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A PATHWISE APPROACH TO DYNKIN GAMES*

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Abstract

We reduce the classical discrete-time game of optimal stopping between two players, known as "Dynkin game", to a *pathwise* (deterministic) game of timing, by addition of a suitable non-adapted compensator (λ_n) to the payoff. This compensator satisfies $\mathbb{E}(\lambda_n|\mathcal{F}_n) \equiv 0$, where $\mathcal{F}_n$ is the information available to the players at time $t = n$, and $\mathbb{E}$ denotes expectation with respect to the underlying probability measure $\mathbb{P}$; the compensator also enforces the *non-anticipativity constraint* that the strategies of both players be stopping times of $(\mathcal{F}_n)$. A pair of such stopping times is identified, which leads to a saddle-point for each of these games; and it is shown that the value V of the stochastic game is obtained by "averaging out" the value $W(\omega)$ of the pathwise game: $V = \int_\Omega W(\omega)\mathbb{P}(d\omega)$.

1. Introduction and Summary. We present a simple approach to the discrete-time *stochastic game of optimal stopping* (or timing) known as "Dynkin game" (Dynkin & Yushkevich (1968)), with payoff from player **A** to player **B** equal to

$$\mathcal{R}(\sigma,\tau) = U_\sigma 1_{\{\sigma<\tau\}} + L_\tau 1_{\{\tau<T,\tau\le\sigma\}} + \xi 1_{\{\sigma=\tau=T\}}. \tag{1.1}$$

Here $U_n \geq L_n (n\epsilon\mathbb{N}_0)$ are integrable random sequences, adapted to the filtration $\mathbb{F} = \{\mathcal{F}_n, n\epsilon\mathbb{N}_0\}; \sigma$ and τ are stopping times of $\mathbb{F}$ with values in $\{0,1,\ldots,T\}$, at the disposal of players **A** and **B**, respectively; $T \leq \infty$ is the "horizon" of the game; ξ is an integrable random variable; and

$$\overline{V} \triangleq \inf_\sigma \sup_\tau \mathbb{E}\mathcal{R}(\sigma,\tau), \qquad \underline{V} \triangleq \sup_\tau \inf_\sigma \mathbb{E}\mathcal{R}(\sigma,\tau) \tag{1.2}$$

are the upper- and lower- values, respectively, of the game. Notice that in the infinite-horizon case $(T = \infty)$ we are allowing stopping times to be extended-valued, i.e., to take the value $+\infty$.

Under reasonably mild conditions, we show that this game has value $V = \overline{V} = \underline{V}$, as well as a saddle-point of stopping times $(\hat{\sigma}, \hat{\tau})$, by looking

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instead at an appropriate *pathwise* (deterministic) game. This new game has payoff

$$Q(s,t;\omega) \triangleq (U_s(\omega)+\lambda_s(\omega))1_{\{s<t\}} + (L_t(\omega)+\lambda_t(\omega))1_{\{t<T,t\le s\}} + (\xi(\omega)+\lambda_T(\omega))1_{\{t=s=T\}} \tag{1.3}$$

for $(s,t)\epsilon\{0,1,\dots,T\}^2$, and value

$$W(\omega) \triangleq \inf_s \sup_t Q(s,t;\omega) = \sup_t \inf_s Q(s,t;\omega) \tag{1.4}$$

for each fixed $\omega\epsilon\Omega$.

For a proper choice of *non-adapted* "compensator" $(\lambda_n(\omega), n=0,1,\dots,T)$ with $I\!E[\lambda_n|\mathcal{F}_n]=0$, one finds very easily a saddle-point $(\hat{\sigma}(\omega),\hat{\tau}(\omega))$ for the deterministic game of (1.3), (1.4); observes that $\omega\mapsto\hat{\sigma}(\omega)$, $\omega\mapsto\hat{\tau}(\omega)$ are *stopping times* which provide also a saddle-point for the stochastic game of (1.1), (1.2); and computes the value in (1.2) simply by "averaging out" the value of the pathwise game: $V=\int W(\omega)I\!P(d\omega)$. Equivalently,

$$\begin{aligned}\inf_\sigma \sup_\tau I\!E[U_\sigma 1_{\{\sigma<\tau\}} + L_\tau 1_{\{\tau<T,\tau\le\sigma\}} + \xi 1_{\{\sigma=\tau=T\}}] &= \\ = I\!E[\inf_s \sup_t ((U_s+\lambda_s)1_{\{s<t\}} + (U_t+\lambda_t)1_{\{t<T,t\le s\}} & \\ + (\xi+\lambda_T)1_{\{s=t=T\}})],&\end{aligned}$$

where the infima and suprema can be interchanged. A similar result for the classical optimal stopping problem appears in Davis & Karatzas (1994), along with an application to a so-called "prophet inequality".

