SCORE TESTS FOR DEPENDENT CENSORING WITH SURVIVAL DATA

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In a standard survival data analysis, the observed time is the minimum of the survival time T and a censoring time independent of T. In this paper, we consider models featuring two censoring times U and V. The distribution of U is possibly related with that of T, while the second censoring time V is independent of both T and U. These models involve an Archimedean copula to incorporate a possible dependency between T and U. Score tests for dependent censoring are derived when a parametric model, e.g. Weibull or exponential, is assumed for T. One is fully parametric; it assumes that the marginal distributions of T, U, and V are either exponential or Weibull. The other is semiparametric; no assumptions are made on neither U nor V. The relative efficiencies of these two tests are compared with that of tests involving uncensored data. A numerical example is presented.

1. Introduction

A standard assumption underlying most statistical models for the analysis of lifetime data is the independence between survival and censoring times. This assumption is acceptable for *administrative* censoring associated with the termination of a study. However it is questionable in follow-up studies with self-selected removal where patients leave for causes possibly associated with the study variable. For such withdrawals, Frangakis and Rubin (2001) use the term *dropout* censoring. It is convenient to distinguish these two kinds of censoring. Administrative censoring is referred to as being "independent" while dropout censoring is called "dependent." The goal of this paper is to develop score tests to ascertain whether dropout censoring is truly dependent.

In a typical study the data set consists of the minimum (X) of survival, dependent censoring and independent censoring times T, U and V, respectively, is observed, with an indicator function δ taking value 1 if $X = T$, 0 if $X = U$ and -1 if $X = V$. Tsiatis (1975) showed that the independence between T, U and V is needed for their marginal distributions to be identifiable. This suggests that it might be difficult, not to say impossible, to test the independence between T and U without assuming a parametric model for at least one of their two marginal distributions. Nonparametric tests of independence might be feasible using additional follow-up of patients that either dropped out (see Lee and Wolfe, 1995, 1998, Lee, 1996) or "died" (see Lin, Robins and Wei, 1996) before the study termination. When only $\{(X_i, \delta_i)\}$ is observed, Zheng and Klein (1995) acknowledge that the level of dependency between T and U is not estimable from the data. It has to be determined a priori before implementing their survival distribution estimator that accounts for this dependency.

Parametric models featuring distributions for T and U and a copula for their association have been considered by Emoto and Matthews (1990) and Bhattacharyya (1997). We consider models similar to theirs. The joint survival function $S_{T,U,V}(t,u,v)$ of (T,U,V) is written in terms of $S_T(t)$, $S_U(u)$ and $S_V(v)$, the marginal survival functions, and a copula $\mathcal{C}\{x,y\}$ for the dependency between T and U (Nelsen, 1999), as

$$S_{T,U,V}(t,u,v) = \mathcal{C}\{S_T(t), S_U(u)\}S_V(v), \quad t,u,v > 0. \tag{1.1}$$

Our derivations use the *Archimedean copulas* introduced, in a statistical context, by Genest and McKay (1986) which are defined by

$$\mathcal{C}_a\{y_1,y_2\} = \phi_a^{-1}\{\phi_a(y_1)+\phi_a(y_2)\}, \quad (y_1,y_2) \in [0,1]^2, \tag{1.2}$$

where $\phi_a(\cdot)$ is a strictly decreasing and convex function defined on $(0,1]$ satisfying $\phi_a(1) = 0$. Parameter $a \in \mathbb{R}$ measures the dependency with $a = 0$ when T and U are independent. Thus at $a = 0$, (1.2) gives $\mathcal{C}_0\{y_1,y_2\} = y_1y_2$ corresponding to $\phi_0(t) = -\ln(t)$.

We construct two kinds of score tests for independence (H_0: $a = 0$). The first one is fully parametric; each marginal distribution is assumed to belong to a parametric family. The second test is semiparametric, a parametric model being assumed only for the distribution of survival time T. The asymptotic normality of these tests is established, and expressions for their respective asymptotic variances are derived. For all the models considered in this paper, the parametric test is shown to have a Pitman efficiency larger than 1 when compared with the semiparametric test.

In Section 2, we motivate the choice of Archimedean copulas to model the dependence in survival analyses. Likelihoods underlying the score tests, parametric and semiparametric, are then constructed. These two tests are described in Sections 3 and 4, respectively. Section 5 is devoted to the study of their relative performances using Pitman efficiencies. A numerical example is presented in Section 6. Section 7 provides some conclusions and perspectives.

2. Models and likelihoods

This section presents background informations about Archimedean copulas. Two parametric copula families, considered for the construction of score tests, are also introduced. The likelihoods for the parametric and semiparametric score tests are presented. A more detailed presentation of Archimedean copulas is available in Genest and McKay (1986) and Nelsen (1999).

2.1. Dependence framework and Archimedean copula

An attractive characteristic of Archimedean copulas is their close relationship to frailty models, as shown in Oakes (1989). Assuming that there exists a latent unobservable variable, say Y, such that the joint survival function of (T,U), conditioned by $Y=y$, is $\{S_T^*(t)\,S_U^*(u)\}^y$, where $S_T^*(t)$ and $S_U^*(u)$ denote baseline survival functions, then the unconditional survival function of (T,U) is $\mathcal{C}\{S_T(t),S_U(u)\}$, where $\mathcal{C}(x,y)$ is an Archimedean copula.

To study the dependency between survival and censoring, one needs to distinguish between *net* and *crude* hazard rates, say $\lambda_T(t)$ and $\lambda_T^{\#}(t)$, defined as

$$\lambda_T(t)=\lim_{h\downarrow 0}\frac{1}{h}\mathbb{P}\{t\le T<t+h\mid T\ge t\},$$
$$\lambda_T^{\#}(t)=\lim_{h\downarrow 0}\frac{1}{h}\mathbb{P}\{t\le T<t+h\mid T\ge t,U\ge t\}.$$

Rivest and Wells (2001) showed that when the joint survival distribution of T and U is given by $\phi_a^{-1}\{\phi_a(S_T(t))+\phi_a(S_U(u))\}$, the net and crude hazard rates are proportional as stated formally in the next proposition.

Proposition 2.1. *Let $\pi_a(t)=\phi_a^{-1}\{\phi_a(S_T(t))+\phi_a(S_U(t))\}$ be the survival function of* $\min(T,U)$ *and $\psi_a(t)=-t\phi_a'(t)$; then*

$$\lambda_{aT}^{\#}(t)=\lambda_T(t)\frac{\psi_a[S_T(t)]}{\psi_a[\pi_a(t)]}.$$

There exists a similar relation for net and crude hazard rates associated with U, say $\lambda_U(u)$ and $\lambda_{aU}^{\#}(u)$.

The next section reviews some basic properties for the two Archimedean copula families used in this paper.

2.2. Two families of Archimedean copulas

Example 2.1. The Clayton family is generated by

$$\phi_a(t)=\frac{1}{a}\{t^{-a}-1\},\quad a\in[-1,\infty)\setminus\{0\}.$$

Independence corresponds to $a\to 0$. For this family,

$$\mathcal{C}_a\{S_T(t),S_U(u)\}=\max\{0,\{[S_T(t)]^{-a}+[S_U(u)]^{-a}-1\}^{-1/a}\} \tag{2.1}$$

and

$$\lambda_{aT}^{\#}(t)=\lambda_T(t)\left\{\frac{S_T(t)}{\pi_a(t)}\right\}^{-a}. \tag{2.2}$$

When a is close to 0

$$\lambda^{\#}_{aT}(t) \approx \lambda_T(t) S_U(t)^a.$$

A dependency "à la Clayton" adds a decreasing component to the risk function when a is positive.

The following results, derived from (2.1), are used in the derivations presented later,

$$\begin{aligned}
\frac{\partial \ln \pi_a(x)}{\partial a} &\xrightarrow{a\to 0} \ln S_T(x) \ln S_U(x),\\
\frac{\partial^2 \ln \pi_a(x)}{\partial a^2} &\xrightarrow{a\to 0} \ln S_T(x) \ln S_U(x)\{\ln S_T(x) + \ln S_U(x)\},\\
\frac{\partial^2 \ln \pi_a(x)}{\partial a \partial \theta} &\xrightarrow{a\to 0} [S_T(x)]^{-1} \dot{S}_T(x) \ln S_U(x) + [S_U(x)]^{-1} \dot{S}_U(x) \ln S_T(x),
\end{aligned}$$

where θ denotes the vector of marginal parameters, and a point denotes a derivative with respect to θ.

Example 2.2. The Ali-Mikhail-Haq family is generated by

$$\phi_a(t) = \frac{1}{1-a} \ln\left\{\frac{1 + a(t-1)}{t}\right\}, \quad a \in [-1, 1),$$

and can be used to model a limited dependence; independence corresponds to $a = 0$. For this family,

$$C_a\{S_T(t), S_U(u)\} = \frac{S_T(t) S_U(u)}{1 - a[1 - S_T(t)][1 - S_U(u)]} \tag{2.3}$$

and

$$\lambda^{\#}_{aT}(t) = \lambda_T(t) \left\{\frac{1 + a[\pi_a(t) - 1]}{1 + a[S_T(t) - 1]}\right\}. \tag{2.4}$$

When a is close to 0,

$$\lambda^{\#}_{aT}(t) \approx \lambda_T(t)[1 - a S_T(t)\{1 - S_U(t)\}].$$

A Ali-Mikhail-Haq dependency multiplies the risk function by a bathtub shape function equal to 1 when t is either small or large.

