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RECONSTRUCTION OF A STATIONARY SPATIAL PROCESS FROM A SYSTEMATIC SAMPLING

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Abstract

We consider the problem of predicting a spatial stationary process over a fixed unit region $[0,1]^d, d > 1$. We derive a linear nonparametric predictor using an extended linear interpolation formula based on a regular sampling design of size m^d. Under some appropriate assumption on the spectral density,we give the rate of convergence of the corresponding integrated mean squared error when the observations get dense in the whole region.

Key words: spatial process, linear interpolation, spectral density, rate of convergence.

1 Introduction and Results

The prediction of a spatial process from its observations at chosen sites is relevant to problems related to geology and environment, known as kriging. Parametric methods have been used to predict a process by means of a linear model. The best linear unbiased estimator of the underlying parameter was studied by many authors such as Cressie (1993), Matern (1986), Sacks, Welch and Mitchell (1989).We wish to predict the process $X(\mathbf{t}), \mathbf{t} \in [0,1]^d$ from observations based on a systematic (regular) sampling design in the unit region $[0,1]^d$ which is divided into m^d grids each of equal size $1/m$. The best linear predictor depends on the inverse of a covariance matrix generated by the m^d observations and thus this maybe subject to serious numerical unstabilities. We consider in this paper a nonparametric approach to predict the process $X(\mathbf{t}), \mathbf{t} \in [0,1]^d$. We consider a weakly stationary process with a spectral density ϕ_X that satisfies for $q \geq 1$

$$\int_{\Re^d} |\omega_i|^{2q}|\omega_j|^{2q}\phi_X(\omega)d\omega < \infty, i,j = 1,\ldots,d.$$

For $q = 1$, the predictor $\hat{X}(\mathbf{t})$ when $\mathbf{t}$ belongs to a given grid is derived by applying an extended Lagrange interpolation formula in each direction. When $\mathbf{t} = (t^1,\ldots,t^d) \in [\frac{\mathbf{k}}{m}, \frac{(\mathbf{k+1})}{m}[$ for some $\mathbf{k} = (k_1,\ldots,k_d) \in \{0,\ldots,m-1\}^d$ then the predictor is obtained by applying the Lagrange interpolation formula for every $t^i \in (t_{k_i}, t_{k_i+1}) = (k_i/m, (k_i+1)/m), k_i = 0,...,m, i = 1,...,d$:

$$\hat{X}(\mathbf{t}) = \sum_{\mathbf{j}=(j_1,\ldots,j_d)\in\{0,1\}^d} C_d(\mathbf{t},\mathbf{k},\mathbf{j})X(t_{\mathbf{k+1-j}})$$

where $C_d(\mathbf{t},\mathbf{k},\mathbf{j}) = m^d\Pi_{i=1}^d(-1)^{j_i} = (t^i - t_{k_i+j_i})$ and $t_{\mathbf{k}} = (t_{k_1},\ldots,t_{k_d})$. The whole process can then be reconstructed for every $\mathbf{t} \in [0,1]^d$ and the error of prediction is measured through an integrated mean squared error:

$$IMSE = \int\ldots\int_{[0,1]^d}(E(X(\mathbf{t}) - \hat{X}(\mathbf{t}))^2 d\mathbf{t}.$$

In the one-dimensional case $d = 1$ we obtain the classical interpolation formula which was used by Su and Cambanis (1993), Muller and Ritter (1997) for predicting a second order stochastic process. For higher dimension d, Stein (1993,1995) considered the prediction of integrals of spatial processes and studied the asymptotic properties of the mean squared error of approximation. In his book, Stein (1999) gives a summary of his results and some discussions on spatial interpolation.

When the observations get dense in the whole region (infill asymptotics), the following theorem gives the rate of convergence of the integrated mean squared error and the corresponding asymptotic constant in terms of the spectral density.

Theorem 1. If a regular sampling design of size $n = m^d$ is used, and if the spectral density ϕ_X of the process X satisfies

$$\int_{\Re^d} |\omega_i|^2|\omega_j|^2\phi_X(\omega)d\omega < \infty, i,j = 1,\ldots,d,$$

then as $m \to \infty$

$$m^4(IMSE) \longrightarrow \frac{1}{120}\int_{\Re^d} h_d(\omega)\phi_X(\omega)d\omega$$

where $h_d(\omega) = (\sum_{l=1}^d \omega_l^2)^2 - \frac{1}{3}\sum\sum_{1\leq l<l'\leq d}\omega_l^2\omega_{l'}^2$.