The approach is carried out first for the finite-horizon case (i.e., $T<\infty$), which is the most transparent and the simplest to present (section 3), and then for the infinite-horizon case $T=\infty$ which requires some additional technicalities (section 4). A similar development for a continuous-time Dynkin game has been carried out by Cvitanić & Karatzas (1996) in connection with the study of Backwards Stochastic Differential Equations. It would be of some interest, to determine whether more general stochastic optimization problems – including stochastic games – might be amenable, and usefully, to such a pathwise approach.

The "canonical" example that the reader may wish to keep in mind throughout, is the situation where $(L_1,U_1),(L_2,U_2),\dots$ are IID observations from a given bivariate distribution, ξ is a given real constant, and $\mathcal{F}_0=\{\emptyset,\Omega\}$, $\mathcal{F}_n=\sigma((L_j,U_j),j=1,\dots,n)$ for $n\epsilon I\!N$. On day $t=n$ $(n<T)$, and after both players have observed the pair (L_n,U_n), player **B** has priority and may decide to stop the game, in which case he receives the amount L_n from player **A**; if player **B** does not stop on that day, player **A** may decide to stop, in which case he pays the amount U_n to player **B**; and if neither

player stops on $t = n$, the game continues into the next period. If neither player stops at some $t = n$ ($n\epsilon\mathbb{N}, n < T$), then player **A** pays the amount ξ to player **B**. A very special game of this type is the "noisy duel", originated by Professor D. Blackwell and his co-workers at the Rand Corporation in the late '40s and discussed in Blackwell & Girshik (1954); see also the paper by T. Radzik (1996) in this volume.

2. The Setup. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an increasing sequence $\mathbb{F} = \{\mathcal{F}_n\}_{n\epsilon\mathbb{N}_0}$ of sub-σ-algebras of $\mathcal{F}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ mod. $\mathbb{P}$, and set $\mathcal{F}_{-1} \triangleq \mathcal{F}_0$, $\mathcal{F}_\infty \triangleq \sigma(\cup_{n\epsilon\mathbb{N}_0}\mathcal{F}_n)$. For any given $n\epsilon\mathbb{N}_0$ and $T\epsilon\mathbb{N}_{n,\infty}$ with $\mathbb{N}_{n,\infty} \triangleq \{n, n+1, \ldots\} \cup \{+\infty\}$, we shall denote by $\mathcal{M}_{n,T}$ the collection of $\mathbb{F}$-stopping times with values in $\mathbb{N}_{n,T} \triangleq \{n, n+1, \ldots, T\}$. Consider also two $\mathbb{F}$-adapted sequences of random variables $\mathbf{U} = \{U_n, n\epsilon\mathbb{N}_0\}$, $\mathbf{L} = \{L_n, n\epsilon\mathbb{N}_0\}$ with

$$L_n \leq U_n \quad (n\epsilon\mathbb{N}_0), \tag{2.1}$$

$$\mathbb{E}(\sup_n L_n^+ + \sup_n U_n^-) < \infty. \tag{2.2}$$

Suppose now that, starting at time $t = n$ and up until time $t = T$, two players **A**, **B** are engaged in the following *game of timing.* Each of them can choose a stopping time in $\mathcal{M}_{n,T}$ (say, σ for player **A**, and τ for player **B**) and the game terminates as soon as one of the players decides to stop, i.e., at the stopping time $\sigma \wedge \tau$. Upon termination, *player* **A** *pays player* **B** the random amount (payoff)

$$\mathcal{R}(\sigma, \tau) \triangleq U_\sigma 1_{\{\sigma<\tau\}} + L_\tau 1_{\{\tau<T, \tau\leq\sigma\}} + \xi 1_{\{\sigma=\tau=T\}} \tag{2.3}$$

where ξ is an integrable, $\mathcal{F}_T$-measurable random variable. In other words, the payoff (which may be positive, or negative) from player **A** to player **B**, equals: L_τ, if player **B** stops strictly before T and no later than **A** does; U_σ, if player **A** stops first; and ξ, if neither player stops before T.