The following results, derived from (2.3), are used in the sequel

$$\begin{aligned}
\left.\frac{\partial \ln \pi_a(x)}{\partial a}\right|_{a=0} &= \{1 - S_T(x)\}\{1 - S_U(x)\},\\
\left.\frac{\partial^2 \ln \pi_a(x)}{\partial a^2}\right|_{a=0} &= \{1 - S_T(x)\}^2\{1 - S_U(x)\}^2,\\
\left.\frac{\partial^2 \ln \pi_a(x)}{\partial a \partial \theta}\right|_{a=0} &= -\dot{S}_T(x)\{1 - S_U(x)\} - \dot{S}_U(x)\{1 - S_T(x)\}.
\end{aligned}$$

This copula is closely related to Frank's (Genest, 1987): these two copulas have the same behavior in the neighborhood of independence. The score test for independence for Frank's family is the same as that for Ali-Mikhail-Haq; thus all the results derived in this paper for Ali-Mikhail-Haq's copula also apply to Frank's copula.

2.3. Likelihoods

This section constructs two likelihoods for inference on dependence parameter a using the sample $\{(x_i, \delta_i); i = 1, \ldots, n\}$, where $x_i = \min(T_i, U_i, V_i)$ and δ_i takes the value 1 if T_i is observed, 0 if $x_i = U_i$ and -1 if $x_i = V_i$.

2.3.1. Fully parametric model

The complete likelihood can be written in terms of the joint survival function $S_{T,U,V}(t, u, v)$ as

$$L(a,\theta) = \prod_{\delta_i=1} \left\{ -\frac{\partial S_{T,U,V}(t, x_i, x_i)}{\partial t}\bigg|_{t=x_i} \right\} \prod_{\delta_i=0} \left\{ -\frac{\partial S_{T,U,V}(x_i, u, x_i)}{\partial u}\bigg|_{u=x_i} \right\} \times \prod_{\delta_i=-1} \left\{ -\frac{\partial S_{T,U,V}(x_i, x_i, v)}{\partial v}\bigg|_{v=x_i} \right\}.$$

For model (1.1), Proposition 2.1 yields

$$-\frac{\partial}{\partial t} S_{T,U,V}(t, x_i, x_i)\bigg|_{t=x_i} = \lambda^{\#}_{aT}(x_i)\, \pi_a(x_i)\, S_V(x_i).$$

The two equations above yield

$$L(a,\theta) = \prod_{\delta_i=1} \lambda^{\#}_{aT}(x_i) \prod_{\delta_i=0} \lambda^{\#}_{aU}(x_i) \prod_{\delta_i=-1} \lambda_V(x_i) \prod_{i=1}^{n} \pi_a(x_i) S_V(x_i). \tag{2.5}$$

Define the crude survival functions, for $W = T$ or $W = U$, as

$$S^{\#}_{aW}(t) = \exp\left\{ -\int_0^t \lambda^{\#}_{aW}(s)\, ds \right\}.$$

Now $\pi_a(\cdot)$ can be expressed as the product of the crude survival functions, $\pi_a(x) = S^{\#}_{aT}(x)\, S^{\#}_{aU}(x)$. Using this relation in (2.5) yields an alternative form for the likelihood,

$$L(a,\theta) = \left\{ \prod_{\delta_i=1} \lambda^{\#}_{aT}(x_i) \prod_{i=1}^{n} S^{\#}_{aT}(x_i) \right\} \left\{ \prod_{\delta_i=0} \lambda^{\#}_{aU}(x_i) \prod_{i=1}^{n} S^{\#}_{aU}(x_i) \right\} \times \left\{ \prod_{\delta_i=-1} \lambda_V(x_i) \prod_{i=1}^{n} S_V(x_i) \right\}. \tag{2.6}$$

Expressions (2.5) and (2.6) for the likelihood do not rely on the assumption that the copula is Archimedean. The likelihood for the parameters of (1.1) has this form even when the dependency between T and U is modeled using a non-Archimedean copula.

2.3.2. Semiparametric model

When nothing is known about the marginal distributions of U and V, inference about the dependence parameter a can be carried out with a *pseudo-likelihood*, comprising only the first term of (2.6). This first term depends on $\pi_a(\cdot)$. The pseudo-likelihood is constructed by replacing $\pi_a(\cdot)$ by a non-parametric estimator, say $\hat{\pi}(\cdot)$. This pseudo-likelihood is then given by

$$L^*(a, \theta_T) = \prod_{\delta_i=1} \lambda^*_{aT}(x_i) \prod_{i=1}^{n} S^*_{aT}(x_i), \tag{2.7}$$

where θ_T is the vector of parameters for the marginal distribution of T,

$$\lambda^*_{aT}(t) = \lambda_T(t) \frac{\psi_a[S_T(t)]}{\psi_a[\hat{\pi}(t)]}, \tag{2.8}$$

and

$$S^*_{aT}(t) = \exp\left\{ - \int_0^t \lambda^*_{aT}(s)\, ds \right\}.$$

Observe that (2.7) is a pseudo-likelihood that can be used to obtain consistent estimators of a and of the marginal parameters for T. This is however not pursued here; we use this pseudo-likelihood in Section 4 to derive tests of independence.

In the derivation of Section 4, $\hat{\pi}(\cdot)$ needs to be a left continuous function. It can be set equal to a left continuous Kaplan-Meier estimator for the survival function of $\min(T, U)$ calculated using the sample $\{(x_i, \delta^*_i) : i = 1, \ldots, n\}$, where $\delta^*_i = 1$ if either $\delta_i = 1$ or $\delta_i = 0$. When only dependent censoring is observed, this estimator reduces to the empirical estimator $\hat{\pi}(x) = (1/n) \sum_{i=1}^{n} \mathbb{1}_{[x_i \geq x]}$.

3. Parametric tests

In this section, we construct score tests for independence, i.e., for H_0: $a = 0$, using expression (2.5) for the likelihood. The marginals for T, U and V are assumed to be known up to a parameter θ. Two marginal models, the exponential and the Weibull, are used in the derivation. The parametrization for each one is given in Table 1. For the Weibull model, the parameter β is assumed to be estimated independently for T, U, and V, however the calculations are carried out assuming, as stated in Table 1, that β is the same for these three distributions. Our primary goal, which is to compare

Table 1. Marginal models

Model	$\lambda_T(t)$	$\lambda_U(t)$	$\lambda_V(t)$
Exponential	γ_T	γ_U	γ_V
Weibull	$\gamma_T\beta\, t^{\beta-1}$	$\gamma_U\beta\, t^{\beta-1}$	$\gamma_V\beta\, t^{\beta-1}$

the Pitman efficiencies of tests for H_0: $a = 0$ for several marginal models for T, explains this somewhat artificial treatment for the Weibull distribution.

General results about the construction of score tests are first reviewed. The results of the derivations for Clayton and Ali-Mikhail-Haq copulas and for the two marginal models of Table 1 are then presented.

3.1. General results

In a parametric framework, with a finite-dimensional parameter, a score test for H_0 is constructed using the statistic

$$U_{\mathrm{P}}(0,\tilde{\theta}) = \frac{\partial}{\partial a}\ell(a,\theta)\bigg|_{a=0,\theta=\tilde{\theta}}, \tag{3.1}$$

where $\ell(a,\theta) = \ln L(a,\theta)$ is the log-likelihood for (2.5) and $\tilde{\theta}$ denotes the estimator of marginal parameter θ under H_0; index "P" stands for parametric test. From (2.6), neglecting terms that do not depend on a, the log-likelihood $\ell(a,\theta)$ is equal to

$$\sum_{\delta_i=1} \ln \lambda^{\#}_{aT}(x_i) + \sum_{\delta_i=0} \ln \lambda^{\#}_{aU}(x_i) + \sum_{i=1}^{n} \ln \pi_a(x_i),$$

where $\lambda^{\#}_{aT}(x_i)$, $\lambda^{\#}_{aU}(x_i)$, and $\pi_a(x_i)$ depend on the models for the marginals and for the copula. Let $T_{\mathrm{P}} = n^{-1}U_{\mathrm{P}}(0,\tilde{\theta})$ denote the normalized score statistic.