Example. In the practical case $d = 2$, the spectral density $\phi_X(\omega) = \frac{a_0}{(1+|\omega|^2)^4}$ satisfies the condition of the theorem for $q = 1$. In this case the rate is of order n^{-2} and the asymptotic constant will have $h_2(\omega_1, \omega_2) = \omega_1^4 + \omega_2^4 + \frac{5}{3}\omega_1^2\omega_2^2$.

The rate of convergence for the predictor $\hat{X}(\mathbf{t})$ cannot be faster than $n^{-4/d}$ for smoother process $q > 1$. However the rate is improved by using a more sophisticated linear predictor which is constructed by applying the extension of the Newton's rules formulae to a spatial case up to some appropriate order, see Benhenni and Cambanis (1992), Benhenni (1998) for $d = 1$.

For any positive integer i, let Δ^i be the increment of the i^{th} order operator

$$\Delta^i g\left(t_k\right) = \sum_{r=0}^{i} \binom{i}{r} (-1)^{i-r} g\left(t_{k+r}\right), 0 \leq k+i \leq m.$$

We define recursively the d-dimensional finite difference operator:

$$\Delta^{i_1,\dots,i_d} g(t_{k_1},\dots,t_{k_d}) = \Delta^{i_d}(\Delta^{i_1,\dots,i_{d-1}} g(t_{k_1},\dots,t_{k_{d-1}},.))(t_{k_d}).$$

When $\mathbf{t} \in [\frac{\mathbf{k}}{m}, \frac{(\mathbf{k+1})}{m}[$ for some $\mathbf{k} = (k_1,\dots,k_d) \in \{0,\dots,m-1\}^d$ then the predictor is:

$$\hat{X}(\mathbf{t}) = \sum_{0 \leq i_1,\dots,i_d \leq 2q-1} C_d(\mathbf{t},\mathbf{k},\mathbf{i}) \Delta^{i_1,\dots,i_d} X(t_{\mathbf{k}})$$

where

$$C_d(\mathbf{t},\mathbf{k},\mathbf{i}) = \Pi_{j=1}^{d} \frac{1}{i_j! m^{i_j}} w_{i_j}(m(t^j - t_{k_j}))$$

and $w_i(u) = u(u-1)\dots(u-i+1), i \geq 1, w_0(u) = 1$

Theorem 2. If a systematic sampling design of size $n = m^d$ is used, and if the spectral density ϕ_X of the process X satisfies for $q > 1$

$$\int_{\Re^d} |\omega_i|^{2q} |\omega_j|^{2q} \phi_X(\omega) d\omega < \infty, i,j = 1,\dots,d,$$

then as $m \to \infty$

$$m^{4q}(IMSE) \longrightarrow \frac{1}{(2q)!^2} \int_{\Re^d} h_{d,q}(\omega) \phi_X(\omega) d\omega$$

where $h_{d,q}(\omega) = \sum_{l=1}^{d} \omega_l^{4q} \int_0^1 w_{2q}^2(u) du + 2\sum\sum_{1 \leq l < l' \leq d} \omega_l^{2q} \omega_{l'}^{2q} (\int_0^1 w_{2q}(u) du)^2$

For high dimension d and low regularity q the rate of convergence $n^{-4q/d}$ becomes slower. Therefore in this case the above predictors may not be very efficient. It would then be interesting to study the interplay between the dimensionality and the regularity of some specific processes. It is a harder problem to study whether these predictors are asymptotically optimal within the class of systematic samplings. That is whether the above predictors have the same asymptotic performance as the optimal linear predictors for any fixed sample size n. This is true for $d = 1$ and $q = 1$, see Su and Cambanis (1993).

The same rules can be applied to predict the partial quadratic mean derivatives $X^{(j_1,\ldots,j_d)}(\mathbf{t})$ of the process under stronger condition on the existing derivatives.

