n+2}^{\infty}(3\mathbf{A})^j(\gamma)\|. \tag{5.16}$$

For each n,

$$\|\mathbf{y}_{n+1} - (I - 3\mathbf{A})^{-1}(\gamma)\| \le D_j \sum_{j=1}^{\infty} c_{n,j}(3\Phi(p/2))^j, \tag{5.17}$$

where

$$c_{n,j} = \begin{cases} \|\gamma_{n-j+1} - \gamma\| & \text{if } j \le n, \\ \|\mathbf{y}_0 - \gamma\| & \text{if } j = n+1 \\ \|\gamma\| & \text{if } j > n+1 \end{cases}.$$

Since the $c_{n,j}$'s are uniformly bounded and $\lim_{n\to\infty} c_{n,j} = 0$, we have

$$\lim_{n\to\infty} \mathbf{y}_{n+1} = (I - 3\mathbf{A})^{-1}(\gamma). \tag{5.18}$$

6. Existence and Uniqueness of a Stable Distribution.

In the preceding two sections we showed that there are numbers $m_{\mathbf{k}}$, $\mathbf{k} \in \mathbf{Z}_+^3$ such that if ν is an initial distribution of velocity with finite moments of all orders, then

$$\lim_{n\to\infty} m_{n,\mathbf{k}} = m_{\mathbf{k}}. \tag{6.1}$$

It is easy to see that $\{T^n(\nu)\}_{n=1}^{\infty}$ is weakly conditionally compact. Therefore, there is some probability measure $\hat{\mu}$ on $\mathbf{R}^3$ such that

$$m_{\mathbf{k}} = \int_{\mathbf{R}^3} \mathbf{x}^{\mathbf{k}} d\hat{\mu}(\mathbf{x}), \tag{6.2}$$

for $\mathbf{k} \in \mathbf{Z}_+^3$.

We will show that

$$\sum_{p=1}^{\infty} \left(\sum_{i=1}^{3} m_{2pe_i}\right)^{-1/2p} = +\infty. \tag{6.3}$$

It follows from (6.3) that there is only one probability measure $\hat{\mu}$ with moments $m_{\mathbf{k}}$. See [6]. It also follows that ν, $T\nu$, $T^2\nu$, . . . converges weakly to $\hat{\mu}$. Finally, since T commutes with rotations, $\hat{\mu}$ is invariant under rotations. So, $\hat{\mu}$ is radially symmetric.

In order to prove (6.3), we will show that there are positive constants L and C that for all p,

$$b_p := \hat{\mu}(\|\mathbf{x}\|^p) \le CL^p p!, \tag{6.4}$$

for $p = 0, 1, 2, \ldots$

Since $\sum_{i=1}^{\infty} m_{2pe_i} \le b_{2p}$, for $p = 0, 1, 2, \ldots$, (6.3) follows from (6.4).

In order to simplify the presentation of the argument for inequality 6.4, we will suppress the measures with respect to which the integrands are being evaluated.

We have

$$b_p = \hat{\mu}(\|\mathbf{x}\|^p) = \int \cdots \int \left\| \frac{\mathbf{S}_1}{3} + \sqrt{\lambda}\sqrt{S_2 - \frac{\|\mathbf{S}_1\|^2}{3}}\mathbf{u}_0 \right\|^p .$$

Let $\tau = sign\,(\langle \mathbf{S}_1, \mathbf{u}_0 \rangle)$. So,

$$b_p \leq \int \cdots \int \left\| \frac{\mathbf{S}_1}{3} + \tau\sqrt{\lambda}\sqrt{S_2 - \frac{\|\mathbf{S}_1\|^2}{3}}\mathbf{u}_0 \right\|^p .$$

Because τ simply flips $\mathbf{u}_0$ to the hemisphere with pole $\mathbf{S}_1$ if $\langle \mathbf{S}_1, \mathbf{u}_0 \rangle < 0$, it follows that if we replace $\sqrt{S_2 - \|\mathbf{S}_1\|^2/3}$ by a larger function, then the integral is larger. Now,

$$S_2 - \frac{\|\mathbf{S}_1\|^2}{3} \leq \frac{2}{3}(\|\mathbf{x}_1\| + \|\mathbf{x}_2\| + \|\mathbf{x}_3\|)^2. \tag{6.5}$$