Classical asymptotic results imply that score statistic T_{P} has a limiting normal distribution as the sample size n goes to ∞ (see Cox and Hinkley, 1979, Chapter 9). To complete the derivation, we have now to estimate its variance. Let $\mathcal{I}(a,\theta)$ be the observed information matrix, partitioned as

$$\mathcal{I}(a,\theta) = \begin{bmatrix} -\frac{\partial^2\ell(a,\theta)}{\partial a^2} & -\frac{\partial^2\ell(a,\theta)}{\partial a\partial\theta} \\ -\frac{\partial^2\ell(a,\theta)}{\partial\theta\partial a} & -\frac{\partial^2\ell(a,\theta)}{\partial\theta^2} \end{bmatrix} = \begin{bmatrix} \mathcal{I}_{aa}(a,\theta) & \mathcal{I}_{a\theta}(a,\theta) \\ \mathcal{I}_{\theta a}(a,\theta) & \mathcal{I}_{\theta\theta}(a,\theta) \end{bmatrix}, \tag{3.2}$$

and let $\mathcal{I}(0,\theta)$ be this matrix evaluated under H_0. To evaluate the variance of (3.1), we use the Fisher's information matrix $I(0,\theta) = \mathbb{E}_{0,\theta}\{\mathcal{I}(0,\theta)\}$. Partitioning $I(a,\theta)$ as (3.2), the variance of $n^{1/2}T_{\mathrm{P}}$ is estimated by

$$\sigma^2_{\mathrm{P}} = n^{-1}\{I_{aa}(0,\theta) - I_{a\theta}(0,\theta)I^{-1}_{\theta\theta}(0,\theta)I_{\theta a}(0,\theta)\}, \tag{3.3}$$

evaluated at $\theta = \tilde{\theta}$. The null hypothesis is rejected at level α if $n^{1/2}|T_{\mathrm{P}}|/\sqrt{\hat{\sigma}^2_{\mathrm{P}}}$ is larger than $z_{1-\alpha/2}$, the $100\alpha/2$ percentile of the normal distribution.

From expression (2.6) for the likelihood, it is clear that $\mathcal{I}_{\theta\theta}(0,\theta)$ depends only on marginal parameters. Note also that this matrix is block diagonal, with blocks $\mathcal{I}_{\theta_T\theta_T}(0,\theta)$, $\mathcal{I}_{\theta_U\theta_U}(0,\theta)$ and $\mathcal{I}_{\theta_V\theta_V}(0,\theta)$, corresponding to components of the null log-likelihood associated with the marginal distributions of T, U and V, respectively. When the marginal distribution for T is exponential, $\theta_T = \gamma_T$ while $\theta_T = (\beta, \gamma_T)$ for a Weibull margin. The parameters θ_U and θ_V are defined in a similar way. The inverse $I^{-1}_{\theta\theta}(0,\theta)$ is also block diagonal.

The next proposition gives $I^{-1}_{\theta_T\theta_T}(0,\theta)$ for the parameters of the distribution of T in Table 1. The blocks for U and V are deduced in a similar way. The proof is technical; a sketch is provided in Appendix A.

Proposition 3.1. *Let T, U and V be independent.*

(i) *Let T, U and V exponentially distributed, with hazard rate γ_T, γ_U and γ_V, respectively; then*

$$n\, I^{-1}_{\theta_T\theta_T}(0,\theta) = \gamma_T(\gamma_T + \gamma_U + \gamma_V).$$

(ii) *Let T, U and V Weibull distributed, with common shape parameter β and scale parameters γ_T, γ_U and γ_V, respectively. Let $c \approx 0.5772$ denote Euler's constant and $b = 1 - c - \ln\gamma$ where $\gamma = \gamma_T + \gamma_U + \gamma_V$; then*

$$n\, I^{-1}_{\theta_T\theta_T}(0,\theta) = \frac{6\gamma}{\pi^2}\left[\begin{array}{c|c} \frac{\beta^2}{\gamma_T} & -b\,\beta \\ \hline -b\,\beta & \gamma_T\left(\frac{\pi^2}{6} + b^2\right) \end{array}\right].$$

3.2. Score test for Clayton family

This section shows how to carry out the score test for Clayton's copula. All the derivations are presented in the appendix.

Example 2.1 (continued). For any marginal model, statistic (3.1) is

$$U_{\mathrm{P}}(0,\theta) = \sum_{\delta_i=0} \ln S_T(x_i) + \sum_{\delta_i=1} \ln S_U(x_i) + \sum_{i=1}^{n} \ln S_T(x_i) \ln S_U(x_i).$$

For exponential margins, the variance is

$$\sigma^2_{\mathrm{P}} = \frac{\gamma_T\gamma_U(\gamma_T + \gamma_U)}{(\gamma_T + \gamma_U + \gamma_V)^3},$$

and for the Weibull model

$$\sigma^2_{\mathrm{P}} = \frac{\gamma_T\gamma_U(\gamma_T + \gamma_U)}{(\gamma_T + \gamma_U + \gamma_V)^3}\left\{1 - \frac{6}{\pi^2}\right\}.$$

3.3. Score test for Ali-Mikhail-Haq family

This section shows how to carry out the score test for Ali-Mikhail-Haq's copula. All the derivations are presented in the appendix.

Example 2.2 (continued). For any marginal model, the statistic (3.1) is

$$U_{\mathrm{P}}(0,\theta) = -\sum_{\delta_i=0}\{1-S_T(x_i)\} - \sum_{\delta_i=1}\{1-S_U(x_i)\} + 2\sum_{i=1}^{n}\{1-S_T(x_i)\}\{1-S_U(x_i)\} - \sum_{\delta_i=-1}\{1-S_T(x_i)\}\{1-S_U(x_i)\}.$$

For this copula the variance formulae are more complex than for Clayton's. They are expressed as functions $k_1(\cdot)$ and $k_2(\cdot)$ of the marginal parameters defined by

$$(3.4)\quad k_1(\gamma_T,\gamma_U,\gamma_V) = \frac{2\gamma_T\gamma_U^2}{(2\gamma_T+\gamma)(2\gamma_T+\gamma_U+\gamma)(2\gamma_T+2\gamma_U+\gamma)} - \frac{\gamma_T\gamma_U^2\gamma}{(\gamma_T+\gamma)^2(\gamma_T+\gamma_U+\gamma)^2}$$

and,

$$(3.5)\quad k_2(\gamma_T,\gamma_U,\gamma_V) = k_1(\gamma_T,\gamma_U,\gamma_V) - \frac{6\gamma\gamma_T}{\pi^2}\left\{\frac{\ln[\gamma/(\gamma_T+\gamma)]}{\gamma_T+\gamma} - \frac{\ln[\gamma/(\gamma_T+\gamma_U+\gamma)]}{\gamma_T+\gamma_U+\gamma}\right\}^2,$$

where $\gamma = \gamma_T+\gamma_U+\gamma_V$.

Lengthy derivations sketched in appendix yield the following variance formulae,

$$\sigma_{\mathrm{P}}^2 = k_1(\gamma_T,\gamma_U,\gamma_V) + k_1(\gamma_U,\gamma_T,\gamma_V),$$

for exponential margins, and

$$\sigma_{\mathrm{P}}^2 = k_2(\gamma_T,\gamma_U,\gamma_V) + k_2(\gamma_U,\gamma_T,\gamma_V)$$

for the Weibull model.

4. Semiparametric tests

In this section, we construct score tests for independence using the pseudo-likelihood (2.7), assuming that the marginal distribution for T is in a parametric family. These tests do not assume any parametric model for censoring times U and V.

4.1. General results

Let $\ell^*(a, \theta_T) = \ln L^*(a, \theta_T)$ be the pseudo log-likelihood for (2.7). The semi-parametric test is constructed using the statistic

$$U_{\mathrm{SP}}(0, \tilde{\theta}_T) = \frac{\partial}{\partial a} \ell^*(a, \theta_T)\Big|_{a=0, \theta_T = \tilde{\theta}_T}, \tag{4.1}$$

where $\tilde{\theta}_T$ denotes the estimator of θ_T under H_0.

To study the asymptotic properties of this statistic, we write the pseudo log-likelihood using counting processes $\overline{N}(t) = \sum_{i=1}^n \mathbb{1}_{[X_i \le t, \delta_i = 1]}$ and $\overline{Y}(t) = \sum_{i=1}^n \mathbb{1}_{[X_i \ge t]}$ as

$$\ell^*(a, \theta_T) = \int_0^\tau \ln \lambda^*_{aT}(t)\, d\overline{N}(t) - \int_0^\tau \lambda^*_{aT}(t) \overline{Y}(t)\, dt, \tag{4.2}$$

where $\tau = \inf\{t : \overline{Y}(t) = 0\}$ and $\lambda^*_{aT}(t)$ is given by (2.8). Let

$$\mathrm{H}^*_{a,\theta_T}(t) = \frac{\partial \ln \lambda^*_{aT}(t)}{\partial a} \tag{4.3}$$

and

$$\overline{M}_{a,\theta_T}(t) = \overline{N}(t) - \int_0^t \lambda^{\#}_{aT}(s) \overline{Y}(s)\, ds \tag{4.4}$$

be the martingale associated with counting process $\overline{N}(t)$ (see Fleming and Harrington, 1991, Theorem 1.3.1).

Using (4.2)–(4.4), statistic (4.1) becomes

$$U_{\mathrm{SP}}(0, \tilde{\theta}_T) = \int_0^\tau H^*_{0,\tilde{\theta}_T}(t)\, d\overline{M}_{0,\tilde{\theta}_T}(t). \tag{4.5}$$

Note that $\overline{M}_{0,\tilde{\theta}_T}(t)$, i.e., $\overline{M}_{a,\theta_T}(t)$ evaluated at $a = 0$ and $\theta_T = \tilde{\theta}_T$, is not a martingale. This fact motivates the proof of the following proposition which is presented in the appendix.