If $\mathbf{t} \in [\frac{\mathbf{k}}{m}, \frac{(\mathbf{k+1})}{m}[$ for some $\mathbf{k} = (k_1, \ldots, k_d) \in \{0, \ldots, m-1\}^d$ then the predictor is:

$$\hat{X}^{(j_1,\ldots,j_d)}(\mathbf{t}) = \sum_{j_1 \leq i_1 \leq 2q-1, \ldots, j_d \leq i_d \leq 2q-1} C_d(\mathbf{t}, \mathbf{k}, \mathbf{i}) \Delta^{i_1,\ldots,i_d} X(t_{\mathbf{k}})$$

Corollary. If a systematic sampling design of size $n = m^d$ is used, and if the spectral density ϕ_X of the process X satisfies for $q > 1$

$$\int_{\Re^d} g_d(\omega)|\omega_i|^{2q}|\omega_j|^{2q}\phi_X(\omega)d\omega < \infty,$$

with $i, j = 1, \ldots, d$ where $g_d(\omega) = \Pi_{i=1}^d \omega_i^{j_i}$, then as $m \to \infty$

$$m^{4q}(IMSE) \longrightarrow \int_{\Re^d} g_d(\omega)h_{d,q}(\omega)\phi_X(\omega)d\omega$$

where $h_{d,q}(\omega) = \sum_{l=1}^d \omega_l^{4q} \int_0^1 w_{2q}^2(u)du + 2\sum\sum_{1 \leq l < l' \leq d} \omega_l^{2q}\omega_{l'}^{2q} (\int_0^1 w_{2q}(u)du)^2$.

2 Proofs

Proof of theorem 1

The stationary spatial process $X(\mathbf{t})$, $\mathbf{t} \in [0,1]^d$ can be expressed by Cramer's representation:

$$X(\mathbf{t}) = \int_{\Re^d} e^{i\omega'\mathbf{t}} dW(\omega)$$

where W is a process with orthogonal increments associated to the spectral measure with Radon-Nikodym derivative ϕ_X with respect to the Lebesgue measure. Then the prediction error can be written as:

$$E(X(\mathbf{t}) - \hat{X}(\mathbf{t}))^2 = \int_{\Re^d} |R_d(\mathbf{t},\mathbf{k})|^2 \phi_X(\omega) d\omega$$

where

$$R_d(\mathbf{t},\mathbf{k}) = \exp(-i\omega'\mathbf{t}) - C_d(\mathbf{t},\mathbf{k},\mathbf{j}) \exp(-i\omega' t_{\mathbf{k+1-j}})$$

The Taylor expansion up to order two for every $t^l \in (t_{k_l}, t_{k_l+1})$ in the neighborhood of t_{k_l} gives:

$$\exp(-i\omega_l t^l) = \exp(-i\omega_l t_{k_l})\{1-(i\omega_l)(t^l - t_{k_l}) + (i\omega_l)^2 \frac{(t^l - t_{k_l})^2}{2} + o((t^l - t_{k_l})^2)\}.$$

Then

$$\begin{aligned}
\exp(-i\omega'\mathbf{t}) &= \exp(-i\omega' t_{\mathbf{k}})\{1 - \sum_{l=1}^{d}(i\omega_l)(t^l - t_{k_l}) + \sum_{l=1}^{d}(i\omega_l)^2 \frac{(t^l - t_{k_l})^2}{2} \\
&+ \sum\sum\nolimits_{1 \le l < l' \le d} (i\omega_l)(i\omega_{l'})(t^l - t_{k_l})(t^{l'} - t_{k_{l'}})(1 + o(1))\}.
\end{aligned}$$

Likewise for $\mathbf{t} = t_{\mathbf{k+1-j}}$, we have:

$$\begin{aligned}
\exp(-i\omega' t_{\mathbf{k+1-j}}) &= \exp(-i\omega' t_{\mathbf{k}})\{1 = -\sum_{l=1}^{d}(i\omega_l)\frac{(1-j_l)}{m} + \sum_{l=1}^{d}(i\omega_l)^2 \frac{(1-j_l)^2}{2m^2} \\
&+ \sum\sum\nolimits_{1 \le l < l' \le = d} (i\omega_l)(i\omega_{l'}) \frac{(1-j_l)}{m}\frac{(1-j_{l'})}{m} + o(\frac{1}{m^2})\}.
\end{aligned}$$