So,

$$\begin{aligned} b_p &\leq \int \cdots \int \left\| \frac{\mathbf{S}_1}{3} + \tau\sqrt{\frac{2\lambda}{3}}(\|\mathbf{x}_1\| + \|\mathbf{x}_2\| + \|\mathbf{x}_3\|)\mathbf{u}_0 \right\|^p \\ &= \int \cdots \int \left\| \sum_{i=1}^{3} \frac{\mathbf{x}_i}{3} + \tau\sqrt{\frac{2\lambda}{3}}\|\mathbf{x}_i\|\mathbf{u}_0 \right\|^p \\ &\leq \int \cdots \int \left(\sum_{i=1}^{3} \left\| \frac{\mathbf{x}_i}{3} + \tau\sqrt{\frac{2\lambda}{3}}\|\mathbf{x}_i\|\mathbf{u}_0 \right\| \right)^p . \end{aligned}$$

Thus, setting $\mathbf{u}_i = \mathbf{x}_i/\|\mathbf{x}_i\|$, we have

$$b_p \leq \int \cdots \int \sum_{wt(\mathbf{j})=p} \binom{p}{\mathbf{j}} \prod_{i=1}^{3} \left\| \frac{\mathbf{x}_i}{3} + \sqrt{\frac{2\lambda}{3}}\|\mathbf{x}_i\|\mathbf{u}_0 \right\|^{j_i}$$

$$= \int \cdots \int \sum_{wt(\mathbf{j})=p} \binom{p}{\mathbf{j}} \prod_{i=1}^{3} \left[\|\mathbf{x}_i\|^{j_i} \left\| \frac{\mathbf{u}_i}{3} + \tau\sqrt{\frac{2\lambda}{3}}\mathbf{u}_0 \right\|^{j_i} \right] . \tag{6.6}$$

The biggest $\tau\sqrt{\frac{2\lambda}{3}}$ can be is $2/3$ and $\mathbf{u}_0$ is independent $\mathbf{u}_i$. So, almost surely $\|\frac{\mathbf{u}_i}{3} + \tau\sqrt{\frac{2\lambda}{3}}\mathbf{u}_0\| < 1$. We replace this function by 1 in (6.6) except when $j = p\mathbf{e}_i$. Thus,

(6.7)

$$b_p \leq \int \cdots \int \left[\sum_{wt(\mathbf{j})=p} \binom{p}{\mathbf{j}} \prod_{i=1}^{3} \|\mathbf{x}_i\|^{j_i} + \sum_{i=1}^{3} \left(\|\mathbf{x}_i\|^p \left\| \frac{\mathbf{u}_i}{3} + \tau_i\mathbf{u}_0 \right\|^p - \|\mathbf{x}_i\|^p \right) \right] ,$$

where $\tau_i = sign < u_i, \mathbf{u}_0 >$. Or,

$$(6.8) \qquad b_p \le \sum_{wt(\mathbf{j})=p} \binom{p}{\mathbf{j}} b_{j_1} b_{j_2} b_{j_3} + \int 3\|\mathbf{x}_i\|^p \left\| \frac{\mathbf{u}_i}{3} + \tau_i \sqrt{\frac{2\lambda}{3}} \mathbf{u}_0 \right\|^p - 3b_p.$$

Consider the middle term of (6.8). Let $\tau^* = sign < \mathbf{e}_1, \mathbf{u}_0 >$. We have

$$(6.9) \qquad b_p \le \sum_{wt(\mathbf{j})=p} \binom{p}{\mathbf{j}} b_{j_1} b_{j_2} b_{j_3} + 3b_p E \left\| \frac{\mathbf{e}_1}{3} + \tau^* \sqrt{\frac{2\lambda}{3}} \mathbf{u}_0 \right\|^p - 3b_p.$$

Or, setting $E_p = E\|\mathbf{e}_1/3 + \tau^* \sqrt{\frac{2\lambda}{3}} \mathbf{u}_0\|^p$,

$$(6.10) \qquad (4 - 3E_p) b_p \le \sum_{wt(\mathbf{j})=p} \binom{p}{\mathbf{j}} b_{j_1} b_{j_2} b_{j_3}.$$