Proposition 4.1. *Let $T_{\mathrm{SP}}(0, \tilde{\theta}_T) = n^{-1} U_{\mathrm{SP}}(0, \tilde{\theta}_T)$, $S_X(\cdot)$ be the survival function of $X = \min(T, U, V)$, and $I_{\theta_T\theta_T}(0, \theta)$ be the Fisher's information matrix associated with θ_T under* H_0. *Suppose that $H^*_{0,\theta_T}(t)$ satisfies the assumptions of Rebolledo's theorem* (Andersen et al., 1993, p. 83). *Then, under* H_0: $a = 0$, $n^{1/2} T_{\mathrm{SP}}(0, \tilde{\theta}_T) \xrightarrow{d} N(0, \sigma^2_{\mathrm{SP}})$, *where*

$$\sigma^2_{\mathrm{SP}} = \int_0^\infty [G_{0,\theta_T}(t)]^2 S_X(t) \lambda_T(t)\, dt, \tag{4.6}$$

where

$$(4.7) \qquad G_{0,\theta_T}(t) = H_{0,\theta_T}(t) - A'[I_{\theta_T\theta_T}(0,\theta)]^{-1}\frac{\dot{\lambda}_T(t)}{\lambda_T(t)},$$

and

$$(4.8) \qquad A = \int_0^\infty H_{0,\theta_T}(t)S_X(t)\dot{\lambda}_T(t)\,dt,$$

where $\dot{\lambda}_T(t) = \partial\lambda_T(t)/\partial\theta_T$.

This semiparametric test is easily implemented for the two copulas considered in this work, for any marginal model for T. For Clayton's, from (4.3) and (2.2),

$$(4.9) \qquad H^*_{0,\theta_T}(t) = \ln\hat{\pi}(t) - \ln S_T(t),$$

while for Ali-Mikhail-Haq's, (4.3) and (2.4) yield

$$(4.10) \qquad H^*_{0,\theta_T}(t) = \hat{\pi}(t) - S_T(t).$$

We estimate σ^2_{SP} nonparametrically by

$$(4.11) \qquad \hat{\sigma}^2_{\mathrm{SP}} = \frac{1}{n}\sum_{\delta_i=1}\{G^*_{0,\tilde{\theta}_T}(x_i)\}^2,$$

where $G^*_{0,\tilde{\theta}_T}(t)$ is obtained from $G_{0,\theta_T}(t)$, replacing $\pi(t)$ by $\hat{\pi}(t)$ and θ_T by $\tilde{\theta}_T$, and

$$\hat{A} = \frac{1}{n}\sum_{\delta_i=1} H^*_{0,\tilde{\theta}_T}(x_i)[\tilde{\lambda}_T(x_i)]^{-1}\tilde{\dot{\lambda}}_T(x_i)$$

For the model with exponential margin, the observed information matrix $\mathcal{I}_{\theta_T\theta_T}(0,\tilde{\theta})$ and the estimated covariance matrix of the score function coincide, so we obtain in this case

$$(4.12) \qquad \hat{\sigma}^2_{\mathrm{SP}} = \frac{1}{n}\left\{\sum_{\delta_i=1}\{H^*_{0,\tilde{\theta}_T}(x_i)\}^2 - \frac{1}{D_T}\left\{\sum_{\delta_i=1} H^*_{0,\tilde{\theta}_T}(x_i)\right\}^2\right\}$$

To calculate relative efficiencies of this test with respect to the fully parametric tests of Section 3, one needs to evaluate the variance σ^2_{SP} for the parametric models of Section 3. This is done next.

4.2. Evaluation of σ^2_{SP} for the copulas of Clayton and of Ali-Mikhail-Haq

Example 2.1 (continued). For Clayton's family with any specified marginal distributions, (4.9) yields $H_{0,\theta_T}(t) = \ln S_U(t)$. Using the expressions for $I^{-1}_{\theta_T\theta_T}(0,\theta)$ given in Proposition 3.1 and evaluating (4.6))–(4.8), with exponential margins for T, U and V as defined in Table 1, yields

$$\sigma^2_{\mathrm{SP}} = \frac{\gamma_T\gamma_U^2}{(\gamma_T+\gamma_U+\gamma_V)^3}.$$

When T, U and V have the Weibull margins given in Table 1, similar manipulations give

$$\sigma^2_{\mathrm{SP}} = \frac{\gamma_T\gamma_U^2}{(\gamma_T+\gamma_U+\gamma_V)^3}\left\{1-\frac{6}{\pi^2}\right\}.$$

Example 2.2 (continued). For the Ali-Mikhail-Haq family with any specified marginal distributions, (4.10) yields $H_{0,\theta_T} = -S_U(t)\{1-S_T(t)\}$. Along the lines of the derivations for Clayton's family, we prove that, when T, U and V have exponential margins,

$$\sigma^2_{\mathrm{SP}} = k_1(\gamma_T,\gamma_U,\gamma_V);$$

with Weibull margins for T, U and V,

$$\sigma^2_{\mathrm{SP}} = k_2(\gamma_T,\gamma_U,\gamma_V),$$

where $k_1(\cdot)$ and $k_2(\cdot)$ are defined by (3.4) and (3.5), respectively. Calculations are more tedious than those for Clayton's copula. They are similar to those underlying the derivation of the parametric tests.

5. Relative efficiencies

5.1. General results

In this section, we compare fully parametric and semiparametric tests of independence, using Pitman efficiencies. Consider two test statistics for H_0: $a=0$, say T_1 and T_2, and suppose that the limiting distribution of $n^{1/2}(T_j-\mu_j(a_n))/\sigma_j(a_n)$ is, for $j=1,2$, a standard normal distribution, where a_n is $O(n^{-1/2})$ and $\mu_j(a_n)$ and $\sigma_j^2(a_n)/n$ are the asymptotic mean and variance of T_j when the dependence parameter is a_n. The *relative efficiency* or *Pitman efficiency* of T_1 with respect to T_2 for testing H_0: $a=0$ is defined by

$$\mathcal{E}(T_1,T_2) = \left(\frac{\mu_1'(0)/\sigma_1(0)}{\mu_2'(0)/\sigma_2(0)}\right)^2.$$

A detailed discussion of the conditions for Pitman's efficiency to be well defined is given in van der Vaart (2000, p. 201–202). The efficiency $\mathcal{E}(T_1,T_2)$

gives the ratio of the sample sizes n_2/n_1 needed for the tests based on T_1 and T_2 to achieve asymptotically the same local power. An efficiency larger than 1 means that the test based on T_1 is locally more powerful than the one based on T_2.

To calculate the efficiencies of score tests, it is convenient to consider estimators $\hat{a}$ of the dependence parameter obtained by solving $U_{\mathrm{P}}(a, \tilde{\theta}) = 0$. When $a = 0$ and as the sample size goes to ∞, the asymptotic distribution of $\hat{a}$ is normal with mean 0 and variance given by the (a, a) entry of $I(0, \theta)^{-1}$. Standard results on the inverses of partitioned matrices can be used to demonstrate that

$$\mathrm{Var}(n^{1/2}\hat{a}) = n\{I_{aa}(0,\theta) - I_{a\theta}(0,\theta)I_{\theta\theta}^{-1}(0,\theta)I_{\theta a}(0,\theta)\}^{-1} = 1/\sigma_{\mathrm{P}}^2(0).$$

According to Kendall and Stuart (1979, Chapter 25), the above variance satisfies $1/\sigma_{\mathrm{P}}^2(0) = [\sigma_{\mathrm{P}}(0)/\mu_{\mathrm{P}}'(0)/]^2$; in other words $\mu_{\mathrm{P}}'(0) = \sigma_{\mathrm{P}}(0)^2$. Therefore the Pitman efficiency of one score test with respect to another is equal to the ratio of their asymptotic variances. For the semiparametric test, $\mu_{\mathrm{SP}}'(0)$ is evaluated in Appendix C and is shown to satisfy $\mu_{\mathrm{SP}}'(0) = \sigma_{\mathrm{SP}}^2(0)$. This implies that the efficiency of the parametric test with respect to the semiparametric test is given by

$$\mathcal{E}(T_{\mathrm{P}}, T_{\mathrm{SP}}) = \sigma_{\mathrm{P}}^2/\sigma_{\mathrm{SP}}^2,$$

where σ_{P}^2 and σ_{SP}^2 are given by (3.3)) and (4.6), respectively.

In the next section, we recall some properties of the standard score tests for independence, constructed with uncensored samples of bivariate observations for (T, U). Pitman efficiencies of the parametric tests for censored sample with respect to those for uncensored data provide information about the loss of information due to censoring.

5.2. Score tests of independence for uncensored data

Let T_{C} be a normalized score statistic to test for independence in an uncensored data set (index "C" stands for complete samples). The score function for a is independent of the score function for the marginal parameters when $a = 0$ (see Genest, Ghoudi and Rivest, 1995). Thus functions $\mu_{\mathrm{C}}(a)$ and $\sigma_{\mathrm{C}}^2(a)$ needed to calculate Pitman efficiencies do not depend on the margins for T and U. For score statistic T_{C}, the (a, θ) submatrix of the Fisher information matrix for uncensored samples is null in (3.3). Thus the asymptotic variance of nT_{C} is the (a, a) term of the above Fisher information matrix. In other words,

$$\sigma_{\mathrm{C}}^2 = \frac{1}{n}\mathbb{E}\left\{-\left.\frac{\partial^2 \ell_{\mathrm{C}}(a)}{\partial a^2}\right|_{a=0}\right\},$$

where $\ell_C(a) = \sum_{i=1}^n \ln f_a(y_{1i}, y_{2i})$, $\{(y_{1i}, y_{2i}); i = 1, \dots, n\}$ is a bivariate sample of independent random variables uniformly distributed on $[0, 1]$, and $f_a(y_1, y_2)$ is the density for copula $\mathcal{C}_a$ defined in (1.2),

$$f_a(y_1, y_2) = -\frac{\phi_a''[\mathcal{C}_a(y_1, y_2)]\, \phi_a'(y_1)\, \phi_a'(y_2)}{\{\phi_a'[\mathcal{C}_a(y_1, y_2)]\}^3}.$$

The arguments of Kendall and Stuart (1979, Chapter 25) mentioned above, to prove $\mu_P'(0) = \sigma_P^2$, can be used here to obtain $\mu_C'(0) = \sigma_C^2$.