The remainder can then be expressed as:

$$R_d(\mathbf{t},\mathbf{k}) = \exp(-i\omega' t_{\mathbf{k}})\{A_d(\mathbf{t},\mathbf{k}) + iB_d(\mathbf{t},\mathbf{k})\},$$

where

$$\begin{aligned}
A_d(\mathbf{t},\mathbf{k}) &= 1 - \sum_{l=1}^{d} \omega_l^2 \frac{(t^l - t_{k_l})^2}{2} + o((t^l - t_{k_l})^2) \\
&- \sum\sum\nolimits_{1 \le l < l' \le d} \omega_l \omega_{l'} (t^l - t_{k_l})(t^{l'} - t_{k_{l'}})(1 + o(1)) \\
&- \sum_{\mathbf{j}=(j_1,\ldots,j_d)\in\{0,1\}^d} C_d(\mathbf{t},\mathbf{k},\mathbf{j}) \\
&\quad \times\{1 - \sum_{l=1}^{d} \omega_l^2 \frac{(1-j_l)^2}{2m^2} - \sum\sum\nolimits_{1 \le l < l' \le d} \omega_l\omega_{l'} \frac{(1-j_l)}{m}\frac{(1-j_{l'})}{m} \\
&+ o(\frac{1}{m^2})\}.
\end{aligned}$$

and

$$B_d(\mathbf{t},\mathbf{k}) = -\sum_{l=1}^{d}\omega_l(t^l - t_{k_l}) + m^d \sum_{\mathbf{j}=(j_1,\dots,j_d)\in\{0,1\}^d} C_d(\mathbf{t},\mathbf{k},\mathbf{j})\sum_{l=1}^{d}\omega_l\frac{(1-j_l)}{m}.$$

Lemma. For every positive integer d, and every $\mathbf{t} = (t^1,\dots,t^d) \in [\frac{\mathbf{k}}{m}, \frac{(\mathbf{k+1})}{m}[$ for some $\mathbf{k} = (k_1,\dots,k_d) \in \{0,\dots,m-1\}^d$ we have:

$$(i) \quad \sum_{\mathbf{j}=(j_1,\dots,j_d)\in\{0,1\}^d} C_d(\mathbf{t},\mathbf{k},\mathbf{j}) = 1.$$

$$(ii) \quad B_d(\mathbf{t},\mathbf{k}) = 0.$$

Proof. (i) and (ii) are proved by using the definition of $C_d(\mathbf{t},\mathbf{k},\mathbf{j})$:

$$\begin{aligned}(i) \sum_{\mathbf{j}=(j_1,\dots,j_d)\in\{0,1\}^d} C_d(\mathbf{t},\mathbf{k},\mathbf{j}) &= m^d\Pi_{i=1}^d \sum_{\mathbf{j}=(j_1,\dots,j_d)\in\{0,1\}^d} (-1)^{j_i}(t^i - t_{k_i+j_i}) \\ &= m^d\Pi_{i=1}^d\{(t^i - t_{k_i}) - (t^i - t_{k_i+1}) = 1.\end{aligned}$$

$$\begin{aligned}(ii) \quad & \sum_{\mathbf{j}=(j_1,\dots,j_d)\in\{0,1\}^d} C_d(\mathbf{t},\mathbf{k},\mathbf{j})\sum_{l=1}^{d}\omega_l\frac{(1-j_l)}{m} \\ &= \sum_{l=1}^{d}\omega_l\{m^d(t^l - t_{k_l}) \sum_{j_i\in\{0,1\}^d, i=1,\dots d} \Pi_{i=1,i\neq l}^d(t^i - t_{k_i+j_i}\} \\ &= \sum_{l=1}^{d}\omega_l\{m^d(t^l - t_{k_l})\frac{1}{m^{d-1}} \\ &= \sum_{l=1}^{d}\omega_l(t^l - t_{k_l}).\end{aligned}$$

from (i).