The objective of player **A** is thus to minimize, and of player **B** to maximize, the conditional expectation of this random payoff (2.3), given the information accumulated up to time $t = n$. Thus, the *upper value* and the *lower value* of this game are given by the random variables

$$\overline{V}_n \triangleq \text{essinf}_{\sigma\epsilon\mathcal{S}_{n,T}} \text{esssup}_{\tau\epsilon\mathcal{T}_{n,T}} \mathbb{E}[\mathcal{R}(\sigma, \tau)|\mathcal{F}_n] \tag{2.4}$$

$$\underline{V}_n \triangleq \text{esssup}_{\tau\epsilon\mathcal{T}_{n,T}} \text{essinf}_{\sigma\epsilon\mathcal{S}_{n,T}} \mathbb{E}[\mathcal{R}(\sigma, \tau)|\mathcal{F}_n], \tag{2.5}$$

respectively. Here $\mathcal{S}_{n,T}$ (resp. $\mathcal{T}_{n,T}$) is the class of stopping times σ (resp. τ) in $\mathcal{M}_{n,T}$ for which the random variable U_σ (resp. L_τ) is integrable. Clearly, the numbers

$$\overline{V}_0 = \inf_{\sigma\epsilon\mathcal{S}_{0,T}} \sup_{\tau\epsilon\mathcal{T}_{0,T}} \mathbb{E}\mathcal{R}(\sigma,\tau), \qquad \underline{V}_0 = \sup_{\tau\epsilon\mathcal{T}_{0,T}} \inf_{\sigma\epsilon\mathcal{S}_{0,T}} \mathbb{E}\mathcal{R}(\sigma,\tau) \tag{2.6}$$

are the upper and lower values, respectively, for a game that starts at $n = 0$.

As we shall see, the upper and lower values of (2.4), (2.5) are actually the same under fairly general conditions; and the common *value of the game*

$$V_n \stackrel{\Delta}{=} \overline{V}_n = \underline{V}_n \qquad (n\epsilon\mathbb{N}_{0,T})$$

satisfies then the *Backwards Induction Equation*

$$X_T = \xi \tag{2.7}$$

$$X_n = \left\{ \begin{array}{ccc} U_n & ; \text{ on } & \{\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geq U_n\} \\ L_n & ; \text{ on } & \{\mathbb{E}(X_{n+1}|\mathcal{F}_n) \leq L_n\} \\ \mathbb{E}(X_{n+1}|\mathcal{F}_n) & ; \text{ on } & \{L_n < \mathbb{E}(X_{n+1}|\mathcal{F}_n) < U_n\} \end{array} \right\} \tag{2.8}$$

$(n\epsilon\mathbb{N}_{0,T}, n < T)$.

We shall do this, first for the finite-horizon case $T < \infty$ in section 3 under minimal assumptions, and then for the infinite-horizon case $T = \infty$ under some additional conditions (section 4).

3. Finite Horizon ($T < \infty$) Let us assume throughout this section, that $T\epsilon\mathbb{N}$ is a given fixed integer, and *the random variables* U_n, L_n $(n = 0, 1, \dots, T)$ *are integrable.* Then it is easy to see that

$$\mathcal{S}_{n,T} = \mathcal{T}_{n,T} = \mathcal{M}_{n,T}$$

and that the Backwards Induction Equation (2.7), (2.8) has a unique solution $\mathcal{X} = \{X_n, n\epsilon\mathbb{N}_{0,T}\}$. This is an integrable, $\mathbb{F}$-adapted random sequence.

Starting from this sequence, let us introduce the indicator random variables

$$\begin{aligned} \zeta_n &\stackrel{\Delta}{=} 1_{\{L_n < \mathbb{E}(X_{n+1}|\mathcal{F}_n) < U_n\}} \\ \eta_n &\stackrel{\Delta}{=} 1_{\{\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geq U_n\}} \qquad\qquad (n\epsilon\mathbb{N}_{0,T-1}) \\ \vartheta_n &\stackrel{\Delta}{=} 1_{\{\mathbb{E}(X_{n+1}|\mathcal{F}_n) \leq L_n\}} \end{aligned} \tag{3.1}$$

which satisfy $\zeta_n + \eta_n + \vartheta_n = 1$, and in terms of which (2.8) becomes

$$X_n = \zeta_n \mathbb{E}(X_{n+1}|\mathcal{F}_n) + \eta_n U_n + \vartheta_n L_n \qquad (n\epsilon\mathbb{N}_{0,T-1}). \tag{3.2}$$