Example 2.1 (continued). For the Clayton family, the joint density is

$$f_a(y_1, y_2) = (1 + a)\, y_1^{-a-1}\, y_2^{-a-1} \{\mathcal{C}_a(y_1, y_2)\}^{2a+1};$$

using results of section 2.2, we prove that

$$-\frac{1}{n} \frac{\partial^2 \ell_C(a)}{\partial a^2} \xrightarrow{a \to 0} 1 - \ln y_1 \ln y_2 \{4 + \ln y_1 + \ln y_2\},$$

and finally, taking expectation under H_0,

$$\sigma_C^2 = 1.$$

Example 2.2 (continued). For the Ali-Mikhail-Haq family, the joint density is

$$f_a(y_1, y_2) = \frac{\mathcal{C}_a(y_1, y_2)\{1 + a[\mathcal{C}_a(y_1, y_2) - 1]\}\{1 + a[2\,\mathcal{C}_a(y_1, y_2) - 1]\}}{y_1 y_2 \{1 + a(y_1 - 1)\}\{1 + a(y_2 - 1\}};$$

using results of Section 2.2 yields

$$\begin{aligned} -\frac{1}{n} \frac{\partial^2 \ell_C(a)}{\partial a^2}\bigg|_{a=0} &= -6 y_1 y_2 (1 - y_1)(1 - y_2) - (1 - y_1)^2 (1 - y_2)^2 \\ &\quad - (1 - y_1)^2 - (1 - y_2)^2 + (y_1 y_2 - 1)^2 + (2\, y_1 y_2 - 1)^2, \end{aligned}$$

and finally

$$\sigma_C^2 = \tfrac{1}{9}.$$

In the next section, we give some numerical results about $\mathcal{E}(T_P, T_{SP})$ and $\mathcal{E}(T_C, T_P)$ for Clayton's and Ali-Mikhail-Haq's families.

5.3. Calculations of relative efficiencies

The efficiencies have the following simple form,

$$\mathcal{E}(T_P, T_{SP}) = \frac{\gamma_T + \gamma_U}{\gamma_U}$$

Table 2. Relative efficiencies for the Clayton family

			Exponential margins		Weibull margins	
γ_T/γ	γ_U/γ	γ_V/γ	$\mathcal{E}(T_{\mathrm{P}}, T_{\mathrm{SP}})$	$\mathcal{E}(T_{\mathrm{C}}, T_{\mathrm{P}})$	$\mathcal{E}(T_{\mathrm{P}}, T_{\mathrm{SP}})$	$\mathcal{E}(T_{\mathrm{C}}, T_{\mathrm{P}})$
0.33	0.33	0.33	2.0	13.5	2.0	34.4
0.75	0.25	0	4.0	5.3	4.0	13.6
0.50	0.50	0	2.0	4.0	2.0	10.2
0.25	0.75	0	1.3	5.3	1.3	13.6
0.50	0.10	0.40	6.0	33.3	6.0	85.0
0.50	0.20	0.30	3.5	14.3	3.5	36.4
0.50	0.30	0.20	2.7	8.3	2.7	21.3
0.50	0.40	0.10	2.3	5.6	2.3	14.2

Table 3. Relative efficiencies for the Ali-Mikhail-Haq family

			Exponential margins		Weibull margins	
γ_T/γ	γ_U/γ	γ_V/γ	$\mathcal{E}(T_{\mathrm{P}}, T_{\mathrm{SP}})$	$\mathcal{E}(T_{\mathrm{C}}, T_{\mathrm{P}})$	$\mathcal{E}(T_{\mathrm{P}}, T_{\mathrm{SP}})$	$\mathcal{E}(T_{\mathrm{C}}, T_{\mathrm{P}})$
0.33	0.33	0.33	2.0	27.5	2.0	156.3
0.75	0.25	0	8.3	18.5	3.5	80.7
0.50	0.50	0	2.0	20.0	2.0	55.0
0.25	0.75	0	1.1	18.5	1.4	80.7
0.50	0.10	0.40	15.3	33.8	10.8	218.0
0.50	0.20	0.30	6.1	25.5	3.9	156.6
0.50	0.30	0.20	3.6	23.0	2.6	108.2
0.50	0.40	0.10	2.5	21.5	2.2	75.1

for both exponential and Weibull margins under Clayton's model. For Ali-Mikhail-Haq's they are written in terms of the functions $k_1(\cdot)$ and $k_2(\cdot)$ (see (3.4)–(3.5)) as

$$\mathcal{E}(T_{\mathrm{P}}, T_{\mathrm{SP}}) = 1 + \frac{k_1(\gamma_U, \gamma_T, \gamma_V)}{k_1(\gamma_T, \gamma_U, \gamma_V)} \quad \text{and} \quad \mathcal{E}(T_{\mathrm{P}}, T_{\mathrm{SP}}) = 1 + \frac{k_2(\gamma_U, \gamma_T, \gamma_V)}{k_2(\gamma_T, \gamma_U, \gamma_V)},$$

for the exponential and Weibull models respectively. Observe that in all cases $\mathcal{E}(T_{\mathrm{P}}, T_{\mathrm{SP}}) > 1$, as expected. The semiparametric test relies on fewer assumptions; this results in a loss of power. Also when $\gamma_T = \gamma_U$, $\mathcal{E}(T_{\mathrm{P}}, T_{\mathrm{SP}}) = 2$.

These efficiencies depend only on the expected proportions for the three possible outcomes, say γ_T/γ, γ_U/γ and γ_V/γ. Tables 2 and 3 give $\mathcal{E}(T_{\mathrm{P}}, T_{\mathrm{SP}})$ and $\mathcal{E}(T_{\mathrm{C}}, T_{\mathrm{P}})$ for Clayton's and Ali-Mikhail-Haq's copula for several patterns of outcomes.

In Table 2, when $\gamma_V/\gamma = 0$ (as defined in Section 3, $\gamma = \gamma_T + \gamma_U + \gamma_V$), $\mathcal{E}(T_{\mathrm{P}}, T_{\mathrm{SP}})$ decreases as the proportion of dependent censoring γ_U/γ increases, and $\mathcal{E}(T_{\mathrm{C}}, T_{\mathrm{P}})$ is minimal when $\gamma_T/\gamma = \gamma_U/\gamma$. When the proportion

γ_T/γ is fixed, $\mathcal{E}(T_{\rm P}, T_{\rm SP})$ and $\mathcal{E}(T_{\rm C}, T_{\rm P})$ decrease as γ_U/γ increases. Comparing $\mathcal{E}(T_{\rm C}, T_{\rm P})$ for exponential and Weibull models shows that, the additional parameter of the Weibull margin requires an increase of $100\{6/(\pi^2 - 6)\} = 155\%$ in the sample size to reach the local power obtained with exponential margins.

In Table 3, the loss of efficiency with respect to tests for complete samples is much more important than in Table 2. A possible explanation for this is that under Ali-Mikhail-Haq's model, when a is small, $\lambda^{\#}_{aT}(t) \approx \lambda_T(t)\{1 - aS_T(t)[1 - S_U(t)]\}$. Thus the crude and net hazard are equal for small and large values of t. They differ only for medium values of t. Discriminating between $\lambda^{\#}_{aT}(t)$ and $\lambda_T(t)$ is therefore more difficult than for Clayton's copula for which $\lambda^{\#}_{aT}(t)/\lambda_T(t)$ decreases in t when a is small.

Remark that $\mathcal{E}(T_{\rm P}, T_{\rm SP})$ is greater for exponential margins as compared to Weibull's. When $\gamma_V/\gamma = 0$, $\mathcal{E}(T_{\rm P}, T_{\rm SP})$ decreases as γ_U/γ increases; a similar behaviour was noted for Clayton's family. For Weibull margins, $\mathcal{E}(T_{\rm C}, T_{\rm P})$ is minimal when $\gamma_T/\gamma = \gamma_U/\gamma$. However, for exponential margins, $\gamma_T/\gamma = \gamma_U/\gamma$ corresponds in fact to a local maximum for $\mathcal{E}(T_{\rm C}, T_{\rm P})$; there are two local minima at $\gamma_U/\gamma \approx 0.24$, 0.76. When the proportion γ_T/γ is fixed, $\mathcal{E}(T_{\rm P}, T_{\rm SP})$ and $\mathcal{E}(T_{\rm C}, T_{\rm P})$ decrease as γ_U/γ increases. Comparing $\mathcal{E}(T_{\rm C}, T_{\rm P})$ for exponential and Weibull models shows that, for Weibull margins, much more data are needed to reach a given level of local power.