Concentrating on $A_d(\mathbf{t},\mathbf{k})$, using the Lemma, we have:

$$\begin{aligned}A_d(\mathbf{t},\mathbf{k}) &= 1 - m^d \sum_{(j_1,\dots,j_d)\in\{0,1\}^d} (-1)^{j_1+\dots+j_d}\Pi_{i=1}^d(t^i - t_{k_i+j_i}) \\ &+ \sum_{l=1}^{d}\omega_l^2\frac{(t^l - t_{k_l})}{2}\{-(t^l - t_{k_l}) + m^{d-2}\Pi_{i\neq l}^d(t^i - t_{k_i+j_i})\} \\ &- \sum\sum_{1\leq l<l'\leq d}\omega_l\omega_{l'}(t^l - t_{k_l})(t^{l'} - t_{k_{l'}}) \\ &\times \{1 - m^{d-2} \sum_{(j_1,\dots,j_d)\setminus(j_l,j_{l'})\in\{0,1\}^{d-2}} (-1)^{j_1+\dots+j_d}\Pi_{i\neq l, i\neq l'}^d(t^i - t_{k_i+j_i})\}\end{aligned}$$

Then again using the Lemma with the appropriate orders, we have

$$A_d(\mathbf{t},\mathbf{k}) = -\sum_{l=1}^{d} \frac{\omega_l^2}{2}(t^l - t_{k_l})\{(t^l - t_{k_l}) - \frac{1}{m} + o((t^l - t_{k_l}) - \frac{1}{m})\}.$$

Now we write:

$$\int \cdots \int_{[0,1]^d} A_d^2(\mathbf{t},\mathbf{k})dt = \sum_{(k_1,\dots,k_d)\in\{0,m-1\}^d} \int \cdots \int_{\{[t_{k_l},t_{k_l}+1],l=1,\dots d\}} A_d^2(\mathbf{t},\mathbf{k})(t)dt$$

Using that $\int_{t_{k_l}}^{t_{k_l}+1} (t^l - t_{k_l})^i dt^l = \frac{1}{(i+1)m^{i+1}}$, we have

$$\begin{aligned} & \int \cdots \int_{\{[t_{k_l},t_{k_l+1}],l=1,\dots d\}} A_d^2(\mathbf{t},\mathbf{k})dt \\ = & \frac{1}{360m^{d+4}}\Big(3\sum_{l=1}^{d}\omega_l^4 + 5\sum\sum\nolimits_{1\le l<l'\le d}\omega_l^2\omega_{l'}^2\Big) + o(\frac{1}{m^{d+4}}) \\ = & \frac{1}{360m^{d+4}}\Big(3(\sum_{l=1}^{d}\omega_l^2)^2 - \sum\sum\nolimits_{1\le l<l'\le d}\omega_l^2\omega_{l'}^2\Big) + o(\frac{1}{m^{d+4}}). \end{aligned}$$

The final result follows from the assumption of the theorem.

Proof of theorem 2. Let $f(\mathbf{t}) = \exp(-i\omega'\mathbf{t})$ and

$$\hat{f}(\mathbf{t}) = \sum_{0\le i_1,\dots,i_d\le 2q-1} C_d(\mathbf{t},\mathbf{k},i_1,\dots,i_d)\Delta^{i_1,\dots,i_d} f(t_{\mathbf{k}})$$

Then the prediction error can be written as:

$$E(X(\mathbf{t}) - \hat{X}(\mathbf{t}))^2 = \int_{\Re^d} |f(\mathbf{t}) - \hat{f}(\mathbf{t})|^2 \phi_X(\omega)d\omega.$$

We apply the Newton's formulae for f with respect to each argument up to order $(2q-1)$ and we obtain for any $d \ge 1$:

$$f(\mathbf{t}) = \hat{f}(\mathbf{t}) + \frac{1}{(2q)!m^{2q}} R_{d,q}(\mathbf{t},\mathbf{k})$$

where putting $C_0(\mathbf{t},\mathbf{k}) = 1$

$$\begin{aligned} R_{d,q}(\mathbf{t},\mathbf{k}) = & \sum_{l=1}^{d} \sum_{0\le i_1,\dots,i_d\le 2q-1} C_{l-1}(\mathbf{t},\mathbf{k},i_1,\dots,i_{l-1}) w_{2q}(m(t^l - t_{k_l}) \\ & \Delta^{i_1,\dots,i_{l-1},0,\dots,0}\frac{\partial^{2q} f}{\partial t^l}(t_{k_1},\dots,t_{k_{l-1}},\xi_l,t^{l+1},\dots,t^d). \end{aligned}$$