Let us also introduce for $n \epsilon I\!N_{1,T}$ the transforms

$$(3.3) \qquad \begin{cases} M_0^{(0)} \triangleq \zeta_0(I\!E(X_1) - X_0), & M_n^{(0)} \triangleq \sum_{j=0}^{n-1} \zeta_j(X_{j+1} - X_j) \\ Y_0 \triangleq \eta_0(U_0 - X_0), & Y_n \triangleq \sum_{j=0}^{n-1} \eta_j(X_{j+1} - X_j) \\ Z_0 \triangleq \vartheta_0(L_0 - X_0), & Z_n \triangleq \sum_{j=0}^{n-1} \vartheta_j(X_{j+1} - X_j) \end{cases}$$

and observe that we have $X_n = X_0 + M_n^{(0)} + Y_n + Z_n$ for every $n \epsilon I\!N_{0,T}$. Clearly from (3.1) - (3.3):

$$\begin{aligned} & I\!E[M_{n+1}^{(0)}|\mathcal{F}_n] - M_n^{(0)} = I\!E[\zeta_n(X_{n+1} - X_n)|\mathcal{F}_n] \\ & = I\!E[1_{\{L_n < I\!E(X_{n+1}|\mathcal{F}_n) < U_n\}}(X_{n+1} - I\!E(X_{n+1}|\mathcal{F}_n))|\mathcal{F}_n] = 0, \end{aligned}$$

so that $\{M_n^{(0)}, n \epsilon I\!N_{0,T}\}$ is a *martingale*. Similarly,

$$\begin{aligned} I\!E(Y_{n+1}|\mathcal{F}_n) - Y_n &= I\!E[\eta_n(X_{n+1} - X_n)|\mathcal{F}_n] \\ &= I\!E[1_{\{I\!E(X_{n+1}|\mathcal{F}_n) \geq U_n\}}(X_{n+1} - U_n)|\mathcal{F}_n] \\ &= (I\!E(X_{n+1}|\mathcal{F}_n) - U_n)^+ \geq 0 \end{aligned}$$

and

$$\begin{aligned} I\!E(Z_{n+1}|\mathcal{F}_n) - Z_n &= I\!E[\vartheta_n(X_{n+1} - X_n)|\mathcal{F}_n] \\ &= I\!E[1_{\{I\!E(X_{n+1}|\mathcal{F}_n) \leq L_n\}}(X_{n+1} - L_n)|\mathcal{F}_n] \\ &= -(L_n - I\!E(X_{n+1}|\mathcal{F}_n))^+ \leq 0. \end{aligned}$$

In other words, $\{Y_n, n \epsilon I\!N_{0,T}\}$ is a submartingale, and $\{Z_n, n \epsilon I\!N_{0,T}\}$ a supermartingale, so that we have the *Doob decompositions*

$$(3.5) \qquad Y_n = M_n^{(A)} + A_n, \quad Z_n = M_n^{(B)} - B_n \qquad (n \epsilon I\!N_{0,T}),$$

where $M^{(A)}, M^{(B)}$ are martingales and $\mathcal{A} = \{A_n, n \epsilon I\!N_{0,T}\}$, $\mathcal{B} = \{B_n, n \epsilon I\!N_{0,T}\}$ with $A_0 \triangleq 0, \quad B_0 \triangleq 0$, and

$$(3.6) \qquad A_n \triangleq \sum_{j=1}^{n} \{I\!E(Y_j|\mathcal{F}_{j-1}) - Y_{j-1}\} = \sum_{j=1}^{n} (I\!E(X_j|\mathcal{F}_{j-1}) - U_{j-1})^+$$

$$(3.7) \qquad B_n \triangleq \sum_{j=1}^{n} \{Z_{j-1} - I\!E(Z_j|\mathcal{F}_{j-1})\} = \sum_{j=1}^{n} (L_{j-1} - I\!E(X_j|\mathcal{F}_{j-1}))^+$$

for $n \epsilon I\!N_{1,T}$, are *predictable, increasing and integrable random sequences* ($I\!E(A_T + B_T) < \infty$). It develops then from (3.4), (3.5) that we have the decomposition

$$(3.8) \qquad X_n = X_0 + M_n + A_n - B_n \qquad (n \epsilon I\!N_{0,T})$$

for the solution of the Backwards Induction equation of (2.7), (2.8), where $M = M^{(0)} + M^{(A)} + M^{(B)}$ is a *martingale.*

The nonnegative, $\mathcal{F}_{n-1}$ - measurable random variables A_n, B_n in the decomposition (3.8) have a nice intuitive interpretation: for the game of (2.6), which starts at $t = 0$ and runs until $t = T$, the random variable A_n (respectively, B_n) represents the "regret" of player **A** (respectively, of player **B**) for not having stopped the game by time $t = n$. *Can we find an intuitive interpretation also for the martingale* (M_n) *in the decomposition (3.8)?*

In order to answer this question, let us introduce the *non-adapted*, integrable random sequence

$$\lambda_n \triangleq M_T - M_n \qquad (n \epsilon \mathbb{N}_{0,T}). \tag{3.9}$$