Again, $\|\frac{\mathbf{e}_1}{3} + \tau^* \sqrt{\frac{2\lambda}{3}} \mathbf{u}_0\| < 1$ almost surely. Thus, $\lim_{p\to\infty} E_p = 0$. Fix p_0 such that if $p \ge p_0$, then $3E_p < 1$. Define $\{B_p\}_{p=1}^{\infty}$ by $B_p = b_p$, if $p < p_0$. and if $p \ge p_0$, then by recursion,

$$(6.11) \qquad B_p := \left(\frac{1}{1 - 3E_{p_0}}\right) \sum_{\substack{wt(\mathbf{j})=p \\ \mathbf{j} \ne p\mathbf{e}_i}} \binom{p}{\mathbf{j}} B_{j_1} B_{j_2} B_{j_3}.$$

Now, it follows by induction that $b_p \le B_p$, $p = 0, 1, 2, \cdots$.

Consider the formal sum

$$\Phi(t) := \sum_{p=0}^{\infty} \frac{B_p}{p!} t^p$$

$$(6.12) \qquad = \sum_{p=0}^{p_0-1} \frac{B_p t^p}{p!} + \sum_{p=p_0}^{\infty} \frac{B_p t^p}{p!}.$$

We have

(6.13)

$$(4 - 3E_{p_0})\Phi(t) = (4 - 3E_{p_0}) \sum_{p=0}^{p_0-1} \frac{B_p t^p}{p!} + \sum_{p=p_0}^{\infty} \sum_{\substack{\mathbf{j}: \\ wt(j)=p}} \frac{B_{j_1} t^{j_1}}{j_1!} \frac{B_{j_2} t^{j_2}}{j_2!} \frac{B_{j_3} t^{j_3}}{j_3!}.$$

Therefore,

$$(6.14) \qquad (4 - 3E_{p_0})\Phi(t) - \Phi^3(t) = p(t),$$

where $p(t)$ is a polynomial. Set

$$(6.15) \qquad g(z) = (4 - 3E_{p_0})z - z^3.$$

So,

$$g(\Phi(t)) = p(t). \tag{6.16}$$

Now, $g'(1) = 1 - 3E_{p_0} > 0$. Thus, g^{-1} is analytic in a neighborhood of $g(1) = p(0)$. So, $\Phi(t) = g^{-1}(p(t))$ is analytic in a neighborhood of 0. Since the coefficients of the power series expansion of Φ about 0 are $B_p/p!$, there is a constant M such that

$$\overline{\lim_{p\to\infty}} \left[\frac{B_p}{p!}\right]^{1/p} = M < \infty. \tag{6.17}$$

Therefore, there is a constant C such that for all p,

$$B_p \le C(M+1)^p p!$$

which establishes (6.4).Thus we have established Theorem 2.1 except for the last sentence. But that follows from the fact the moment recursion formulas are only functions of the moments of the marginals. See Remark 7.1 and Corollary 7.4

Interestingly enough, there is a partial converse to the Theorem 2.1.

Theorem 6.1. *If $\hat{\mu}$ is the unique fixed point of both $T_{\tilde{\mu}_1}$ and $T_{\tilde{\mu}_2}$ iff $\mu_1 = \mu_2$ (i.e., $\tilde{\mu}_1$ and $\tilde{\mu}_2$ have equal marginals.).*

Proof. That the invariant measure It is enough to show that knowing the moments of $\hat{\mu}$ and that $\hat{\mu} = T_{\tilde{\mu}}(\hat{\mu})$ uniquely determines the moments of μ, the marginal of $\tilde{\mu}$, since μ is defined on the bounded interval $[0, \frac{2}{3}]$. To this end note that (2.1) implies $\hat{\mu}\left(\|\mathbf{x}\|^{2n}\right)$ equals

$$\int \left[\left\|\frac{\mathbf{S}_1}{3}\right\|^2 + 2\sqrt{\lambda}\left\langle\frac{\mathbf{S}_1}{3},\mathbf{u}\right\rangle\sqrt{S_2 - \frac{\|\mathbf{S}_1\|^2}{3}} + \frac{\mathbf{S}_1}{3} + \lambda\left(S_2 - \frac{\|\mathbf{S}_1\|^2}{3}\right)\right]^n d\hat{\mu}^3 d\pi_2 d\mu(\lambda)$$