where $\xi_l \in (t_{k_l}, t^l), l = 1, \ldots, d$. This can also be written as:

$$
\begin{aligned}
& R_{d,q}(\mathbf{t},\mathbf{k}) \\
= & \sum_{l=1}^{d} w_{2q}(m(t^l - t_{k_l})) \frac{\partial^{2q} f}{\partial t^l}(t_{k_1}, \ldots, t_{k_{l-1}}, \xi_l, t^{l+1}, \ldots, t^d) \\
+ & \sum_{l=1}^{d} \sum_{0 \le i_1, \ldots, i_d \le 2q-1, \bigvee_{j=1}^{d}\{i_j \neq 0\}} C_{l-1}(\mathbf{t},\mathbf{k}, i_1, \ldots, i_{l-1}) w_{2q}(m(t^l - t_{k_l})) \\
& \times \Delta^{i_1, \ldots, i_{l-1}, 0, \ldots, 0} \frac{\partial^{2q} f}{\partial t^l}(t_{k_1}, \ldots, t_{k_{l-1}}, \xi_l, t^{l+1}, \ldots, t^d).
\end{aligned}
$$

It is clear that for some $i_j \neq 0$, $C_{l-1}(\mathbf{t},\mathbf{k}, i_1, \ldots, i_{l-1}) w_{2q}(m(t^l - t_{k_l})) = O(\frac{1}{m})$. This implies that:

$$
R_{d,q}(\mathbf{t},\mathbf{k}) = \sum_{l=1}^{d} w_{2q}(m(t^l - t_{k_l})) \frac{\partial^{2q} f}{\partial t^l}(t_{k_1}, \ldots, t_{k_{l-1}}, \xi_l, t^{l+1}, \ldots, t^d) + O(\frac{1}{m})
$$

Now we have that $f(\mathbf{t}) = \Pi_{j=1}^{d} \exp(i\omega_j t^j)$ so that $\frac{\partial^{2q} f}{\partial t^l}(\mathbf{t}) = (i\omega_l)^{2q} f(\mathbf{t})$. Using the Taylor expansion we have: $\frac{\partial^{2q} f}{\partial t^l}(\mathbf{t}) = (i\omega_l)^{2q} \exp(i\sum_{l=1}^{d} \omega_l t_{k_l})\{1 - \sum_{l=1}^{d}(i\omega_l)(t^l - t_{k_l})(1 + o(1))\}$. Moreover since the intermediate points satisfy $\xi_l = t_{k_l} + o(1)$, it follows that:

$$
R_{d,q}(\mathbf{t},\mathbf{k}) = \sum_{l=1}^{d} w_{2q}(m(t^l - t_{k_l}))(i\omega_l)^{2q} \exp(i\sum_{l=1}^{d} \omega_l t_{k_l}) + o(1).
$$

Therefore

$$
|f(\mathbf{t}) - \hat{f}(\mathbf{t})| = \frac{1}{(2q)!m^{2q}} \sum_{l=1}^{d} w_{2q}(m(t^l - t_{k_l}))\omega_l^{2q} + o(\frac{1}{m^{2q}}).
$$

Now we write:

$$
\begin{aligned}
& \int \cdots \int_{[0,1]^d} |f(\mathbf{t}) - \hat{f}(\mathbf{t})|^2 dt \\
= & \sum_{(k_1,\ldots,k_d) \in \{0,m-1\}^d} \int \cdots \int_{\{[t_{k_l}, t_{k_l}+1], l=1,\ldots d\}} |f(\mathbf{t}) - \hat{f}(\mathbf{t})|^2 dt
\end{aligned}
$$

Since $\int_{t_{k_l}}^{t_{k_l}+1} w_{2q}(m(t^l - t_{k_l}))dt^l = \frac{1}{m}\int_0^1 w_{2q}(u)du$, we obtain

$$
\int \cdots \int_{\{[t_{k_l}, t_{k_l+1}], l=1,\ldots d\}} |f(\mathbf{t}) - \hat{f}(\mathbf{t})|^2 dt = \frac{1}{(2q)!^2 m^{4q}} h_{d,q}(\omega) + o(\frac{1}{m^{4q+d}}).
$$

The final result follows from the assumption of the theorem.

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