Also, for each fixed $\omega \epsilon \Omega$, let us consider a new, *deterministic game of timing*, with upper - and lower - values

$$\begin{aligned} \overline{W}_n(\omega) &\triangleq \inf_{s \epsilon \mathbb{N}_{n,T}} \sup_{t \epsilon \mathbb{N}_{n,T}} Q(s,t;\omega) \\ \underline{W}_n(\omega) &\triangleq \sup_{t \epsilon \mathbb{N}_{n,T}} \inf_{s \epsilon \mathbb{N}_{n,T}} Q(s,t;\omega) \end{aligned} \qquad (n \epsilon \mathbb{N}_{0,T}, \quad n < T) \tag{3.10}$$

respectively, and payoff (from player **A** to player **B**)

$$\begin{aligned} Q(s,t;\omega) &\triangleq U_s(\omega) 1_{\{s<t\}} + L_t(\omega) 1_{\{t<T, t \le s\}} \\ &\qquad + \xi(\omega) 1_{\{s=t=T\}} + \lambda_{s \wedge t}(\omega) \\ &= [U_s(\omega) + \lambda_s(\omega)] 1_{\{s<t\}} + [L_t(\omega) + \lambda_t(\omega)] 1_{\{t<T, t\le s\}} \\ &\qquad + [\xi(\omega) + \lambda_T(\omega)] 1_{\{s=t=T\}} \end{aligned} \tag{3.11}$$

for s, t in $\mathbb{N}_{n,T}$ (of course, $\lambda_T(\omega) \equiv 0$).

In other words, we create the new payoff $Q(s,t;\omega)$ by replacing the stopping times σ, τ in (2.3) by the *non-random times* $s \epsilon \mathbb{N}_{n,T}, t \epsilon \mathbb{N}_{n,T}$, and adding the "compensator" $\lambda_{s\wedge t}(\omega)$, for each fixed $\omega \epsilon \Omega$ – or alternatively, replacing $U_{\sigma(\omega)}(\omega)$, $L_{\tau(\omega)}(\omega)$, $\xi(\omega)$ by their "compensated counterparts" $U_s(\omega) + \lambda_s(\omega)$, $L_t(\omega) + \lambda_t(\omega)$ and $\xi(\omega) + \lambda_T(\omega)$, respectively.

Let us introduce also the $\mathbb{F}$-stopping times

$$\begin{aligned} \hat{\sigma}_n &\triangleq \min\{t \epsilon \mathbb{N}_{n,T} / X_t = U_t\} \wedge T \\ &= \min\{t \epsilon \mathbb{N}_{n,T} / \mathbb{E}(X_{t+1} | \mathcal{F}_t) \ge U_t\} \wedge T \\ \hat{\tau}_n &\triangleq \min\{t \epsilon \mathbb{N}_{n,T} / X_t = L_t\} \wedge T \\ &= \min\{t \epsilon \mathbb{N}_{n,T} / \mathbb{E}(X_{t+1} | \mathcal{F}_t) \le L_t\} \wedge T \end{aligned} \tag{3.12}$$

(with the convention $\min\emptyset = \infty$).

3.1 Theorem: For each fixed $\omega\epsilon\Omega$ and $n\epsilon\mathbb{N}_{0,T}, n < T$, the deterministic game of (3.10), (3.11) has *saddle - point* $(\hat{\sigma}_n(\omega), \hat{\tau}_n(\omega))$, i.e.,

$$\begin{aligned} Q(\hat{\sigma}_n(\omega), t; \omega) &\le Q(\hat{\sigma}_n(\omega), \hat{\tau}_n(\omega); \omega) = X_n(\omega) + \lambda_n(\omega) \\ &\le Q(s, \hat{\tau}_n(\omega); \omega), \quad \forall (s,t)\epsilon(\mathbb{N}_{n,T})^2, \end{aligned} \tag{3.13}$$

and thus its *value* $W_n(\omega) = \overline{W}_n(\omega) = \underline{W}_n(\omega)$ is given as

$$\begin{aligned} W_n(\omega) &= X_n(\omega) + \lambda_n(\omega) = Q(\hat{\sigma}_n(\omega), \hat{\tau}_n(\omega); \omega) \\ &= \mathcal{R}(\hat{\sigma}_n(\omega), \hat{\tau}_n(\omega)) + \lambda_{\hat{\sigma}_n(\omega)\wedge\hat{\tau}_n(\omega)}(\omega). \end{aligned} \tag{3.14}$$