which equals upon applying an obvious multinomial expansion

$$\sum_{i+j+k=n}\binom{n}{i,j,k}\int\left\|\frac{\mathbf{S}_1}{3}\right\|^{2i}\left\langle\frac{\mathbf{S}_1}{3},\mathbf{u}\right\rangle^j\left(S_2 - \frac{\|\mathbf{S}_1\|^2}{3}\right)^{k+\frac{j}{2}} \lambda^{k+\frac{j}{2}} d\hat{\mu}^3 d\pi_2 d\mu(\lambda).$$

In the last sum, if j is odd, then the term $\langle\frac{\mathbf{S}_1}{3},\mathbf{u}\rangle^j$ forces the term to be zero, so only whole powers of λ appear in the sum. Further, there is only one term which has λ^n, namely $i = j = 0; k = n$, and the coefficient $\int\left(S_2 - \frac{\|\mathbf{S}_1\|^2}{3}\right)^n d\hat{\mu}^3$ is positive. Thus, the moment $\int \lambda^n d\mu(\lambda)$ can be written as the fraction

$$\frac{\hat{\mu}\left(\|\mathbf{x}\|^{2n}\right) - \sum \binom{n}{i,j,k} \int \left\|\frac{\mathbf{S}_1}{3}\right\|^{2i} \langle \frac{\mathbf{S}_1}{3}, \mathbf{u}\rangle^j \left(S_2 - \frac{\|\mathbf{S}_1\|^2}{3}\right)^{k+\frac{j}{2}} \lambda^{k+\frac{j}{2}} d\hat{\mu}^3 d\pi_2 d\mu(\lambda)}{\int \left(S_2 - \frac{\|\mathbf{S}_1\|^2}{3}\right)^n d\hat{\mu}^3},$$

where the sum is over $\{(i,j,k) : k \neq n \text{ and } j \text{ even}\}$.■

7. The Normal Distribution

In this section we show that there is a redistribution of energy law, $\widetilde{\mu}$, such that the normal distribution on $\mathbf{R}^3$, $N(O, I_3)$ is its attractive invariant distribution of velocity. In order to demonstrate this, we fix the following notation. Let I_n denote the $n \times n$ identity matrix, S^n the unit sphere in $\mathbf{R}^{n+1}$, and π_n the uniform distribution on S^n. Let A be the 3×9 matrix $[I_3, I_3, I_3]$. Let O be an isometry of $\mathbf{R}^9$ given by the orthogonal matrix

$$O = \begin{bmatrix} B \\ \frac{1}{\sqrt{3}} A \end{bmatrix}. \tag{7.1}$$

For example, if U, V, and W are orthogonal 3×3 matrices each with last row $(1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3})$, then

$$\begin{bmatrix} u_{11} & 0 & 0 & u_{12} & 0 & 0 & u_{13} & 0 & 0 \\ u_{21} & 0 & 0 & u_{22} & 0 & 0 & u_{23} & 0 & 0 \\ 0 & v_{11} & 0 & 0 & v_{12} & 0 & 0 & v_{13} & 0 \\ 0 & v_{21} & 0 & 0 & v_{22} & 0 & 0 & v_{23} & 0 \\ 0 & 0 & w_{11} & 0 & 0 & w_{21} & 0 & 0 & w_{13} \\ 0 & 0 & w_{21} & 0 & 0 & w_{22} & 0 & 0 & w_{23} \\ \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \tag{7.2}$$

is such a matrix O.