6. A numerical example

This section considers the survival times, in weeks, of 61 patients with unoperable lung cancer treated with cyclophosphamide considered in Lagakos and Williams (1978) and in Lee and Wolfe (1998). The data set is: 0.14^+, 0.14^+, 0.29^+, 0.43^+, 0.43, 0.57^+, 0.57^+, 1.86^+, 2.86, 3.00^+, 3.00^+, 3.14, 3.14, 3.29^+, 3.29^+, 3.43, 3.43, 3.71, 3.86, 6.00^+, 6.00^+, 6.14^+, 6.14, 6.86, 8.71^+, 9.00, 9.43, 10.57^+, 10.71, 10.86, 11.14, 11.86^+, 13.00, 14.43, 15.57^+, 15.71, 16.57^+, 17.29^+, 18.43, 18.57, 18.71^+, 20.71, 21.29^+, 23.86^+, 26.00^+, 27.57^+, 29.14, 29.71, 32.14^+, 33.14^+, 40.57, 47.29^+, 48.57, 49.43, 53.86, 61.86, 66.57, 68.71, 68.96, 72.86, 72.86. The exponent $+$ identifies the 28 censored observations. They correspond to patients whose treatment was terminated because of a deteriorating condition.

Following Lagakos and Williams (1978), we fit exponential distributions to both the survival and the censoring distributions. The parameter estimates are $\tilde{\gamma}_T = 0.0276$, (s.e. $= 0.0048$) and $\tilde{\gamma}_U = 0.0234$, (s.e. $= 0.0044$). This section investigates a possible dependency between these two variables. All the results of the previous sections apply; the lack of independent censoring simplifies the calculations.

The form of the statistic $U_{\rm P}(0, \tilde{\theta})$ and of the estimated variance $\hat{\sigma}^2_{\rm P}$ for the parametric tests are easily deduced from the results of Sections 3.2 and 3.3.

For the semiparametric tests with exponential margin, the estimator of σ^2_{SP} is given by (4.12), but some technical manipulations are needed to evaluate $U_{\mathrm{SP}}(0,\tilde{\theta}_T)$ from (4.1). From (4.5), we obtain

$$U_{\mathrm{SP}}(0,\tilde{\theta}_T) = \sum_{\delta_i=1} H^*_{0,\tilde{\theta}_T}(x_i) - \tilde{\gamma}_T \int_0^{\tau} H^*_{0,\tilde{\theta}_T}(t)\overline{Y}(t)\,dt, \tag{6.1}$$

where $H^*_{0,\tilde{\theta}_T}(t) = f[\hat{\pi}(t)] - f[\exp(-\tilde{\gamma}_T t)]$, with $f(y) = \log y$ for Clayton's copula, and $f(y) = y$ for Ali-Mikhail-Haq's copula.

Assume that all the X_i's are distinct, and let $0 = X_{(0)} < X_{(1)} < \cdots < X_{(n)}$ denote the corresponding order statistics. Since $\hat{\pi}(x) = [n-(i-1)]/n$, $\forall x \in (X_{(i-1)}, X_{(i)}]$, the following holds,

$$\begin{aligned}
\int_0^{\tau} f[\hat{\pi}(t)]\,\overline{Y}(t)\,dt &= \sum_{i=1}^{n} \int_0^{X_{(i)}} f[\hat{\pi}(t)]\,dt \\
&= \sum_{i=1}^{n}\sum_{j=1}^{i} f\left[\frac{n-(j-1)}{n}\right](X_{(j)} - X_{(j-1)}) \\
&= \sum_{i=1}^{n} (n-i+1) f\left[\frac{n-(i-1)}{n}\right](X_{(i)} - X_{(i-1)}) \\
&= n\sum_{i=1}^{n} \hat{\pi}(X_{(i)}) f[\pi(X_{(i)})](X_{(i)} - X_{(i-1)})
\end{aligned}$$

This allows to simplify score statistics (6.1). For Clayton's copula,

$$\begin{aligned}
U_{\mathrm{SP}}(0,\tilde{\theta}_T) &= \sum_{\delta_i=1} \log \hat{\pi}(X_i) + \tilde{\gamma}_T X_i \\
&\quad - n\,\tilde{\gamma}_T \sum_{i=1}^{n} \hat{\pi}(X_{(i)}) \log[\pi(X_{(i)})]\big(X_{(i)} - X_{(i-1)}\big) + \frac{\tilde{\gamma}_T^2}{2}\sum_{i=1}^{n} x_i^2,
\end{aligned}$$

while for Ali-Mikhail-Haq,

$$\begin{aligned}
U_{\mathrm{SP}}(0,\tilde{\theta}_T) &= \sum_{\delta_i=1} \hat{\pi}(X_i) - \exp(\tilde{\gamma}_T X_i) \\
&\quad - n\,\tilde{\gamma}_T \sum_{i=1}^{n} \hat{\pi}(X_{(i)})^2\big(X_{(i)} - X_{(i-1)}\big) + \sum_{i=1}^{n}[1 - \exp(-\tilde{\gamma}_T X_i)].
\end{aligned}$$

Table 4 below gives the p-values for the four tests considered here. There is a significant dependency for the parametric test constructed using Ali-Mikhail-Haq's copula.

Table 4. P-values of four tests of independence for Lagakos and Williams data

Model	Parametric test	Semiparametric test
Clayton	0.441	0.384
Ali-Mikhail-Haq	0.011	0.129

Given the poor showing of the score test for Ali-Mikhail-Haq family in the efficiency comparisons presented in Section 5, it is unexpected that this test gives the only significant result. To investigate this further, one can estimate the failure rate of the survival time by fitting a piecewise exponential distribution. This reveals that the hazard rate has a bathtub shape: it is high in the first 20 weeks and after the 60th and relatively low in between. According to the discussion in Section 2.2 Ali-Mikhail-Haq copula is more appropriate to model this type of discrepancy than Clayton's. This explains the significant outcome of Table 4. The increase in p-value when one goes to the semiparametric test is in line with the findings of Section 5.

7. Discussion

In many instances a positive dependence between T and U is the most interesting alternative to H_0: $a = 0$. One-sided versions of the tests presented here could then be considered. These tests would reject the null hypothesis if the score statistics is too large. This would permit a much needed increase in power for the independence tests. The efficiency comparisons of Section 5 suggest that dependent censoring is hard to detect using standard lifetime data; additional follow-ups of censored or dead units might be needed to make such a detection.

Testing for a dependency between T and U amounts to testing the validity of the parametric models for the margins. As shown in Section 2, a Clayton's dependency multiplies the failure rate by a decreasing function while for Ali-Mikhail-Haq, the multiplicative term has a bathtub shape.

Tables 2 and 3, and some simulations not presented here, suggest that a dependency is much more difficult to detect for Ali-Mikhail-Haq's model that for Clayton's. The expressions for $\lambda^{\#}_{aT}(t)$ presented in section shed some light on this finding: for Ali-Mikhail-Haq's model $\lambda^{\#}_{aT}(t)$ is much closer to a Weibull failure rate that for Clayton's. This highlights that in competing risk models certain types of dependency might be nearly impossible to detect. Still, as noted by Zheng and Klein (1995) and Rivest and Wells (2001), once a dependency is detected and its level ascertained the estimator of the marginal survival function does not depend too much on the family of copulas chosen to model the dependency.

This paper leaves the door open for future work. Improvements over the semiparametric tests presented in Section 4 could possibly be achieved by using the full score statistics evaluated at the Kaplan-Meier estimator of $S_U(t)$. The investigation of the estimators of the parameters of the marginal distribution of T and of the dependence parameter a obtained by maximizing the pseudo-likelihood (2.7) would be of interest. The sampling distribution of the resulting estimators could possibly be derived following the arguments presented in Section 4.

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APPENDIX

A. Proofs for the parametric test

Recall that $X = \min(T, U, V)$ and δ indicates what kind of event is observed, say $\delta = 1$ if $X = T$, $\delta = 0$ if $X = U$ and $\delta = -1$ if $X = V$. Let D_T, D_U and D_V the respective frequencies of each kind of event among the n units in the sample. The evaluations of the Fisher information matrices needed to construct the score tests rely on the following lemma.

Lemma A.1. *Let T, U and V be independent.*

(i) *If T, U and V are exponentially distributed, with hazard rate γ_T, γ_U and γ_V, respectively, then X and (D_T, D_U, D_V) are independent, and*

  (a) $X \sim \text{Exponential}(\gamma)$, *where* $\gamma = \gamma_T + \gamma_U + \gamma_V$;

  (b) $(D_T, D_U, D_V) \sim \text{Multinomial}(\gamma_T/\gamma, \gamma_U/\gamma, \gamma_V/\gamma)$.