3.2 Theorem: For each fixed $n\epsilon\mathbb{N}_{0,T}$, $n < T$ the original stochastic game of (2.3), (2.5) has *saddle - point* $(\hat{\sigma}_n, \hat{\tau}_n)\epsilon(\mathcal{M}_{0,T})^2$, i.e.,

$$\mathbb{E}[\mathcal{R}(\hat{\sigma}_n, \tau)|\mathcal{F}_n] \le \mathbb{E}[\mathcal{R}(\hat{\sigma}_n, \hat{\tau}_n)|\mathcal{F}_n] = X_n \le \mathbb{E}[\mathcal{R}(\sigma, \hat{\tau}_n)|\mathcal{F}_n] \tag{3.15}$$

for every $(\sigma, \tau)\epsilon(\mathcal{M}_{n,T})^2$, and *value* $V_n = \overline{V}_n = \underline{V}_n$ given by

$$V_n = X_n = \mathbb{E}[\mathcal{R}(\hat{\sigma}_n, \hat{\tau}_n)|\mathcal{F}_n] = \mathbb{E}[W_n|\mathcal{F}_n]. \tag{3.16}$$

In particular, for the game of (2.6) which starts at $n = 0$, Theorem 3.2 gives $(\overline{V}_0 = \underline{V}_0 =)\quad V_0 = \mathbb{E}(W_0) \quad (= \mathbb{E}(\overline{W}_0) = \mathbb{E}(\underline{W}_0))$, or equivalently

$$\begin{aligned} &\inf_{\sigma\epsilon\mathcal{S}_{0,T}} \sup_{\tau\epsilon\mathcal{T}_{0,T}} \mathbb{E}[U_\sigma 1_{\{\sigma<\tau\}} + L_\tau 1_{\{\tau<T,\tau\le\sigma\}} + \xi 1_{\{\sigma=\tau=T\}}] = \\ &= \mathbb{E}[\inf_{s\epsilon\mathbb{N}_{0,T}} \sup_{t\epsilon\mathbb{N}_{0,T}} ((U_s + \lambda_s)1_{\{s<t\}} + (L_t + \lambda_t)1_{\{t<T,t\le s\}} \\ &\qquad + (\xi + \lambda_T)1_{\{s=t=T\}})], \end{aligned} \tag{3.17}$$

where "infima" and "suprema" can be interchanged. In other words, the random sequence (λ_n) of (3.9) is a non-adapted *compensator* which, when added to the payoff structure of the original stochastic game as in (3.11), allows its solution to be carried out *pathwise*, i.e., for each $\omega\epsilon\Omega$ separately. At the same time, this compensator *enforces the non-anticipativity constraint* inherent in the stochastic game, in that it leads to a saddle - point $(\hat{\sigma}_0(\omega), \hat{\tau}_0(\omega))$ in (3.13) such that both $\hat{\sigma}_0, \hat{\tau}_0$ are *stopping times* (recall (3.12)). Finally, the value of the stochastic game is obtained by "averaging out" the value of the pathwise game:

$$V_0 = \int_\Omega W_0(\omega)\mathbb{P}(d\omega).$$

Proof of Theorem 3.1: From (3.12) and (3.6), we obtain

$$A_{\hat{\sigma}_n}(\omega) = A_n(\omega), \quad B_{\hat{\tau}_n}(\omega) = B_n(\omega) \tag{3.18}$$

for every $\omega\epsilon\Omega$; we shall use these facts repeatedly in what follows.

Let us take an arbitrary $t\epsilon\mathbb{N}_{n,T}$ and $s = \hat{\sigma}_n(\omega)$.

(i) If $s = \hat{\sigma}_n(\omega) < t$, we have from (3.18)

$$\begin{aligned}
&\mathcal{R}(\hat{\sigma}_n(\omega), t) = U_{\hat{\sigma}_n(\omega)}(\omega) = X_{\hat{\sigma}_n(\omega)}(\omega) \\
&= X_n(\omega) + (M_{\hat{\sigma}_n(\omega)}(\omega) - M_n(\omega)) \\
&\qquad\qquad + (A_{\hat{\sigma}_n(\omega)}(\omega) - A_n(\omega)) - (B_{\hat{\sigma}_n(\omega)}(\omega) - B_n(\omega)) \\
&= X_n(\omega) + (\lambda_n(\omega) - \lambda_{\hat{\sigma}_n(\omega)}(\omega)) - (B_{\hat{\sigma}_n(\omega)}(\omega) - B_n(\omega)) \\
&\leq X_n(\omega) + \lambda_n(\omega) - \lambda_{\hat{\sigma}_n(\omega)}(\omega)),
\end{aligned}$$

with equality if $t = \hat{\tau}_n(\omega)$.