We will employ the three 3×9 matrices

$$\begin{aligned} M_1 &= [I_3, \mathbf{0}, \mathbf{0}] \\ M_2 &= [\mathbf{0}, I_3, \mathbf{0}] \\ M_3 &= [\mathbf{0}, \mathbf{0}, I_3]. \end{aligned} \tag{7.3}$$

Let O be an orthogonal 9×9 matrix of the form of (7.1). Let φ_0 be the map from $\mathbf{R}^6$ into $\mathbf{R}^3$ given by $\varphi_0(\mathbf{y}) = \langle \lambda_i \rangle_{i=1}^3$, where

$$\lambda_i = \left\| M_i O^T \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} \right\|^2, i = 1, 2, 3, \tag{7.4}$$

for $\mathbf{y} \in \mathbf{R}^6$, and

$$\mathbf{y} * 0 = \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} := \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_6 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Lemma 7.1. *For each O of the form of (7.1), φ_O maps S^5 onto the disk D.*

Proof. Let $\mathbf{c}_1, ..., \mathbf{c}_9$ be the column vectors of B. We have

$$O^T \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \langle \mathbf{c}_1, \mathbf{y} \rangle \\ \vdots \\ \langle \mathbf{c}_9, \mathbf{y} \rangle \end{bmatrix}. \tag{7.5}$$

So,

$$\begin{aligned} \sum_{i=1}^{3} \lambda_i &= \sum_{i=1}^{3} \left\| M_i O^T \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} \right\|^2 \\ &= \sum_{i=1}^{9} \langle \mathbf{c}_i, \mathbf{y} \rangle^2 . \end{aligned} \tag{7.6}$$

But, for each i, $\langle \mathbf{c}_i, \mathbf{y} \rangle = \langle \mathbf{k}_i, \mathbf{y} * 0 \rangle$, where $\mathbf{k}_i$ is the i^{th} column of O. Thus,(1.4)

$$\sum_{i=1}^{3} \lambda_i = \|\mathbf{y} * \mathbf{0}\|^2 = 1.$$

By the orthogonality of the rows of O, we have:

$$\begin{aligned} \mathbf{c}_1 + \mathbf{c}_4 + \mathbf{c}_7 &= 0 \\ \mathbf{c}_2 + \mathbf{c}_5 + \mathbf{c}_8 &= 0 \\ \mathbf{c}_3 + \mathbf{c}_6 + \mathbf{c}_9 &= 0. \end{aligned} \tag{7.7}$$

Also, from (7.5),

$$(M_1 + M_2 + M_3) O^T \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} = (1/\sqrt{3}) \begin{bmatrix} \langle \mathbf{c}_1 + \mathbf{c}_4 + \mathbf{c}_7, \mathbf{y} \rangle \\ \langle \mathbf{c}_2 + \mathbf{c}_5 + \mathbf{c}_8, \mathbf{y} \rangle \\ \langle \mathbf{c}_3 + \mathbf{c}_6 + \mathbf{c}_9, \mathbf{y} \rangle \end{bmatrix} = \mathbf{0}. \tag{7.8}$$

For each i, set $w_i = \frac{1}{\sqrt{\lambda_i}} M_i O^T \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix}$, if $\lambda_i \neq 0$ and $w_i = 0$ if $\lambda_i = 0$. We have (1.5)

$$\sqrt{\lambda_1} w_1 + \sqrt{\lambda_2} w_2 + \sqrt{\lambda_3} w_3 = 0.$$

Thus according to the results of section 1, φ_O maps S^5 into D. It can be checked that φ_O actually maps S^5 onto D.

For each O of the form of (7.1), let $\widetilde{\mu}_0$ be the probability measure on D defined by

$$\widetilde{\mu}_0(E) = \pi_5(\varphi_0^{-1}(E)).$$

Lemma 7.2. *If O is of the form of (7.1), then $\widetilde{\mu}_O$ is uniform measure on D.*

Proof. We begin by showing that $\widetilde{\mu}_{O_1} = \widetilde{\mu}_{O_2}$ for any two matrices O_1, O_2 of form of (7.1). First note for any orthogonal matrix $O = \begin{bmatrix} B \\ \frac{1}{\sqrt{3}}A \end{bmatrix}$,

$$I_9 = OO^T = \begin{bmatrix} BB^T & \frac{1}{\sqrt{3}}BA^T \\ \frac{1}{\sqrt{3}}AB^T & I_3 \end{bmatrix}.$$