(ii) *If T, U and V are Weibull distributed, with common shape parameter β, and respective scale parameter γ_T, γ_U and γ_V, then X and (D_T, D_U, D_V) are independent and*

  (a) $X \sim \text{Weibull}(\beta, \gamma)$, *where* $\gamma = \gamma_T + \gamma_U + \gamma_V$;

  (b) $(D_T, D_U, D_V) \sim \text{Multinomial}(\gamma_T/\gamma, \gamma_U/\gamma, \gamma_V/\gamma)$;

(c) $Y = \gamma X^{\beta} \sim$ Exponential(1), *and* $\ln Y$ *has an extreme value distribution, so* $\mathbb{E}\{\ln Y\} = -c$, *where* $c \approx 0.5772$ *is Euler's constant, and* $\mathrm{Var}\{\ln Y\} = \pi^2/6$ (see Lawless, 1982, Section 1.3)*; in addition, we have*

$$\mathbb{E}\{Y \ln Y\} = 1 - c,$$
$$\mathbb{E}\{Y^2 \ln Y\} = 3 - 2c,$$
$$\mathbb{E}\{Y(\ln Y)^2\} = \frac{\pi^2}{6} + c(c-2),$$

and, for any $\lambda > 0$,

$$\mathbb{E}\{e^{-(\lambda/\gamma)Y}\} = \frac{\gamma}{\lambda+\gamma},$$
$$\mathbb{E}\{Ye^{-(\lambda/\gamma)Y}\} = \left(\frac{\gamma}{\lambda+\gamma}\right)^2,$$
$$\mathbb{E}\{Y \ln Y e^{-(\lambda/\gamma)Y}\} = \left(\frac{\gamma}{\lambda+\gamma}\right)^2 \left\{1 - c + \ln\left(\frac{\gamma}{\lambda+\gamma}\right)\right\}.$$

A.1. Proof of Proposition 3.1

Using the complete likelihood (2.6), at $a = 0$, we have the general form

$$\begin{aligned}\text{(A.1)} \qquad \mathcal{I}_{\theta\theta}(0,\theta) = &-\sum_{\delta_i=1}\left\{\frac{\partial^2 \ln \lambda_T(x_i)}{\partial\theta^2}\right\} - \sum_{i=1}^{n}\left\{\frac{\partial^2 \ln S_T(x_i)}{\partial\theta^2}\right\} \\ &-\sum_{\delta_i=0}\left\{\frac{\partial^2 \ln \lambda_U(x_i)}{\partial\theta^2}\right\} - \sum_{i=1}^{n}\left\{\frac{\partial^2 \ln S_U(x_i)}{\partial\theta^2}\right\} \\ &-\sum_{\delta_i=-1}\left\{\frac{\partial^2 \ln \lambda_V(x_i)}{\partial\theta^2}\right\} - \sum_{i=1}^{n}\left\{\frac{\partial^2 \ln S_V(x_i)}{\partial\theta^2}\right\}.\end{aligned}$$

From (A.1), it is clear that $\mathcal{I}_{\theta\theta}(0,\theta)$ is block diagonal, with blocks $\mathcal{I}_{\theta_T\theta_T}(0,\theta)$, $\mathcal{I}_{\theta_U\theta_U}(0,\theta)$ and $\mathcal{I}_{\theta_V\theta_V}(0,\theta)$.

A.1.1. Exponential model

The three blocks of $\mathcal{I}_{\theta\theta}(0,\gamma)$ are 1×1 matrices. From (A.1), the component for survival time T is $\mathcal{I}_{\theta_T\theta_T}(0,\theta) = D_T/\gamma_T^2$ (see Lawless, 1982, p. 107). Part (i) of Proposition 3.1 follows immediately from Lemma A.1(i).

A.1.2. Weibull model

From (A.1), the entries of the 2×2 observed information matrix for the parameters of survival time T are

$$-\frac{\partial^2 \ell(0,\theta)}{\partial\beta^2} = \frac{D_T}{\beta^2} + \gamma_T \sum_{i=1}^{n} x_i^{\beta} (\ln x_i)^2,$$

$$-\frac{\partial^2 \ell(0,\theta)}{\partial \gamma_T^2} = \frac{D_T}{\gamma_T^2},$$
$$-\frac{\partial^2 \ell(0,\theta)}{\partial \beta \partial \gamma_T} = \sum_{i=1}^{n} x_i^{\beta} \ln x_i.$$

Using Lemma A.1(ii), and noting $b = 1 - c - \ln\gamma$ ($c \approx 0.5772$), we have $\mathbb{E}\{X^{\beta}\ln X\} = b/(\beta\gamma)$ and $\mathbb{E}\{X^{\beta}(\ln X)^2\} = \{\pi^2/6 - 1 + b^2\}/(\beta^2\gamma)$, and we deduce

$$n^{-1} I_{\theta_T\theta_T}(0,\theta) = \frac{1}{\gamma}\left[\begin{array}{c|c} \frac{\gamma_T}{\beta^2}\left(\frac{\pi^2}{6} + b^2\right) & \frac{b}{\beta} \\ \hline \frac{b}{\beta} & \frac{1}{\gamma_T} \end{array}\right].$$

Inverting this matrix yields part (ii) of Proposition 3.1.

A.2. Derivations underlying tests for Clayton's family

From expression (2.5) for the likelihood, and using (2.1)–(2.2), we have

$$\text{(A.2)}\quad U_{\mathrm{P}}(a,\theta) = -\sum_{\delta_i=1} \ln S_T(x_i) - \sum_{\delta_i=0} \ln S_U(x_i) + \sum_{\delta_i\neq -1} \ln \pi(x_i) + \sum_{i=1}^{n}\left\{\frac{\partial \ln \pi(x_i)}{\partial a}\right\} + a\sum_{\delta_i\neq -1}\left\{\frac{\partial \ln \pi(x_i)}{\partial a}\right\}.$$

Using the results of Section 2.2, Example 2.1, completes the calculations of $U_{\mathrm{P}}(0,\theta)$ for Clayton's family. Taking the partial derivatives of (A.2), at $a = 0$, and the results in Section 2.2 yields, for any marginal model,

$$\text{(A.3)}\quad \mathcal{I}_{aa}(0,\theta) = -\sum_{i=1}^{n} \ln S_T(x_i)\ln S_U(x_i)\{\ln S_T(x_i) + \ln S_U(x_i)\} - 2\sum_{\delta_i\neq -1} \ln S_T(x_i)\,\ln S_U(x_i),$$

$$\text{(A.4)}\quad \mathcal{I}_{\theta a}(0,\theta) = -\sum_{\delta_i=0}[S_T(x_i)]^{-1}\dot S_T(x_i) - \sum_{\delta_i=1}[S_U(x_i)]^{-1}\dot S_U(x_i) - \sum_{i=1}^{n}[S_T(x_i)]^{-1}\dot S_T(x_i)\ln S_U(x_i) - \sum_{i=1}^{n}[S_U(x_i)]^{-1}\dot S_U(x_i)\ln S_T(x_i).$$

A.2.1. Exponential model

Using Lemma A.1(i), (A.3)–(A.4) yields

$$n^{-1} I_{aa}(0,\theta) = \frac{2\gamma_T\gamma_U(\gamma_T+\gamma_U)}{\gamma^3}$$

and

$$n^{-1}I_{a\theta}(0,\theta) = \left(\frac{-\gamma_U}{\gamma^2}, \frac{-\gamma_T}{\gamma^2}, 0\right),$$

where $\gamma = \gamma_T + \gamma_U + \gamma_V$, and Proposition 3.1(i) yields σ_{P}^2 for Clayton's family with exponential margins.

A.2.2. Weibull model

Using Lemma A.1(ii), we deduce from (A.3)–(A.4) that

$$n^{-1}I_{aa}(0,\theta) = \frac{2\gamma_T\gamma_U(\gamma_T + \gamma_U)}{\gamma^3}$$

and

$$n^{-1}I_{a\theta}(0,\theta) = -\frac{1}{\gamma^2}\left(\frac{\gamma_T\,\gamma_U(1+b)}{\beta}, \gamma_U, \frac{\gamma_T\,\gamma_U(1+b)}{\beta}, \gamma_T, 0, 0\right),$$

where $\gamma = \gamma_T + \gamma_U + \gamma_V$ and $b = 1 - c - \ln\gamma$, and using Proposition 3.1(ii), we deduce σ_{P}^2 for Clayton's family with Weibull margins.