(ii) If $s = \hat{\sigma}(\omega) \geq t$, we have:

$$\begin{aligned}
&\mathcal{R}(\hat{\sigma}_n(\omega), t) = L_t(\omega)1_{\{t<T\}} + \xi(\omega)1_{\{t=T\}} \leq X_t(\omega) \\
&= X_n(\omega) + M_t(\omega) - M_n(\omega) + A_t(\omega) - A_n(\omega) - B_t(\omega) + B_n(\omega) \\
&= X_n(\omega) + (\lambda_n(\omega) - \lambda_t(\omega)) - (B_t(\omega) - B_n(\omega)) \\
&\leq X_n(\omega) + \lambda_n(\omega) - \lambda_t(\omega)
\end{aligned}$$

with equality if $t = \hat{\tau}_n(\omega)$, again from (3.18).

In either case,

(3.19)
$$\left\{ \begin{array}{c} Q(\hat{\sigma}_n(\omega), t; \omega) = \mathcal{R}(\hat{\sigma}_n(\omega), t) + \lambda_{\hat{\sigma}_n(\omega)\wedge t}(\omega) \leq X_n(\omega) + \lambda_n(\omega) \\ \text{for all } \ t\epsilon\mathbb{N}_{n,T}, \text{ with equality if } \ t = \hat{\tau}_n(\omega) \end{array} \right\}.$$

A similar analysis yields

(3.20)
$$\left\{ \begin{array}{c} Q(s, \hat{\tau}_n(\omega); \omega) = \mathcal{R}(s, \hat{\tau}_n(\omega)) + \lambda_{s\wedge\hat{\tau}_n(\omega)}(\omega) \geq X_n(\omega) + \lambda_n(\omega) \\ \text{for all } \ s\epsilon\mathbb{N}_{n,T}, \text{ with equality if } \ s = \hat{\sigma}_n(\omega) \end{array} \right\},$$

and (3.19), (3.20) lead directly to (3.13) and to (3.14).

Proof of Theorem 3.2: From (3.9) and the optional sampling theorem, we have $\mathbb{E}(\lambda_\rho|\mathcal{F}_n) = 0, \ \forall \ \rho\epsilon\mathcal{M}_{n,T}$. Now let $s = \sigma(\omega), \ \ t = \tau(\omega)$ in (3.20), (3.19) for arbitrary stopping times $\sigma\epsilon\mathcal{S}_{n,T}$ and $\tau\epsilon\mathcal{T}_{n,T}$, and take conditional expectations with respect to $\mathcal{F}_n$; thanks to the above observations, we obtain (3.15), which then leads directly to (3.16) in conjunction with (3.14).

4. The Infinite-Horizon Case ($T = \infty$). We shall describe now, briefly, how Theorems 3.1 and 3.2 can be extended to the infinite-horizon case $T = \infty$. We shall take here $\xi = 0$ (that is, if neither player ever decides to stop the game, the amount paid is zero), so that the payoff of (2.3) becomes

$$\mathcal{R}(\sigma, \tau) = U_\sigma 1_{\{\sigma<\tau\}} + L_\tau 1_{\{\tau<\infty, \tau\leq\sigma\}}.$$

In this case it is known (cf. Neveu (1975), pp.139-144) that *there is a unique* $\mathbb{F}$*-adapted random sequence* $\mathcal{X} = \{X_n, n\epsilon\mathbb{N}_0\}$, *which satisfies the equation (2.8) and the double inequality*

$$\tilde{U}_n \leq X_n \leq \tilde{L}_n \qquad (n\epsilon\mathbb{N}_0). \tag{4.1}$$

Here $\{\tilde{L}_n, n\epsilon\mathbb{N}_0\}$ and $\{-\tilde{U}_n, n\epsilon\mathbb{N}\}$ are the smallest nonnegative supermartingales that dominate the random sequences $\{L_n, n\epsilon\mathbb{N}_0\}$ and $\{-U_n, n\epsilon\mathbb{N}_0\}$, respectively.

In particular, let us notice (with Neveu (1975)) that $\{\mathbb{E}(\sup_{k\geq n} L_k^+|\mathcal{F}_n), n\epsilon\mathbb{N}_0\}$ and $\{\mathbb{E}(\sup_{k\geq n} U_k^-|\mathcal{F}_n), n\epsilon\mathbb{N}_0\}$ are nonnegative supermartingales, and that they dominate the random sequences $\{L_n, n\epsilon\mathbb{N}_0\}$ and $\{-U_n, n\epsilon\mathbb{N}_0\}$ respectively. We deduce from (4.1) that

$$X_n \leq \tilde{L}_n \leq \mathbb{E}(\sup_{k\geq n} L_k^+|\mathcal{F}_n), \quad -X_n \leq -\tilde{U}_n \leq \mathbb{E}(\sup_{k\geq n} U_k^-|\mathcal{F}_n),$$

thus also

$$-\mathbb{E}\Big(\sup_{k\geq \ell} U_k^- \;\Big|\; \mathcal{F}_n\Big) \leq X_n \leq \mathbb{E}\Big(\sup_{k\geq \ell} L_k^+ \;\Big|\; \mathcal{F}_n\Big); \quad n\epsilon\mathbb{N}_{\ell,\infty}, \quad \ell\epsilon\mathbb{N}_0. \tag{4.2}$$