Thus

$$BB^T = I_6, AB^T = \mathbf{0}, \text{ and } BA^T = \mathbf{0}. \tag{7.9}$$

Let $O_i = \begin{bmatrix} B_i \\ \frac{1}{\sqrt{3}}A \end{bmatrix}$, for $i = 1, 2$. Then, $O_1O_2^T = \begin{bmatrix} B_1B_2^T & \frac{1}{\sqrt{3}}B_1A^T \\ \frac{1}{\sqrt{3}}AB_2^T & I_3 \end{bmatrix} = \begin{bmatrix} B_1B_2^T & \mathbf{0} \\ \mathbf{0} & I_3 \end{bmatrix}$ which is an orthogonal 9×9 matrix. And so $B_1B_2^T$ is an orthogonal 6×6 matrix. Thus $O_1O_2^T \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix}$ has the same distribution as $\begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix}$ for uniform $\mathbf{y}$. Easily, $\widetilde{\mu}_{O_1} = \widetilde{\mu}_{O_2}$.

Next we show $\widetilde{\mu}_O$ is invariant under the rotation group of the disk D. Let O be chosen to be the matrix

$$\begin{bmatrix}
\sqrt{\frac{2}{3}} & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & \sqrt{\frac{2}{3}} & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & -\frac{1}{\sqrt{6}} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\
0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}}
\end{bmatrix}. \tag{7.10}$$

For $0 \le \gamma_1, \gamma_2, \gamma_3 \le \frac{\pi}{2}$ with $\sum_{i=1}^{3} \cos^2 \gamma_i = 1$ and $0 \le \theta_1, \theta_2, \theta_3 < 2\pi$, the vector

$$\mathbf{y} = \begin{bmatrix} \cos\gamma_1 \cos\theta_1 \\ \cos\gamma_1 \sin\theta_1 \\ \cos\gamma_2 \cos\theta_2 \\ \cos\gamma_2 \sin\theta_2 \\ \cos\gamma_3 \cos\theta_3 \\ \cos\gamma_3 \sin\theta_3 \end{bmatrix} \tag{7.11}$$

represents an arbitrary vector of S^5. Defining $\lambda_1, \lambda_2, \lambda_3$ by (7.4), using double angle formulas

$$\lambda_1 = \frac{2}{3} \sum_{i=1}^{3} \cos^2\gamma_i \cos^2\theta_i = \frac{1}{3} + \frac{1}{3} \sum_{i=1}^{3} \cos^2\gamma_i \cos 2\theta_i$$

and similarly

$$\begin{aligned} \gamma_2 &= \frac{1}{3} - \frac{1}{6} \sum_{i=1}^{3} \cos^2\gamma_i \cos 2\theta_i + \frac{1}{2\sqrt{3}} \sum_{i=1}^{3} \cos^2\gamma_i \sin 2\theta_i \\ \gamma_3 &= \frac{1}{3} - \frac{1}{6} \sum_{i=1}^{3} \cos^2\gamma_i \cos 2\theta_i - \frac{1}{2\sqrt{3}} \sum_{i=1}^{3} \cos^2\gamma_i \sin 2\theta_i. \end{aligned}$$

Thus the representation of $[\lambda_1, \lambda_2, \lambda_3]$ of the form of display (1.12) is

$$\begin{aligned} r\cos\theta &= \sum_{i=1}^{3} \cos^2\gamma_i \cos 2\theta_i \\ r\sin\theta &= \sum_{i=1}^{3} \cos^2\gamma_i \sin 2\theta_i. \end{aligned} \tag{7.12}$$

Multiplying (7.12) by the rotation matrix $\begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$ we obtain

$$\begin{aligned} r\cos(\theta - \psi) &= \sum_{i=1}^{3} \cos^2\gamma_i \cos(2\theta_i - \psi) \\ r\sin(\theta - \psi) &= \sum_{i=1}^{3} \cos^2\gamma_i \sin(2\theta_i - \psi). \end{aligned} \tag{7.13}$$

Since *given* γ_1, γ_2, and γ_3, the π_5-conditional distribution of the $\theta_i's$ are independent uniforms on $[0, 2\pi)$, it follows that (7.13) has the same distribution as (7.12). Thus, $\tilde{\mu}_O$ is rotation invariant.