A.3. Derivations underlying tests for Ali-Mikhail-Haq's family

From likelihood (2.5), and using (2.3)–(2.4), we have

$$\text{(A.5)}\quad U_{\mathrm{P}}(a,\theta) = \sum_{\delta_i=1} \frac{1 - S_T(x_i)}{1 + a[S_T(x_i) - 1]} + \sum_{\delta_i=0} \frac{1 - S_U(x_i)}{1 + a[S_U(x_i) - 1]} + \sum_{i=1}^{n} \frac{\partial \ln \pi(x_i)}{\partial a} + \sum_{\delta_i \neq -1} \frac{\pi(x_i) - 1 + a(\partial\pi(x_i)/\partial a)}{1 + a[\pi(x_i) - 1]}.$$

Using the results of Section 2.2, Example 2.2, completes the proof of $U_{\mathrm{P}}(0,\theta)$ for the Ali-Mikhail-Haq's family. Expressing the partial derivatives of (A.5), at $a = 0$, in terms of $1 - S_T(x_i)$ and $1 - S_U(x_i)$, and using the results of Section 2.2 yields, for any marginal model,

$$\text{(A.6)}\quad \mathcal{I}_{aa}(0,\theta) = \sum_{\delta_i=0}\{1 - S_T(x_i)\}^2 + \sum_{\delta_i=1}\{1 - S_U(x_i)\}^2 - 2\sum_{i=1}^{n}\{1 - S_T(x_i)\}^2\{1 - S_U(x_i)\}^2 + \sum_{\delta_i=-1}\{1 - S_T(x_i)\}^2\{1 - S_U(x_i)\}^2,$$

$$\text{(A.7)}\quad \mathcal{I}_{\theta a}(0,\theta) = -\sum_{\delta_i=0} \dot{S}_T(x_i) - \sum_{\delta_i=1} \dot{S}_U(x_i) + 2\sum_{i=1}^{n}\{\dot{S}_T(x_i)[1-S_U(x_i)] + \dot{S}_U(x_i)[1-S_T(x_i)]\} - \sum_{\delta_i=-1}\{\dot{S}_T(x_i)[1-S_U(x_i)] + \dot{S}_U(x_i)[1-S_T(x_i)]\}.$$

A.3.1. Exponential model

Using Lemma A.1(i), we deduce from (A.6)–(A.7) that

$$n^{-1}I_{aa}(0,\theta) = \frac{2\gamma_T\gamma_U^2}{(2\gamma_T+\gamma)(2\gamma_T+\gamma_U+\gamma)(2\gamma_T+2\,\gamma_U+\gamma)} + \frac{2\gamma_T^2\gamma_U}{(2\,\gamma_U+\gamma)(\gamma_T+2\gamma_U+\gamma)(2\gamma_T+2\,\gamma_U+\gamma)}$$

and

$$n^{-1}I_{a\theta}(0,\theta) = \frac{-1}{\gamma_T+\gamma_U+\gamma}\left(\frac{\gamma_U}{\gamma_T+\gamma}, \frac{\gamma_T}{\gamma_U+\gamma}, 0\right),$$

where $\gamma = \gamma_T + \gamma_U + \gamma_V$, and using the Proposition 3.1(i), we deduce σ_{P}^2 for Ali-Mikhail-Haq's family with exponential margins. Note that proof $I_{a\theta}(0,\theta)$ is straightforward, however tedious calculations are needed to obtain $I_{aa}(0,\theta)$.

A.3.2. Weibull model

Using Lemma A.1(ii), we deduce easily from (A.6)–(A.7) that $I_{aa}(0,\theta)$ is the same as that for the exponential model. The contribution, $I_{a\theta_T}(0,\theta)$, of T to $I_{a\theta}(0,\theta) = (I_{a\theta_T}(0,\theta), I_{a\theta_U}(0,\theta), 0, 0)$ is

$$n\begin{bmatrix} -\frac{\gamma_T}{\beta}\left\{\frac{1-c-\ln(\gamma_T+\gamma)}{\gamma_T+\gamma} - \frac{1-c-\ln(\gamma_T+\gamma_U+\gamma)}{\gamma_T+\gamma_U+\gamma}\right\} \\ -\frac{\gamma_U}{(\gamma_T+\gamma)(\gamma_T+\gamma_U+\gamma)} \end{bmatrix},$$

where $\gamma = \gamma_T + \gamma_U + \gamma_V$. Using the Proposition 3.1(ii), we deduce σ_{P}^2 for Ali-Mikhail-Haq's family with Weibull margins. Lengthy derivations are required to obtain the first term of $I_{a\theta_T}(0,\theta)$; calculation of the final variance σ_{P}^2 requires additional technical manipulations.

B. Proof of Proposition 4.1

Writing $\overline{M}_{0,\tilde{\theta}_T}(t) = \overline{M}_{0,\theta_T}(t) - \int_0^t\{\tilde{\lambda}_T(s) - \lambda_T(s)\}\overline{Y}(s)\,ds$, where $\overline{M}_{0,\theta_T}(t)$ is a martingale under H_0 (see (4.4)), and using first order expansions for

$H^*_{0,\tilde{\theta}_T}(t) - H^*_{0,\theta_T}(t)$ and $\tilde{\lambda}_T(s) - \lambda_T(s)$, (4.5) becomes

$$\text{(B.1)} \qquad T_{\text{SP}}(0,\tilde{\theta}_T) = \frac{1}{n}\int_0^{\mathcal{T}} H^*_{0,\theta_T}(t) d\overline{M}_{0,\theta_T}(t) - (\tilde{\theta}_T - \theta_T)'\frac{1}{n}\int_0^{\mathcal{T}} H^*_{0,\theta_T}(t)\overline{Y}(t)\dot{\lambda}_T(t)\,dt + (\tilde{\theta}_T - \theta_T)'\frac{1}{n}\int_0^{\mathcal{T}} \dot{H}_{0,\theta_T}(t)\,d\overline{M}_{0,\theta_T}(t).$$

Under classical regularity conditions (see Andersen et al., 1993, pp. 420–421), one has $\sqrt{n}(\tilde{\theta}_T - \theta_T) \xrightarrow{d} N(0, [I_{\theta_T\theta_T}(0,\theta)/n]^{-1})$; furthermore the strong law of large numbers yields that $n^{-1}\int_0^{\mathcal{T}} \dot{H}_{0,\theta_T}(t)\,d\overline{M}_{0,\theta_T}(t) \xrightarrow{\mathbb{P}} 0$. Therefore the third term in (B.1) is negligible.

Note that $\overline{Y}(t)/n$ is the empirical estimator of the survival distribution $S_X(\cdot)$ of $X = \min(T,U,V)$. Assuming that $H^*_{0,\theta_T}(t)$ is uniformly bounded (see (4.3) and (2.8)), one has

$$\frac{1}{n}\int_0^{\mathcal{T}} H^*_{0,\theta_T}(t)\overline{Y}(t)\dot{\lambda}_T(t)\,dt \xrightarrow{\mathbb{P}} A = \int_0^{\infty} H_{0,\theta_T}(t)S_X(t)\dot{\lambda}_T(t)\,dt,$$

and (B.1) becomes

$$\text{(B.2)} \qquad T_{\text{SP}}(0,\tilde{\theta}_T) \approx \frac{1}{n}\int_0^{\mathcal{T}} H^*_{0,\theta_T}(t)\,d\overline{M}_{0,\theta_T}(t) - A'(\tilde{\theta}_T - \theta_T),$$

where under H_0,

$$\text{(B.3)} \qquad \tilde{\theta}_T - \theta_T = [I_{\theta_T\theta_T}(0,\theta)]^{-1}\int_0^{\mathcal{T}} [\lambda_T(t)]^{-1}\,\dot{\lambda}_T(t)\,d\overline{M}_{0,\theta_T}(t) + R,$$

with $n^{1/2}R \xrightarrow{\mathbb{P}} 0$. By (B.2)–(B.3), we have now that

$$\text{(B.4)} \qquad T_{\text{SP}}(0,\tilde{\theta}_T) \approx n^{-1}\int_0^{\mathcal{T}} G^*_{0,\theta_T}(t)\,d\overline{M}_{0,\theta_T}(t),$$

where

$$G^*_{0,\theta_T}(t) = H^*_{0,\theta_T}(t) - A'[I_{\theta_T\theta_T}(0,\theta)]^{-1}\frac{\dot{\lambda}_T(t)}{\lambda_T(t)}.$$

Convergence in law of $n^{1/2}T_{\text{SP}}(0,\tilde{\theta}_T)$ results now from the classical martingale limit central theorem (see Andersen et al., 1993, Theorem II.5.3).

The asymptotic variance given in Proposition 4.1 comes from the evaluation of $\mathbb{E}\{[T_{\text{SP}}(0,\tilde{\theta}_T)]^2\}$, using expression (B.4) (see Fleming and Harrington, 1991, Theorem 2.5.4).

C. Evaluation of $\mu'_{\mathrm{SP}}(0)$

Writing $\overline{M}_{0,\theta_T}(t) = \overline{M}_{a,\theta_T}(t) - \int_0^t \{\lambda_T(s) - \lambda^{\#}_{aT}(s)\}\overline{Y}(s)\,ds$, where $\overline{M}_{a,\theta_T}(t)$ is defined by (4.4), when the value of the dependence parameter is $a \neq 0$, one can write (B.4) as

$$n^{-1}U_{\mathrm{SP}}(0,\tilde{\theta}_T) \approx \frac{1}{n}\int_0^\tau G^*_{0,\theta_T}(t)\,d\overline{M}_{a,\theta_T}(t) \\ - \frac{1}{n}\int_0^\tau G^*_{0,\theta_T}(t)\{\lambda_T(t) - \lambda^{\#}_{aT}(t)\}\overline{Y}(t)\,dt,$$

where $\overline{M}_{a,\theta_T}(t)$ is a martingale when a is nonzero.

The expectation of the first term of the right hand side is null. Since $\overline{Y}(t)/n$ is an estimator for $S_X(t)$, the second term yields

$$\mu'_{\mathrm{SP}}(0) = \int_0^\infty G_{0,\theta_T}(t)\,H_{0,\theta_T}(t)\lambda_T(t)S_X(t)\,dt,$$

where $G_{0,\theta_T}(t) = H_{0,\theta_T}(t) - A'[I_{\theta_T\theta_T}(0,\theta)]^{-1}\dot{\lambda}_T(t)/\lambda_T(t)$. Evaluating the above expression yields $\mu'_{\mathrm{SP}}(0) = \sigma^2_{\mathrm{SP}}(0)$.

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