To proceed with a minimum of technical fuss, let us impose from now on the additional conditions

$$\overline{\lim}_n L_n \leq 0 \leq \underline{\lim}_n U_n \tag{4.3}$$

$$\mathbb{E}\sum_{n=1}^{\infty}(L_{n-1} - \mathbb{E}(L_n|\mathcal{F}_{n-1}))^+ < \infty \tag{4.4}$$

$$\mathbb{E}\sum_{n=1}^{\infty}(\mathbb{E}(U_n|\mathcal{F}_{n-1}) - U_{n-1})^+ < \infty. \tag{4.5}$$

From (4.2) we have then

$$\overline{\lim}_n X_n \leq \inf_\ell\Big(\lim_n \mathbb{E}\Big(\sup_{k\geq\ell} L_k^+ \;\Big|\; \mathcal{F}_n\Big)\Big) = \inf_\ell(\sup_{k\geq\ell} L_k^+) = \overline{\lim}_n L_n^+$$

$$-\underline{\lim}_n X_n \leq \inf_\ell\Big(\lim_n \mathbb{E}\Big(\sup_{k\geq\ell} U_k^- \;\Big|\; \mathcal{F}_n\Big)\Big) = \inf_\ell(\sup_{k\geq\ell} U_k^-) = \overline{\lim}_n U_n^-,$$

and in conjunction with the assumption (4.3) these imply that

(4.6) the limit $X_\infty \triangleq \lim_n X_n$ exists and equals $\xi = 0$, a.s.

In other words, *the terminal condition (2.7) is satisfied here as well.*

We can proceed now as in section 3, all the way up to the decomposition (3.8) which now holds for $n \epsilon \mathbb{N}_0$. The increasing, predictable random sequences $\mathcal{A} = \{A_n, n\epsilon\mathbb{N}_0\}, \mathcal{B} = \{B_n, n\epsilon\mathbb{N}_0\}$ as in (3.6), (3.7) are now dominated by the random variables $A_\infty \triangleq \lim_n \uparrow A_n$ and $B_\infty \triangleq \lim_n \uparrow B_n$ given by

$$A_\infty = \sum_{n=1}^{\infty} (\mathbb{E}(X_n|\mathcal{F}_{n-1}) - U_{n-1})^+ \leq \sum_{n=1}^{\infty} (\mathbb{E}(U_n|\mathcal{F}_{n-1}) - U_{n-1})^+$$
$$B_\infty = \sum_{n=1}^{\infty} (L_{n-1} - \mathbb{E}(X_n|\mathcal{F}_{n-1}))^+ \leq \sum_{n=1}^{\infty} (L_{n-1} - \mathbb{E}(L_n|\mathcal{F}_{n-1}))^+$$

respectively, which are integrable, thanks to the assumptions (4.4), (4.5).

Therefore, $\mathcal{A}$ and $\mathcal{B}$ are uniformly integrable; but so is also $\mathcal{X}$, as it is bounded from above and from below by two uniformly integrable martingales (recall (4.2) with $\ell = 0$, as well as the assumption (2.2)). Thus, the martingale

$$M_n = X_n - X_0 + B_n - A_n \qquad (n\epsilon\mathbb{N}_0)$$

of the decomposition (3.8) is also uniformly integrable; in particular

$$M_n \underset{n\to\infty}{\longrightarrow} M_\infty = B_\infty - A_\infty - X_0, \quad \text{both } a.s. \text{ and in } \mathbb{L}^1$$

(recall (4.6)), and $M_n = \mathbb{E}(M_\infty|\mathcal{F}_n)$ for $n\epsilon\mathbb{N}_{0,\infty}$.

We deduce from all this, that the non-adapted random sequence

$$\lambda_n \triangleq M_\infty - M_n \qquad (n\epsilon\mathbb{N}_{0,\infty})$$

is well-defined, by analogy with (3.9), and satisfies $\mathbb{E}[\lambda_\rho|\mathcal{F}_n] = 0$, for every $\rho\epsilon\mathcal{M}_{n,\infty}$ by the optional sampling theorem. It is then straightforward, to verify that *Theorems 3.1, 3.2 are still valid in this case* ($T = \infty$), *under the additional assumptions (4.3) - (4.5).*

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