Next the distribution of the variable '$r\cos\theta$' of the (1.12) representation will be calculated. We choose the following parameterization of S^5 : for

$\gamma \in [0, \frac{\pi}{2}]$, β, $\alpha \in [0, \pi]$, ϕ, and $\psi \in [0, 2\pi)$,

$$\mathbf{y} = \begin{bmatrix} \cos\gamma\cos\beta \\ \sin\gamma\cos\alpha \\ \cos\gamma\sin\beta\cos\phi \\ \sin\gamma\sin\alpha\cos\psi \\ \cos\gamma\sin\beta\sin\phi \\ \sin\gamma\sin\alpha\sin\psi \end{bmatrix}. \tag{7.14}$$

Using the orthogonal matrix (7.10) with the parameterization (7.14), and easy calculation using double angle formulas reveals $\lambda_1 = \frac{1}{3} + \frac{1}{3}\cos 2\gamma$. Thus $r\cos\theta$ of representation 1.11 is distributed like $\cos 2\gamma$. For parameterization (7.14), the probability density function associated with π_5 has form $C\cos^2\gamma\sin\beta\sin^2\gamma\sin\alpha$ for some constant C. Thus for constants C', C'':

$$\begin{aligned} \pi_5(r\cos\theta \le a) &= C' \int_{\frac{1}{2}\arccos(a)}^{\pi} \cos^2\gamma\sin^2\gamma d\gamma \\ &= \int_{-1}^{a} \sqrt{1-v^2}dv. \end{aligned} \tag{7.15}$$

But the only probability measure on the unit disk which is rotation invariant and which has (7.15)-distributed x-coordinate is the uniform distribution. ■

Theorem 7.3. *The normal distribution, $N(\mathbf{0}, I_3)$, on $\mathbf{R}^3$ is the attractive invariant distribution of velocity under the redistribution of energy law which is the uniform distribution on D.*

Proof. Let $\tilde{v}$ distribution on D and v is its projection onto the x-axis. We calculate

$$\begin{aligned} \int_{\mathbf{R}^3} f dN(\mathbf{0}, I_3) &= \int_{\mathbf{R}^9} f((x_1, x_2, x_3))dN(\mathbf{0}, I_9) \\ &= \int_0^\infty \int_{S^8} f(\sqrt{r}(w_1, w_2, w_3))d\pi_8(\mathbf{w})dp(r). \end{aligned} \tag{7.16}$$

Set $\mathbf{c} = \mathbf{A}w \in \mathbf{A}(S^8)$ and let $\tau(\mathbf{c})$ be the marginal distribution of $\mathbf{c}$. The form of O and the fact $\frac{[\mathbf{c},c,c]}{3}$ is perpendicular to $O^T \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix}$ implies: for each $\mathbf{c} \in \mathbf{A}(S^8)$ we have $\mathbf{A}w = \mathbf{c}$ with $\|\mathbf{w}\| = 1$ if and only if there is some $\mathbf{y} \in S^5$ such that

$$\mathbf{w} = \frac{[\mathbf{c}, c, c]}{3} + \sqrt{\left(1 - \frac{\|\mathbf{c}\|^2}{3}\right)} O^T \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix}.$$

Thus,

$$[w_1, w_2, w_3] = (\mathbf{c}/3) + \sqrt{1 - \frac{\|\mathbf{c}\|^2}{3}} M_1 O^T \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix}.$$

So, (7.16) =

$$\int_0^\infty \int_{\mathbf{A}(S^8)} \int_{S^5} f\left(\sqrt{r}((\mathbf{c}/3) + \sqrt{\left(1 - \frac{\|\mathbf{c}\|^2}{3}\right)} M_1 O^T \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix}\right) d\pi_5(\mathbf{y}) d\tau(\mathbf{c}) dp(r).$$

Now, $M_1 O^T \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix}$ is rotation invariant on $\mathbf{R}^3$. So,

$$M_1 O^T \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} d\pi_5(\mathbf{y}) = \left\| M_1 O^T \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} \right\| \mathbf{u} d\pi_5(\mathbf{y}) d\pi(\mathbf{u}).$$

Thus, (7.16) =

$$\int_0^\infty \int_{\mathbf{A}(S^8)} \int_{[0,\frac{2}{3}]} \int_{S^2} f\left[\sqrt{r}\left(\frac{\mathbf{c}}{3} + \sqrt{\lambda\left(1 - \frac{\|\mathbf{c}\|^2}{3}\right)} \mathbf{u}\right)\right] d\pi_2(\mathbf{u}) dv(\lambda) d\tau(\mathbf{c}) dp(r).$$

Putting in a dummy integral over S^5, (7.16) =

$$\int_{[0,\frac{2}{3}]} \int_{S^2} \int_0^\infty \int_{\mathbf{A}(S^8)} \int_{S^5} f\left[\sqrt{r}\left(\frac{\mathbf{c}}{3} + \sqrt{\lambda\left(1 - \frac{\|\mathbf{c}\|^2}{3}\right)}\right)\mathbf{u}\right] d\pi_5 d\tau(\mathbf{c}) dp(r) d\pi_2(\mathbf{u}) dv(\lambda)$$

Now, $\mathbf{c} = \mathbf{A}(\mathbf{x}/\|\mathbf{x}\|) = (1/\|\mathbf{x}\|)\mathbf{A}x = (1/\sqrt{S_2})\mathbf{S}_1$. So, (7.16) =

$$\int_{[0,\frac{2}{3}]} \int_{S^2} \int_{\mathbf{R}^9} f\left[\sqrt{S_2}\left(\frac{\mathbf{S}_1}{3\sqrt{S_2}}) + \sqrt{\lambda\left(1 - \frac{\|\mathbf{S}_1\|^2}{3}\right)} \mathbf{u}\right)\right] dN(O, I_9) d\pi_2 dv,$$

which equals (2.1). ■

This choice $\tilde{v}$, uniform distribution on D, is not the only choice of energy redistribution law that attracts the Gaussian distribution of velocity. Let *Wedge* $= \{\mathbf{x} \in D : \exists$ a $\theta \in [0, \frac{\pi}{6}]$ and an $r \in [0,1]$ s.t. equation 1.12 holds$\}$, let $\mathcal{L} = \{\mathfrak{a} : \textit{Wedge} \to [0,1] : \mathfrak{a}$ is a Borel function$\}$, and $\tilde{\mu}_{\mathfrak{a}}$ be the measure defined through the following representation

$$\tilde{\mu}_{\mathfrak{a}} = \int \mathfrak{a}(\mathbf{x})\tau_{\mathbf{x}} + (1 - \mathfrak{a}(\mathbf{x}))\, \tau_{Ref(\mathbf{x})} dv(x),$$

where v is uniform distribution on *Wedge*, $\tau_{\mathbf{x}}$ is the τ for $\mathbf{x}$ in (2.8) and $Ref : D \to D$ is defined by $Ref(\mathbf{x}) = \left[\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right] - \mathbf{x}$. Let $\mu_{\mathfrak{a}}$ denote the projection measure of $\tilde{\mu}_{\mathfrak{a}}$ with respect to the x-axis.

Corollary 7.4. *For all $\mathfrak{a} \in \mathcal{L}$, $\mu_{\mathfrak{a}} = v$, and therefore the attractive invariant distribution of velocity associated with $\tilde{\mu}_{\mathfrak{a}}$ is $N(\mathbf{0}, I_3)$.*

Proof. It is easy to check that $\tilde{\mu}_{\mathfrak{a}} \circ Ref^{-1}$ is also a redistribution of energy law and that it has the same marginal projection in the x-direction as $\tilde{\mu}_{\mathfrak{a}}$. But the measure $\frac{1}{2}\left(\tilde{\mu}_{\mathfrak{a}} + \tilde{\mu}_{\mathfrak{a}} \circ Ref^{-1}\right) = \tilde{v}$, uniform on D, and so the marginal of $\tilde{\mu}_{\mathfrak{a}}$ on the x-axis is the same as v, the marginal of $\tilde{v}$. Apply Theorem 2.1 and Theorem 7.3.■

Obviously $\tilde{\mu}_{\mathfrak{a}} \neq \tilde{v}$ unless $\mathfrak{a}$ is essentially identically $\frac{1}{2}$.

Remark 7.1. *The only redistribution of energy law to produce the attractive invariant distribution of velocity law associated with the redistribution law* $\delta_{[\frac{1}{3},\frac{1}{3},\frac{1}{3}]}$ *is the energy redistribution law* $\delta_{[\frac{1}{3},\frac{1}{3},\frac{1}{3}]}$ *itself.*

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